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REGULARIZATION OF A DISCRETE BACKWARD PROBLEM USING COEFFICIENTS OF TRUNCATED LAGRANGE POLYNOMIALS

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Abstract. We consider the problem of finding the initial temperature $u(x,0)$, from a countable set of measured values $\{u(x_j,1)\}$. The problem is severely ill-posed and a regularization is in order. Using the Hermite polynomials and coefficients of truncated Lagrange polynomials, we shall change the problem into an analytic interpolation problem and give explicitly a stable approximation. Error estimates and some numerical examples are given.

1. Introduction

Let $u = u(x,t)$ represent a temperature distribution satisfying the heat equation

$$u_t - \Delta u = 0 \quad (x,t) \in \mathbb{R} \times (0,1). \tag{1.1}$$

The backward problem is of finding the initial temperature $u(x,0)$ from the final temperature $u(x,T)$. For simplicity, we shall assume that $T=1$. This is an ill-posed problem and has a long history [3]. This problem has been considered by many authors, using different approaches. The problem was studied intensively by the semi-group method associated with the quasi-reversibility method and the quasi-boundary value method; see for example [1, 2, 5, 7, 15, 22, 23, 12, 16, 13, 9, 26]. Using the Green function, we can transform the heat equation into

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} u(\xi,0) e^{-\frac{(x-\xi)^2}{4t}} d\xi, \quad x \in \mathbb{R},\ t > 0.$$

Hence

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} u(2\xi,0) e^{-(x-\xi)^2} d\xi = u(2x,1).$$

In this form, we can consider the backward problem as the inversion Gaussian convolution (or Weierstrass transform) problem of finding $u(2x,0)$ from its image $u(2x,1)$. Many inversion formulae for the Gauss transform were given in [18, 19, 20, 21]. In [13], using the reproducing kernel theory, the authors gave analytical inversion formulas which is optimal in an appropriate sense. In the latter paper, the case of nonexact L^2-data was studied and some sharp error estimates were given.

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Very recently, in [14], using the Paley-Wiener space and sinc approximation, the authors established a powerful practical numerical and analytical inversion formulas for the Gaussian convolution that is realized by computers. In [6, 27], the inversion Weierstrass transform for generalized functions was studied.

In practical situations, we get temperature measurements only at a discrete set of points, i.e.

$$u(x_j, 1) = \mu_j. \tag{1.2}$$

So, the problem of finding the initial temperature from discrete final values is necessary. In this case, the problem is severely ill-posed. Hence, a regularization is in order. However, the literature on this direction is very scarce. In [17], the authors used the shifted-Legendre polynomial to regularize a discrete form of the backward problem on the plane. However, the assumption that the temperature $u(x, y)$ is of order $e^{-(x^2+y^2)^{\alpha(x,y)}}$ ($\lim_{x,y\to-\infty} \alpha(x, y) = +\infty$) is very restrictive. In the present paper, the condition is removed completely.

As discussed, in the present paper, we shall consider a discrete form of the inversion problem for the Weierstrass transform

$$Wv(x_j) \equiv \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} v(\xi) e^{-(\frac{x_j}{2}-\xi)^2} d\xi = \mu_j. \tag{1.3}$$

where $v(\xi) = u(2\xi, 0)$. For the rest of this paper, we shall denote by Wv the sequence $(Wv(x_j))$.

Before going to the content of our paper, we shall give some definitions. In this paper, we denote

$$L^2_\rho(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is Lebesgue measurable and } e^{-x^2/2} f \in L^2(\mathbb{R})\}.$$

The latter space is a Hilbert space with the norm

$$\|f\| = \Big(\int_{-\infty}^{\infty} |f(x)|^2 e^{-x^2} dx \Big)^{1/2}$$

and the inner product

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x) g(x) e^{-x^2} dx, \quad \text{for } f, g \in L^2_\rho(\mathbb{R}).$$

We also denote

$$\ell^\infty = \{\mu = (\mu_j) : \mu_j \in \mathbb{R}, \sup_j |\mu_j| < \infty\}$$

with the norm $\|\mu\|_\infty = \sup_j |\mu_j|$.

For $R > 0$,we denote $B_R = \{z \in C : |z| < R\}$ and $C_R = \{z \in C : |z| = R\}$. We also denote by $H^1(B_R)$ the Hardy space of functions

$$\psi(z) = \sum_{n=0}^{\infty} \alpha_n z^n$$

analytic on the disc B_R with the norm

$$\|\psi\|^2_{H^1(B_R)} = \sum_{n=0}^{\infty} |\alpha_n R^n|^2 < \infty.$$

Using the Parseval equality, we can rewrite the latter norm in another form

$$\|\psi\|^2_{H^1(B_R)} = \frac{1}{2\pi} \int_0^{2\pi} |\psi(Re^{i\theta})|^2 d\theta.$$

If $|\psi(Re^{i\theta})| \le M$ for every $\theta \in [0, 2\pi]$ then the latter equality gives

$$\|\psi\|^2_{H^1(B_R)} \le M.$$

Let v be an exact solution of (1.3), we recall that a sequence of linear operator $T_n : \ell^\infty \to L^2_\rho(\mathbb{R})$ is a regularization sequence (or a regularizer) of Problem (1.3) if (T_n) satisfies two following conditions (see, [10])

(R1) For each n, T_n is bounded,
(R2) $\lim_{n\to\infty} \|T_n(Wv) - v\| = 0$.

