

## ASYMPTOTIC BEHAVIOR OF BLOWUP SOLUTIONS FOR HÉNON TYPE PARABOLIC EQUATIONS WITH EXPONENTIAL NONLINEARITY

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ABSTRACT. This article concerns the blow up behavior for the Hénon type parabolic equation with exponential nonlinearity,

$$u_t = \Delta u + |x|^\sigma e^u \quad \text{in } B_R \times \mathbb{R}_+,$$

where  $\sigma \geq 0$  and  $B_R = \{x \in \mathbb{R}^N : |x| < R\}$ . We consider all cases in which blowup of solutions occurs, i.e.  $N \geq 10 + 4\sigma$ . Grow up rates are established by a certain matching of different asymptotic behaviors in the inner region (near the singularity) and the outer region (close to the boundary). For the cases  $N > 10 + 4\sigma$  and  $N = 10 + 4\sigma$ , the asymptotic expansions of stationary solutions have different forms, so two cases are discussed separately. Moreover, different inner region widths in two cases are also obtained.

### 1. INTRODUCTION

**1.1. Background.** In this article, we consider the parabolic problem

$$\begin{aligned} u_t &= \Delta u + \lambda |x|^\sigma e^u \quad \text{in } B_{R'} \times \mathbb{R}_+, \\ u(x, t) &= 0 \quad \text{on } S_{R'} \times \mathbb{R}_+, \\ u(x, 0) &= u_0(x) \quad \text{in } B_{R'}, \end{aligned} \tag{1.1}$$

where  $\sigma \geq 0$ ,  $\lambda > 0$ ,  $B_{R'} = \{x \in \mathbb{R}^N : |x| < R'\}$  is the unit ball with the boundary  $S_{R'} = \{x \in \mathbb{R}^N : |x| = R'\}$ . Problem (1.1) is known in stellar structure and combustion theory [18, 30].

The geometric background of (1.1) is presented in [24, 53]. Let  $M$  be a compact manifold and  $p_i$  ( $i = 1, 2, \dots, k$ ) be the puncture points. If there exists a nonsingular conformal map  $\phi_i$  from  $U_i$  (a neighborhood of  $p_i$ , and  $U_i \cap U_j = \emptyset$ ,  $i \neq j$ ) to  $B$  (the unit ball) such that  $\phi_i(p_i) = 0$  and  $ds^2 = \rho(\phi_i)|\phi_i|^{2\beta_i}|d\phi_i|^2$ , then for any continuous function  $\rho$ , the point  $p_i$  is a conical singularity of order  $\beta_i$  of the metric  $ds^2$ . In particular, a singular Riemannian metric on  $U_i^* = U_i \setminus \{p_i\}$  is defined as

$$ds^2 = |x|^{2\beta_i} ds_0^2, \quad \beta_i > -1, \quad x \in B \setminus \{0\},$$

where  $ds_0$  is the Euclidean metric. We extended to  $M^0 = M \setminus \{p_1, p_2, \dots, p_k\}$  and denote it by a conical metric. Researchers also look at the asymptotic behaviors at

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the singular point  $p_i$  of solutions to the elliptic equation with the conical metric

$$-\Delta_g u = u^p, \quad x \in M^0, \quad (1.2)$$

where  $\frac{N}{N-2} < p \leq \frac{N+2}{N-2}$ . At each singular point  $p_i$ , the conical metric gives the Laplace-Beltrami operator  $\Delta_g$  in the form of

$$\Delta_g = \frac{1}{|x|^{N\beta_i}} \sum_{j=1}^N \frac{\partial}{\partial x_j} \left( |x|^{(N-2)\beta_i} \frac{\partial}{\partial x_j} \right), \quad x \in U_i \setminus \{p_i\}.$$

Hence, (1.2) in  $U_i \setminus \{p_i\}$  can be written as the weighted equation

$$-\operatorname{div}(|x|^\gamma \nabla u) = |x|^\sigma u^p, \quad x \in B \setminus \{0\}, \quad (1.3)$$

where  $\gamma = (N-2)\beta_i$ ,  $\sigma = N\beta_i$ , and  $\sigma - \gamma = 2\beta_i > -2$ . Since the function  $e^u$  can be regarded as the limit of  $u^p$  when  $p \rightarrow \infty$ , thus, (1.1) is a special case of (1.3) when  $\gamma = 0$  and  $p \rightarrow \infty$ . In this article, we discuss the impact of the singularity at  $x = p_i$  (i.e.  $x = 0$ ) on the asymptotic behaviors of the solutions.

The weight  $|x|^\sigma$  is essential. To describe the dynamics of globular cluster of stars, Matukuma [37] in 1930 first proposed the model with weighted term:  $\Delta u = \frac{1}{1+|x|^2} u^p$ . Based on numerical simulations, Hénon [25] in 1973 considered the equation

$$-\Delta u = |x|^\sigma u^p, \quad \sigma > 0, \quad p > 1. \quad (1.4)$$

For the works on the existence and qualitative properties of solutions to (1.4), we refer to [4, 3, 7, 21, 35, 39, 42, 45, 46, 47]. In 1982, Ni [39] first realized that the weight  $|x|^\sigma$  impacts the global homogeneity of (1.4) and widths the critical exponent between existence and nonexistence. Moreover, he proved that (1.4) admits at least one radial solution for  $p \in (1, \frac{N+2+2\alpha}{N+2})$ . Recently, Barboza et al. [4] studied the Hénon-type Dirichlet problem, and proved existence of at least one radial solution using variational methods. Because the weight  $r^\sigma$  is increasing with respect to  $r$ , the classical moving plane method cannot be applied to (1.4), nonradial solutions appear naturally. In [47], Smets et al. proved that ground state is nonradial for  $p \in (1, \frac{N+2}{N-2})$  and sufficiently large  $\sigma$ . The property that  $|x|^\sigma$  prohibits concentration phenomena at zero is applied in the case  $p = \frac{N+2}{N-2}$  for sufficiently large  $\sigma$  in [45], and it is used to obtain multiplicity results in [3]. The problems with weighted exponential source are also studied in [9, 41, 12].

**1.2. Known results.** For the stationary problem of (1.1),

$$\begin{aligned} -\Delta U &= \lambda |x|^\sigma e^U && \text{in } B_{R'}, \\ U(x) &= 0 && \text{on } S_{R'}, \end{aligned} \quad (1.5)$$

there exists a critical value of  $\lambda$  in the form of

$$\lambda^* = \sup \{ \lambda > 0 : (1.5) \text{ admits at least one solution} \}.$$

Thus,

- (1) If  $\lambda > \lambda^*$ , (1.5) admits no solution, that is, any solution of (1.1) blows up in the finite time.
- (2) If  $\lambda < \lambda^*$ , (1.5) admits a bounded classical solution, that is, the solution of (1.1) converges to stationary solution for sufficiently small initial data  $u_0$ .

- (3) If  $\lambda = \lambda^*$ , (1.5) admits the extremal solution  $U^*(x) = U(x, \lambda^*)$ . The behavior of  $U^*(x)$  depends on the dimension  $N$ . When  $N < 10 + 4\sigma$ ,  $U^*$  is bounded, corresponding to the case of a closed spectrum. When  $N \geq 10 + 4\sigma$ , (1.5) admits no classical stationary solution and  $\|U(x, \lambda)\|_\infty \rightarrow \infty$  as  $\lambda \rightarrow \lambda^*$ , corresponding to the case of an open spectrum.

In this paper, we focus on the case  $\lambda = \lambda^*$ . We study the relation between the asymptotic behavior of solutions and dimension  $N$ . The similar relation also holds for the equations with nonlinearity  $\lambda e^u$  and  $\lambda(1+s)^p$  with  $p > 1$  in [18, 30, 33, 40, 11, 6] and  $\frac{\lambda f(x)}{(1-u)^2}$  in [19, 22].

