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UNIQUENESS OF SOLUTIONS TO DIRICHLET PROBLEMS FOR GENERALIZED LAVRENT'EV-BITSADZE EQUATIONS WITH A FRACTIONAL DERIVATIVE

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Abstract. In this article we study the uniqueness of the solution of the Dirichlet problem for an equation of Lavrent'ev-Bitsadze type with a fractional derivative. The equation studied becomes the regular Lavrent'ev-Bitsadze equation when the order of the derivative is an integer.

1. Introduction

We consider the equation

$$Lu \equiv \frac{\partial^2 u}{\partial x^2} - D_{0y}^{\gamma}\frac{\partial u}{\partial y} = 0, \quad 0 < \gamma < 1, \tag{1.1}$$

in the domain $\Omega = \{(x,y) : 0 < x < r, \alpha < y < \beta\}$, $\alpha < 0, \beta > 0$, where D_{0y}^{γ} is the Riemann-Liouville differential operator of order γ [8, p. 37].

In [2, 3] the Dirichlet problem for second order partial differential equations with a Caputo derivative has been studied. The equations become the Laplace equation and a vibrating string equation when the order of differentiation in the equation is an integer. The Dirichlet problem for the Lavrent'ev-Bitsadze equation has been studied in [1, 9].

Here with the *abc* method a uniqueness of the solution to the Dirichlet problem is proved for equation (1.1) in the domain Ω. Uniqueness conditions for the solution of the problem has been found in terms of the upper limits for the zeros of a Mittag-Leffler type function.

Let us set $\Omega^- = \Omega \cap \{y < 0\}$, $\Omega^+ = \Omega \cap \{y > 0\}$. Let the function $u = u(x,y)$ be such that $u \in C^1(\bar{\Omega})$, $u_{xy} \in C(\bar{\Omega}^+)$, $u_{xx}, D_{0y}^{\gamma}u_y \in C(\Omega^- \cup \Omega^+)$, satisfying (1.1) at all points $(x,y) \in \Omega^- \cup \Omega^+$ be a regular solution of equation (1.1) in the domain Ω.

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2. Dirichlet problem

We try to find a regular solution to (1.1), satisfying the conditions

$$u(0,y)=\psi_0(y), \quad u(r,y)=\psi_r(y), \quad \alpha<y<\beta, \tag{2.1}$$

$$u(x,\alpha)=\varphi_\alpha(x), \quad u(x,\beta)=\varphi_\beta(x), \quad 0<x<r. \tag{2.2}$$

where $\psi_0(y)$, $\psi_r(y)$, $\varphi_\alpha(x)$, $\varphi_\beta(x)$ are given functions. We consider the Mittag-Leffler type function

$$E_{\rho,\,\mu}\,(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\rho k+\mu)}, \quad \rho>0,\, \mu\in\mathbb{C}. \tag{2.3}$$

It is known that this function can have only a finite number of real zeros for all $\rho<2$, $\mu\in\mathbb{C}$ [5, p. 372]. Also it is known that the set of real zeros of (2.3) is not empty for $1<\rho<2$, $\mu=\rho$ and $\mu=1$; see [6].

3. Uniqueness

Theorem 3.1. *Let* $t_1=\max\{t\in\mathbb{R}:E_{\nu,\nu}(-t)=0\}$, $t_2=\max\{t\in\mathbb{R}:E_{\nu,1}(-t)=0\}$, $\nu=\gamma+1$, $h=\max\{t_1,t_2\}$ *and*

$$\frac{\beta^\nu}{r^2}\geq\frac{h}{\pi^2}. \tag{3.1}$$

Then the homogeneous Dirichlet problem (1.1), (2.1), (2.2) *has only the trivial solution.*

Proof. First, we consider equation (1.1) in Ω^-. According to the definition of the Riemann-Liouville fractional derivative in Ω^- equation (1.1) can be written as

$$Lu=u_{xx}+\frac{\partial}{\partial y}D_{0y}^{\gamma-1}\frac{\partial}{\partial y}u=0. \tag{3.2}$$

