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STRUCTURE OF GROUP INVARIANTS OF A QUASIPERIODIC FLOW

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ABSTRACT. It is shown that the multiplier representation of the generalized symmetry group of a quasiperiodic flow induces a semidirect product structure on certain group invariants (including the generalized symmetry group) of the flow's smooth conjugacy class.

1. Introduction

The generalized symmetry group, S_ϕ, of a smooth flow $\phi : \mathbb{R} \times T^n \to T^n$ is the collection of all diffeomorphisms of T^n that map the generating vector field of ϕ to a uniformly scaled copy of itself (see next section for definitions). The multiplier representation of S_ϕ is the one-dimensional linear representation

$$\rho_\phi : S_\phi \to \mathbb{R}^* \equiv \mathrm{GL}(\mathbb{R})$$

that takes a generalized symmetry $R \in S_\phi$ to its unique multiplier $\rho_\phi(R)$ (Theorem 2.8 in [5]), the multiplier being the scalar by which the generating vector field of ϕ is uniformly scaled by R. For each subgroup Λ of the multiplier group $\rho_\phi(S_\phi)$, the multiplier representation induces the short exact sequence of groups,

$$\mathrm{id}_{T^n} \to \ker \rho_\phi \to \rho_\phi^{-1}(\Lambda) \xrightarrow{j_\Lambda} \Lambda \to 1,$$

in which id_{T^n} is the identity diffeomorphism of T^n, $\ker \rho_\phi \to \rho_\phi^{-1}(\Lambda)$ is the canonical monomorphism, and $j_\Lambda : \rho_\phi^{-1}(\Lambda) \to \Lambda \cong \rho_\phi^{-1}(\Lambda)/\ker \rho_\phi$ is $\rho_\phi|\rho_\phi^{-1}(\Lambda)$. This short exact sequence indicates that $\rho_\phi^{-1}(\Lambda)$ is a group extension of $\ker \rho_\phi$ by the Abelian group Λ. When ϕ is a quasiperiodic flow on T^n, it will be shown that

(i) every element of $\rho_\phi(S_\phi)$ is a real algebraic integer of degree at most n (Corollary 4.4),
(ii) $\ker \rho_\phi \cong T^n$ (Corollary 4.7),
(iii) every $R \in S_\phi$ with $\rho_\phi(R) = -1$ is an involution (Corollary 4.8),
(iv) $\rho_\phi(S_\phi)$ is isomorphic to an Abelian subgroup of $\mathrm{GL}(n, \mathbb{Z})$ (Theorem 5.3), and
(v) for each subgroup $\Lambda < \rho_\phi(S_\phi)$ there is a splitting map $h_\Lambda : \Lambda \to \rho_\phi^{-1}(\Lambda)$ for the extension (Theorem 5.4).

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The main result (Theorem 5.5) is that

$$\rho_\phi^{-1}(\Lambda) = \ker\rho_\phi \rtimes_\Gamma h_\Lambda(\Lambda)$$

for every $\Lambda < \rho_\phi(S_\phi)$; that is, $\rho_\phi^{-1}(\Lambda)$ is the semidirect product of $\ker\rho_\phi$ by $h_\Lambda(\Lambda)$ corresponding to the conjugating homomorphism $\Gamma : h_\Lambda(\Lambda) \to \mathrm{Aut}(\ker\rho_\phi)$.

2. Multipliers and Quasiperiodic Flows

A generalized symmetry of a (smooth, i.e. C^∞) flow ϕ on the n-torus T^n ($n \geq 2$) is an $R \in \mathrm{Diff}(T^n)$ (the group of smooth diffeomorphisms on T^n) for which there exists an $\alpha \in \mathbb{R}^*$ such that

$$R\phi(t,\theta) = \phi\big(\alpha t, R(\theta)\big) \quad \text{for all } t \in \mathbb{R} \text{ and all } \theta \in T^n.$$

This condition is $R\phi_t = \phi_{\alpha t}R$ for all $t \in \mathbb{R}$, where ϕ_t is the diffeomorphism of T^n defined by $\phi_t(\theta) = \phi(t,\theta)$. A generalized symmetry of ϕ is characterized by its action on the generating vector field X of ϕ, which vector field is defined by

$$X(\theta) = \frac{d}{dt}\phi_t(\theta)\Big|_{t=0}, \quad \theta \in T^n.$$

(In what follows, $\mathbf{T}$ is the tangent functor, and $R_*X = \mathbf{T}RXR^{-1}$ is the push-forward of X by R.)

Theorem 2.1. *An $R \in \mathrm{Diff}(T^n)$ is a generalized symmetry of a flow ϕ on T^n if and only if there exists a unique $\alpha \in \mathbb{R}^*$ such that $R_*X = \alpha X$.*

For the proof of this theorem, see Proposition 1.4 and Lemma 2.7 in [5].

The generalized symmetry group, S_ϕ, of a flow ϕ on T^n is the collection of all the generalized symmetries of ϕ. The Abelian group $F_\phi = \{\phi_t : t \in \mathbb{R}\} \subset \mathrm{Diff}(T^n)$ generated by ϕ is a subgroup of the normal subgroup $\ker\rho_\phi$ of S_ϕ. On the other hand, S_ϕ is the group theoretic normalizer of F_ϕ in $\mathrm{Diff}(T^n)$ (Theorem 2.5 [5]).

The unique α attached to an $R \in S_\phi$ in Theorem 2.1 is $\rho_\phi(R)$, the multiplier of R. An $R \in S_\phi$ with $\rho_\phi(R) = 1$ is known as a (classical) symmetry of ϕ (p.8 [10]); the symmetry group of ϕ is $\ker\rho_\phi = \rho_\phi^{-1}(\{1\})$. An $R \in S_\phi$ with $\rho_\phi(R) = -1$ is called a reversing symmetry (p.4 [10]); if $R^2 = \mathrm{id}_{T^n}$, then R is a reversing involution or a classical time-reversing symmetry of ϕ; the reversing symmetry group of ϕ is $\rho_\phi^{-1}(\{1,-1\})$ (p.8 [10]). An $R \in S_\phi$ with $\rho_\phi(R) \neq \pm 1$, if it exists, is another type of symmetry of ϕ. Two flows ϕ and ψ are smoothly conjugate if and only if there is a $V \in \mathrm{Diff}(T^n)$ such that $V\phi_t = \psi_t V$ for all $t \in \mathbb{R}$. (This is equivalent to $V_*X = Y$ where X is the generating vector field for ϕ, and Y is the generating vector field for ψ.) A flow ϕ on T^n with generating vector field X is quasiperiodic if and only if there exists a $V \in \mathrm{Diff}(T^n)$ such that V_*X is a constant vector field whose coefficients are independent over $\mathbb{Q}$ (see pp.79-80 [7]). (Recall that real numbers $a_1, a_2, \dots, a_n$ are independent over $\mathbb{Q}$ if for $m = (m_1, m_2, \dots, m_n) \in \mathbb{Z}^n$, the equation $\sum_{j=1}^n m_j a_j = 0$ implies that $m_j = 0$ for all $j = 1, 2, \dots, n$.) The frequencies of a quasiperiodic flow ϕ generated by a constant vector field X are the components of X.