During the past several decades, a lot of works regarding the existence and the asymptotic behavior of blowup solutions have emerged in [1, 2, 10, 57, 51, 44], see also [36, 43, 54, 55, 56, 48, 49, 50] for gradient blowup studies. Results include blowup criteria, blowup locations, blowup rates, and blowup profiles. Blowup rates are usually determined by the self-similar rates, which are related to the scaling invariance of the equation. For the classical semilinear heat equation

$$\begin{aligned} u_t &= \Delta u + u^p \quad \text{in } B_R \times \mathbb{R}_+, \\ u &= 0 \quad \text{on } \partial B_R \times \mathbb{R}_+, \\ u(x, 0) &= u_0(x) \quad \text{in } B_R, \end{aligned} \tag{1.6}$$

the blowup rate of solutions is  $(T-t)^{-\frac{1}{p-1}}$  for  $1 < p < \frac{N+2}{N-2}$ . This result shows that the self-similar rate is not the only rate, there are other rates which are usually referred to as Type II rate, a vast literatures exist [14, 20, 26, 27, 28, 29]. If the large time behavior of solutions is not self-similar, the blowup profiles could be obtained by applying the matched asymptotic method, see [5, 13, 14, 15, 16, 17, 23, 32, 38, 50, 52]. For (1.6), the cases  $p = \frac{N+2}{N-2}$  with  $N = 3, 4, 5$  and  $p \geq p_u = \frac{N-2\sqrt{N-1}}{N-4-2\sqrt{N-1}}$  with  $N > 10$  were studied in [16] and [13], respectively. Galaktionov et al. [13] also considered the semilinear Frank-Kamenetskii equation

$$\begin{aligned} u_t &= \Delta u + e^u \quad \text{in } B_R \times \mathbb{R}_+, \\ u &= 0 \quad \text{on } \partial B_R \times \mathbb{R}_+, \\ u(x, 0) &= u_0(x) \quad \text{in } B_R. \end{aligned} \tag{1.7}$$

It was proved that for  $N > 10$  and  $u_0$  belows the singular stationary solution  $U_s(x)$ , then

$$\|u(x, t)\|_\infty = \alpha_0 t + O(1), \quad t \rightarrow \infty,$$

where  $\alpha_0 = \alpha_0(N) > 0$ . Galaktionov and King [17] studied the critical case  $N = 10$ , and showed that

$$\|u(0, t)\|_\infty = \alpha_0 t + O(\log t), \quad t \rightarrow \infty,$$

where  $\alpha_0$  is given by the first eigenvalue of an associated linear differential operator. Matched asymptotic expansions can also be applied to study other PDE models, see [50, 34, 8, 31, 32, 52].

**1.3. Main results.** Motivated by [13, 17], we consider the blowup rates of solutions of (1.1) in the critical case  $\lambda = \lambda^*$ . By rescaling  $x \mapsto (\lambda^*)^{\frac{1}{2+\sigma}} x$ , (1.1) with  $\lambda = \lambda^*$  can be written as

$$\begin{aligned} u_t &= \Delta u + |x|^\sigma e^u \quad \text{in } B_R \times \mathbb{R}_+, \\ u(x, t) &= 0 \quad \text{on } S_R \times \mathbb{R}_+, \\ u(x, 0) &= u_0(x) \quad \text{in } B_R. \end{aligned} \tag{1.8}$$

It admits the explicit singular stationary solution

$$U_s(x) = \log \frac{(2 + \sigma)(N - 2)}{|x|^{2+\sigma}}, \quad N > 2. \quad (1.9)$$

By the boundary condition, we have

$$R = [(2 + \sigma)(N - 2)]^{\frac{1}{2+\sigma}}.$$

The initial data  $u_0 \in L^1(B_R)$  satisfies

$$u_0(x) \leq U_s(x), \quad u_0(x) \not\equiv U_s(x). \quad (1.10)$$

Our main results read as follows.

**Theorem 1.1.** *Let  $N > 10 + 4\sigma$  and condition (1.10) hold. Then the global solution to (1.8) satisfies*

$$\|u(\cdot, t)\|_{L^\infty(B_R)} = \frac{(2 + \sigma)\lambda_1}{|\gamma_+|} t + O(1), \quad t \rightarrow \infty, \quad (1.11)$$

where

$$\gamma_+ = \frac{1}{2} \left[ 2 - N + \sqrt{(N - 2)(N - 10 - 4\sigma)} \right] < 0,$$

and  $\lambda_1$  is defined in Lemma 2.5 as the first eigenvalue of an associated linearized problem.

**Theorem 1.2.** *Let  $N = 10 + 4\sigma$  and condition (1.10) hold. Then the global solution to (1.8) satisfies*

$$u(0, t) = \frac{\pi_1^2 t}{2(4 + 2\sigma)^{\frac{4}{2+\sigma}}} + O(\log t), \quad t \rightarrow \infty, \quad (1.12)$$

where  $\pi_1$  is the first zero of the zeroth-order Bessel's function, i.e.  $J_0(\pi_1) = 0$ .

Based on Theorems 1.1 and 1.2, we find that the asymptotic behavior of solutions depends on  $N$  and  $\sigma$ . When  $N > 10 + 4\sigma$ , the lower order term of asymptotic expansion of stationary solution is in the form of exponential function times power function, i.e.  $e^{\frac{\mu\gamma_+}{2+\sigma}} r^{\gamma_+}$  (see (2.9)). When  $N = 10 + 4\sigma$ , it is in the form of exponential function times power function times logarithmic function, i.e.  $e^{-2\mu} r^{-4-2\sigma} \log(re^{\frac{\mu}{2+\sigma}})$  (see (2.10)). In the matching process, logarithmical term makes it difficult to estimate the upper bound of solution. Following the general strategy of [17, 20, 23], we find the term which matches the logarithmical term.

In the outer region (away from  $r = 0$ ), the term  $r^\sigma$  is a function with upper and lower bounds and strictly greater than 0, so it is evaluated as a constant. But this is not applied in the inner region (near  $r = 0$ ), the degeneration of weight  $r^\sigma$  will lead some difficulties as follows.

Firstly, when the maximum of the solution is attained, the term  $r^\sigma e^u$  in (1.1) would be removed due to  $u(0, t) = \sup_r u(r, t)$ , which makes it impossible to apply (1.1) to estimate. To solve this problem, we adopt the idea of limit (consider the case  $r \rightarrow 0$ ), and use the inner region width to characterize  $r$ , i.e.  $r \leq C e^{-\frac{u(0, t)}{2+\sigma}}$  ( $r \rightarrow 0$  as  $t \rightarrow \infty$ ). However, when  $\sigma = 0$ , (1.1) can be used directly to obtain the estimate due to the existence of the term  $e^u$ .

Secondly, the weight  $|x|^\sigma$  generates complex calculations in the asymptotic expansion of stationary solution (see Subsection 3.1). When  $N = 10 + 4\sigma$ , the asymptotic expansions in the inner and outer regions do not match directly, see Remark

3.2. To solve this problem, we apply the method in [15] to obtain a more accurate asymptotic expansion (exact coefficients and lower order terms) in the inner region. An ODE of order  $(2+\sigma)$  is obtained during the calculation (see (3.3)). To solve this ODE, we assume the solution has a perturbation term  $\Omega$  (i.e. (3.4)). We apply Taylor expansion to extract the terms with only  $\Omega''$ ,  $\Omega'$  and  $\Omega$ , and then solve the ODE composed of these three terms to obtain the accurate expressions of  $\Omega$  (i.e. (3.8), (3.9)). Later on, we put the expressions of  $\Omega$  into the remaining term  $G(\xi, N, \sigma)$  to verify that  $G(\xi, N, \sigma) \rightarrow 0$  as  $\xi \rightarrow \infty$ . However, when  $\sigma = 0$ , Galaktionov and King [17] obtained a quadratical ODE, which can be solved directly without applying Taylor expansion.

This article is organized as follows. In Section 2, we consider the case  $N > 10+4\sigma$  and prove Theorem 1.1. In Section 3, we study the case  $N = 10 + 4\sigma$  and prove Theorem 1.2.