Let

$$\omega^-(y)=(y-\alpha)^\gamma E_{\gamma+1,\gamma+1}\left(\lambda_n(y-\alpha)^{\gamma+1}\right), \quad \lambda_n=\big(\frac{\pi n}{r}\big)^2,$$

be the solution to the Cauchy problem

$$D_{\alpha y}^{\gamma}\frac{dw^-(y)}{dy}+\lambda_n w^-(y)=0,$$

$$\lim_{y\to\alpha}D_{\alpha y}^{\gamma-1}\frac{dw^-(y)}{dy}=1, \quad w^-(\alpha)=0.$$

We multiply (3.2) by $v(x,y)=\omega^-(y)\sin\left(\sqrt{\lambda_n}x\right)$ and rewrite it as

$$vLu=vu_{xx}+v\frac{\partial}{\partial y}D_{0y}^{\gamma-1}\frac{\partial}{\partial y}u=(vu_x-uv_x)_x+uv_{xx}+\left(vD_{0y}^{\gamma-1}u_y\right)_y-v_yD_{0y}^{\gamma-1}u_y.$$

Then we consider the integral

$$\begin{aligned}
&\int_\varepsilon^{r-\varepsilon}\int_{\alpha+\varepsilon}^{-\varepsilon}vLu\,dx\,dy\\
&=\int_{\alpha+\varepsilon}^{-\varepsilon}(vu_x-uv_x)\big|_{x=\varepsilon}^{x=r-\varepsilon}dy+\int_\varepsilon^{r-\varepsilon}\int_{\alpha+\varepsilon}^{-\varepsilon}uv_{xx}\,dx\,dy\\
&\quad+\int_\varepsilon^{r-\varepsilon}[vD_{0y}^{\gamma-1}u_y]\big|_{y=\alpha+\varepsilon}^{y=-\varepsilon}dx-\int_\varepsilon^{r-\varepsilon}\int_{\alpha+\varepsilon}^{-\varepsilon}v_yD_{0y}^{\gamma-1}u_y\,dx\,dy,
\end{aligned} \tag{3.3}$$

where $\varepsilon > 0$.

Since $u(x,y) \in C^1(\bar\Omega)$, we have $D_{0y}^{\gamma-1}u_y \in C(\bar\Omega)$. Therefore, in (3.3) we can make ε tend to zero

$$
\begin{aligned}
0 &= \int_\alpha^0 [v(r,y)u_x(r,y) - u(r,y)v_x(r,y)]dy \\
&\quad - \int_\alpha^0 [v(0,y)u_x(0,y) - u(0,y)v_x(0,y)]dy + \int_0^r \int_\alpha^0 uv_{xx}\,dx\,dy \\
&\quad + \int_0^r \Big(v(x,0)[D_{0y}^{\gamma-1}u_y]_{y=0} - v(x,\alpha)[D_{0y}^{\gamma-1}u_y]_{y=\alpha} \Big)\,dx \\
&\quad - \int_0^r \int_\alpha^0 v_y D_{0y}^{\gamma-1}u_y\,dx\,dy.
\end{aligned} \tag{3.4}
$$

Since $v|_{\{x=0\}\cup\{x=r\}\cup\{y=\alpha\}} = 0$ and $u|_{\{x=0\}\cup\{x=r\}} = 0$ from (3.4) we obtain

$$
0 = \int_0^r \int_\alpha^0 uv_{xx}\,dx\,dy + \int_0^r v(x,0)[D_{0y}^{\gamma-1}u_y]_{y=0}dx - \int_0^r \int_\alpha^0 v_y D_{0y}^{\gamma-1}u_y dx\,dy.
$$

Applying here the formula of fractional integration by parts [8, p. 34],

$$
\int_c^d h(t)D_{dt}^{\delta}g(t)dt = \int_c^d g(t)D_{ct}^{\delta}h(t)dt, \quad \delta \le 0, \tag{3.5}
$$

we can obtain

$$
0 = \int_0^r \int_\alpha^0 uv_{xx}\,dx\,dy + \int_0^r v(x,0)[D_{0y}^{\gamma-1}u_y]_{y=0}dx - \int_0^r \int_\alpha^0 u_y D_{\alpha y}^{\gamma-1}v_y dx\,dy. \tag{3.6}
$$