The number "n" is called the regularization parameter. From (R1), (R2), we can obtain

(R3) For $\epsilon > 0$, there exists the functions $n(\epsilon)$ and $\delta(\epsilon)$ such that $\lim_{\epsilon\to 0} n(\epsilon) = \infty$, $\lim_{\epsilon\to 0} \delta(\epsilon) = 0$ and that

$$\|T_{n(\epsilon)}(\mu) - v\| \le \delta(\epsilon)$$

for every $\mu \in \ell^\infty$ such that $\|\mu - Wv\|_\infty < \epsilon$.

The number ϵ is the error between the exact data Wv and the measured data μ. For a given error ϵ, there are infinitely many ways of choosing the regularization parameter $n(\epsilon)$. In the present paper, we give an explicit form of $n(\epsilon)$.

The remainder of the paper is divided into three sections. In Section 2, we shall transform the problem into an analytic interpolation problem and prove a uniqueness result. In Section 3, we shall find regularization functions by an association between Hermite polynomials and coefficients of Lagrange polynomials. Finally, in Section 4, some numerical examples are given.

2. Reformulation of the problem and the uniqueness

Using Hermite polynomials (see [4, P. 65]) we can write

$$e^{-(z-\xi)^2} = \sum_{n=0}^\infty \frac{1}{n!} e^{-\xi^2} H_n(\xi) z^n,$$

where we recall that

$$H_n(\xi) = (-1)^n e^{\xi^2} \frac{d^n}{d\xi^n} e^{-\xi^2},$$
$$\langle H_n, H_m \rangle = \delta_{mn} \sqrt{\pi} 2^n n!$$

where $\delta_{mn} = 0$ when $n \ne m$ and $\delta_{nn} = 1$. We shall find a sequence (a_n) such that

$$v(\xi) = u(2\xi, 0) = \sum_{n=0}^\infty a_n H_n(\xi)$$

satisfies (1.3). From the orthogonality of $\{H_n\}$ in the space $L^2_\rho(R)$,we can substitute the latter expansion into (1.3) to get

$$\mu_j = \sum_{n=0}^\infty a_n x_j^n.$$

Now, if we put

$$\phi(v)(z) = \sum_{n=0}^\infty a_n z^n, \tag{2.1}$$

then we have

$$\phi(v)(x_j) = \mu_j. \tag{2.2}$$

Hence, the problem is reformulated to the classical one of finding the sequence (a_n) (and of constructing a function v) from the prescribed values (μ_j) such that $\phi(v)(z)$ satisfies (2.2). We first give some properties of the function $\phi(v)$.

Lemma 2.1. *Let $v(x) = u(2x, 0)$ be in $L^2_\rho(\mathbb{R})$. If v has the expansion*

$$v(\xi) = \sum_{n=0}^{\infty} a_n H_n(\xi)$$

then

$$\sqrt{\pi} \sum_{n=0}^{\infty} |a_n|^2 2^n n! < \infty \tag{2.3}$$

and that the function $\phi(v)(.)$ is an entire function of order $\rho \le 2$. Here we recall that the order of an entire function f is the number

$$\rho = \limsup_{r\to\infty} \frac{\ln\ln M_f(r)}{\ln r}$$

where $M_f(r) = \max_{|z|=r} |f(z)|$.

Proof. As mentioned before, $\langle H_n, H_m\rangle = \delta_{mn}\sqrt{\pi}2^n n!$ where $\delta_{mn} = 0$ for $m \ne n$ and $\delta_{mm} = 1$. Since

$$v(\xi) = \sum_{n=0}^{\infty} a_n H_n(\xi)$$

we get

$$\sqrt{\pi} \sum_{n=0}^{\infty} |a_n|^2 2^n n! = \|v\|^2 < \infty. \tag{2.4}$$

Now we prove that $\phi(v)$ is an entire function. In fact, we consider the power series

$$\phi(v)(z) = \sum_{n=0}^{\infty} a_n z^n.$$

From (2.4), one has

$$|a_n|^2 \le \frac{\|v\|^2}{2^n n!}.$$

It follows that

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = 0.$$

Hence, the power series has the convergent radius $R = \infty$, i.e., ϕ is an entire function. Now, we estimate the order of the entire function ϕ. We note that the order ρ of ϕ can be calculated by the following formula (see [11, P. 6])