## 2. THE CASE $N > 10 + 4\sigma$

2.1. **Preliminary estimates.** By (1.10), the maximum principle implies that

$$u(x, t) \leq U_s(x), \quad u(x, t) \not\equiv U_s(x).$$

Set

$$w(x, t) = U_s(x) - u(x, t). \quad (2.1)$$

Clearly,  $w(x, t) \rightarrow 0$  as  $t \rightarrow \infty$ . We find that  $w$  satisfies

$$\begin{aligned} w_t &= \Delta w + \frac{\nu}{|x|^2}(1 - e^{-w}) \quad \text{in } B_R \times \mathbb{R}_+, \\ w(x, t) &= 0 \quad \text{on } S_R \times \mathbb{R}_+, \\ w(x, 0) &= w_0(x) \quad \text{in } B_R, \end{aligned} \quad (2.2)$$

where  $\nu = (2 + \sigma)(N - 2)$  and  $w_0(x) \in L^1(B_R)$ . Applying a standard regularity theory, we deduce that  $w(x, t) \in C^\infty(B_R \setminus \{0\} \times \mathbb{R}_+)$ , it makes sense that  $w_0(x) \in L^2(B_R)$ .

Next, we give useful lemmas which are similar to the ones in [13], the proofs are valid to ours with few modifications, so we omit them.

**Lemma 2.1.** *Let  $N \geq 10 + 4\sigma$ . Then as  $t \rightarrow \infty$ , we have*

$$\|w(\cdot, t)\|_2 \leq ce^{-mt},$$

where  $m = m(N) > 0$ .

By Lemma 2.1, the following lemma provides a linear lower bound of  $u$ , which plays a key role in the inner analysis.

**Lemma 2.2.** *Let  $N \geq 10 + 4\sigma$  and condition (1.10) hold. Then*

$$\|u(\cdot, t)\|_\infty \geq \frac{2m}{N}t(1 + o(1)), \quad (2.3)$$

where  $m = m(N) > 0$ .

2.2. **Asymptotic behavior in the inner region.**

2.2.1. *Properties of radial stationary solutions for  $N \geq 10 + 4\sigma$ .* We consider the radially symmetric stationary equation

$$\Delta U + r^\sigma e^U = 0, \quad U = U(r), \quad r > 0. \quad (2.4)$$

Set

$$S(U) = \Delta U + r^\sigma e^U.$$

Let  $U_0(r)$  be the solution of (2.4) under the assumptions

$$U_0(0) = 0, \quad U_0'(0) = 0.$$

Clearly,  $U_0(r) < 0$  and  $U_0'(r) < 0$  for all  $r > 0$ . In fact, it follows from (2.4) that  $(r^{N-1}U_0')' = -r^{N-1+\sigma}e^{U_0} < 0$ , that is, the term  $r^{N-1}U_0'$  is decreasing with respect to  $r$ . Recalling that  $U_0'(0) = 0$ , we have  $U_0'(r) < 0$ . By  $U_0(0) = 0$ , we obtain  $U_0(r) < 0$ .

When  $N \geq 10 + 4\sigma$ , for  $r > 0$ , we have

$$U_0(r) < U_s(r) = \log \frac{(2 + \sigma)(N - 2)}{|x|^{2+\sigma}}. \quad (2.5)$$

Moreover, if  $N > 10 + 4\sigma$ , then as  $r \rightarrow \infty$ ,

$$U_0(r) = U_s(r) - b_0 r^{\gamma_+} (1 + o(1)), \quad (2.6)$$

where  $b_0 = b_0(\sigma, N) > 0$ ,  $\gamma_+ = \frac{1}{2} \left[ 2 - N + \sqrt{(N - 2)(N - 10 - 4\sigma)} \right] < 0$ . If  $N = 10 + 4\sigma$ , then as  $r \rightarrow \infty$ ,

$$U_0(r) = U_s(r) - b_0 r^{-4-2\sigma} \log r (1 + o(1)) \quad (2.7)$$

where  $b_0 = b_0(\sigma, N) > 0$ . Notice that the asymptotic expansion of  $U_0(r)$  includes the logarithmic term.

For any given  $\mu \in \mathbb{R}$ , let  $U_\mu(r)$  be the solution of (2.4) under the assumptions

$$U_\mu(0) = \mu, \quad U_\mu'(0) = 0.$$

By the scaling invariance of (2.4), we deduce

$$U_\mu(r) = \mu + U_0(re^{\frac{\mu}{2+\sigma}}). \quad (2.8)$$

The maximum principle implies that  $U_\mu(r) < U_s(r)$ . If  $N > 10 + 4\sigma$ , there exists a sufficiently small  $\delta$  such that, for  $r \geq \delta$ ,

$$U_\mu(r) = U_s(r) - b_0 e^{\frac{\mu}{2+\sigma}} r^{\gamma_+} (1 + o(1)), \quad \mu \rightarrow \infty. \quad (2.9)$$

If  $N = 10 + 4\sigma$ , we have for  $r \geq \delta$ ,

$$U_\mu(r) = U_s(r) - b_0 e^{-2\mu} r^{-4-2\sigma} \log(re^{\frac{\mu}{2+\sigma}}) (1 + o(1)), \quad \mu \rightarrow \infty. \quad (2.10)$$

Summing up all the above cases, we find that as  $\mu \rightarrow \infty$ ,

$$U_\mu(r) \rightarrow U_s(r) \quad \text{uniformly on } [\delta, \infty).$$

Moreover, the solution  $U_\mu(r)$  is strictly monotone increasing with respect to  $\mu$  for all  $r \geq 0$ .

In Subsection 3.1, we will give more accurate asymptotic expansions of stationary solutions. In particular, we obtain the lower order term which is in the form of power functions, and the precise expression of coefficient  $b_0$  in (2.10).

2.2.2. *Inner analysis.* We consider the asymptotic behavior of solutions in the inner region, which is a small region near  $x = 0$  for sufficiently large  $t$ . Based on symmetrization and comparison argument, we suppose that  $u = u(r, t) \geq 0$  is symmetric and decreasing with respect to  $r$  for all  $t \geq 0$ . It follows from Lemma 2.2 that

$$\alpha(t) = \sup_r u(r, t) = u(0, t) \rightarrow \infty, \quad t \rightarrow \infty. \quad (2.11)$$

By intersection comparison with stationary, we find that  $\alpha(t)$  is strictly monotonous increasing with respect to  $t$ , that is,

$$\alpha'(t) > 0, \quad t \rightarrow \infty. \quad (2.12)$$

**Theorem 2.3.** *Let  $N \geq 10 + 4\sigma$ . Then as  $t \rightarrow \infty$ ,*

$$u(r, t) = U_{\alpha(t)}(r)(1 + o(1)) \quad (2.13)$$

*uniformly on compact subsets  $\{\xi = re^{\frac{\alpha(t)}{2+\sigma}} \leq C\}$  with  $C > 0$ .*

To prove Theorem 2.3, we introduce the rescaled function  $\theta$ , which satisfies

$$u(r, t) = \alpha(t) + \theta(\xi, \tau), \quad \xi = re^{\frac{\alpha(t)}{2+\sigma}}. \quad (2.14)$$

It follows from (2.11) that

$$\theta(0, \tau) \equiv 0, \quad \theta \leq 0. \quad (2.15)$$

We set the new time variable  $\tau$  in the form of

$$\tau = \int_0^t e^{\frac{2\alpha(s)}{2+\sigma}} ds \rightarrow \infty, \quad t \rightarrow \infty. \quad (2.16)$$

Substituting (2.14) into (1.1), we obtain that  $\theta(\xi, \tau)$  satisfies

$$\theta_\tau = S(\theta) + g(\tau) \left[ \frac{1}{2+\sigma} \theta_\xi \xi + 1 \right], \quad (2.17)$$

where the operator  $S$  is defined in (2.4) and

$$g(\tau) = -\alpha'(t) e^{-\frac{2\alpha(t)}{2+\sigma}} = \left[ \frac{2+\sigma}{2} e^{-\frac{2\alpha(t)}{2+\sigma}} \right]' < 0. \quad (2.18)$$

Equation (2.17) can be viewed as the time-dependent perturbation of (1.1). It follows from (2.18) and (2.16) that

$$\begin{aligned} \int^\infty g(\tau) d\tau &= \int^\infty g(\tau) \frac{d\tau}{dt} dt = \int^\infty -\alpha'(t) e^{-\frac{2\alpha(t)}{2+\sigma}} e^{\frac{2\alpha(t)}{2+\sigma}} dt \\ &= -\int^\infty \alpha'(t) dt = -\infty, \quad \text{i.e. } g \notin L^1(\mathbb{R}_+), \end{aligned}$$

that is, the perturbation  $g(\tau)$  is not integrable in time. However, the following lemma ensures that the perturbation vanishes as  $\tau \rightarrow \infty$ .