Example 2.2. Identify T^3 with $S^1 \times S^1 \times S^1$ where $S^1 = \mathbb{R}/\mathbb{Z}$. Let $\theta = (\theta_1, \theta_2, \theta_3)$ be global coordinates on T^3. The quasiperiodic flow ϕ on T^3 generated by vector field

$$X = \frac{\partial}{\partial\theta_1} + 7^{1/3}\frac{\partial}{\partial\theta_2} + 7^{2/3}\frac{\partial}{\partial\theta_3}$$

is
$$\phi_t(\theta) = \phi(t,\theta_1,\theta_2,\theta_3) = \big(\theta_1 + t, \theta_2 + 7^{1/3}t, \theta_3 + 7^{2/3}t\big),$$
where the addition in the components of ϕ is mod 1. For each $c = (c_1, c_2, c_3) \in T^3$, the translation
$$R_c(\theta_1,\theta_2,\theta_3) = \big(\theta_1 + c_1, \theta_2 + c_2, \theta_3 + c_3\big)$$
of T^3 is a symmetry of ϕ because
$$R_c\phi(t,\theta_1,\theta_2,\theta_3) = \big(\theta_1 + c_1 + t, \theta_2 + c_2 + 7^{1/3}t, \theta_3 + c_3 + 7^{2/3}t\big) = \theta(t, R_c(\theta_1,\theta_2,\theta_3).$$
The involution $N(\theta_1,\theta_2,\theta_3) = (-\theta_1,\theta_2,\theta_3)$ of T^3 is a reversing symmetry of ϕ because
$$N\phi(t,\theta_1,\theta_2,\theta_3) = (-\theta_1 - t, -\theta_2 - 7^{1/3}t, -\theta_3 - 7^{2/3}t\big) = \phi\big(-t, N(\theta_1,\theta_2,\theta_3)\big).$$

Theorem 2.3. *If ϕ is a quasiperiodic, then $\{1,-1\} < \rho_\phi(S_\phi)$.*

Proof. Suppose ϕ is quasiperiodic. Then there is a $V \in \mathrm{Diff}(T^n)$ such that $Y = V_*X$ is a constant vector field. Let ψ be the flow generated by Y. For any $t \in \mathbb{R}$, the diffeomorphism ψ_t satisfies $(\psi_t)_*Y = Y$, so that $1 \in \rho_\psi(S_\psi)$. On the other hand, the map $N : T^n \to T^n$ defined by $N(\theta) = -\theta$ satisfies $N_*Y = -Y$, so that $-1 \in \rho_\psi(S_\psi)$. The flows ϕ and ψ are smoothly conjugate because $Y = V_*X$. This implies that $\rho_\phi(S_\phi) = \rho_\psi(S_\psi)$ (Theorem 4.2 [5]), and so $\{1,-1\} < \rho_\phi(S_\phi)$. □

Theorem 2.4. *If ϕ is quasiperiodic and Λ is a nontrivial subgroup of $\rho_\phi(S_\phi)$, then $\rho_\phi^{-1}(\Lambda)$ is non-Abelian, and hence the generalized symmetry group of ϕ and the reversing symmetry group of ϕ are non-Abelian.*

Proof. Suppose ϕ is quasiperiodic and Λ is a nontrivial subgroup of $\rho_\phi(S_\phi)$. Then there is an $R \in S_\phi$ such that $\alpha = \rho_\phi(R) \neq 1$. Thus $R\phi_1 = \phi_\alpha R$. If $\phi_1 = \phi_\alpha$, then ϕ would be periodic. Thus, $\rho_\phi^{-1}(\Lambda)$ is non-Abelian. By Theorem 2.3, both $\rho_\phi(S_\phi)$ and $\rho_\phi\big(\rho_\phi^{-1}(\{1,-1\})\big)$ contain -1, so that $S_\phi = \rho_\phi^{-1}(\rho_\phi(S_\phi))$ and $\rho_\phi^{-1}(\{1,-1\})$ are both non-Abelian. □

For any $\Lambda < \rho_\phi(S_\phi)$, $\rho_\phi^{-1}(\Lambda)$ is an invariant of the smooth conjugacy class of ϕ in the sense that if ϕ and ψ are smoothly conjugate, then $\rho_\phi^{-1}(\Lambda)$ and $\rho_\psi^{-1}(\Lambda)$ are conjugate subgroups of $\mathrm{Diff}(T^n)$ (Theorem 4.3 [5]). Because a quasiperiodic flow ϕ is smoothly conjugate to a quasiperiodic flow ψ generated by a constant vector field, the group structure of $\mathrm{id}_{T_n} \to \ker\rho_\phi \to \rho_\phi^{-1}(\Lambda) \to \Lambda \to 1$ is determined by that of $\mathrm{id}_{T^n} \to \ker\rho_\psi \to \rho_\psi^{-1}(\Lambda) \to \Lambda \to 1$. Attention is therefore restricted to a quasiperiodic flow ϕ generated by a constant vector field X.

3. Lifting the Generalized Symmetry Equation

The generalized symmetry equation of a flow ϕ on T^n is the equation $R_*X = \alpha X$ that appears in Theorem 2.1. Lifting it from $\mathbf{T}T^n$ to $\mathbf{T}\mathbb{R}^n$, the universal cover of $\mathbf{T}T^n$, requires lifting the diffeomorphism R of T^n to a diffeomorphism of $\mathbb{R}^n$, and lifting the vector field X on T^n to a vector field on $\mathbb{R}^n$. The covering map $\pi : \mathbb{R}^n \to T^n$ is a local diffeomorphism for which
$$\pi(x + m) = \pi(x)$$
for any $x \in \mathbb{R}^n$ and any $m \in \mathbb{Z}^n$. Let $R : T^n \to T^n$ be a continuous map. A *lift* of $R\pi : \mathbb{R}^n \to T^n$ is a continuous map $Q : \mathbb{R}^n \to \mathbb{R}^n$ for which $R\pi = \pi Q$. Since π

is a fixed map, Q is also said to be a lift of R. Any two lifts of R differ by a deck transformation of π, which is a translation of $\mathbb{R}^n$ by an $m \in \mathbb{Z}^n$.

Theorem 3.1. *Let $R : T^n \to T^n$ and $Q : \mathbb{R}^n \to \mathbb{R}^n$. Then Q is a lift of a diffeomorphism R of T^n if and only if Q is a diffeomorphism of $\mathbb{R}^n$ such that* a) *for any $m \in \mathbb{Z}^n$, $Q(x+m) - Q(x)$ is independent of $x \in \mathbb{R}^n$, and* b) *the map $l_Q(m) = Q(x+m) - Q(x)$ is an isomorphism of $\mathbb{Z}^n$.*

The proof of this theorem uses standard arguments in topology, we omit it.

The canonical projections $\tau_{\mathbb{R}^n} : \mathbf{T}\mathbb{R}^n \to \mathbb{R}^n$ and $\tau_{T^n} : \mathbf{T}T^n \to T^n$ are smooth. The former is a lift of the latter,

$$\tau_{T^n}\mathbf{T}\pi = \pi\tau_{\mathbb{R}^n},$$

which lift sends $w \in \mathbf{T}_x\mathbb{R}^n$ to $x \in \mathbb{R}^n$. The covering map $\mathbf{T}\pi : \mathbf{T}\mathbb{R}^n \to \mathbf{T}T^n$ is a local diffeomorphism.A vector field on T^n is a smooth map $Y : T^n \to \mathbf{T}T^n$ such that $\tau_{T^n}Y = \mathrm{id}_{T^n}$. A vector field on $\mathbb{R}^n$ is a smooth map $Z : \mathbb{R}^n \to \mathbf{T}\mathbb{R}^n$ such that $\tau_{\mathbb{R}^n}Z = \mathrm{id}_{\mathbb{R}^n}$.

