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Nonexistence of Positive Singular Solutions for a Class of Semilinear Elliptic Systems *

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Abstract

We study nonexistence and removability results for nonnegative subsolutions to

$$\left.\begin{array}{rcl}\Delta u &=& a(x)v^p \\ \Delta v &=& b(x)u^q\end{array}\right\} \text{ in } \Omega\subset\mathbb{R}^N, \quad N\geq 3\,,$$

where $p\geq 1$, $q\geq 1$, $pq>1$, and a and b are nonnegative functions. As a consequence of this work, we obtain new results for biharmonic equations.

1 Introduction

The aim of this paper is to study nonexistence and removability results for nonnegative solutions of the inequality system

$$\left.\begin{array}{rcl}\Delta u &\geq& a(x)v^p \\ \Delta v &\geq& b(x)u^q\end{array}\right\} \text{ in } \Omega\subset\mathbb{R}^N, \quad N\geq 3 \tag{1.1}$$

where $p\geq 1$, $q\geq 1$ and $pq>1$. We assume that the functions a and b are nonnegative functions defined in $L^\infty_{\mathrm{loc}}(\Omega)$.

We will give a unified treatment for the cases $\Omega=\mathbb{R}^N$, $\Omega=B_1(0)\backslash\{0\}$ and $\Omega=\mathbb{R}^N\backslash\{0\}$ in (1.1). For this purpose we will base our arguments essentially on a priori bounds results for (1.1) in the one-dimensional case in exterior domains (Theorem 2.1, Theorem 2.2 and Corollary 2.1 below).

One reason for tackling this type of problem is the study of nonnegative solutions for the semilinear biharmonic equation

$$\Delta^2 u = u^q \quad \text{in} \quad \mathbb{R}^N,\ N\geq 3\,. \tag{1.2}$$

As a consequence of our results for system (1.1) we will prove that all the nonnegative nontrivial solutions of (1.2) are super-harmonic functions in $\mathbb{R}^N$ (Corollary 3.1). Then, for instance, nonexistence results of positive super-harmonic

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functions for (1.2) proved by Mitidieri in [9, 10] are now nonexistence results of positive solutions for the biharmonic equation.

Moreover, the system

$$\left.\begin{array}{rcl} -\Delta u & = & |v|^{p-1}v \\ -\Delta v & = & u^{q-1}u \end{array}\right\} \quad \text{in } \Omega \subset \mathbb{R}^N, \quad N \geq 3 \tag{1.3}$$

with u positive and v negative can be treated as a particular case of (1.1). For the system (1.3) we refer to [13, 16] and the references therein.

In the case that $\Omega = \mathbb{R}^N$, we will assume that a and b in (1.1) satisfy the following condition at infinity:

$$\begin{array}{rcl} a_p(|x|) := \left(\frac{1}{|S_{N-1}|}\int_{S_{N-1}} a(|x|\sigma)^{-1/(p-1)}\,d\sigma\right)^{1-p} & \geq & c|x|^{-\alpha} \\ b_q(|x|) := \left(\frac{1}{|S_{N-1}|}\int_{S_{N-1}} b(|x|\sigma)^{-1/(q-1)}\,d\sigma\right)^{1-q} & \geq & c|x|^{-\beta}, \end{array} \tag{1.4}$$

for some positive constant c. Let us define

$$\gamma_1(\alpha,\beta) = \frac{\alpha - 2 + (\beta-2)p}{pq-1} \text{ and } \gamma_2(\alpha,\beta) = \frac{\beta - 2 + (\alpha-2)q}{pq-1}\,. \tag{1.5}$$

Our main result for the system (1.1) in $\mathbb{R}^N$ reads as follows

Theorem 3.4 *Let* $(u,v) \in \left(C\left(\mathbb{R}^N\right)\right)^2$ *be a positive solution of* (1.1). *Let* $p \geq 1, q \geq 1$ *and* $pq > 1$. *Assume* a *and* b *are nonnegative functions defined in* $\mathbb{R}^N$ *satisfying* (1.4) *for* $|x|$ *near infinity with* α, β *such that*

$$\min\{\gamma_1(\alpha,\ \beta), \gamma_2(\alpha,\ \beta)\} \leq 0.$$

Then $u \equiv 0$ *and* $v \equiv 0$.