**Lemma 2.4.** *It holds  $\lim_{\tau \rightarrow \infty} g(\tau) = 0$ .*

*Proof.* We claim that  $g(\tau)$  is uniformly bounded on compact subsets of  $\xi$ . In fact, by (2.11) and (2.14),

$$\alpha'(t) = u_t(0, t) \leq \lim_{r \rightarrow 0} r^\sigma e^u \leq \lim_{r \rightarrow 0} r^\sigma e^{\alpha(t)} = \lim_{r \rightarrow 0} \xi^\sigma e^{-\frac{\sigma \alpha(t)}{2+\sigma}} e^{\alpha(t)} \leq C^\sigma e^{\frac{2\alpha(t)}{2+\sigma}}$$

on any compact subsets of  $\xi$ . It follows from (2.18) that  $|g(\tau)| \leq C^\sigma$ .

Next, we prove that  $g(\tau) \rightarrow 0$  as  $\tau \rightarrow \infty$  by contradiction. Since  $g(\tau)$  is uniformly bounded, we may assume that there exists a sequence  $\{\tau_k\}_{k \in \mathbb{N}} \rightarrow \infty$  such that

$g(\tau_k) \rightarrow -\gamma_0 < 0$ . By the standard regularity, we deduce that  $\theta(\cdot, \tau_k + s) \rightarrow h(\cdot, s)$ , where  $h$  satisfies

$$h_s = S(h) - \gamma_0 \left[ \frac{1}{2+\sigma} h_\xi \xi + 1 \right], \quad s \geq 0, \quad (2.19)$$

and

$$h(0, s) \equiv 0, \quad h(\xi, s) \leq U_s(\xi).$$

Let  $V_0$  be the solution of the stationary problem of (2.19), that is,

$$S(V_0) - \gamma_0 \left[ \frac{1}{2+\sigma} V_{0\xi} \xi + 1 \right] = 0, \quad V_0(0) = 0.$$

The maximum principle implies that  $V_0 \leq U_s$ . The function  $V_0$  comes from the self-similar solution  $u_*$  of (1.1) with finite time  $T$  in the form of

$$u_*(x, t) = -\frac{2+\sigma}{2} \log \left[ \frac{2\gamma_0}{2+\sigma} (T-t) \right] + V_0(\eta), \quad \eta = \frac{x}{\sqrt{\frac{2\gamma_0}{2+\sigma} (T-t)}}. \quad (2.20)$$

We shall obtain that  $V_0$  must intersect  $U_s$  by contradiction. Assume that  $V_0 < U_s$ . It follows from (1.10) and the maximum principle that  $u_* \leq U_s$  and  $u_* \not\equiv U_s$ . Recalling (2.20),  $V_0 < U_s$  and the fact  $-\frac{2+\sigma}{2} \log \left[ \frac{2\gamma_0}{2+\sigma} (T-t) \right] \rightarrow +\infty$  as  $t \rightarrow T$ , we deduce that  $u_* \leq U_s$  is not valid. Thus,  $g(\tau)$  vanishes at infinity.  $\square$

*Proof of Theorem 2.3.* Fix a sequence  $\{\tau_k\}_{k \in \mathbb{N}} \rightarrow \infty$ . Let  $\theta = \theta(\cdot, \tau_k + s)$ . Applying the standard interior regularity, we obtain that  $\theta, \theta_\xi, \theta_{\xi\xi}, \theta_\tau, \theta_{\tau\xi}$  are uniformly bounded and  $\theta(\cdot, \tau_k + s) \rightarrow f(\cdot, s)$  uniformly on any compact sets of  $\xi$ . Then  $f$  satisfies the limit equation of (2.17), i.e.

$$f_s = S(f) \quad \text{in } \mathbb{R}_+ \times \mathbb{R}_+, \quad (2.21)$$

$$f(0, s) = 0, \quad f \leq 0. \quad (2.22)$$

Since coefficients in (2.21) are analytic, by the standard regularity theory, we know that  $f$  is a  $C^\infty$  function and analytic in  $\xi$ . Notice that (2.13) is equivalent to

$$f(\xi, s) \equiv U_0(\xi).$$

We proceed by contradiction. Assume that  $f(\xi, s) \not\equiv U_0(\xi)$ . By (2.22) and  $U_0(0) = 0$ , we have that  $f(\xi, s)$  intersects  $U_0(\xi)$  infinitely many times for all  $s \geq 0$ . On the other hand, based on the Sturmian argument, the number of intersections between  $\theta(\xi, \tau)$  and  $U_0(\xi)$  cannot increase with respect to time, and it is finite initially since the solutions are analytic. This leads to a contradiction.  $\square$

The following lemma shows that the stabilization in (2.13) is from above, see details in [13, Lemma 3.2].

**Lemma 2.5.** *Let  $N \geq 10 + 4\sigma$ . Then*

$$\theta(\xi, t) \geq U_0(\xi), \quad t \rightarrow \infty. \quad (2.23)$$

**2.3. Asymptotic behavior in the outer region.** We consider the asymptotic behavior of solutions in the outer region, which is the region away from  $x = 0$  for  $t$  sufficiently large.

2.3.1. *Linearized analysis.* (2.2) can be written as

$$w_t = -\mathcal{A}w - F(w), \quad (2.24)$$

where  $\mathcal{A}$  is the linear operator in the form of

$$\mathcal{A}w = -\Delta w - \frac{\nu}{|x|^2}w \quad (2.25)$$

with  $\nu = (2 + \sigma)(N - 2)$ , and  $F$  is the nonlinear operator in the form of

$$F(w) = \frac{\nu}{|x|^2}(e^{-w} - 1 + w) \geq 0 \quad \text{for } w \geq 0. \quad (2.26)$$

The corresponding radially homogeneous problem of (2.24) is given by

$$\mathcal{A}\psi = 0 \quad \text{in } B_R.$$

If  $N > 10 + 4\sigma$ , we obtain two linearly independent solutions:

$$\psi_+ = r^{\gamma_+} \quad \text{and} \quad \psi_- = r^{\gamma_-},$$

where

$$\gamma_{\pm} = \frac{1}{2} \left[ 2 - N \pm \sqrt{(N-2)(N-10-4\sigma)} \right] < 0,$$

which are roots of the quadratic equation

$$\gamma^2 + (N-2)\gamma + (2+\sigma)(N-2) = 0.$$

If  $N = 10 + 4\sigma$ , we obtain two linearly independent solutions:

$$\tilde{\psi}_+ = r^{-4-2\sigma} \quad \text{and} \quad \tilde{\psi}_- = r^{-4-2\sigma} \log\left(\frac{r}{4+2\sigma}\right).$$

Let  $\psi_1$  and  $\tilde{\psi}_1$  be the first eigenfunctions of operate  $\mathcal{A}$  in the cases  $N > 10 + 4\sigma$  and  $N = 10 + 4\sigma$ , respectively. As shown in [13, 17], we find, if  $N > 10 + 4\sigma$ ,

$$\psi_1 = ar^{\gamma_+}(1 + o(1)), \quad r \rightarrow 0, \quad (2.27)$$

and if  $N = 10 + 4\sigma$ ,

$$\tilde{\psi}_1 = \tilde{a}r^{-4-2\sigma}(1 + o(1)), \quad r \rightarrow 0, \quad (2.28)$$

where  $a = a(\sigma, N) > 0$ ,  $\tilde{a} = \tilde{a}(\sigma, N) > 0$ .