$$\rho = \limsup_{n\to\infty} \frac{n \ln n}{\ln(1/|a_n|)}.$$

From (2.4), one has

$$1/|a_n|^2 \ge C 2^n n!$$

where $C = \|v\|^{-2}$. On the other hand, we have the Stirling formula (see [24, P. 688])

$$n! = \sqrt{2\pi n} n^n e^{-n} e^{\theta_n},$$

where

$$\frac{1}{12(n+1)} \le \theta_n \le \frac{1}{12(n-1)}.$$

Hence

$$2^n n! \ge \sqrt{2\pi n}(2/e)^n n^n.$$

It follows that

$$\ln(1/|a_n|^2) \ge C_1(1 + \ln\ln n + n\ln(2/e)) + n\ln n$$

where C_1 is a generic constant. Hence

$$\rho = \limsup_{n\to\infty} \frac{2n\ln n}{\ln(1/|a_n|^2)} \le \limsup_{n\to\infty} \frac{2n\ln n}{C_1(\ln\ln n + n\ln(2/e)) + n\ln n} = 2.$$

This completes the proof of Lemma 2.1. □

Now we have a uniqueness result.

Theorem 2.2. *Let $\delta > 0$. If*

$$\sum_{n=1}^{\infty} \frac{1}{|x_n|^{2+\delta}} = \infty$$

then Problem (1.3) *has at most one solution $v \in L^2_\rho(\mathbb{R})$.*

The latter condition means that the sequence (x_n) has an accumulation point on the extended real axis $\mathbb{R} \cup \{\pm\infty\}$. Moreover, if the accumulation point is ∞ then the sequence (x_n) has to be "dense enough" near ∞.

Proof. Let $v_1, v_2 \in L^2_\rho(\mathbb{R})$ be two solutions of (1.3). Putting $v = v_1 - v_2$ and assuming that $v = \sum_{n=1}^{\infty} a_n H_n$, we shall get as in the beginning of Section 2

$$\phi(v)(x_j) = 0, \quad j = 1, 2, \dots$$

where $\phi(v) = \sum_{n=1}^{\infty} a_n z^n$. It follows that x_j's are zeroes of the entire function ϕ. If x_j's has a finite accumulation point then the identity theorem shows that $\phi(v) \equiv 0$. If x_j's do not have any finite accumulation points, we can assume, without loss of generality, that $|x_1| \le |x_2| < \dots.$ and $\lim_{j\to\infty} |x_j| = \infty$. Since the order of $\phi(v)$ is ≤ 2, we get (see [11, P. 18])

$$\inf\Big\{\lambda \Big| \sum_{n=1}^{\infty} \frac{1}{|x_n|^\lambda} < \infty\Big\} \le \rho \le 2.$$

It follows that

$$\sum_{n=1}^{\infty} \frac{1}{|x_n|^{2+\delta}} < \infty$$

which is a contradiction. Hence, in either cases, we have $\phi(v) \equiv 0$. It follows that $a_n = 0,\ n = 1, 2, \dots.$ This completes the proof. □

3. Regularization and error estimate

For the rest of this article, we shall assume that there exists an $R > 0$ such that $\sup_j |x_j| < R$. Put $\omega_n(z) = (z - x_0) \dots (z - x_n)$ and $\mu = (\mu_j) \in \ell^\infty$. We denote by $L_n(\mu)$ the Lagrange polynomial of degree (at most) n,i.e.,

$$L_n(\mu)(z) = \sum_{j=0}^{n} \mu_j \frac{\omega_n(z)}{\omega_n'(x_j)(z - x_j)}$$

which satisfies $L_n(\mu)(x_j) = \mu_j$. Now, we denote by $l_j^{(n)}(\mu)$ the coefficient of z^j in the expansion of the Lagrange polynomial $L_n(\mu)$, i.e.

$$L_n(z)(\mu) = \sum_{j=0}^{n} l_j^{(n)}(\mu) z^j. \tag{3.1}$$

We shall construct a regularization sequence. We denote by k_{0n} the greatest integer satisfying

$$n \ln\big(\frac{3}{2}\big) > (2k_{0n} + 1) \ln k_{0n}. \tag{3.2}$$

We can verify easily that $\lim_{n\to\infty} k_{0n} = \infty$. We choose a sequence (k_n) such that

$$0 < k_n \le k_{0n}, \quad \lim_{n\to\infty} k_n = \infty. \tag{3.3}$$

For each n, we shall approximate the function $v(x) = u(2x, 0)$ by the function

$$T_n(\mu)(x) = \sum_{j=0}^{k_n} l_j^{(n)}(\mu) H_j(x). \tag{3.4}$$

We shall verify that T_n is a regularization sequence. We first note that $T_n : \ell^\infty \to L^2_\rho(\mathbf{R})$ is bounded, i.e., Condition (R1) (in Section 1) holds. In Theorem 3.1 below, we shall prove that (T_n) satisfies (R2) and, in Theorem 3.6, we shall prove that (T_n) satisfies (R3). In fact, we get the following regularization result for the case of exact data.