We substitute the expression

$$
u_y D_{\alpha y}^{\gamma-1}v_y = (uD_{\alpha y}^{\gamma-1}v_y)_y - u\frac{\partial}{\partial y}D_{\alpha y}^{\gamma-1}v_y
$$

in (3.6), then we have

$$
\begin{aligned}
0 &= \int_0^r \int_\alpha^0 u(v_{xx} + D_{\alpha y}^{\gamma}v_y)\,dx\,dy + \int_0^r v(x,0)[D_{0y}^{\gamma-1}u_y]_{y=0}dx \\
&\quad - \int_0^r u(x,0)[D_{\alpha y}^{\gamma-1}v_y]_{y=0}dx + \int_0^r u(x,\alpha)[D_{\alpha y}^{\gamma-1}v_y]_{y=\alpha}dx.
\end{aligned}
$$

Hence, as $u(x,\alpha) = 0$ and $v_{xx} + D_{\alpha y}^{\gamma}v_y = 0$, we have

$$
\int_0^r v(x,0)[D_{0y}^{\gamma-1}u_y]_{y=0}dx - \int_0^r u(x,0)[D_{\alpha y}^{\gamma-1}v_y]_{y=0}dx = 0. \tag{3.7}
$$

In Ω^+ we have

$$
Lu = u_{xx} - \frac{\partial}{\partial y}D_{0y}^{\gamma-1}\frac{\partial}{\partial y}u. \tag{3.8}
$$

Denote by $\omega^+(y) = (\beta - y)^\gamma E_{\gamma+1,\gamma+1}\left(-\lambda_n(\beta-y)^{\gamma+1}\right)$ the solution of the Cauchy problem

$$
\begin{gathered}
D_{\beta y}^{\gamma}\frac{dw^+(y)}{dy} - \lambda_n w^+(y) = 0, \\
\lim_{y\to\beta} D_{\beta y}^{\gamma-1}\frac{dw^+(y)}{dy} = -1, \quad w^+(\beta) = 0.
\end{gathered}
$$

Multiplying (3.8) by $v(x,y)=\omega^+(y)\sin\big(\sqrt{\lambda_n}x\big)$, we obtain

$$vLu=(vu_x-uv_x)_x+uv_{xx}-\big(vD_{0y}^{\gamma-1}u_y\big)_y+v_yD_{0y}^{\gamma-1}u_y.$$

Then

$$\begin{aligned}
&\int_\varepsilon^{r-\varepsilon}\int_\varepsilon^{\beta-\varepsilon} v\,Lu\,dx\,dy\\
&=\int_\varepsilon^{\beta-\varepsilon}(vu_x-uv_x)\big|_{x=\varepsilon}^{x=r-\varepsilon}dy+\int_\varepsilon^{r-\varepsilon}\int_\varepsilon^{\beta-\varepsilon} uv_{xx}\,dx\,dy\\
&\quad-\int_\varepsilon^{r-\varepsilon}(vD_{0y}^{\gamma-1}u_y)\big|_{y=\varepsilon}^{y=\beta-\varepsilon}dx+\int_\varepsilon^{r-\varepsilon}\int_\varepsilon^{\beta-\varepsilon} v_yD_{0y}^{\gamma-1}u_y\,dx\,dy.
\end{aligned}$$

Making ε tend to zero, then in view of $u(0,y)=u(r,y)=0$, $v(0,y)=v(r,y)=0$, $v(x,\beta)=0$, and formula (3.5), we have

$$0=\int_0^r\int_0^\beta uv_{xx}\,dx\,dy+\int_0^r v(x,0)[D_{0y}^{\gamma-1}u_y]_{y=0}dx+\int_0^r\int_0^\beta u_yD_{\beta y}^{\gamma-1}v_y\,dx\,dy. \tag{3.9}$$

