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Removable singular sets of fully nonlinear elliptic equations *

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Abstract

In this paper we consider fully nonlinear elliptic equations, including the Monge-Ampere equation and the Weingarden equation. We assume that

$$\begin{aligned} F(D^2u, x) &= f(x) \quad x \in \Omega\,, \\ u(x) &= g(x) \quad x \in \partial\Omega \end{aligned}$$

has a solution u in $C^2(\Omega) \cap C(\bar{\Omega})$, and

$$\begin{aligned} F(D^2v(x), x) &= f(x) \quad x \in \Omega \setminus S\,, \\ v(x) &= g(x) \quad x \in \partial\Omega \end{aligned}$$

has a solution v in $C^2(\Omega\setminus S)\cap\mathrm{Lip}(\Omega)\cap C(\bar{\Omega})$. We prove that under certain conditions on S and v, the singular set S is removable; i.e., $u = v$.

1 Introduction

Removability of singularities of solutions to elliptic equations has studied extensible. Known results include the fact that isolated singularities of bounded harmonic functions are removable. Jörgens [4] stated the related result that the isolated singularity of the Monge-Ampere equation, in two dimensions, is removable if the solution is C^1 along a curve passing though the singularity. Jörgens' result was extended in 1995 by Beyerstedt [1] who considered isolated singularity for general equations in n-dimensions.

In this paper, we use rather elementary tools to prove removability of singular sets in arbitrary dimensions . Our result for the Monge-Ampere equation is optimal, as shown by the examples in [2].

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The Maximum Principle. In this paper, we use a generalized version of the Aleksandroff Maximum Principle (see Lemma 2 below). Let us start out with the following lemma.

Lemma 1 *Let $B = \{x : \Gamma v(x) = v(x)\}$, where*

$$\Gamma u(x) = \sup\{w(x) : w \text{ is convex and } w \le v \text{ on } \bar{\Omega}\}\,.$$

If $v \in \mathrm{Lip}(\Omega)$ and $v|_{\partial\Omega} \ge 0$, then $\{p : |p| < M/D\}$ is contained in the set

$$\{p : p \text{ is normal of the tangent plane of } z(x) = v(x) \text{ at some } x_0 \in B\}\,.$$

Proof. For each p satisfying $|p| \le M/D$, suppose that v take its minimum at x_0, and $v(x_0) = -M$. Consider the plane π defined by

$$x_{n+1} = -M + p \cdot (x - x_0)\,.$$

When $x \in \partial\Omega$, we have

$$\begin{aligned} x_{n+1} &\le -M + |p \cdot (x - x_0)| \\ &\le -M + |p|D \le 0\,. \end{aligned}$$

But $\min_{\ partial\Omega} v(x) \ge 0$, so that, $-M * p \cdot (x - x_0)|_{\partial\Omega} \le v(x)|_{\partial\Omega}$. We can take $M_0 \le -M$ such that for all $x \in \bar{\Omega}$ we have

$$M_0 + p \cdot (x - x_0) \le v(x)$$

and for all $M' > M_0$, there exist $x_1 \in \bar{\Omega}$, such that

$$M' + p \cdot (x_1 - x_0) > v(x_1)\,.$$

We can also prove that the set

$$G = \{x : M_0 + p \cdot (x - x_0) = v(x)\}$$

satisfies $G \subset B$. In fact, if there is a point $y \in G$ with $y \notin B$, then $\Gamma v(y) < v(y) = M_0 + p \cdot (y - x_0)$. The set $G_1 = \{y : \Gamma v(y) < v(y), y \in \bar{\Omega}\}$ is open in $\bar{\Omega}$. Since $v(y) \ge v(y), y \in G_1$, we can take

$$\Gamma' v(x) = \begin{cases} \Gamma v(x) & x \notin G_1 \\ M_0 + p \cdot (x - x_0) & x \in \bar{G}_1\,. \end{cases}$$

Then $\Gamma' v$ is convex, and $\Gamma' v \le v, \Gamma' v(x) > \Gamma v(x)$ for $x \in G_1$, which is a contradiction to the definition of Γv. Therefore, $G \subset B$ and the present proof is complete.