Ni [12] has proven that, for $\alpha < 2$, the equation

$$\Delta u = a(x)u^q \qquad \text{in} \quad \mathbb{R}^N \tag{1.6}$$

does not have any positive solution. This result was improved by F.H. Lin [6] for $\alpha \leq 2$. On the other hand for $\alpha > 2$, Ni [12], and Naito [11], among others, have proven existence results. In this case, there is no sign restriction for the function a, but now $|a(x)| \leq c|x|^{-\alpha}$. Thus $\alpha = 2$ is a *critical exponent* for the equation (1.6) in $\mathbb{R}^N$. We point out that for equation (1.6) we have $\alpha = \beta$, $p = q$, and $\gamma_1 = \gamma_2 = \frac{\alpha-2}{p-1}$. Thus, the critical exponent $\alpha = 2$ is represented now by $\min\{\gamma_1, \gamma_2\} = 0$. Therefore, Theorem 3.4 generalizes the early works [12] and [6] to the nonlinear system (1.1). In exterior domains the behavior near infinity of any solution u of

$$\Delta u = |u|^{q-1}u, \tag{1.7}$$

has been given by Véron [17].

In the case the $\Omega = B_1(0)\backslash\{0\}$, we are interested in removability results for system (1.1), that is, when all nonnegative solutions of (1.1) are bounded at zero and satisfy (1.1) in the sense of distributions in $\mathcal{D}'(B_1(0))$. The main result that we will prove in this direction is the following.

Theorem 4.3 *Let $(u,v) \in (C(B_1(0)\backslash\{0\}))^2$ be a positive solution of* (1.1) *in $B_1(0)\backslash\{0\}$. Let $p \geq 1, q \geq 1$ and $pq > 1$. Assume a and b are nonnegative functions defined in $B_1(0)\backslash\{0\}$ satisfying* (1.4) *for $|x|$ near 0, with α, β such that, either*

(i) $\min\{\gamma_1(\alpha,\ \beta),\ \gamma_2(\alpha,\ \beta)\} \geq 2 - N$, *or*

(ii) $\max\{\gamma_1(\alpha,\ \beta),\ \gamma_2(\alpha,\ \beta)\} \geq 2 - N$, $p \geq (2-\alpha)/(N-2)$, and $q \geq (2-\beta)/(N-2)$.

Then u and v are bounded near zero, and (u,v) satisfies (1.1) in $\mathcal{D}'(B_1(0))$.

Loewner and Nirenberg [8] proved removability results for (1.7) with $p = (N+2)/(N-2)$. Later, Brèsis and Véron [3] improved the Loewner-Nirenberg result for $p \geq N/(N-2)$. If $1 < p < N/(N-2)$, there are solutions of (1.7) with isolated singularities. Therefore, for equation (1.7), the critical exponent for removability results in a ball is $p = N/(N-2)$, which is exactly the condition (i) (or (ii)) in Theorem 4.3.

Finally, in the case that $\Omega = \mathbb{R}^N\backslash\{0\}$, we prove nonexistence of nonnegative solutions (singular or not) for the system (1.1). We remark that for the equation

$$\Delta u - V(|x|)u = a(x)u^p, \tag{1.8}$$

nonexistence of nonnegative sub-solutions was proven in [1] under decay conditions on $a(x)$ for x near zero and infinity. For existence results for (1.8) see also [15] .

The rest of the paper is organized as follows: In Section 2, we give some preliminary results for the one dimensional case in (1.1) in exterior domains. Section 3 is devoted to the cases where Ω in (1.1) is either the whole space or an exterior domain and in Section 4 we study removability results for (1.1). Finally, in Section 5 we prove nonexistence results in $\mathbb{R}^N\backslash\{0\}$.