Lemma 3.2. *If Y is a vector field on T^n, then there is only one lift of Y that is a vector field on $\mathbb{R}^n$.*

Proof. Let $x_0 \in \mathbb{R}^n$, $\theta_0 \in T^n$ be such that $Y\pi(x_0) = Y(\theta_0)$. Let $w_{x_0} \in \mathbf{T}_{x_0}\mathbb{R}^n$ be the only vector such that $\mathbf{T}\pi(w_{x_0}) = Y(\theta_0)$. By the Lifting Theorem (Theorem 4.1, p.143 [6]), there exists a unique lift $Z : \mathbb{R}^n \to \mathbf{T}\mathbb{R}^n$ such that $Y\pi = \mathbf{T}\pi Z$ and $Z(x_0) = w_{x_0}$. It needs only be checked that this Z is a vector field. Because Y is a vector field on T^n, Z is a lift of $Y\pi$, and $\tau_{\mathbb{R}^n}$ is a lift of τ_{T^n}, it follows that

$$\pi(x) = \tau_{T_n}Y\pi(x) = \tau_{T^n}\mathbf{T}\pi Z(x) = \pi\tau_{\mathbb{R}^n}Z(x).$$

So the difference $x - \tau_{\mathbb{R}^n}Z(x)$ is a discrete valued map. Because $\mathbb{R}^n$ is connected, this difference is a constant (see Proposition 4.5, p.10 [6]). This constant is zero because $\tau_{\mathbb{R}^n}Z(x_0) = x_0$, and so $\tau_{\mathbb{R}^n}Z = \mathrm{id}_{\mathbb{R}^n}$. The equation $Y\pi = \mathbf{T}\pi Z$ implies that Z is smooth because π and $\mathbf{T}\pi$ are local diffeomorphisms and because Y is smooth. The choice of the only vector $w \in \mathbf{T}_{x_0+m}\mathbb{R}^n$ for any $0 \neq m \in \mathbb{Z}^n$ such that $\mathbf{T}\pi(w) = Y(\theta_0)$ would lead to a lift Z_m of Y that is not a vector field on $\mathbb{R}^n$ because $\tau_{\mathbb{R}^n}Z_m(x) = x + m$. The collection $\{Z_m : m \in \mathbb{Z}\}$, with $Z_0 = Z$, accounts for all the lifts of Y by the uniqueness of the lift and the uniqueness of the vector w. Therefore Z is the only lift of Y that is a vector field on $\mathbb{R}^n$. □

For a vector field X on T^n, let $\hat{X}$ denote the only lift of X that is a vector field on $\mathbb{R}^n$ as described in Lemma 3.2; $\hat{X}$ satisfies $X\pi = \mathbf{T}\pi\hat{X}$. For a diffeomorphism R of T^n, let $\hat{R}$ be a lift of R; the lift $\hat{R}$ is a diffeomorphism of $\mathbb{R}^n$ (by Theorem 3.1) for which $R\pi = \pi\hat{R}$.

Lemma 3.3. *The only lift of the vector field R_*X on T^n that is a vector field on $\mathbb{R}^n$ is $\hat{R}_*\hat{X}$.*

Proof. A lift of R_*X is $\hat{R}_*\hat{X}$ because

$$\begin{aligned}\mathbf{T}\pi\hat{R}_*\hat{X} &= \mathbf{T}\pi\mathbf{T}\hat{R}\hat{X}\hat{R}^{-1} = \mathbf{T}(\pi\hat{R})\hat{X}\hat{R}^{-1} = \mathbf{T}(R\pi)\hat{X}\hat{R}^{-1}\\ &= \mathbf{T}R\mathbf{T}\pi\hat{X}\hat{R}^{-1} = \mathbf{T}RX\pi\hat{R}^{-1} = \mathbf{T}RXR^{-1}\pi = R_*X\pi.\end{aligned}$$

By definition, $\hat{R}_*\hat{X}$ is a vector field on $\mathbb{R}^n$. By Lemma 3.2, it is the only lift of R_*X that is a vector field on $\mathbb{R}^n$. □

Lemma 3.4. *For any $\alpha \in \mathbb{R}^*$, the only lift of the vector field αX on T^n that is a vector field on $\mathbb{R}^n$ is $\alpha\hat{X}$.*

Proof. A lift of αX is $\alpha\hat{X}$ because $\mathbf{T}\pi(\alpha\hat{X}) = \alpha\mathbf{T}\pi\hat{X} = \alpha X\pi$. Only one lift of αX is a vector field (Lemma 3.2), and $\alpha\hat{X}$ is this lift. □

Theorem 3.5. *Let X be a vector field on T^n, $\hat{X}$ the lift of X that is a vector field on $\mathbb{R}^n$, R a diffeomorphism of T^n, $\hat{R}$ a lift of R, and α a nonzero real number. Then $R_*X = \alpha X$ if and only if $\hat{R}_*\hat{X} = \alpha\hat{X}$.*

Proof. Suppose that $R_*X = \alpha X$. By Lemma 3.3, $\hat{R}_*\hat{X}$ is a lift of R_*X: $\mathbf{T}\pi\hat{R}_*\hat{X} = R_*X\pi$. By Lemma 3.4, $\alpha\hat{X}$ is a lift of αX: $\mathbf{T}\pi(\alpha\hat{X}) = \alpha X\pi$. Then

$$\mathbf{T}\pi\big(\hat{R}_*\hat{X} - \alpha\hat{X}\big) = \big(R_*X - \alpha X\big)\pi = \mathbf{0}_{T^n}\pi,$$

where $\mathbf{0}_{T^n}$ is the zero vector field on T^n. So $\hat{R}_*\hat{X} - \alpha\hat{X}$ is a lift of $\mathbf{0}_{T^n}$. The only lift of $\mathbf{0}_{T^n}$ that is a vector field on $\mathbb{R}^n$ is $\mathbf{0}_{\mathbb{R}^n}$, the zero vector field on $\mathbb{R}^n$. By Lemma 3.3 and Lemma 3.4, the difference $\hat{R}_*\hat{X} - \alpha\hat{X}$ is a vector field on $\mathbb{R}^n$. By Lemma 3.2, $\hat{R}_*\hat{X} - \alpha\hat{X} = \mathbf{0}_{\mathbb{R}^n}$. Thus, $\hat{R}_*\hat{X} = \alpha\hat{X}$.Suppose that $\hat{R}_*\hat{X} = \alpha\hat{X}$. Then

$$\begin{aligned} R_*X\pi &= \mathbf{T}RXR^{-1}\pi = \mathbf{T}RX\pi\hat{R}^{-1} = \mathbf{T}R\mathbf{T}\pi\hat{X}\hat{R}^{-1} \\ &= \mathbf{T}(R\pi)\hat{X}\hat{R}^{-1} = \mathbf{T}(\pi\hat{R})\hat{X}\hat{R}^{-1} = \mathbf{T}\pi\mathbf{T}\hat{R}\hat{X}\hat{R}^{-1} \\ &= \mathbf{T}\pi\hat{R}_*\hat{X} = \mathbf{T}\pi(\alpha\hat{X}) = \alpha\mathbf{T}\pi\hat{X} = \alpha X\pi. \end{aligned}$$

The surjectivity of π implies that $R_*X = \alpha X$. □

4. Solving the Lifted Generalized Symmetry Equation

The lift of $R_*X = \alpha X$ is an equation on $\mathbf{T}\mathbb{R}^n$ of the form $Q_*\hat{X} = \alpha\hat{X}$ for $Q \in \mathrm{Diff}(\mathbb{R}^n)$. With global coordinates $x = (x_1, x_2, \dots, x_n)$ on $\mathbb{R}^n$, the diffeomorphism Q has the form

$$Q(x_1, x_2, \dots, x_n) = (f_1(x_1, x_2, \dots, x_n), \dots, f_n(x_1, x_2, \dots, x_n))$$

for smooth functions $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, n$. Let $\theta = (\theta_1, \theta_2, \dots, \theta_n)$ be global coordinates on T^n such that $\theta_i = x_i \bmod 1$, $i = 1, 2, \dots, n$. If

$$X(\theta) = a_1\frac{\partial}{\partial\theta_1} + a_2\frac{\partial}{\partial\theta_2} + \dots + a_n\frac{\partial}{\partial\theta_n}$$

for constants $a_i \in \mathbb{R}$, $i = 1, \dots, n$, then

$$\hat{X}(x) = a_1\frac{\partial}{\partial x_1} + a_2\frac{\partial}{\partial x_2} + \dots + a_n\frac{\partial}{\partial x_n},$$

so that $Q_*\hat{X} = \alpha\hat{X}$ has the form

$$\sum_{j=1}^{n} a_j\frac{\partial f_i}{\partial x_j} = \alpha a_i,\ \ i = 1, \dots, n.$$

This is an uncoupled system of linear, first order equations which is readily solved for its general solution.