Theorem 3.1. *Let (k_n) be as in (3.3), let $R \ge 1$ and let $v \in L^2_\rho(\mathbb{R})$ be as in Theorem 2.2. Put $F_n = T_n(Wv)$. Then $\|v - F_n\| \to 0$ as $n \to \infty$. Moreover, if $v' \in L^2_\rho(\mathbb{R})$ then we can find an n_0 such that*

$$\|v - F_n\|^2 \le \|v\|^2 e^{8R^2} \big(\frac{2}{3}\big)^n + \frac{1}{k_n} \|v'\|^2 \quad \text{for } n \ge n_0.$$

If we choose $k_n = k_{0n}$, $n = 1, 2, \dots.$ then the latter inequality can be rewritten as follows

$$\|v - F_n\|^2 \le \|v\|^2 e^{8R^2} \big(\frac{2}{3}\big)^n + \frac{1}{\sqrt{n}} \|v'\|^2 \quad \text{for } n \ge n_0.$$

Before proving this theorem, some remarks are in order. We note that the coefficients of z^j $(j \ge k_n + 1)$ in the expansion of the Lagrange polynomial (3.1) are truncated in (3.4). If we use coefficients of z^j's (for j large) of L_n in (3.4) then we shall get functions which are unstable approximation of v. To illustrate the latter fact, in Section 4, we shall give a numerical example. In fact, we can say that the polynomial

$$L_{nk_n}(z) = \sum_{j=0}^{k_n} l_j^{(n)} z^j$$

is a truncated Lagrange polynomial (see [25] for a similar definition). Hence, our method of regularization is of using the coefficients of truncated Lagrange polynomials. We shall give an estimate for $l_j^{(n)}$ and the proof of Theorem 3.1. To this end, some lemmas will be established.

Lemma 3.2. *Let $v, \phi(v)$ be as in Lemma 2.1. Then $\phi : L^2_\rho(\mathbb{R}) \to H^1(B_R)$ is a bounded linear operator satisfying*

$$\|\phi(v)\|^2_{H^1(B_R)} \le e^{R^2/2}\|v\|^2.$$

Proof. We have

$$\begin{aligned}
\|\phi(v)\|^2_{H^1(B_R)} &= \sum_{n=0}^{\infty} |a_n|^2 R^{2n} \le \sqrt{\pi}\sum_{n=0}^{\infty} |a_n|^2 n2^n n! \frac{R^{2n}}{n2^n n!}\\
&\le \|v\|^2 \sum_{n=0}^{\infty} \frac{R^{2n}}{2^n n!}\\
&= e^{R^2/2}\|v\|^2.
\end{aligned}$$

This completes the proof $\square$

Lemma 3.3. *Let $v, \phi(v)$ and (a_n) be as in Lemma 2.1. Assume that (x_j) is in the disc B_R. Then one has*

$$\sum_{j=0}^{n} R^{2j}|a_j - l_j^{(n)}|^2 + \sum_{j=n+1}^{\infty} R^{2j}|a_j|^2 \le \frac{1}{9}\Big(\frac{2}{3}\Big)^{2n} e^{8R^2}\|v\|^2.$$

Proof. In the present proof, we shall denote $\phi(v)$ by ϕ. We have the Hermitian representation (see [8, P. 58])

$$\phi(z) - L_n(z) = \frac{1}{2\pi i}\int_{C_{4R}} \frac{\omega_n(z)}{\omega_n(t)}.\frac{\phi(t)}{t-z}dt.$$

Now, for every $t \in C_{4R}$ one has $|\omega_n(t)| \ge (3R)^n$. On the other hand, one has for every $|z| \le R$

$$|\omega_n(z)| \le (2R)^n.$$

We claim that

$$\|\phi - L_n\|_{H^1(B_R)} \le \frac{1}{3}\Big(\frac{2}{3}\Big)^n \|\phi\|_{H^1(B_{4R})}.$$

In fact, we have for $|z| = R$, $t = 4Re^{i\theta}$

$$\begin{aligned}
|\phi(z) - L_n(z)| &\le \frac{1}{2\pi}\int_0^{2\pi} \frac{(2R)^n}{(3R)^n}.\frac{|\phi(4Re^{i\theta})|}{4R - R} R d\theta\\
&\le \frac{1}{3}\Big(\frac{2}{3}\Big)^n \frac{1}{2\pi}\int_0^{2\pi} |\phi(4Re^{i\theta})| d\theta\\
&\le \frac{1}{3}\Big(\frac{2}{3}\Big)^n \Big(\frac{1}{2\pi}\int_0^{2\pi} |\phi(4Re^{i\theta})|^2 d\theta\Big)^{1/2}\\
&= \frac{1}{3}\Big(\frac{2}{3}\Big)^n \|\phi\|_{H^1(B_{4R})}.
\end{aligned}$$