Taking into account the equality $u_yD_{\beta y}^{\gamma-1}v_y=(uD_{\beta y}^{\gamma-1}v_y)_y-u\frac{\partial}{\partial y}D_{\beta y}^{\gamma-1}v_y$, and using (3.9), we obtain

$$\begin{aligned}
&\int_0^r\int_0^\beta uv_{xx}\,dx\,dy+\int_0^r v(x,0)[D_{0y}^{\gamma-1}u_y]_{y=0}dx+\int_0^r u(x,\beta)[D_{\beta y}^{\gamma-1}v_y]_{y=\beta}dx\\
&-\int_0^r u(x,0)[D_{\beta y}^{\gamma-1}v_y]_{y=0}dx-\int_0^r\int_0^\beta u\frac{\partial}{\partial y}D_{\beta y}^{\gamma-1}v_y\,dx\,dy=0.
\end{aligned}$$

Hence, $D_{\beta y}^{\gamma}v_y=-\frac{\partial}{\partial y}D_{\beta y}^{\gamma-1}v_y$, $u(x,\beta)=0$, which lead us to

$$\begin{aligned}
&\int_0^r\int_0^\beta u(v_{xx}+D_{\beta y}^{\gamma}v_y)\,dx\,dy+\int_0^r v(x,0)[D_{0y}^{\gamma-1}u_y]_{y=0}dx\\
&-\int_0^r u(x,0)[D_{\beta y}^{\gamma-1}v_y]_{y=0}dx=0.
\end{aligned}\tag{3.10}$$

Note that $v_{xx}+D_{\beta y}^{\gamma}v_y=0$. Therefore, using (3.10), we obtain

$$\int_0^r v(x,0)[D_{0y}^{\gamma-1}u_y]_{y=0}dx-\int_0^r u(x,0)[D_{\beta y}^{\gamma-1}v_y]_{y=0}dx=0. \tag{3.11}$$

Considering that the function $u(x,y)$ satisfies the conditions

$$\lim_{y\to0-}u(x,y)=\lim_{y\to0+}u(x,y),\quad \lim_{y\to0-}D_{0y}^{\gamma-1}u_y=\lim_{y\to0+}D_{0y}^{\gamma-1}u_y,$$

using (3.7) and (3.11), one finds the values of the functions $u(x,0)$ and $[D_{0y}^{\gamma-1}u_y]_{y=0}$.

Now let us consider the system of the algebraic equations

$$\begin{aligned}
&\omega^-(0)u_\gamma-[D_{\alpha y}^{\gamma-1}\frac{d}{dy}\omega^-]_{y=0}u_0=0,\\
&-[D_{\beta y}^{\gamma-1}\frac{d}{dy}\omega^+]_{y=0}u_0+\omega^+(0)u_\gamma=0,
\end{aligned}\tag{3.12}$$

where

$$u_0=\int_0^r u(x,0)\sin(\sqrt{\lambda_n}x)dx,\quad u_\gamma=\int_0^r[D_{0y}^{\gamma-1}u_y]_{y=0}\,\sin(\sqrt{\lambda_n}x)dx.$$

From the definition of the Riemann-Liouville fractional integro-differentiation, it follows that

$$-\frac{d}{dy}w^{+}=D^{1}_{\beta y}w^{+},\quad \frac{d}{dy}w^{-}=D^{1}_{\alpha y}w^{-}.$$

Using the formula of fractional integro-differentiation for the Mittag-Leffler type function,

$$D^{\delta}_{st}|t-s|^{\mu-1}E_{\rho,\,\mu}(\lambda|t-s|^{\rho})=|t-s|^{\mu-\delta-1}E_{\rho,\,\mu-\delta}(\lambda|t-s|^{\rho}),\quad \delta\in\mathbb{R},$$