Lemma 2 *For $u \in \mathrm{Lip}(\Omega)$, $u|_{\partial\Omega} \ge 0$, and $\min_{\bar{\Omega}} u = M < 0$, there is a constant C depending only on the domain Ω and n, such that*

$$-\min_{\bar{\Omega}} u \le C\Big[\big(\int_{B\setminus S} \det D^2 u(x)\,dx\big)^{1/n} + \big(\mathrm{meas}\{\nabla u(x) | x \in S \cap B\}\big)^{1/n}\Big]\,,$$

where B is the set $\{x : \Gamma u(x) = u(x)\}$, $S = \{x : D^2 u(x) \text{does not exist }\}$, and $\nabla u(x_0)$ denotes all $p \in \mathbb{R}^n$ satisfying

$$p \cdot (x - x_0) + u(x_0) \le u(x)\,.$$

Proof. By Lemma 1, we have

$$
\begin{aligned}
-\min_{\bar\Omega} u &\le \frac{D}{K_n^{1/n}}\,[\text{meas}\{p : p \text{ is normal to the tangent plane at } x \in \{\Gamma u = u\}\}]^{1/n} \\
&= \frac{D}{K_n^{1/n}}\Big(\text{ meas}\{\nabla u(x) : x \in \{\Gamma u = u\}, D^2u(x) \text{ exits }\}\Big)^{1/n} \\
&\quad +\text{meas}\{\nabla u(x) : x \in \{\Gamma u = u\}, D^2u(x) \text{ does not exist }\}^{1/n} \\
&= \frac{D}{K_n^{1/n}}\Big(\int_{\{\Gamma u = u\}\setminus S} \det D^2u\,dx\Big)^{1/n} \\
&\quad +\frac{D}{K_n^{1/n}}\Big(\text{ meas}\{\nabla u(x) : x \in \{\Gamma u = u\}, D^2u(x) \text{ does not exist}\}\Big)^{1/n}
\end{aligned}
$$

where $D =$ dim Ω, and K_n is the volume of the unit ball in $\mathbb{R}^n$.

2 Main Theorem

Using the Lemmas 1 and 2, we can prove the following theorem.

Theorem 1 *Let $F(A,x)$ be a function defined on a convex cone C of symmetric matrices S^n, which satisfies the following conditions:*

1. *For A and B in C with $A > B$, $F(A,x) > F(B,x)$.*
2. *The equation*

$$
\begin{gathered}
F(D^2u(x),x) = 0 \quad x \in \Omega\,, \\
u(x) = g(x) \quad x \in \partial\Omega
\end{gathered}
$$

   has a solution u in $C^2(\Omega) \cap C(\bar\Omega)$.

Also assume that $v \in C^2(\Omega \setminus S) \cap \mathrm{Lip}(\Omega) \cap C(\bar\Omega)$ is a solution to

$$
\begin{gathered}
F(D^2v(x),x) = 0 \quad x \in \Omega \setminus S\,, \\
v(x) = g(x) \quad x \in \partial\Omega\,,
\end{gathered}
$$

where $S \subset\subset \Omega$ satisfies

1. *The dimension of S is l with $l < n$.*
2. *For every $x \in S$, there are $l+1$ independent C^2 curves $\{r_{xi}\}$ through x, with $i \in \{1, 2, \cdots, l+1\}$, such that $v(r_{xi}) \in C^1$.*

Then v is in C^2, satisfies the equation in Ω, and $u(x) = v(x)$.