Lemma 4.1. *For real numbers $a_1, a_2, \dots, a_n$ and α with $a_n \neq 0$, the general solution of the system of n linear partial differential equations*

$$\sum_{j=1}^{n} a_j\frac{\partial f_i}{\partial x_j} = \alpha a_i,\ \ i = 1, \dots, n$$

is

$$f_i(x) = \alpha \frac{a_i}{a_n} x_n + h_i\Big(x_1 - \frac{a_1}{a_n} x_n, x_2 - \frac{a_2}{a_n} x_n, \dots, x_{n-1} - \frac{a_{n-1}}{a_n} x_n\Big),$$

for arbitrary smooth functions $h_i : \mathbb{R}^{n-1} \to \mathbb{R}$, $i = 1, \dots, n$.

Proof. For each $i = 1, \dots, n$, consider the initial value problem

$$\begin{gathered}
\sum_{j=1}^{n} a_j \frac{\partial f_i}{\partial x_j} = \alpha a_i \\
x_j(0, s_1, s_2, \dots, s_{n-1}) = s_j \text{ for } j = 1, \dots, n-1 \\
x_n(0, s_1, s_2, \dots, s_{n-1}) = 0 \\
f_i(0, s_1, s_2, \dots, s_{n-1}) = h_i(s_1, s_2, \dots, s_{n-1})
\end{gathered}$$

for parameters $(s_1, s_2, \dots, s_{n-1}) \in \mathbb{R}^{n-1}$ and initial data $h_i : \mathbb{R}^{n-1} \to \mathbb{R}$. Using the method of characteristics (see [9] for example), the solution of the initial value problem in parametric form is

$$\begin{gathered}
x_j(t, s_1, s_2, \dots, s_{n-1}) = a_j t + s_j \text{ for } j = 1, \dots, n-1 \\
x_n(t, s_1, s_2, \dots, s_{n-1}) = a_n t \\
f_i(t, s_1, s_2, \dots, s_{n-1}) = \alpha a_i t + h_i(s_1, s_2, \dots, s_{n-1}).
\end{gathered}$$

The coordinates $(x_1, x_2, \dots, x_n)$ and the parameters $(t, s_1, s_2, \dots, s_{n-1})$ are related by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} a_1 & 1 & 0 & 0 & \dots & 0 \\ a_2 & 0 & 1 & 0 & \dots & 0 \\ a_3 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & 0 & 0 & 0 & \dots & 1 \\ a_n & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} t \\ s_1 \\ s_2 \\ \vdots \\ s_{n-2} \\ s_{n-1} \end{bmatrix}$$

The determinant of the $n \times n$ matrix is $(-1)^n a_n$, which is nonzero by hypothesis. Inverting the matrix equation gives

$$\begin{bmatrix} t \\ s_1 \\ s_2 \\ \vdots \\ s_{n-2} \\ s_{n-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 & 1/a_n \\ 1 & 0 & \dots & 0 & 0 & -a_1/a_n \\ 0 & 1 & \dots & 0 & 0 & -a_2/a_n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & -a_{n-2}/a_n \\ 0 & 0 & \dots & 0 & 1 & -a_{n-1}/a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

Substitution of the expressions for t and the s_i's in terms of the x_i's into

$$f_i(x_1, x_2, \dots, x_n) = \alpha a_i t + h_i(s_1, s_2, \dots, s_{n-1})$$

gives the desired form of the general solution. □

Lemma 4.2. *If $a_1, a_2, \dots, a_n$ are independent over $\mathbb{Q}$, then*

$$J = \Big\{\Big(m_1 - \frac{a_1}{a_n} m_n, \dots, m_{n-1} - \frac{a_{n-1}}{a_n} m_n\Big) : m_1, dots, m_n \in \mathbb{Z}\Big\}$$

is a dense subset of $\mathbb{R}^{n-1}$.

Proof. Suppose $a_1, a_2, \dots, a_n$ are independent over $\mathbb{Q}$. This implies that none of the a_i's are zero. In particular, $a_n \neq 0$. Consider the flow

$$\psi_t(\theta_1, \dots, \theta_{n-1}, \theta_n) = (\theta_1 - (a_1/a_n)t, \dots, \theta_{n-1} - (a_{n-1}/a_n)t, \theta_n - t)$$

on T^n which is generated by the vector field

$$Y = -\frac{a_1}{a_n}\frac{\partial}{\partial\theta_1} - \frac{a_2}{a_n}\frac{\partial}{\partial\theta_2} - \dots - \frac{a_{n-1}}{a_n}\frac{\partial}{\partial\theta_{n-1}} - \frac{\partial}{\partial\theta_n}.$$

The coefficients of Y are independent over $\mathbb{Q}$ because $a_1, a_2, \dots, a_n$ are independent over $\mathbb{Q}$ and

$$m_1a_1 + \dots + m_na_n = 0 \Leftrightarrow -m_1\frac{a_1}{a_n} - \dots - m_{n-1}\frac{a_{n-1}}{a_n} - m_n = 0.$$

So the orbit of ψ through any point $\theta_0 \in T^n$,

$$\gamma_\psi(\theta_0) = \{\psi_t(\theta_0) : t \in \mathbb{R}\},$$

is dense in T^n (Corollary 1, p. 287 [2]).The submanifold

$$P = \{(\theta_1, \dots, \theta_{n-1}, \theta_n) : \theta_n = 0\}$$

of T^n, which is diffeomorphic to T^{n-1}, is a global Poincaré section for ψ because $X(\theta) \notin \mathbf{T}_\theta P$ for every $\theta \in P$ and because $\gamma_\psi(\theta_0) \cap P \neq \emptyset$ for every $\theta_0 \in T^n$. Define the projection $\wp : T^n \to T^{n-1}$ by

$$\wp(\theta_1, \theta_2, \dots, \theta_{n-1}, \theta_n) = (\theta_1, \theta_2, \dots, \theta_{n-1})$$

and the injection $\imath : T^{n-1} \to T^n$ by

$$\imath(\theta_1, \theta_2, \dots, \theta_{n-1}) = (\theta_1, \theta_2, \dots, \theta_{n-1}, 0).$$

The Poincaré map induced on $\wp(P)$ by ψ is given by $\bar\psi = \wp\psi_1\imath$ because $\psi_1(\theta_0) \in P$ when $\theta_0 \in P$. For any $\kappa \in \mathbb{Z}$, $\bar\psi^\kappa = \wp\psi_\kappa\imath$. So, for instance, with $0 = (0, 0, \dots, 0) \in T^n$ and $\bar 0 = \wp(0)$,

$$\wp\big(\gamma_\psi(0) \cap P\big) = \{\bar\psi^\kappa(\bar 0) : \kappa \in \mathbb{Z}\} = \Big\{\Big(-\frac{a_1}{a_n}\kappa, -\frac{a_2}{a_n}\kappa, \dots, -\frac{a_{n-1}}{a_n}\kappa\Big) : \kappa \in \mathbb{Z}\Big\},$$

where for each $i = 1, \dots, n-1$, the quantity $-(a_i/a_n)\kappa$ is taken mod 1. With $\bar\pi : \mathbb{R}^{n-1} \to T^{n-1}$ as the covering map,

$$J = \bar\pi^{-1}\big(\wp(\gamma_\psi(0) \cap P)\big).$$

If $\wp(\gamma_\psi(0) \cap P)$ were dense in $\wp(P)$, then J would be dense in R^{n-1} because $\bar\pi$ is a covering map. (That is, if $\wp(\gamma_\psi(0) \cap P) \cap [0,1)^{n-1}$ is dense in the fundamental domain $[0,1)^{n-1}$ of the covering map $\bar\pi$, then by translation, it is dense in $\mathbb{R}^{n-1}$.)Define $\chi : \mathbb{R} \times T^{n-1} \to T^n$ by

$$\chi(t, \theta_1, \theta_2, \dots, \theta_{n-1}) = \psi\big(t, \imath(\theta_1, \theta_2, \dots, \theta_{n-1})\big).$$