It follows that

$$\|\phi - L_n\|^2_{H^1(B_R)} = \frac{1}{2\pi}\int_0^{2\pi} |\phi(Re^{i\theta}) - L_n(Re^{i\theta})|^2 d\theta \le \frac{1}{9}\Big(\frac{2}{3}\Big)^{2n}\|\phi\|^2_{H^1(B_{4R})}$$

as claimed. In view of Lemma 3.2, it follows that

$$\sum_{j=0}^{n} R^{2j}|a_j - l_j^{(n)}|^2 + \sum_{j=n+1}^{\infty} R^{2j}|a_j|^2 \le \frac{1}{9}\big(\frac{2}{3}\big)^{2n} e^{8R^2}\|v\|^2.$$

This completes the proof. □

Lemma 3.4. *Let $f \in L^2_\rho(\mathbb{R})$ satisfy $f' \in L^2_\rho(\mathbb{R})$ and $f = \sum_{n=0}^{\infty} c_n H_n$ Then we have*

$$\sum_{n=0}^{\infty} 2nc_n^2\sqrt{\pi}2^n n! = \|f'\|^2.$$

Proof. We note that H_n satisfies the differential equation

$$y" - 2xy' + 2ny = 0,$$

(see [4, P. 66]). It follows that H_n satisfies

$$(e^{-x^2}y')' + 2nye^{-x^2} = 0.$$

Hence we have

$$(e^{-x^2}f')' = \sum_{n=0}^{\infty} c_n(e^{-x^2}H_n')' = \sum_{n=0}^{\infty} -2nc_nH_ne^{-x^2}.$$

Hence, taking the inner product in $L^2(\mathbb{R})$ with respect H_n, we get in view of the orthogonality

$$\int_{-\infty}^{\infty} e^{-x^2} f'(x)c_nH_n'(x)dx = 2nc_n^2\sqrt{\pi}2^n n!.$$

It follows that

$$\int_{-\infty}^{\infty} e^{-x^2} f'(x)f'(x)dx = \sum_{n=0}^{\infty} 2nc_n^2\sqrt{\pi}2^n n!.$$

This completes the proof. □

Lemma 3.5. *For (k_{0n}) as in (3.2), there exist $a_0, n_0 > 0$ such that*

$$\big(\frac{3}{2}\big)^n \ge \sqrt{\pi}j!2^j \quad \text{for } 0 \le j \le k_{0n}$$

for every $n > a_0$ and that $k_{0n} \ge \sqrt{n}$ for every $n > n_0$.

Proof. For every $k > 4\pi^2e^2$, we have

$$\begin{aligned}\ln\Big(2\pi e\sqrt{k}\,(2ek)^k\Big) &= \ln(2\pi e) + \frac{1}{2}\ln k + k(1 + \ln 2 + \ln k)\\ &\le \ln(2\pi e) + k(1 + \ln 2) + \frac{1}{2}\ln k + k\ln k\\ &\le (2k+1)\ln k \equiv g(k)\end{aligned}$$

For every $n > a_0 = g(576)\ln^{-1}\left(\frac{3}{2}\right)$, one has in view of the definition of k_{0n} that $k_{0n} \ge 576 > 4\pi^2e^2$. Hence, we have for $n > a_0$

$$(2k_{0n}+1)\ln k_{0n} \ge \ln\Big(2\pi e\sqrt{k_{0n}}\,(2ek_{0n})^{k_{0n}}\Big).$$

Now, since k_{0n} satisfies

$$n\ln\big(\frac{3}{2}\big) > (2k_{0n}+1)\ln k_{0n},$$

we have for $n > a_0$

$$n \ln\big(\frac{3}{2}\big) > \ln\Big(2\pi e\sqrt{k_{0n}}\,(2ek_{0n})^{k_{0n}}\Big).$$

Using Stirling formula we get

$$\sqrt{\pi}k_{0n}!2^{k_{0n}} \le 2\pi e\sqrt{k_{0n}}\,(2ek_{0n})^{k_{0n}}.$$

It follows that

$$\big(\frac{3}{2}\big)^n > 2\pi e\sqrt{k_{0n}}\,(2ek_{0n})^{k_{0n}} \ge \sqrt{\pi}k_{0n}!2^{k_{0n}}.$$