$\mu>0$, if $\delta\notin\mathbb{N}\cup\{0\}$, and $\mu\in\mathbb{R}$, if $\delta\in\mathbb{N}\cup\{0\}$, then we find that

$$\begin{aligned}D^{\gamma-1}_{\beta y}\frac{d}{dy}\omega^{+}&=-E_{\gamma+1,1}\left(-\lambda_n(\beta-y)^{\gamma+1}\right),\\ D^{\gamma-1}_{\alpha y}\frac{d}{dy}\omega^{-}&=E_{\gamma+1,1}\left(\lambda_n(y-\alpha)^{\gamma+1}\right).\end{aligned}$$

Then the determinant of (3.12) has the form

$$\begin{aligned}\Delta&=\beta^{\gamma}E_{\gamma+1,\gamma+1}\left(-\lambda_n\beta^{\gamma+1}\right)E_{\gamma+1,1}\left(\lambda_n|\alpha|^{\gamma+1}\right)\\&\quad+|\alpha|^{\gamma}E_{\gamma+1,\gamma+1}\left(\lambda_n|\alpha|^{\gamma+1}\right)E_{\gamma+1,1}\left(-\lambda_n\beta^{\gamma+1}\right),\quad n=1,2,\dots\end{aligned}$$

Let us show that $\Delta\neq 0$. Since

$$E_{\gamma+1,1}(\lambda_n|\alpha|^{\gamma+1})>0,\quad E_{\gamma+1,\gamma+1}(\lambda_n|\alpha|^{\gamma+1})>0,$$

the existence of roots of the equation $\Delta=0$ depends on

$$E_{\gamma+1,\gamma+1}(-\lambda_n\beta^{\gamma+1}),\quad E_{\gamma+1,1}(-\lambda_n\beta^{\gamma+1}).$$

Next, we use the asymptotic expansion (2.3) at $\rho\in(1,2)$. As $|z|\to\infty$, [5, p. 219], the following formula holds

$$E_{\rho,\,\mu}(z)=1/\rho z^{(1-\mu)/\rho}e^{z^{1/\rho}}-\sum_{k=1}^{m}z^{-k}/\Gamma(\mu-\rho k)+O\left(|z|^{-m-1}\right),\tag{3.13}$$

for $|\arg z|\le\pi$. When $z\in\mathbb{R}$ and $z\to-\infty$,

$$E_{\rho,\,\mu}(z)=-\sum_{k=1}^{m}z^{-k}/\Gamma(\mu-\rho k)+O(|z|^{-m-1}).\tag{3.14}$$

From (3.13), we obtain the expansions

$$\begin{aligned}E_{\gamma+1,1}(\lambda_n|\alpha|^{\gamma+1})&=\frac{1}{\gamma+1}e^{\lambda_n^{\frac{1}{\gamma+1}}|\alpha|}+O\left(\lambda_n^{-1}\right),\\ E_{\gamma+1,\gamma+1}(\lambda_n|\alpha|^{\gamma+1})&=\frac{1}{\gamma+1}\lambda_n^{-\frac{\gamma}{\gamma+1}}|\alpha|^{-\gamma}e^{\lambda_n^{\frac{1}{\gamma+1}}|\alpha|}+O\left(\lambda_n^{-2}\right).\end{aligned}$$

By (3.14) at $m=2$ and $m=1$ we have the representations

$$\begin{aligned}E_{\gamma+1,\gamma+1}(-\lambda_n\beta^{\gamma+1})&=-\frac{\beta^{-2\gamma-2}}{\lambda_n^2\Gamma(-\gamma-1)}+O\left(\lambda_n^{-3}\right),\\ E_{\gamma+1,1}(-\lambda_n\beta^{\gamma+1})&=\frac{\beta^{-\gamma-1}}{\lambda_n\Gamma(-\gamma)}+O\left(\lambda_n^{-2}\right).\end{aligned}$$

Taking into account $\Gamma(-\gamma)<0$, $\Gamma(-\gamma-1)>0$, we obtain

$$\begin{aligned}&\lim_{n\to\infty}E_{\gamma+1,\gamma+1}(-\lambda_n\beta^{\gamma+1})E_{\gamma+1,1}(\lambda_n|\alpha|^{\gamma+1})=-\infty,\\ &\lim_{n\to\infty}E_{\gamma+1,\gamma+1}(\lambda_n|\alpha|^{\gamma+1})E_{\gamma+1,1}(-\lambda_n\beta^{\gamma+1})=-\infty.\end{aligned}$$