The map χ is a local diffeomorphism by the Inverse Function Theorem because

$$\mathbf{T}\chi = \begin{bmatrix} -a_1/a_n & 1 & 0 & \dots & 0 \\ -a_2/a_n & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1}/a_n & 0 & 0 & \dots & 1 \\ -1 & 0 & 0 & \dots & 0 \end{bmatrix}$$

has determinant of $(-1)^{n+1}$. Let O be a small open subset of $\wp(P)$. For $\epsilon > 0$, the set $O_\epsilon = (-\epsilon, \epsilon) \times O$ is an open subset in the domain of χ. For ϵ small enough, the image $\chi(O_\epsilon)$ is open in T^n because χ is a local diffeomorphism. By the denseness of $\gamma_\psi(0)$ in T^n, there is a point θ_0 in $\chi(O_\epsilon)\cap\gamma_\psi(0)$. By the definition of $\chi(O_\epsilon)$, there is an $\bar\epsilon\in(-\epsilon,\epsilon)$ and a $\bar\theta_0\in O$ such that $\chi(\bar\epsilon,\bar\theta_0)=\theta_0$. Thus $\imath(\bar\theta_0)\in\gamma_\psi(0)$, and so $\wp(\gamma_\psi(0)\cap P)$ intersects O at $\bar\theta_0$. Since O is any small open subset of $\wp(P)$, the set $\wp(\gamma_\psi(0)\cap P)$ is dense in $\wp(P)$. $\square$

Theorem 4.3. *If $\alpha\in\mathbb{R}^*$ and the coefficients of $X=\sum_{i=1}^n a_i\partial/\partial\theta_i$ are independent over $\mathbb{Q}$, then for each $R\in\mathrm{Diff}(T^n)$ that satisfies $R_*X=\alpha X$ there exist $B=(b_{ij})\in\mathrm{GL}(n,\mathbb{Z})$ and $c\in\mathbb{R}^n$ such that*

$$\hat R(x)=Bx+c$$

for $x=(x_1,x_2,\dots,x_n)$, in which

$$b_{in}=\alpha\frac{a_i}{a_n}-\sum_{j=1}^{n-1}b_{ij}\frac{a_j}{a_n},\ i=1,\dots,n.$$

Proof. Suppose that the $a_1,a_2,\dots,a_n$ are independent over $\mathbb{Q}$. For $\alpha\in\mathbb{R}^*$, suppose that $R\in\mathrm{Diff}(T^n)$ is a solution of $R_*X=\alpha X$. A lift $\hat R$ of R is a diffeomorphism of $\mathbb{R}^n$ by Theorem 3.1. The lift of X that is a vector field on $\mathbb{R}^n$ is $\hat X=\sum_{i=1}^n a_i(\partial/\partial x_i)$. By Theorem 3.5, $\hat R$ is a solution of $\hat R_*\hat X=\alpha\hat X$. With global coordinates $(x_1,x_2,\dots,x_n)$ on $\mathbb{R}^n$ write

$$\hat R(x)=(f_1(x_1,\dots,x_n),\dots,f_n(x_1,\dots,x_n)).$$

In terms of this coordinate description, the equation $\hat R_*\hat X=\alpha X$ written out is

$$\sum_{j=1}^n a_j\frac{\partial f_i}{\partial x_j}=\alpha a_i,\ i=1,\dots,n.$$

The independence of the coefficients of $\hat X$ over $\mathbb{Q}$ implies that $a_n\neq0$. By Lemma 4.1, there are smooth functions $h_i:\mathbb{R}^{n-1}\to\mathbb{R}$, $i=1,\dots,n$, such that

$$f_i(x_1,\dots,x_n)=\alpha\frac{a_i}{a_n}x_n+h_i(s_1,s_2,\dots,s_{n-1})$$

where

$$s_i=x_i-\frac{a_i}{a_n}x_n,\ i=1,\dots,n-1.$$

By Theorem 3.1, $\hat R(x+m)-\hat R(x)$ is independent of x for each $m\in\mathbb{R}^n$. This implies for each $i=1,\dots,n$ that

$$\begin{aligned}&f_i(x+m)-f_i(x)\\&=f_i(x_1+m_1,x_2+m_2,\dots,x_n+m_n)-f_i(x_1,x_2,\dots,x_n)\\&=\alpha\frac{a_i}{a_n}m_n+h_i\big(s_1+m_1-\frac{a_1}{a_n}m_n,\dots,s_{n-1}+m_{n-1}-\frac{a_{n-1}}{a_n}m_n\big)-h_i(s_1,\dots,s_{n-1})\end{aligned}$$

is independent of x for every $m=(m_1,m_2,\dots,m_n)\in\mathbb{Z}^n$. This independence means that $f_i(x+m)-f_i(x)$ is a function of m only. So for each $j=1,\dots,n-1$,

$$\begin{aligned}0&=\frac{\partial}{\partial x_j}\big[f_i(x_1+m_1,x_2+m_2,\dots,x_n+m_n)-f_i(x_1,x_2,\dots,x_n)\big]\\&=\frac{\partial h_i}{\partial s_j}\Big(s_1+m_1-\frac{a_1}{a_n}m_n,\dots,s_{n-1}+m_{n-1}-\frac{a_{n-1}}{a_n}m_n\Big)-\frac{\partial h_i}{\partial s_j}\big(s_1,\dots,s_{n-1}\big).\end{aligned}$$

So, in particular

$$\frac{\partial h_i}{\partial s_j}\Big(m_1 - \frac{a_1}{a_n}m_n, \dots, m_{n-1} - \frac{a_{n-1}}{a_n}m_n\Big) = \frac{\partial h_i}{\partial s_j}(0,\dots,0)$$

for all $(m_1,\dots,m_n)\in\mathbb{Z}^n$. By Lemma 4.2, the set

$$\Big\{\Big(m_1 - \frac{a_1}{a_n}m_n, \dots, m_{n-1} - \frac{a_{n-1}}{a_n}m_n\Big) : m_1,\dots,m_n\in\mathbb{Z}\Big\}$$

is dense in $\mathbb{R}^{n-1}$, which together with the smoothness of h_i implies that $\partial h_i/\partial s_j$ is a constant. Let this constant be b_{ij} for $i=1,\dots,n$, $j=1,\dots,n-1$. By Taylor's Theorem,

$$h_i(s_1,\dots,s_{n-1}) = c_i + \sum_{j=1}^{n-1} b_{ij}s_j$$

for constants $c_i\in\mathbb{R}$. Thus,

$$\begin{aligned} f_i(x_1,\dots,x_n) &= c_i + \alpha\frac{a_i}{a_n}x_n + \sum_{j=1}^{n-1} b_{ij}\Big(x_j - \frac{a_j}{a_n}x_n\Big)\\ &= c_i + \sum_{j=1}^{n-1} b_{ij}x_j + \Big(\alpha\frac{a_i}{a_n} - \sum_{j=1}^{n-1} b_{ij}\frac{a_j}{a_n}\Big)x_n. \end{aligned}$$