2 Preliminary results

In this section we prove some results that are needed later in the proof of our main theorems. The first two lemmas are proven in [1] (see also [12] for the second one). We also need the spherical average of a function f, which is defined by

$$\bar{f}(r) = \frac{1}{|S_{N-1}|}\int_{S_{N-1}} f(r\sigma)d\sigma,$$

where $d\sigma$ *denotes the invariant measure on the sphere*

$$S_{N-1} = \left\{ x \in \mathbb{R}^N \ : \ \sum_{i=1}^{N} x_i^2 = 1 \right\}.$$

Here, $|S_{N-1}|$ *denotes the volume of the unit sphere. We denote by* $\mathbb{R}_0^N$ *the set* $\mathbb{R}^N \backslash \{0\}$*. We will say that* $(u,v) \in (C(\Omega))^2$ *is a nonnegative solution of (1.1) if* u *and* v *are nonnegative in* Ω *and* (u,v) *satisfies (1.1) in* $\mathcal{D}'(\Omega)$*.*

The following lemma is a nonexistence result for positive sub-harmonic functions with prescribed behavior at zero and at infinity (see [1])

Lemma 2.1 *Let* $u \geq 0 \in L^1_{\mathrm{loc}}(\mathbb{R}_0^N)$ *such that* $\Delta u \geq 0$ *and assume*

$$\lim_{r \to 0} r^{N-2} \bar{u}(r) = 0 \tag{2.1}$$

and

$$\lim_{r \to \infty} \bar{u}(r) = 0. \tag{2.2}$$

Then $u \equiv 0$ *in* $\mathbb{R}^N$*.*

The next lemma is used to reduce the study of a partial differential problem to the study of an ordinary differential one (see [1] and [12])

Lemma 2.2 *Let* $f(x,t) = a(x)t^p$*,* $a(x) \geq 0$*,* $p \geq 1$ *and let* v *be a nonnegative function. Then*

$$\overline{av^p}(|x|) \geq a_p(|x|)\bar{v}^p(|x|)\,, \tag{2.3}$$

where

$$a_p(r) = \left(\frac{1}{|S_{N-1}|} \int_{S_{N-1}} a(r\sigma)^{-1/(p-1)} \, d\sigma \right)^{1-p} \qquad \text{for} \quad p > 1$$

and $a_1(r) = \min_{\sigma \in S_{N-1}} a(r\sigma)$ *for* $p = 1$*. If* $\int_{S_{N-1}} a(r\sigma)^{-1/(p-1)} \, d\sigma = \infty$*, we put* $a_1(r) = 0$*.*

Having reduced the partial differential problem to an ordinary differential one, we need some previous results for solutions of system (1.1) in one dimension. To begin with, we give some power solutions for the system

$$\begin{aligned} (r^{N-1}u')' &= ar^{N-1-\alpha}v^p \\ (r^{N-1}v')' &= br^{N-1-\beta}u^q\,, \end{aligned} \tag{2.4}$$

with a *and* b *positive constants, which will play an important role in determining the regions of nonexistence as well as bounds for the solutions of (1.1). This is not surprising, since for the equation*

$$(r^{N-1}u')' = ar^{N-1-\alpha}u^q \tag{2.5}$$

those solutions have an outstanding role, too. If we try to get solutions to (2.4) of power type, that is

$$\begin{array}{rcl} u(r) & = & l_1 r^{\gamma_1(\alpha,\beta)} \\ v(r) & = & l_2 r^{\gamma_2(\alpha,\beta)}\,. \end{array} \tag{2.6}$$

We then find that l_1, l_2, γ_1 and γ_2 must satisfy

$$\begin{array}{rcl} l_1\gamma_1(\gamma_1+N-2) & = & al_2{}^p \\ l_2\gamma_2(\gamma_2+N-2) & = & bl_1{}^q \end{array} \tag{2.7}$$

and

$$\gamma_1(\alpha,\beta) = \frac{\alpha-2+(\beta-2)p}{pq-1}\,, \quad \gamma_2(\alpha,\beta) = \frac{\beta-2+(\alpha-2)q}{pq-1}\,. \tag{2.8}$$

We write at our convenience $\gamma_1(\alpha,\beta)$ and $\gamma_2(\alpha,\beta)$, but γ_1, γ_2 certainly depend also on p and q.

The existence of positive constants l_1, l_2 which satisfy (2.7) is equivalent to

$$\gamma_i(\gamma_i+N-2) > 0, \quad \text{for} \quad i = 1, 2\,.$$

We observe that for $N \geq 3$ and $\min\{\gamma_1,\gamma_2\} > 0$, we get the existence of power solutions for the system (2.4) in the whole space. This fact is very relevant in view of Theorem 3.4. Moreover, for some values of α, β, p and q we have existence of a solution of (2.4) satisfying (2.6) in $\mathbb{R}^N\backslash\{0\}$, with u bounded near zero and v going to infinity and vice versa.