Since

$$(2k_{0n}+3)\ln(k_{0n}+1) \ge n\ln\big(\frac{3}{2}\big) > (2k_{0n}+1)\ln k_{0n}$$

one has $k_{0n} \to \infty$ as $n \to \infty$ and that

$$\lim_{n\to\infty} \frac{n\ln\big(\frac{3}{2}\big)}{k_{0n}\ln k_{0n}} = 2.$$

It follows that

$$\lim_{n\to\infty} \frac{\sqrt{n}}{k_{0n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{2k_{0n}\ln k_{0n}}}.\frac{\sqrt{2k_{0n}\ln k_{0n}}}{k_{0n}} = 0.$$

Hence, we can find an $n_0 > a_0$ such that $k_{0n} \ge \sqrt{n}$ for every $n \ge n_0$. This completes the proof. $\square$

Proof of Theorem 3.1. For $R \ge 1$, it follows in view of the orthogonality of (H_n) that

$$\begin{aligned}
\|v - F_n\|^2 &= \sum_{j=0}^{k_n} |a_j - l_j^{(n)}|^2\sqrt{\pi}j!2^j + \sum_{j=k_n+1}^{\infty} |a_n|^2\sqrt{\pi}j!2^j \\
&= \sum_{j=0}^{k_n} R^{2j}|a_j - l_j^{(n)}|^2\frac{\sqrt{\pi}j!2^j}{R^{2j}} + \sum_{j=k_n+1}^{\infty} |a_j|^2\sqrt{\pi}j!2^j \\
&\le \sum_{j=0}^{k_n} R^{2j}|a_j - l_j^{(n)}|^2\sqrt{\pi}j!2^j + \sum_{j=k_n+1}^{\infty} |a_j|^2\sqrt{\pi}j!2^j.
\end{aligned}$$

Using Lemma 3.3, we have

$$\|v - F_n\|^2 \le \|v\|^2 e^{8R^2}\frac{1}{9}\big(\frac{2}{3}\big)^{2n}\sqrt{\pi}k_n!2^{k_n} + \sum_{j=k_n+1}^{\infty} |a_j|^2\sqrt{\pi}j!2^j.$$

In view of Lemma 3.5,

$$\|v - F_n\|^2 \le \|v\|^2 e^{8R^2}\big(\frac{2}{3}\big)^{n} + \sum_{j=k_n+1}^{\infty} |a_j|^2\sqrt{\pi}j!2^j.$$

Using Lemma 2.1, one gets

$$\lim_{n\to\infty} \sum_{j=k_n+1}^{\infty} |a_j|^2\sqrt{\pi}j!2^j = 0.$$

It follows that

$$\lim_{n\to\infty} \|v - F_n\| = 0.$$

Now, if $v' \in L^2_\rho(\mathbb{R})$ then we get in view of Lemma 3.4

$$\sum_{j=k_n+1}^{\infty} |a_j|^2 \sqrt{\pi} j! 2^j \le \frac{1}{k_n} \|v'\|^2,$$

which gives

$$\|v - F_n\|^2 \le \|v\|^2 \big(\frac{2}{3}\big)^n + \frac{1}{k_n} \|v'\|^2.$$

This proves the first estimate of Theorem 3.1. Now if $k_n = k_{0n}$ then Lemma 3.5 gives $k_n \ge \sqrt{n}$ for $n \ge n_0$. Hence

$$\|v - F_n\|^2 \le \|v\|^2 \big(\frac{2}{3}\big)^n + \frac{1}{\sqrt{n}} \|v'\|^2.$$

This completes the proof. □

Now, we consider the case of nonexact data. Let $\epsilon > 0$ and let $\mu^\epsilon = (\mu^\epsilon_j)$ be a nonexact data of $(Wv(x_j)) = (u(x_j, 1))$ satisfying

$$\sup_j |u(x_j, 1) - \mu^\epsilon_j| < \epsilon.$$

We first put

$$D_m = \max_{1 \le n \le m} \Big(\max_{|z| \le R} \big| \frac{\omega_m(z)}{(z - x_n)\omega'_m(x_n)} \big| \Big)$$

and

$$F^\epsilon_n = T_n(\mu^\epsilon) = \sum_{j=0}^{k_n} l^{(n)}_{j\epsilon} H_j,$$

where $l^{(n)}_{j\epsilon}$ is the coefficient of z^j in the expansion of the Lagrange polynomial

$$L_{\epsilon n} = \sum_{j=0}^{n} \mu^\epsilon_j \frac{\omega_n(z)}{\omega'_n(z_j)(z - z_j)}.$$

Let ψ be an increasing function such that

$$\psi(n) \ge (n+1) D_n \big(\frac{3}{2}\big)^{n/2}, \ \lim_{x \to \infty} \psi(x) = \infty$$

and

$$n(\epsilon) = [\psi^{-1}(\epsilon^{-\frac{1}{2}})] + 1$$

where $[x]$ is the greatest integer $\le x$. Using the latter function, we shall prove that (T_n) satisfies the condition (R3).