For each $i=1,2,\dots,n$, set

$$b_{in} = \alpha\frac{a_i}{a_n} - \sum_{j=1}^{n-1} b_{ij}\frac{a_j}{a_n}$$

Then for each $i=1,2,\dots,n$,

$$f_i(x_1,x_2,\dots,x_n) = c_i + \sum_{j=1}^{n} b_{ij}x_j.$$

So $\hat{R}$ has the form $\hat{R}(x) = Bx+c$ where $B=(b_{ij})$ is an $n\times n$ matrix, and $c\in\mathbb{R}^n$.By Theorem 3.1, the map $l_{\hat{R}}(m) = \hat{R}(x+m)-\hat{R}(x)$ is an isomorphism of $\mathbb{Z}^n$. By the formula for f_i derived above,

$$f_i(x_1+m_1,\dots,x_n+m_n) - f_i(x_1,x_2,\dots,x_m) = \sum_{j=1}^{n} b_{ij}m_j$$

for each $i=1,2,\dots,n$. This implies that $l_{\hat{R}}(m) = Bm$. Since $l_{\hat{R}}$ is an isomorphism of $\mathbb{Z}^n$, it follows that $B\in \mathrm{GL}(n,\mathbb{Z})$. $\square$

Theorem 4.3 restricts the search for lifts of generalized symmetries of a quasiperiodic flow on T^n to affine maps on $\mathbb{R}^n$ of the form $Q(x) = Bx+c$ for $B\in\mathrm{GL}(n,\mathbb{Z})$ and $c\in\mathbb{R}^n$. For an affine map of this form, the difference

$$Q(x+m) - Q(x) = B(x+m)+c-(Bx+c) = Bm$$

is independent of x, and the map $l_Q(m) = Q(x+m)-Q(x)$ is an isomorphism of $\mathbb{Z}^n$, so that Q is a lift of a diffeomorphism R on T^n by Theorem 3.1. If Q is a solution of $Q_*\hat{X} = \alpha\hat{X}$, then by Theorem 3.5, R is a solution of $R_*X=\alpha X$, so that by Theorem 2.1, $R\in S_\phi$.The following two corollaries of Theorem 4.3 restrict the possibilities for the multipliers of the generalized symmetries of a quasiperiodic flow on T^n. One restriction employs the notion of an *algebraic integer*, which is a complex number that is a root of a monic polynomial in the polynomial ring $\mathbb{Z}[z]$. If m is the smallest degree of a monic polynomial in $\mathbb{Z}[z]$ for which an algebraic integer is a root, then m is the *degree* of that algebraic integer (Definition 1.1, p.1 [11]).

Corollary 4.4. *If ϕ is a quasiperiodic flow on T^n with generating vector field $X = \sum_{i=1}^n a_i\partial/\partial\theta_i$, then each $\alpha \in \rho_\phi(S_\phi)$ is a real algebraic integer of degree at most n, and $\rho_\phi(S_\phi) \cap \mathbb{Q} = \{1, -1\}$.*

Proof. For each $\alpha \in \rho_\phi(S_\phi)$ (which is real) there is an $R \in S_\phi$ such that $\rho_\phi(R) = \alpha$. By Theorem 4.3 there is a $B \in \mathrm{GL}(n, \mathbb{Z})$ such that $\mathbf{T}\hat{R} = B$. Then by Theorem 2.1 and Theorem 3.5,

$$B\hat{X} = \hat{R}_*\hat{X} = \alpha\hat{X}.$$

So, α is an eigenvalue of B (and $\hat{X}$ is an eigenvector of B.) The characteristic polynomial of B is an n-degree monic polynomial in $\mathbb{Z}[z]$:

$$z^n + d_{n-1}z^{n-1} + \cdots + d_1 z + d_0.$$

Thus α is a real algebraic integer of degree at most n. The value of d_0 is $\det(B)$, which is a unit in $\mathbb{Z}$ (Theorem 3.5, p.351 [8]). The only units in $\mathbb{Z}$ are ± 1. So the only possible rational roots of the characteristic polynomial of B are ± 1 (Proposition 6.8, p.160 [8]). This means that $\rho_\phi(S_\phi) \cap \mathbb{Q} \subset \{1, -1\}$. But $\rho_\phi(S_\phi) \cap \mathbb{Q} \supset \{1, -1\}$ by Theorem 2.3. Thus, $\rho_\phi(S_\phi) \cap \mathbb{Q} = \{1, -1\}$. □

The other restriction on the possibilities for the multipliers of any generalized symmetries of ϕ employs linear combinations over $\mathbb{Z}$ of pair wise ratios of the entries of the "eigenvector" $\hat{X}$ (which entries are the frequencies of ϕ).

Corollary 4.5. *If ϕ is a quasiperiodic flow on T^n with generating vector field $X = \sum_{i=1}^n a_i\partial/\partial\theta_i$, then for any $\alpha \in \rho_\phi(S_\phi)$ there exists a $B = (b_{ij}) \in \mathrm{GL}(n, \mathbb{Z})$ such that*

$$\alpha = \sum_{j=1}^n b_{ij}\frac{a_j}{a_i}, \quad i = 1, \dots, n.$$

Proof. Suppose that $\alpha \in \rho_\phi(S_\phi)$. Then there is an $R \in S_\phi$ such that $\alpha = \rho_\phi(R)$. By Theorem 4.3, there is a $B = (b_{ij}) \in \mathrm{GL}(n, \mathbb{Z})$ such that $\mathbf{T}\hat{R} = B$ with

$$b_{in} = \alpha\frac{a_i}{a_n} - \sum_{j=1}^{n-1} b_{ij}\frac{a_j}{a_n}, \quad i = 1, \dots, n.$$

Solving this equation for α gives

$$\alpha = \sum_{j=1}^n b_{ij}\frac{a_j}{a_i}, \quad i = 1, \dots, n.$$

□

The multiplier group of any quasiperiodic flow ϕ always contains $\{1, -1\}$ as stated in Theorem 2.3. For each $t \in \mathbb{R}$, the diffeomorphism ϕ_t is in S_ϕ by definition. A lift of ϕ_t is $\hat{\phi}_t(x) = Ix + t\hat{X}$, where $I = \delta_{ij}$ is the $n \times n$ identity matrix, so that by Corollary 4.5,

$$\alpha = \sum_{j=1}^n \delta_{ij}\frac{a_j}{a_i} = \frac{a_i}{a_i} = 1$$

for each $i = 1, \ldots, n$. A lift of the reversing involution N defined in the proof of Theorem 2.3 is $\hat{N}(x) = -Ix$, so that by Corollary 4.5,

$$\alpha = -\sum_{j=1}^{n} \delta_{ij} \frac{a_j}{a_i} = -\frac{a_i}{a_i} = -1$$

for each $i = 1, \ldots, n$. Corollary 4.5 enables a complete description of all symmetries and reversing symmetries of ϕ.

Theorem 4.6. *Suppose that ϕ is a quasiperiodic flow on T^n with generating vector field $X = \sum_{i=1}^{n} a_i \partial/\partial\theta_i$. If $\rho_\phi(R) = \pm 1$ for an $R \in S_\phi$, then there is $c \in \mathbb{R}^n$ such that $\hat{R}(x) = \rho_\phi(R)Ix + c$.*

Proof. Let $R \in S_\phi$. By Theorem 4.3 there exists a $B = (b_{ij}) \in \mathrm{GL}(n, \mathbb{Z})$ and a $c \in \mathbb{R}^n$ such that $\hat{R}(x) = Bx + c$. By Corollary 4.5, the entries of B satisfy

$$\rho_\phi(R) = \sum_{j=1}^{n} b_{ij} \frac{a_j}{a_i}$$

for each $i = 1, 2, \ldots, n$. By hypothesis, $\rho_\phi(R) = \pm 1$. Then for each $i = 1, 2, \ldots, n$,

$$b_{i1}a_1 + \cdots + (b_{ii} \mp 1)a_i + \cdots + b_{in}a_n = 0.$$

By the independence of $a_1, a_2, \ldots, a_n$ over $\mathbb{Q}$, $b_{ij} = 0$ when $i \neq j$ and $b_{ii} = \rho_\phi(R)$ for all $i = 1, 2, \ldots, n$. Therefore, $\hat{R}(x) = \rho_\phi(R)Ix + c$. □

Corollary 4.7. *If ϕ is a quasiperiodic flow on T^n, then* $\ker \rho_\phi \cong T^n$.