Proof. Let $w(x)=u(x)-v(x)$. Then $w(x)|_{x\in\partial\Omega}=0$. Suppose $\min_{\bar\Omega} w<0$. Then

$$\begin{aligned} -\inf_{\bar\Omega} w &\leq C\Big[\int_{\{\Gamma w=w\}\setminus S}\det(D^2w(x))\,dx\Big]^{1/n} \\ &\quad +C\left[\text{meas}\{\nabla w(x): x\in S\cap\{\Gamma w=w\}\}\right]^{1/n}. \end{aligned}$$

If there is $x_0\in\{\Gamma w=w\}\setminus S$ such that $\det(D^2w(x_0))\neq 0$, then by the convexity of Γw, $D^2w(x_0)\geq D^2\Gamma w(x_0)\geq 0$. So $D^2w(x_0)>0$, or $D^2u(x_0)>D^2v(x_0)$. By the structure conditions on F we have

$$0=F(D^2u(x_0),x_0)>F(D^2v(x_0),x_0)=0$$

which is a contradiction.

Next, for $x_0\in S\cap\{\Gamma w=w\}$, there are $l+1$ independent C^2 curves through x_0 satisfying $v(r_{x_0i}(t))\in C^1$, with $i=1,2,\cdots,l+1$. Without loss of generality, we can assume that $r_{x_0i}(0)=x_0$ for $i=1,2,\cdots,l+1$. Then for any $p\in\{\nabla w(x_0)\}=\{p: w(x_0)+p\cdot(x-x_0)\leq w(x)\}$ we have

$$p\cdot\frac{d}{dt}\big(r_{x_0i}(0)\big)=c_i(x_0)\quad \text{for } i=1,2,\cdots,l+1\,.$$

Since $r_{x_0i}(t)$ are independent, we obtain that $\{\nabla w(x_0)\}$ is a subset in the $n-(l+1)$ dimensional space. We have that

$$\begin{aligned} &\text{meas}_n\{\nabla w(x): x\in S\cap\{\Gamma w=w\}\} \\ &\leq \text{meas}_n[\{x\in S\cap\{\Gamma w=w\}\}\times\{\nabla w(x)\}]\,. \end{aligned}$$

From

$$\dim S+\dim\{\nabla w\}=l+(n-l-1)=n-1<n\,,$$

and the boundedness of $\|\nabla w(x)\|$ and of S, we conclude that

$$\text{meas}\{\nabla w(x)|x\in S\cap\{\Gamma w=w\}\}=0\,,$$

which implies that

$$-\inf_{\bar\Omega} w\leq 0\,,$$

which, in turn, allows us to see that $w\geq 0$ or $u\geq v$. In a similar way, we can prove that

$$u\leq v\,.$$

Thus $u=v$. This completes the present proof.

For the Monge-Ampere equation, we have the following corollary

Corollary 1 *Suppose that*

$$\begin{aligned} \det(D^2u(x)) &= f(x)\quad x\in\Omega\,,\\ u(x) &= g(x)\quad x\in\partial\Omega \end{aligned}$$

has a convex solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$*, and suppose that*

$$\begin{aligned} \det(D^2 v(x)) = f(x) \quad x \in \Omega \setminus S\,, \\ v(x) = g(x) \quad x \in \partial\Omega \end{aligned}$$

has a convex solution $v \in C^2(\Omega \setminus S) \cap \mathrm{Lip}(\Omega) \cap C(\bar{\Omega})$*. Also assume that* $S \subset\subset \Omega$ *satisfies*

1. *The dimension of* S *is* l *with* $l < n$*.*
2. *For every* $x \in S$*, there are* $l+1$ *independent* C^2 *curves* $\{r_{xi}\}$ *through* x*, with* $i \in \{1, 2, \cdots l+1\}$*, such that* $v(r_{xi}) \in C^1$*.*

Then v *is in* C^2*, satisfies the above equations in* Ω*, and* $u(x) = v(x)$*.*

Remark It is straight forward to prove this Corollary the above equation with a ∇u term added.

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