Theorem 3.6. *Let $R > 1$ and let $v \in L^2_\rho(\mathbb{R})$. Let $\epsilon > 0$ and let (μ^ϵ_j) be a measured data of $(u(x_j, 1))$ satisfying*

$$\sup_j |u(x_j, 1) - \mu^\epsilon_j| < \epsilon.$$

Then

$$\|v - F^\epsilon_{n(\epsilon)}\| \le \delta(\epsilon) = \|v - F_{n(\epsilon)}\| + \sqrt{\epsilon}.$$

Moreover, if $v' \in L^2_\rho(\mathbb{R})$ then

$$\|v - F^\epsilon_{n(\epsilon)}\|^2 \le 2\|v\|^2 \big(\frac{2}{3}\big)^{n(\epsilon)} + \frac{2}{k_{n(\epsilon)}} \|v'\|^2 + 2\epsilon.$$

In the latter inequality, if $k_n = k_{0n}$ *then there exists an* $\epsilon_0 > 0$ *such that*

$$\|v - F^{\epsilon}_{n(\epsilon)}\|^2 \le 2\|v\|^2\big(\frac{2}{3}\big)^{n(\epsilon)} + \frac{2}{\sqrt{n(\epsilon)}}\|v'\|^2 + 2\epsilon.$$

for $0 < \epsilon < \epsilon_0$.

Proof. We first claim that

$$\|F_n - F^{\epsilon}_n\| \le (n+1)\big(\frac{3}{2}\big)^{n/2}\epsilon D_n.$$

In view of Lemma 3.5,

$$\begin{aligned} \|F_n - F^{\epsilon}_n\|^2 &= \sum_{j=0}^{k_n} |l^{(n)}_j - l^{(n)}_{j\epsilon}|^2\sqrt{\pi}j!2^j \\ &= \sum_{j=0}^{k_n} R^{2j}|l^{(n)}_j - l^{(n)}_{j\epsilon}|^2\frac{\sqrt{\pi}j!2^j}{R^{2j}} \\ &\le \big(\frac{3}{2}\big)^n\sum_{j=0}^{k_n} R^{2j}|l^{(n)}_j - l^{(n)}_{j\epsilon}|^2. \end{aligned}$$

Hence

$$\|F_n - F^{\epsilon}_n\|^2 \le \big(\frac{3}{2}\big)^n\|L_n - L_{\epsilon n}\|^2_{H^1(B_R)}. \tag{3.5}$$

On the other hand

$$L_n(z) - L_{\epsilon n}(z) = \sum_{j=0}^{n}(\mu_j - \mu^{\epsilon}_j)\frac{\omega_n(z)}{\omega'_n(x_j)(z - x_j)}.$$

It follows that

$$\|L_n - L_{\epsilon n}\|_{H^1(B_{1R})} \le \sum_{j=0}^{n}|\mu_j - \mu^{\epsilon}_j|D_n \le (n+1)\epsilon D_n.$$

So that

$$\|F_n - F^{\epsilon}_n\| \le (n+1)\big(\frac{3}{2}\big)^{n/2}\epsilon D_n.$$

Now, we have

$$\|v - F^{\epsilon}_n\| \le \|v - F_n\| + \|F^{\epsilon}_n - F_n\|.$$

Hence

$$\|v - F^{\epsilon}_n\| \le \|v - F_n\| + \epsilon(n+1)D_n\big(\frac{3}{2}\big)^{n/2}.$$

For $n = n(\epsilon)$, we get in view of the definition of $n(\epsilon)$ that

$$\|v - F^{\epsilon}_{n(\epsilon)}\| \le \|v - F_{n(\epsilon)}\| + \sqrt{\epsilon}.$$

Since $n(\epsilon) \to \infty$ as $\epsilon \to 0$, we can get from Theorem 3.1 and the latter inequality that

$$\lim_{\epsilon\to 0}\|v - F^{\epsilon}_{n(\epsilon)}\| = 0.$$

Now if $v' \in L^2_{\rho}(\mathbb{R})$, Theorem 3.1 and (3.5) give

$$\|v - F^{\epsilon}_n\|^2 \le 2\|v\|^2\big(\frac{2}{3}\big)^n + \frac{2}{k_n}\|v'\|^2 + 2(n+1)^2\epsilon^2D^2_n\big(\frac{3}{2}\big)^n.$$

From the definition of $n(\epsilon)$, one has

$$\|v - F^\epsilon_{n(\epsilon)}\|^2 \le 2\|v\|^2\big(\frac{2}{3}\big)^{n(\epsilon)} + \frac{2}{k_{n(\epsilon)}}\|v'\|^2 + 2\epsilon.$$

Finally, if $k_n = k_{0n}$ then Lemma 3.5 shows that, there exists an $\epsilon_0 > 0$ such that $k_{n(\epsilon)} \ge \sqrt{n(\epsilon)}$ for every $0 < \epsilon < \epsilon_0$. Hence, we shall get the desired estimate. This completes the proof. □