Proof. Let $R \in S_\phi$ such that $\rho_\phi(R) = 1$. By Theorem 4.6, $\hat{R}(x) = Ix + c$ for some $c \in \mathbb{R}^n$. Now, for any $c \in \mathbb{R}^n$, the $Q \in \mathrm{Diff}(T^n)$ induced by $\hat{Q}(x) = Ix + c$ satisfies $Q_*X = X$ by Theorem 3.5 because $\hat{Q}_*\hat{X} = \hat{X}$. So, by Theorem 2.1, $Q \in \ker \rho_\phi$. Since c is arbitrary, $Q\pi = \pi\hat{Q}$, and $\pi(\mathbb{R}^n) = T^n$, it follows that $\ker \rho_\phi \cong T^n$. □

Corollary 4.8. *If ϕ is a quasiperiodic flow on T^n, then every reversing symmetry of ϕ is an involution.*

Proof. Suppose $R \in S_\phi$ is a reversing symmetry. By Theorem 4.6, $\hat{R}(x) = -Ix + c$ for some $c \in \mathbb{R}^n$, and so $\hat{R}^2(x) = Ix$. This implies that $R^2 = \mathrm{id}_{T^n}$. □

Example 4.9. Recall the quasiperiodic flow ϕ on T^3 and its generating vector field

$$X = \frac{\partial}{\partial\theta_1} + 7^{1/3}\frac{\partial}{\partial\theta_2} + 7^{2/3}\frac{\partial}{\partial\theta_3}$$

from Example 2.2. By Corollary 4.7, the symmetry group of ϕ is exactly the group of translations $\{R_c : c \in T^n\}$ on T^n, where $R_c(\theta) = \theta + c$. By Corollary 4.8, every reversing symmetry of ϕ is an involution. In particular, this implies that the reversing symmetry group of ϕ is a semidirect product of the symmetry group of ϕ by the $\mathbb{Z}_2$ subgroup generated by reversing involution $N(\theta) = -\theta$ (see p.8 in [10]). Are there symmetries of ϕ with multipliers other than ± 1? The $\mathrm{GL}(3, \mathbb{Z})$ matrix

$$B = (b_{ij}) = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 7 & 0 & -2 \end{bmatrix}$$

induces a $Q \in \mathrm{Diff}(T^3)$ by Theorem 3.1. Since

$$\hat{Q}_* \hat{X} = \mathbf{T}\hat{Q}\hat{X} = B\hat{X} = \big(-2 + 7^{1/3}\big)\hat{X},$$

Theorem 3.5 implies that $Q_* X = (-2 + 7^{1/3})X$. Hence, by Theorem 2.1, $Q \in S_\phi$. The number $-2+7^{1/3}$ is $\rho_\phi(Q)$, the multiplier of Q, is an algebraic integer of degree at most 3 by Corollary 4.4, and satisfies

$$-2 + 7^{1/3} = \sum_{j=1}^{3} b_{ij} \frac{a_j}{a_i}, \quad i = 1, 2, 3,$$

by Corollary 4.5. (The matrix B was found by using Theorem 3.1 in [3], a result which characterizes the matrices in $\mathrm{GL}(3,\mathbb{Z})$ inducing generalized symmetries of a quasiperiodic flow generated by a vector field of a certain type, of which X above is.) Since S_ϕ is a group and $\rho_\phi : S_\phi \to \mathbb{R}^*$ is a homomorphism, it follows for each $k \in \mathbb{Z}$ that $Q^k \in S_\phi$ with $\rho_\phi(Q^k) = \big(\rho_\phi(Q)\big)^k = (-2 + 7^{1/3})^k$, and that $NQ^k \in S_\phi$ with $\rho_\phi(NQ^k) = -(-2 + 7^{1/3})^k$.

5. A Splitting Map for the Extension

For a quasiperiodic flow ϕ on T^n, Theorem 4.3 implies that $\mathbf{T}\hat{R} \in \mathrm{GL}(n,\mathbb{Z})$ for every $R \in S_\phi$. Set

$$\Pi_\phi = \{B \in \mathrm{GL}(n,\mathbb{Z}) : \text{ there is } R \in S_\phi \text{ for which } B = \mathbf{T}\hat{R}\},$$

and define a map $\nu_\phi : \Pi_\phi \to \rho_\phi(S_\phi)$ by $\nu_\phi(B) = \rho_\phi(R)$ where $R \in S_\phi$ with $\mathbf{T}\hat{R} = B$.

Lemma 5.1. *If ϕ is a quasiperiodic flow on T^n with generating vector field X, then ν_ϕ is well-defined.*

Proof. Let $B \in \Pi_\phi$, and suppose there are $R, Q \in S_\phi$ with $\mathbf{T}\hat{R} = B = \mathbf{T}\hat{Q}$. Then $RQ^{-1} \in S_\phi$ and $\hat{R}\hat{Q}^{-1}$ is a lift of RQ^{-1} for which $\mathbf{T}(\hat{R}\hat{Q}^{-1}) = BB^{-1} = I$. Hence $\hat{R}\hat{Q}^{-1}(x) = Ix + c$ for some $c \in \mathbb{R}^n$. This implies that $(\hat{R}\hat{Q}^{-1})_*\hat{X} = \hat{X}$, so that by Theorem 3.5, $(RQ^{-1})_* X = X$. By Theorem 2.1, $\rho_\phi(RQ^{-1}) = 1$. Because ρ_ϕ is a homomorphism, $\rho_\phi(R) = \rho_\phi(Q)$. □

Lemma 5.2. *If ϕ is a quasiperiodic flow on T^n with generating vector field X, then Π_ϕ is a subgroup of $\mathrm{GL}(n,\mathbb{Z})$.*

Proof. Let $B, C \in \Pi_\phi$. Then there are $R, Q \in S_\phi$ such that $\mathbf{T}\hat{R} = B$ and $\mathbf{T}\hat{Q} = C$. The latter implies that $\mathbf{T}\hat{Q}^{-1} = (\mathbf{T}\hat{Q})^{-1} = C^{-1}$. Then $BC^{-1} = \mathbf{T}\hat{R}\mathbf{T}\hat{Q}^{-1} = \mathbf{T}(\hat{R}\hat{Q}^{-1})$. The diffeomorphism $x \to \hat{R}\hat{Q}^{-1}x$ of $\mathbb{R}^n$ satisfies conditions a) and b) of Theorem 3.1, and so is a lift of a diffeomorphism V of T^n. Let $\alpha = \rho_\phi(R)$ and $\beta = \rho_\phi(Q)$. Then $\rho_\phi(Q^{-1}) = \beta^{-1}$ because ρ_ϕ is a homomorphism, and so $(\hat{Q}^{-1})_*\hat{X} = \beta^{-1}\hat{X}$. Thus, $\mathbf{T}(\hat{R}\hat{Q}^{-1})\hat{X} = (\hat{R}\hat{Q}^{-1})_*\hat{X} = \alpha\beta^{-1}\hat{X}$. By Theorem 3.5, $V_* X = \alpha\beta^{-1}X$, so that by Theorem 2.1, $V \in S_\phi$. The lifts $\hat{R}\hat{Q}^{-1}$ and $\hat{V}$ of V differ by a deck transformation of π, so that $BC^{-1} = \mathbf{T}(\hat{R}\hat{Q}^{-1}) = \mathbf{T}\hat{V}$. Therefore, $BC^{-1} \in \Pi_\phi$. □