Therefore,

$$
\begin{aligned}
&E_{\gamma+1,\gamma+1}(-\lambda_n\beta^{\gamma+1})E_{\gamma+1,1}(\lambda_n|\alpha|^{\gamma+1})<0, \quad \lambda_n\beta^{\gamma+1}>t_1,\\
&E_{\gamma+1,\gamma+1}(\lambda_n|\alpha|^{\gamma+1})E_{\gamma+1,1}(-\lambda_n\beta^{\gamma+1})<0, \quad \lambda_n\beta^{\gamma+1}>t_2.
\end{aligned}
$$

Next by (3.1), for $\lambda_n\beta^{\gamma+1}\ge h$, $n=1,2,\dots$, we have $\Delta<0$, $n=1,2,\dots$. Thus, from (3.12), it follows that

$$
u_0=0, \quad u_\gamma=0. \tag{3.15}
$$

Since the functions $\{\sin(\frac{\pi n}{r}x)\}$ form a dense system, using (3.15) we conclude that

$$
[D_{0y}^{\gamma-1}u_y]_{y=0}=0, \quad u(x,0)=0, \quad x\in(0,r). \tag{3.16}
$$

With this result we prove that $\Omega^- u=0$. We have

$$
uLu=(uu_x)_x-u_x^2+(uD_{0y}^{\gamma-1}u_y)_y-u_yD_{0y}^{\gamma-1}u_y, \quad (x,y)\in\Omega^-.
$$

We consider the integral

$$
\begin{aligned}
&\int_\varepsilon^{r-\varepsilon}\int_{\alpha+\varepsilon}^{-\varepsilon} uLu\,dx\,dy\\
&=-\int_\varepsilon^{r-\varepsilon}\int_{\alpha+\varepsilon}^{-\varepsilon}(u_x^2+u_yD_{0y}^{\gamma-1}u_y)\,dx\,dy+\int_{\alpha+\varepsilon}^{-\varepsilon}(uu_x)\Big|_{x=\varepsilon}^{x=r-\varepsilon}dy\\
&\quad+\int_\varepsilon^{r-\varepsilon}(uD_{0y}^{\gamma-1}u_y)\Big|_{y=\alpha+\varepsilon}^{y=-\varepsilon}dx,
\end{aligned}
$$

As $Lu=0$, we obtain

$$
\begin{aligned}
&-\int_\varepsilon^{r-\varepsilon}\int_{\alpha+\varepsilon}^{-\varepsilon}(u_x^2+u_yD_{0y}^{\gamma-1}u_y)\,dx\,dy+\int_{\alpha+\varepsilon}^{-\varepsilon}(uu_x)\Big|_{x=\varepsilon}^{x=r-\varepsilon}dy\\
&\quad+\int_\varepsilon^{r-\varepsilon}(uD_{0y}^{\gamma-1}u_y)\Big|_{y=\alpha+\varepsilon}^{y=-\varepsilon}dx=0,
\end{aligned}
$$

Making ε tend to zero we get

$$
\int_0^r\int_\alpha^0(u_x^2+u_yD_{0y}^{\gamma-1}u_y)\,dx\,dy=0. \tag{3.17}
$$

Since the fractional integral operator is positive [4],

$$
\int_0^r\int_\alpha^0 u_yD_{0y}^{\gamma-1}u_y\,dx\,dy\ge 0,
$$

then from (3.17) it follows that $u_x=0$, $u_y=0$. So, $u=\text{const}$ in Ω^-. Namely, due to $u\in C(\bar\Omega)$, we can obtain $u(x,y)=0$ for all $(x,y)\in\Omega^-$.