We state some properties of symmetric Sturm-Liouville operator  $\mathcal{A}$ . The proof can be referred to [13, Lemma 4.1] and [17, Lemma 2.2].

**Lemma 2.6.** *The operator  $\mathcal{A}$  defined in (2.25) satisfies the following properties:*

- (i) *If  $N > 10 + 4\sigma$ , the operator  $\mathcal{A}$  admits a unique self-adjoint Friedrichs extension which is positive definite with a purely discrete spectrum. Moreover, the first eigenvalue is strictly positive and satisfies*

$$\lambda_1 \geq m = \mu_1 \frac{N - (10 + 4\sigma)}{N - 2} > 0,$$

where  $\mu_1 > 0$  is the first eigenvalue of  $-\Delta$  in  $B_R$ .

- (ii) *If  $N = 10 + 4\sigma$ , the operator  $\mathcal{A}$  admits a unique self-adjoint Friedrichs extension with a purely discrete spectrum of simple eigenvalues  $\sigma(\mathcal{A}) = \{\dots < \tilde{\lambda}_2 < \tilde{\lambda}_1 < 0\}$ . Moreover, the orthonormal set of eigenfunctions  $\{\tilde{\psi}_k\}$  for  $\mathcal{A}$  is complete.*

2.3.2. *Outer analysis.* The following result gives the asymptotic behavior in the outer region, which is proved as in [13, Lemma 5.1].

**Theorem 2.7.** *Assume  $N > 10 + 4\sigma$ . Let  $\lambda_1$  and  $\psi_1$  be the first eigenpair of operator  $\mathcal{A}$ . Then there exists a positive constant  $C_0 = C_0(u_0)$  such that as  $t \rightarrow \infty$ ,*

$$w(r, t) = C_0 e^{-\lambda_1 t} \psi_1(r) (1 + o(1)) \quad \text{uniformly in } \{\delta \leq |x| \leq R\}, \quad (2.29)$$

where  $\delta > 0$ .

2.4. **Matching process.** Combining Subsections 2.2 and 2.3, we obtain the desired result by matching the asymptotic behaviors of solutions in the inner and outer regions.

*Proof of Theorem 1.1.* The proof is divided into three steps. In step 1, we observe the formal matching expansion of the global solution for (1.1). In steps 2 and 3, we give the analytic proof. Step 2 gives the estimate of upper bound. Step 3 gives the estimate of the lower bound.

**Step 1:**  $\alpha(t) \approx \frac{(2+\sigma)\lambda_1}{|\gamma_+|} t + O(1)$  as  $t \rightarrow \infty$ . Set  $r = \delta$  with  $\delta \ll 1$ . By (2.13) and (2.9), the function  $w = U_s - u$  satisfies

$$w(\delta, t) \approx b_0 e^{\frac{\alpha(t)\gamma_+}{2+\sigma}} \delta^{\gamma_+}, \quad t \rightarrow \infty. \quad (2.30)$$

Substituting (2.27) into (2.29), we obtain

$$w(\delta, t) = aC_0 e^{-\lambda_1 t} \delta^{\gamma_+} (1 + o(1)), \quad t \rightarrow \infty. \quad (2.31)$$

Combining (2.30) and (2.31), we deduce

$$\alpha(t) = \|u(x, t)\|_\infty \approx \frac{(2+\sigma)\lambda_1}{|\gamma_+|} t + O(1), \quad t \rightarrow \infty.$$

Thus (1.11) is valid.

**Step 2:**  $\alpha(t) \leq \frac{(2+\sigma)\lambda_1}{|\gamma_+|} t + O(1)$  as  $t \rightarrow \infty$ . For a fixed positive  $r \ll 1$ , by (2.23), (2.8), and (2.9), we obtain

$$u(r, t) \geq U_s(r) - b_0 e^{\frac{\alpha(t)\gamma_+}{2+\sigma}} r^{\gamma_+} (1 + o(1)), \quad t \rightarrow \infty. \quad (2.32)$$

It follows from (2.31) and (2.32) that

$$e^{\frac{\alpha(t)\gamma_+}{2+\sigma}} \geq \frac{aC_0}{b_0} e^{-\lambda_1 t} (1 + o(1)), \quad t \rightarrow \infty. \quad (2.33)$$

Based on the continuity and positivity of  $e^x$ , there exists  $C' > 0$  such that  $\frac{aC_0}{b_0} = e^{C'}$ . By (2.33), we find

$$\alpha(t) \leq \frac{(2+\sigma)\lambda_1}{|\gamma_+|} t + \frac{C'(2+\sigma)}{\gamma_+}, \quad t \rightarrow \infty.$$

Therefore,

$$\alpha(t) \leq \frac{(2+\sigma)\lambda_1}{|\gamma_+|} t + O(1), \quad t \rightarrow \infty. \quad (2.34)$$

**Step 3:**  $\alpha(t) \geq \frac{(2+\sigma)\lambda_1}{|\gamma_+|} t + O(1)$  as  $t \rightarrow \infty$ . A direct calculation shows that  $\bar{w}(r, t) = C_0 e^{-\lambda_1(t-T)} \psi_1(r)$  is a supersolution of (2.24), where we fix  $T \gg 1$  such

that  $w_0(r) = U_s(r) - u_0(r) \leq \bar{w}(r, 0)$ . Hence,  $\underline{u}(r, t) = U_s(r) - \bar{w}(r, t)$  is a subsolution of (1.1), i.e.

$$u(r, t) \geq \underline{u}(r, t) \quad \text{in } B_R \times \mathbb{R}_+.$$

By the monotonicity of  $u(r, t)$  with respect to  $r$ , we deduce for  $t \geq T$ ,

$$\alpha(t) = \sup_r u(r, t) \geq \sup_r \underline{u}(r, t) \equiv \frac{(2 + \sigma)\lambda_1}{|\gamma_+|} t + O(1), \tag{2.35}$$

where the supremum is attained at  $r \approx [\frac{\sigma+2}{aC_0|\gamma_+}]^{\frac{1}{\gamma_+}} e^{\frac{\lambda_1 t}{\gamma_+}}$ . □

### 3. THE CASE $N = 10 + 4\sigma$

This section concerns the asymptotic expansion of solutions for (1.1) in the critical case  $N = 10 + 4\sigma$ . The estimates in the inner and outer regions are similar to the ones in the case  $N > 10 + 4\sigma$ . The estimate in the inner region is given by Theorem 2.3, we give the estimate in the outer region as follows. The proof can be referred to [17, Lemma 2.4].

**Lemma 3.1.** *Let  $N = 10 + 4\sigma$ . Then there exists a constant  $C_0 > 0$  such that, for sufficiently large  $t$ ,*

$$w(r, t) \leq \min\{U_s(r), C_0 e^{\tilde{\lambda}_1 t} \tilde{\psi}_1(r)\}. \tag{3.1}$$

**Remark 3.2.** Using the proof of (2.34), we cannot deduce the estimate of upper bound of  $\alpha(t)$ . In fact, by (2.23), (2.8), and (2.10), we have for  $r \ll 1$ ,

$$u(r, t) \geq U_s(r) - b_0 e^{-2\alpha(t)} r^{-4-2\sigma} \log\left(r e^{\frac{\alpha(t)}{2+\sigma}}\right) (1 + o(1)), \quad t \rightarrow \infty.$$

It follows from Lemma 3.1 and (2.28) that

$$w(r, t) \leq C_0 e^{\tilde{\lambda}_1 t} \tilde{\psi}_1(r) = \tilde{a} C_0 e^{\tilde{\lambda}_1 t} r^{-4-2\sigma} (1 + o(1)), \quad t \rightarrow \infty.$$

Then

$$b_0 e^{-2\alpha(t)} \log\left(r e^{\frac{\alpha(t)}{2+\sigma}}\right) \geq \tilde{a} C_0 e^{\tilde{\lambda}_1 t} (1 + o(1)), \quad t \rightarrow \infty.$$

Clearly, there are no terms which matches the term  $\log\left(r e^{\frac{\alpha(t)}{2+\sigma}}\right)$ .