Theorem 5.3. *If ϕ is a quasiperiodic flow on T^n with generating vector field X, then ν_ϕ is an isomorphism and Π_ϕ is an Abelian subgroup of $\mathrm{GL}(n,\mathbb{Z})$.*

Proof. Let $B, C \in \Pi_\phi$. Then there are $R, Q \in S_\phi$ such that $\mathbf{T}\hat{R} = B$ and $\mathbf{T}\hat{Q} = C$. Let $\alpha = \rho_\phi(R)$ and $\beta = \rho_\phi(Q)$. By Theorem 2.1 and Theorem 3.5, $\mathbf{T}\hat{R}\hat{X} = \alpha\hat{X}$ and $\mathbf{T}\hat{Q}\hat{X} = \beta\hat{X}$. By Lemma 5.2, $BC \in \Pi_\phi$, so that there is a $V \in S_\phi$ such that $\mathbf{T}\hat{V} = BC$. Hence, $\hat{V}_*\hat{X} = \mathbf{T}\hat{V}\hat{X} = BC\hat{X} = \alpha\beta\hat{X}$. By Theorem 3.5 and Theorem 2.1, $\rho_\phi(V) = \alpha\beta$. Thus, $\nu_\phi(BC) = \alpha\beta = \nu_\phi(B)\nu_\phi(C)$. By definition, ν_ϕ is surjective, and by Theorem 4.6, $\ker\nu_\phi = \{I\}$. Therefore, ν_ϕ is an isomorphism. The multiplier group $\rho_\phi(S_\phi)$ is Abelian because it is a subgroup of the Abelian group $\mathbb{R}^*$. Thus Π_ϕ is Abelian. □

A splitting map for the short exact sequence,

$$\mathrm{id}_{T^n} \to \ker\rho_\phi \to \rho_\phi^{-1}(\Lambda) \xrightarrow{j_\Lambda} \Lambda \to 1,$$

is a homomorphism $h_\Lambda : \Lambda \to \rho_\phi^{-1}(\Lambda)$ such that $j_\Lambda h_\Lambda$ is the identity isomorphism on Λ. Take for h_Λ the map where for each $\alpha \in \Lambda$, the image $h_\Lambda(\alpha)$ is the diffeomorphism in $\rho_\phi^{-1}(\Lambda)$ induced by the $\mathrm{GL}(n,\mathbb{Z})$ matrix $\nu_\phi^{-1}(\alpha)$.

Theorem 5.4. *If ϕ is a quasiperiodic flow on T^n, then h_Λ is a splitting map for the extension $\mathrm{id}_{T^n} \to \ker\rho_\phi \to \rho_\phi^{-1}(\Lambda) \to \Lambda \to 1$ for each $\Lambda < \rho_\phi(S_\phi)$.*

Proof. For arbitrary $\alpha, \beta \in \Lambda$, set $R = h_\Lambda(\alpha)$, $Q = h_\Lambda(\beta)$, and $V = h_\Lambda(\alpha\beta)$. Then $\hat{R}(x) = \nu_\phi^{-1}(\alpha)x$, $\hat{Q}(x) = \nu_\phi^{-1}(\beta)x$, and $\hat{V}(x) = \nu_\phi^{-1}(\alpha\beta)x$. By Theorem 5.3, ν_ϕ^{-1} is an isomorphism, so that $\hat{V}(x) = \nu_\phi^{-1}(\alpha)\nu_\phi^{-1}(\beta)x$. Because

$$\begin{aligned} h_\Lambda(\alpha)h_\Lambda(\beta)\pi(x) &= RQ\pi(x) = \pi\hat{R}\hat{Q}(x) = \pi\nu_\phi^{-1}(\alpha)\nu_\phi^{-1}(\beta)x \\ &= \pi\nu_\phi^{-1}(\alpha\beta)x = \pi\hat{V}(x) = V\pi(x) = h_\Lambda(\alpha\beta)\pi(x), \end{aligned}$$

and because π is surjective, $h_\Lambda(\alpha)h_\Lambda(\beta) = h_\Lambda(\alpha\beta)$. Let $B = \mathbf{T}\hat{R} = \nu_\phi^{-1}(\alpha)$. Then $\nu_\phi(B) = \rho_\phi(R)$, so that

$$j_\Lambda h_\Lambda(\alpha) = j_\Lambda(R) = \rho_\phi(R) = \nu_\phi(B) = \nu_\phi(\nu_\phi^{-1}(\alpha)) = \alpha.$$

Therefore, h_Λ is a splitting map for the extension. □

Theorem 5.5. *If ϕ is a quasiperiodic flow on T^n, then*

$$\rho_\phi^{-1}(\Lambda) = \ker\rho_\phi \rtimes_\Gamma h_\Lambda(\Lambda)$$

for each $\Lambda < \rho_\phi(S_\phi)$, where $\Gamma : h_\Lambda(\Lambda) \to \mathrm{Aut}(\ker\rho_\phi)$ is the conjugating homomorphism. Moreover, if Λ is a nontrivial subgroup of $\rho_\phi(S_\phi)$, then Γ is nontrivial.

Proof. By Theorem 5.4, h_Λ is a splitting map for the extension

$$\mathrm{id}_{T^n} \to \ker\rho_\phi \to \rho_\phi^{-1}(\Lambda) \xrightarrow{j_\Lambda} \Lambda \to 1.$$

Thus, $\rho_\phi^{-1}(\Lambda) = \big(\ker\rho_\phi\big)\big(h_\Lambda(\Lambda)\big)$ and $\ker\rho_\phi \cap h_\Lambda(\Lambda) = \mathrm{id}_{T^n}$ (Theorem 9.5.1, p.240 [12]). Since $\ker\rho_\phi$ is a normal subgroup of $\rho_\phi^{-1}(\Lambda)$, then $\rho_\phi^{-1}(\Lambda) = \ker\rho_\phi \rtimes_\Gamma h_\Lambda(\Lambda)$ where $\Gamma : h_\Lambda(\Lambda) \to \mathrm{Aut}(\ker\rho_\phi)$ is the conjugating homomorphism (see p.21 in [1]). If Γ is the trivial homomorphism, then $\rho_\phi^{-1}(\Lambda)$ is Abelian since $\ker\rho_\phi$ is Abelian by Corollary 4.7 and $h_\Lambda(\Lambda)$ is Abelian by Theorem 5.3 (see p.21 in [1]). But $\rho_\phi^{-1}(\Lambda)$ is non-Abelian by Theorem 2.4 whenever Λ is a nontrivial subgroup of $\rho_\phi(S_\phi)$. □

Example 5.6. For the quasiperiodic flow ϕ on T^3 with frequencies 1, $7^{1/3}$, and $7^{2/3}$, it was shown in Example 4.9 that $\alpha = -2 + 7^{1/3} \in \rho_\phi(S_\phi)$. The set

$$\Lambda = \{(-1)^j \alpha^k : j \in \{0,1\}, k \in \mathbb{Z}\}$$

is a nontrivial subgroup of $\rho_\phi(S_\phi)$ that is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}$. By Theorem 5.5 and Corollary 4.7,

$$\rho_\phi^{-1}(\Lambda) \cong T^3 \rtimes_\Gamma \big(\mathbb{Z}_2 \times \mathbb{Z}\big),$$

where Γ is the (nontrivial) conjugating homomorphism. In particular, every element of $\rho_\phi^{-1}(\Lambda)$ can be written uniquely as $R_c N^j Q^k$ where $R_c \in \ker\rho_\phi$ is a translation by c on T^n (as defined in Example 2.2), N is the reversing involution (as defined Example 2.2), and Q is the generalized symmetry of ϕ whose multiplier is α (as defined in Example 4.9). Thus

$$\rho_\phi^{-1}(\Lambda) = \{R_c N^j Q^k : c \in T^n, j \in \{0,1\}, k \in \mathbb{Z}\}.$$

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