In Ω^+, we have

$$
D_{0y}^{\gamma-1}u_y\cdot Lu=\frac{\partial}{\partial x}\big(u_xD_{0y}^{\gamma-1}u_y\big)-u_x\frac{\partial}{\partial x}D_{0y}^{\gamma-1}u_y-2^{-1}\frac{\partial}{\partial y}\big(D_{0y}^{\gamma-1}u_y\big)^2. \tag{3.18}
$$

Integrating (3.18), we obtain

$$\begin{aligned} &\int_0^\beta u_x(r,y) D_{0y}^{\gamma-1} u_y(r,y)\,dy - \int_0^\beta u_x(0,y) D_{0y}^{\gamma-1} u_y(0,y)\,dy \\ &- \int_0^r \int_0^\beta u_x D_{0y}^{\gamma-1} u_{yx}\,dx\,dy - \frac{1}{2}\int_0^r \big(D_{0y}^{\gamma-1} u_y\big)^2\Big|_{y=\beta} dx \\ &+ \frac{1}{2}\int_0^r \big(D_{0y}^{\gamma-1} u_y\big)^2\Big|_{y=0} dx = 0. \end{aligned} \tag{3.19}$$

Since $u(0,y) = u(r,y) = 0$, we have $D_{0y}^{\gamma-1} u_y(r,y) = D_{0y}^{\gamma-1} u_y(0,y) = 0$. So from (3.19) it follows that

$$- \int_0^r \int_0^\beta u_x D_{0y}^{\gamma-1} u_{yx}\,dx\,dy - \frac{1}{2}\int_0^r \big(D_{0y}^{\gamma-1} u_y\big)^2\Big|_{y=\beta} dx = 0. \tag{3.20}$$

We have

$$D_{0y}^{\gamma-1} u_{yx} = D_{0y}^{\gamma} u_x - \frac{y^{-\gamma}}{\Gamma(1-\gamma)} u_x(x,0) = D_{0y}^{\gamma} u_x.$$

Substituting the above formula in (3.20), we obtain

$$\int_0^r \int_0^\beta u_x D_{0y}^{\gamma} u_x\,dx\,dy + \frac{1}{2}\int_0^r \big(D_{0y}^{\gamma-1} u_y\big)^2\Big|_{y=\beta} dx = 0. \tag{3.21}$$

Assume $f = D_{0y}^{\gamma} u_x$. Then

$$u_x = D_{0y}^{-\gamma} f + \frac{y^{\gamma-1}}{\Gamma(\gamma)} \lim_{y\to 0} D_{0y}^{\gamma-1} u_x. \tag{3.22}$$

From [7], we know that $\lim_{y\to 0} D_{0y}^{\gamma-1} u_x = \Gamma(\gamma) \lim_{y\to 0} y^{1-\gamma} u_x(x,y)$. Therefore, as $u_x(x,0) = 0$, then $\lim_{y\to 0} D_{0y}^{\gamma-1} u_x = 0$. So from (3.22), it follows that $u_x = D_{0y}^{-\gamma} f$. Thereby,

$$\int_0^r \int_0^\beta u_x D_{0y}^{\gamma} u_x\,dx\,dy = \int_0^r \int_0^\beta f D_{0y}^{-\gamma} f\,dx\,dy \geq 0,$$

accordingly, (3.21) makes possible the conclusion $u_x = 0$, i.e. $u = u(y)$. Then according to (1.1), we have

$$D_{0y}^{\gamma} u_y = 0.$$

Applying the operator $D_{0y}^{-\gamma}$ to both sides of this equation, we have

$$D_{0y}^{-\gamma} D_{0y}^{\gamma} u_y = u_y - \frac{y^{\gamma-1}}{\Gamma(\gamma)} \lim_{y\to 0} D_{0y}^{\gamma-1} u_y = 0.$$

Considering the first formula of (3.16), we have $u_y = 0$. Consequently, $u(x,y) =$ const. As u from the class $C(\bar\Omega^+)$ and $u|_{\partial\Omega^+} = 0$, then $u(x,y) = 0\ \forall (x,y) \in \Omega^+$. Thus, $u(x,y) \equiv 0$ for all points $(x,y) \in \Omega$. This proves the theorem. □

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