The following lemma presents the estimate of lower bound of  $\alpha(t)$ .

**Lemma 3.3.** *Suppose  $N = 10 + 4\sigma$ . Then  $\alpha(t)$  defined in (2.11) satisfies*

$$\alpha(t) \geq \frac{|\tilde{\lambda}_1|}{2} t + C_1, \quad t \rightarrow \infty, \tag{3.2}$$

where  $C_1 > 0$ ,  $\tilde{\lambda}_1 < 0$  is the first eigenvalue of operator  $\mathcal{A}$  defined in (2.25).

*Proof.* The proof is similar to (2.35). By (3.1), we have for sufficiently large  $T$ ,

$$u(r, t) \geq U_s(r) - C_0 e^{\tilde{\lambda}_1(t-T)} \tilde{\psi}_1(r) \quad \text{in } B_R \times (T, \infty),$$

where  $\tilde{\lambda}_1 < 0$  and  $\tilde{\psi}_1(r)$  are the first eigenpair of operator  $\mathcal{A}$ . Therefore,

$$\alpha(t) \geq \max_r \left\{ \log \frac{(2 + \sigma)(8 + 4\sigma)}{r^{2+\sigma}} - C_0 e^{\tilde{\lambda}_1 t} \tilde{\psi}_1(r) \right\},$$

where the supermum is attained at  $r \approx e^{\frac{\tilde{\lambda}_1 t}{4+2\sigma}}$ . □

From the later calculation, we obtain that  $\tilde{\lambda}_1 = -\frac{\pi_1^2}{(4+2\sigma)^{\frac{4}{2+\sigma}}}$ , where  $\pi_1$  is the first zero of the zeroth-order Bessel's function, i.e.  $J_0(\pi_1) = 0$ . In other words, Lemma 3.3 gives the optimal coefficient of  $t$  in (1.12). Moreover, we find that the second term  $C_1$  on the right-hand side of (3.2) is not optimal, and it turns out to be a logarithmically growing function in the later proof.

**3.1. The inner problem.** As shown in Sub-subsection 2.2.2, we introduce the rescaled function  $\Phi_0$ , which satisfies

$$u(r, t) = \alpha(t) + \Phi_0(\xi, t), \quad \xi = re^{\frac{\alpha(t)}{2+\sigma}},$$

where  $\alpha(t) = \sup_r u(r, t) = u(0, t)$ . Recalling the proof of Theorem 2.3, we find that the leading-order term of expansion of  $\Phi_0$  is quasi-steady as  $t \rightarrow \infty$ , and  $\Phi_0$  satisfies

$$\begin{aligned} \Phi_0'' + \frac{N-1}{\xi} \Phi_0' + |\xi|^\sigma e^{\Phi_0} &= 0, \quad \xi \in (0, R), \\ \Phi_0(0) = \Phi_0'(0) &= 0. \end{aligned}$$

Indeed,  $\Phi_0$  corresponds to  $U_0$  in Sub-subsection 2.2.1. Set

$$\Phi_0 = \log(2 + \sigma) - (2 + \sigma) \log \Psi_0.$$

A simple calculation implies that

$$\Psi_0'' \Psi_0^{1+\sigma} - (\Psi_0')^2 \Psi_0^\sigma + \frac{N-1}{\xi} \Psi_0' \Psi_0^{1+\sigma} = \xi^\sigma, \quad (3.3)$$

which is a nonlinear differential equation of order  $(2 + \sigma)$ . Set

$$\Psi_0(\xi) \sim \frac{\xi}{(N-2)^{\frac{1}{2+\sigma}}} + \Omega(\xi), \quad \xi \rightarrow \infty, \quad (3.4)$$

where  $\Omega(\xi) \rightarrow 0$  as  $\xi \rightarrow \infty$ . Substituting (3.4) into (3.3), we obtain

$$\begin{aligned} 0 &= \Omega'' \left[ \frac{\xi}{(N-2)^{\frac{1}{2+\sigma}}} + \Omega \right]^{1+\sigma} - \left[ \frac{1}{(N-2)^{\frac{1}{2+\sigma}}} + \Omega' \right]^2 \left[ \frac{\xi}{(N-2)^{\frac{1}{2+\sigma}}} + \Omega \right]^\sigma \\ &\quad + \frac{N-1}{\xi} \left[ \frac{1}{(N-2)^{\frac{1}{2+\sigma}}} + \Omega' \right] \left[ \frac{\xi}{(N-2)^{\frac{1}{2+\sigma}}} + \Omega \right]^{1+\sigma} - \xi^\sigma. \end{aligned}$$

Using Taylor expansion on the term  $\left[ \frac{\xi}{(N-2)^{\frac{1}{2+\sigma}}} + \Omega \right]^\beta$  at the point  $\frac{\xi}{(N-2)^{\frac{1}{2+\sigma}}}$ , we deduce that

$$\begin{aligned} 0 &= \Omega'' \left[ \left( \frac{\xi}{(N-2)^{\frac{1}{2+\sigma}}} \right)^{1+\sigma} + (1+\sigma) \left( \frac{\xi}{(N-2)^{\frac{1}{2+\sigma}}} \right)^\sigma \Omega + o(\Omega) \right] \\ &\quad - \left[ \left( \frac{1}{(N-2)^{\frac{1}{2+\sigma}}} \right)^2 + (\Omega')^2 + \frac{2\Omega'}{(N-2)^{\frac{1}{2+\sigma}}} \right] \\ &\quad \times \left[ \left( \frac{\xi}{(N-2)^{\frac{1}{2+\sigma}}} \right)^\sigma + \sigma \left( \frac{\xi}{(N-2)^{\frac{1}{2+\sigma}}} \right)^{-1+\sigma} \Omega + o(\Omega) \right] \\ &\quad + \frac{N-1}{\xi} \left[ \frac{1}{(N-2)^{\frac{1}{2+\sigma}}} + \Omega' \right] \\ &\quad \times \left[ \left( \frac{\xi}{(N-2)^{\frac{1}{2+\sigma}}} \right)^{1+\sigma} + (1+\sigma) \left( \frac{\xi}{(N-2)^{\frac{1}{2+\sigma}}} \right)^\sigma \Omega + o(\Omega) \right] - \xi^\sigma. \end{aligned} \quad (3.5)$$

By sorting out the terms with  $\Omega''$ ,  $\Omega'$  and  $\Omega$ , we rewrite (3.5) in the form

$$0 = \left[ \frac{\xi}{(N-2)^{\frac{1}{2+\sigma}}} \right]^{1+\sigma} \Omega'' + \left[ \frac{\xi}{(N-2)^{\frac{1}{2+\sigma}}} \right]^\sigma \frac{N-3}{(N-2)^{\frac{1}{2+\sigma}}} \Omega' + \frac{N\sigma - 2\sigma + N - 1}{(N-2)^{\frac{2}{2+\sigma}}} \left[ \frac{\xi}{(N-2)^{\frac{1}{2+\sigma}}} \right]^{-1+\sigma} \Omega - G(\xi, N, \sigma), \quad (3.6)$$

where

$$\begin{aligned} G(\xi, N, \sigma) = & -\Omega'' \left[ (1+\sigma) \left( \frac{\xi}{(N-2)^{\frac{1}{2+\sigma}}} \right)^\sigma \Omega + o(\Omega) \right] \\ & + \left( \frac{1}{(N-2)^{\frac{1}{2+\sigma}}} \right)^2 \left[ \left( \frac{\xi}{(N-2)^{\frac{1}{2+\sigma}}} \right)^\sigma + o(\Omega) \right] \\ & + (\Omega')^2 \left[ \left( \frac{\xi}{(N-2)^{\frac{1}{2+\sigma}}} \right)^\sigma + \sigma \left( \frac{\xi}{(N-2)^{\frac{1}{2+\sigma}}} \right)^{-1+\sigma} \Omega + o(\Omega) \right] \\ & + \frac{2\Omega'}{(N-2)^{\frac{1}{2+\sigma}}} \left[ \sigma \left( \frac{\xi}{(N-2)^{\frac{1}{2+\sigma}}} \right)^{-1+\sigma} \Omega + o(\Omega) \right] \\ & - \frac{N-1}{\xi} \frac{1}{(N-2)^{\frac{1}{2+\sigma}}} \left[ \left( \frac{\xi}{(N-2)^{\frac{1}{2+\sigma}}} \right)^{1+\sigma} + o(\Omega) \right] \\ & - \frac{N-1}{\xi} \Omega' \left[ (1+\sigma) \left( \frac{\xi}{(N-2)^{\frac{1}{2+\sigma}}} \right)^\sigma \Omega + o(\Omega) \right] + \xi^\sigma. \end{aligned} \quad (3.7)$$