4. Numerical examples

We shall give two numerical examples. In the first example, we consider $x_j = \frac{1}{1+j}$, $j = 0, 1, \dots, 100$. We choose the exact function $v(\xi) = 1$ and the non-exact data $\mu^\epsilon_j = 1 + \frac{1}{2.10^{20}(j+1)}$. From the latter data, we can calculate (using MAPLE) the first six coefficients of the corresponding Lagrange polynomial $[l_0^{(100)}, l_1^{(100)}, l_2^{(100)}, l_3^{(100)}, l_4^{(100)}, l_5^{(100)}]$ which are

$$\begin{aligned} s := [&1 + 2575.000000 \times 10^{-20}, -6.546062500 \times 10^{-14}, 1.094478041 \times 10^{-10},\\ &- 1.354054633 \times 10^{-7}, 1.322015356 \times 10^{-4}, -1.060903238 \times 10^{-1}]. \end{aligned}$$

Using the first five coefficients of the corresponding Lagrange polynomial, we get the approximation $F_1 = \sum_{j=0}^4 l_j^{(100)} H_j$ of v

$$\begin{aligned} F_1 :=\ &1.001586418 + 0.000001624865429x - 0.006345673271x^2\\ &- 0.000001083243706x^3 + 0.002115224570x^4. \end{aligned}$$

We have

$$\int_{-20}^{20} |F_1(x) - v(x)| e^{-x^2}\,dx \simeq 0.003448971524.$$

We have the graphs of two functions v and F_1. The approximation is very good in the interval $[-2, 2]$.

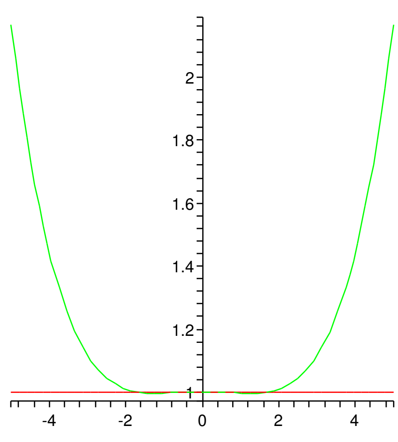

Figure 1. the graphs of v and F_1 on $[-4, 4]$

If we use the first six coefficients of the Lagrange polynomial, we get the approximation $F_2 = \sum_{j=0}^{5} l_j^{(100)} H_j$ of v

$$\begin{aligned} F_2 &:= 1.001586418 - 12.73083724x - 0.006345673271x^2 \\ &\quad + 16.97445073x^3 + 0.002115224570x^4 - 3.394890362x^5. \end{aligned}$$

We have

$$\int_{-20}^{20} |F_2(x) - v(x)| e^{-x^2} dx \simeq 8.752434897.$$

In this case, we can see that the error is larger than the foregoing case.

In the second example, we consider $x_j = \frac{1}{1+j}$, $j = 0, 1, \dots, 140$. We choose the exact function $v(\xi) = 1$, $\mu_j^{\epsilon} = 1 + \frac{1}{2.10^{20}(j+1)}$. ¿From the latter data, we can calculate the first six coefficients of Lagrange polynomial $[l_0^{(140)}, l_1^{(140)}, l_2^{(140)}, l_3^{(140)}, l_4^{(140)}, l_5^{(140)}]$ which are

$$\begin{aligned} s &:= [1 + 5005.000000 \times 10^{-20}, -2.481893750 \times 10^{-13}, 8.126181478 \times 10^{-10}, \\ &\quad - 0.1976424306 \times 10^{-5}, 0.3808576622 \times 10^{-2}, -6.056645660]. \end{aligned}$$

Using the first five coefficients of the corresponding Lagrange polynomial, we get the approximation $F_3 = \sum_{j=0}^{4} l_j^{(140)} H_j$ of v

$$\begin{aligned} F_3 &:= 1.045702918 + 0.00002371709117x - 0.1828116746x^2 \\ &\quad - 0.00001581139445x^3 + 0.06093722595x^4. \end{aligned}$$

We have

$$\int_{-20}^{20} |F_3(x) - v(x)| e^{-x^2} dx \simeq 0.09936096138.$$

On the other hand, if we use the first six coefficients of the Lagrange polynomial, we have the function $F_4 = \sum_{j=0}^{5} l_j^{(140)} H_j$

$$\begin{aligned} F_4 &:= 1.045702918 - 726.7974555x - 0.1828116746x^2 + 969.0632898x^3 \\ &\quad + 0.06093722595x^4 - 193.8126611x^5. \end{aligned}$$

We have an error estimate

$$\int_{-20}^{20} |F_4(x) - v(x)| e^{-x^2} dx \simeq 499.6722779.$$

This case shows that the error is very large if we use too many coefficients of the Lagrange polynomial.

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