To solve  $\Omega(\xi)$ , we consider the homogeneous equation of (3.6):

$$\xi^2 \Omega'' + \xi(N-3)\Omega' + (N\sigma - 2\sigma + N - 1)\Omega = 0.$$

If  $N > 10 + 4\sigma$ ,

$$\Omega(\xi) = C_1 \xi^{q_+} + C_2 \xi^{q_-}, \quad (3.8)$$

where  $C_1, C_2 > 0$ , and

$$q_\pm = \frac{1}{2} \left[ 4 - N \pm \sqrt{(N-2)(N-10-4\sigma)} \right],$$

which are the roots of the quadratic equation

$$q^2 + (N-4)q + (N\sigma - 2\sigma + N - 1) = 0.$$

If  $N = 10 + 4\sigma$ ,

$$\Omega(\xi) = C_3 \xi^{-3-2\sigma} \log \xi + C_4 \xi^{-3-2\sigma}, \quad (3.9)$$

where  $C_3, C_4 > 0$ .

Next, we consider the case  $N = 10 + 4\sigma$ . Since

$$\lim_{\xi \rightarrow \infty} \xi^{-3-2\sigma} \log \xi = \lim_{\xi \rightarrow \infty} \frac{1}{(3+2\sigma)\xi^{3+2\sigma}} = 0$$

and

$$\lim_{\xi \rightarrow \infty} \frac{\xi^{-3-2\sigma} \log \xi}{\xi^{-3-2\sigma}} = \lim_{\xi \rightarrow \infty} \log \xi = \infty,$$

we find that the convergence rate of term  $\xi^{-3-2\sigma}$  to 0 is faster than that of term  $\xi^{-3-2\sigma} \log \xi$  to 0. Hence,

$$\Omega(\xi) = C_3 \xi^{-3-2\sigma} \log \xi, \quad \xi \rightarrow \infty. \quad (3.10)$$

We compute

$$\Omega'(\xi) = -(3+2\sigma)C_3 \xi^{-4-2\sigma} \log \xi + C_3 \xi^{-4-2\sigma} = -(3+2\sigma)C_3 \xi^{-4-2\sigma} \log \xi, \quad (3.11)$$

and

$$\begin{aligned}\Omega''(\xi) &= (3 + 2\sigma)(4 + 2\sigma)C_3\xi^{-5-2\sigma} \log \xi - (7 + 4\sigma)C_3\xi^{-5-2\sigma} \\ &= (3 + 2\sigma)(4 + 2\sigma)C_3\xi^{-5-2\sigma} \log \xi.\end{aligned}\quad (3.12)$$

Substituting (3.10)-(3.12) into (3.7), we obtain

$$G(\xi, N, \sigma) = \frac{N - 1 + (N - 4)\sigma}{(N - 2)^{\frac{1}{2+\sigma}}} \xi^{-8-3\sigma} (\log \xi)^2 + \frac{\sigma}{(N - 2)^{\frac{-1+\sigma}{2+\sigma}}} \xi^{-12-5\sigma} (\log \xi)^3 + o(1).$$

Since

$$\begin{aligned}\lim_{\xi \rightarrow \infty} \xi^{-8-3\sigma} (\log \xi)^2 &= \lim_{\xi \rightarrow \infty} \frac{2}{(8 + 3\sigma)^2 \xi^{8+3\sigma}} = 0, \\ \lim_{\xi \rightarrow \infty} \xi^{-12-5\sigma} (\log \xi)^3 &= \lim_{\xi \rightarrow \infty} \frac{6}{(12 + 5\sigma)^3 \xi^{12+5\sigma}} = 0,\end{aligned}$$

we have

$$G(\xi, N, \sigma) \rightarrow 0, \quad \xi \rightarrow \infty. \quad (3.13)$$

Thus,  $\Omega(\xi)$  can be given by (3.8) and (3.9).

By (3.4), (3.9), and the fact  $\log(1 + x) \sim x$  as  $x \rightarrow 0$ , we have as  $\xi \rightarrow \infty$ ,

$$\begin{aligned}\Phi_0(\xi) &= \log(2 + \sigma) - (2 + \sigma) \log \Psi_0 \\ &\sim \log(2 + \sigma) - \log \left[ \frac{\xi}{(8 + 4\sigma)^{\frac{1}{2+\sigma}}} + C_3 \xi^{-3-2\sigma} \log \xi + C_4 \xi^{-3-2\sigma} \right]^{2+\sigma} \\ &= \log(2 + \sigma) - \log \left[ \frac{\xi}{(8 + 4\sigma)^{\frac{1}{2+\sigma}}} \right]^{2+\sigma} \\ &\quad - (2 + \sigma) \log \left[ 1 + (8 + 4\sigma)^{\frac{1}{2+\sigma}} (C_3 \xi^{-4-2\sigma} \log \xi + C_4 \xi^{-4-2\sigma}) \right] \\ &\sim \log \frac{(4 + 2\sigma)^2}{\xi^{2+\sigma}} - (2 + \sigma)(8 + 4\sigma)^{\frac{1}{2+\sigma}} \xi^{-4-2\sigma} (C_3 \log \xi + C_4).\end{aligned}$$

Therefore,

$$\Phi_0(\xi) \sim \log \frac{(4 + 2\sigma)^2}{\xi^{2+\sigma}} - \xi^{-4-2\sigma} (A_0 \log \xi + B_0), \quad \xi \rightarrow \infty, \quad (3.14)$$

where  $A_0 = (2 + \sigma)(8 + 4\sigma)^{\frac{1}{2+\sigma}} C_3$  and  $B_0 = (2 + \sigma)(8 + 4\sigma)^{\frac{1}{2+\sigma}} C_4$ .

For the case  $N > 10 + 4\sigma$ , we can also get an accurate estimate of lower order term, which is not needed in this paper. Notice that the term  $e^{\frac{\mu\gamma}{2+\sigma}} r^{\gamma+}$  in (2.9) is sufficient to match the resulting terms in the outer analysis.

**3.2. The outer problem.** The leading-order term of expansion of  $u$  in the outer region is  $\log \frac{(4+2\sigma)^2}{r^{2+\sigma}}$  as  $t \rightarrow \infty$ , which is given by the leading-order term in (3.14). In order to study the correction term, we give the behavior of  $u$  in the form of

$$u(r, t) \sim \log \frac{(4 + 2\sigma)^2}{r^{2+\sigma}} - v(r, t), \quad t \rightarrow \infty,$$

where  $v$  corresponds to  $w$  in Subsection 2.1. The linearized problem of  $v$  is given by

$$v_t = v_{rr} + \frac{9 + 4\sigma}{r}v_r + \frac{(4 + 2\sigma)^2}{r^2}v \quad \text{in } (0, (4 + 2\sigma)^{\frac{1}{2+\sigma}}) \times \mathbb{R}_+, \tag{3.15}$$

$$v((4 + 2\sigma)^{\frac{1}{2+\sigma}}, t) \equiv 0.$$

To match logarithmic term in (3.14), we set

$$v(r, t) = e^{-\frac{\pi^2 t}{(4+2\sigma)^{\frac{4}{2+\sigma}}}} \left[ \beta(t)\phi_0(r) + \dot{\beta}(t)\phi_1(r) + \dots \right], \tag{3.16}$$

where  $\pi$  and  $\beta(t)$  will be chosen later. Substituting(3.16) into (3.15), we obtain

$$\phi_0''(r) + \frac{9 + 4\sigma}{r}\phi_0'(r) + \left[ \left(\frac{4 + 2\sigma}{r}\right)^2 + \frac{\pi^2}{(4 + 2\sigma)^{\frac{4}{2+\sigma}}} \right]\phi_0(r) = 0,$$

$$\phi_1''(r) + \frac{9 + 4\sigma}{r}\phi_1'(r) + \left[ \left(\frac{4 + 2\sigma}{r}\right)^2 + \frac{\pi^2}{(4 + 2\sigma)^{\frac{4}{2+\sigma}}} \right]\phi_1(r) = \phi_0(r).$$

Let

$$\phi_0(r) = r^{-4-2\sigma}P_0(r), \quad \phi_1(r) = r^{-4-2\sigma}P_1(r).$$

Then

$$P_0''(r) + \frac{1}{r}P_0'(r) + \frac{\pi^2}{(4 + 2\sigma)^{\frac{4}{2+\sigma}}}P_0(r) = 0, \tag{3.17}$$

$$P_1''(r) + \frac{1}{r}P_1'(r) + \frac{\pi^2}{(4 + 2\sigma)^{\frac{4}{2+\sigma}}}P_1(r) = P_0(r). \tag{3.18}$$

Equation (3.17) is the zeroth-order Bessel's equation and  $P_0(r) = J_0\left(\frac{\pi r}{(4+2\sigma)^{\frac{2}{2+\sigma}}}\right)$ .

On the other hand, by (3.14), we have

$$v(r, t) \sim \frac{A_0}{2 + \sigma}\alpha(t)e^{-2\alpha(t)}r^{-4-2\sigma} + e^{-2\alpha(t)}(A_0r^{-4-2\sigma} \log r + B_0r^{-4-2\sigma}), \tag{3.19}$$

as  $t \rightarrow \infty$ . To match (3.16) and (3.19), it suffices to show that

$$\beta(t)e^{-\frac{\pi_1^2 t}{(4+2\sigma)^{\frac{4}{2+\sigma}}}} \sim \frac{A_0}{2 + \sigma}\alpha(t)e^{-2\alpha(t)}, \quad t \rightarrow \infty. \tag{3.20}$$

We choose  $\pi_1$  to be the first zero of zeroth-order Bessel's function, and  $J_0(\pi_1) = 0$ . It follows from (3.17) and (3.18) that

$$r(P_1'P_0 + P_0'P_1)$$

$$= - \int_s^{(4+2\sigma)^{\frac{2}{2+\sigma}}} rP_0^2(s)dr \tag{3.21}$$

$$= -\frac{(4 + 2\sigma)^{\frac{4}{2+\sigma}}}{2}J_1^2(\pi_1) + \frac{r^2}{2} \left[ J_0^2\left(\frac{\pi_1 r}{(4 + 2\sigma)^{\frac{2}{2+\sigma}}}\right) + J_1^2\left(\frac{\pi_1 r}{(4 + 2\sigma)^{\frac{2}{2+\sigma}}}\right) \right].$$

Since  $P_0(0)$  is a constant, we deduce from (3.21) that

$$P_1(r) = -\frac{(4 + 2\sigma)^{\frac{4}{2+\sigma}}}{2}J_1^2(\pi_1) \log r + O(1), \quad r \rightarrow 0.$$

By (3.16) and (3.19), we require

$$-\frac{(4+2\sigma)^{\frac{4}{2+\sigma}}}{2} J_1^2(\pi_1) \dot{\beta}(t) e^{-\frac{\pi_1^2 t}{(4+2\sigma)^{\frac{4}{2+\sigma}}}} \sim A_0 e^{-2\alpha(t)}. \quad (3.22)$$

Combining (3.20) and (3.22), we find

$$\alpha(t) = \frac{\pi_1^2 t}{2(4+2\sigma)^{\frac{4}{2+\sigma}}} + \alpha_1(t), \quad t \rightarrow \infty. \quad (3.23)$$

The function  $\beta(t)$  is determined by

$$\beta(t) \sim \frac{A_0 \pi^2 t}{(4+2\sigma)^{\frac{6+\sigma}{2+\sigma}}} e^{-2\alpha_1(t)}, \quad \dot{\beta}(t) \sim -\frac{2A_0}{(4+2\sigma)^{\frac{4}{2+\sigma}} J_1^2(\pi_1)} e^{-2\alpha_1(t)}.$$

Therefore,

$$\beta(t) \sim \beta_\infty t^{-\frac{8+4\sigma}{\pi_1^2 J_1^2(\pi_1)}} \quad t \rightarrow \infty,$$

where  $\beta_\infty > 0$  depends only on  $u_0$ . Then

$$\alpha_1(t) \sim \frac{1}{2} \left[ 1 + \frac{8+4\sigma}{\pi_1^2 J_1^2(\pi_1)} \right] \log t + \frac{1}{2} \log \left[ \frac{A_0 \pi^2 t}{(4+2\sigma)^{\frac{6+\sigma}{2+\sigma}}} \right], \quad t \rightarrow \infty. \quad (3.24)$$

We deduce from (3.23) and (3.24) that, as  $t \rightarrow \infty$ ,

$$u(0, t) \sim \frac{\pi_1^2 t}{2(4+2\sigma)^{\frac{4}{2+\sigma}}} + \frac{1}{2} \left[ 1 + \frac{8+4\sigma}{\pi_1^2 J_1^2(\pi_1)} \right] \log t + \frac{1}{2} \log \left[ \frac{A_0 \pi^2 t}{(4+2\sigma)^{\frac{6+\sigma}{2+\sigma}}} \right].$$

It follows that  $u(0, t)$  grows linearly as  $t \rightarrow \infty$  with a logarithmic correction term.

**Remark 3.4.** Combining the estimates in the inner and outer regions, we derive the widths of the inner layer, which are estimated as  $O(e^{-\frac{\lambda_1}{|\gamma+1|}t})$  if  $N > 10 + 4\sigma$  and  $O(e^{-\frac{\pi_1^2 t}{(4+2\sigma)^{\frac{6+\sigma}{2+\sigma}}}} t^{-\frac{1}{2+\sigma}})$  if  $N = 10 + 4\sigma$ . Indeed, since inner analysis is studied on compact set  $\{\xi = r e^{\frac{\alpha(t)}{2+\sigma}} \leq C\}$  with  $C > 0$ , we have that  $r = O(e^{-\frac{\alpha(t)}{2+\sigma}})$ . It follows from Theorem 1.1 that

$$r = O\left(e^{-\frac{\alpha(t)}{2+\sigma}}\right) = O\left(e^{-\frac{1}{2+\sigma} \left[ \frac{(2+\sigma)\lambda_1}{|\gamma+1|} t + O(1) \right]}\right) = O\left(e^{-\frac{\lambda_1}{|\gamma+1|}t}\right).$$

It follows from Theorem 1.2 that

$$r = O\left(e^{-\frac{\alpha(t)}{2+\sigma}}\right) = O\left(e^{-\frac{1}{2+\sigma} \left[ \frac{\pi_1^2 t}{2(4+2\sigma)^{\frac{4}{2+\sigma}}} + O(\log t) \right]}\right) = O\left(e^{-\frac{\pi_1^2 t}{(4+2\sigma)^{\frac{6+\sigma}{2+\sigma}}} t^{-\frac{1}{2+\sigma}}}\right).$$

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