Now, we state the main results of this section, that belong to the case $N = 1$ for the system (1.1). Theorem 2.1 and Theorem 2.2, or their equivalents in higher dimensions (Theorem 3.1 and Theorem 3.2), will be the key to demonstrate nonexistence results for the coming sections. The proof will be shown at the end of this section because some preliminary lemmas are required.

Theorem 2.1 *Let (w_1, w_2) be a nonnegative solution of*

$$\left.\begin{array}{rcl} \ddot{w}_1(s) & \geq & c_1 s^{-\delta_1} w_2^p \\ \ddot{w}_2(s) & \geq & c_2 s^{-\delta_2} w_1^q \end{array}\right\} \qquad \text{for all} \qquad s \geq s_0, \tag{2.9}$$

for some s_0 positive. Assume that $p, q > 0$ and $pq > 1$. Moreover, we assume that either

(i) *$\gamma_1(\delta_1,\delta_2) \leq 1$, or*

(ii) *$\gamma_2(\delta_1,\delta_2) \leq 1$ and $\delta_2 \leq q+2$.*

Then w_1 is bounded.

Similarly, we have the following

Theorem 2.2 *Let (w_1, w_2) be a nonnegative solution of (2.9) with $p, q > 0$ and $pq > 1$. Moreover, we assume that either*

(i) $\gamma_2(\delta_1, \delta_2) \leq 1$, *or*

(ii) $\gamma_1(\delta_1, \delta_2) \leq 1$ *and* $\delta_1 \leq p + 2$.

Then w_2 is bounded.

Corollary 2.1 *Let (w_1, w_2) be a nonnegative solution of (2.9) with $p, q > 0$ and $pq > 1$. Moreover, we assume that*

$$\min\{\gamma_1(\delta_1, \delta_2),\ \gamma_2(\delta_1, \delta_2)\} \leq 1. \tag{2.10}$$

Then w_1 or w_2 is bounded.

Corollary 2.2 *Let (w_1, w_2) be a nonnegative solution of (2.9) with $p, q > 0$ and $pq > 1$. Moreover, assume that either*

(i) $\max\{\gamma_1(\delta_1, \delta_2), \gamma_2(\delta_1,\ \delta_2)\} \leq 1$, *or*

(ii) $\delta_1 \leq p + 2$, $\delta_2 \leq q + 2$ *and* $\min\{\gamma_1(\delta_1, \delta_2), \gamma_2(\delta_1,\ \delta_2)\} \leq 1$.

Then w_1 and w_2 are bounded.

The next lemma is a generalization of Lemma 2.4 in [1] for a systems.

Lemma 2.3 *Let p and q be two positive real numbers such that $pq > 1$, and let (w_1, w_2) be a nonnegative solution of*

$$\begin{array}{rcl} \ddot{w}_1(s) & \geq & X_1(s) w_2^p \\ \ddot{w}_2(s) & \geq & X_2(s) w_1^q, \end{array} \tag{2.11}$$

for all $s \geq s_0$, for some $s_0 > 0$. Here $X_1(s) \geq 0$, $X_2(s) \geq 0$ are continuous and non-increasing functions on $s \geq s_0$. Moreover, we assume the following hypotheses:

(H1) $\int^\infty X_1(s)\, s^p\, ds = \infty$ *and* $\int^\infty X_2(s)\, s^q\, ds = \infty$,

(H2) *There exist three positive constants $\alpha_1 > 1$, $\alpha_2 > 1$, and c such that*

$$\frac{\alpha_1}{p+1} + \frac{\alpha_2}{q+1} = 1,$$

and, for all s large enough

$$\max\{s^{-\alpha_1+1}, s^{-\alpha_2+1}\} \leq c \int_s^\infty X_1(s)^{\alpha_1/(2(p+1))} X_2(s)^{\alpha_2/(2(q+1))}\, ds\,.$$

Then w_1 and w_2 are bounded.

Remarks *In the above lemma we have that $\alpha_1 = \alpha_2 = (p+1)/2$ for the equation (2.5). Lemma 2.3 can be generalized for more general functions than t^p and t^q.*

Proof of Lemma 2.3. *First, we will show that it is enough to consider the case in which w_1 and w_2 are both unbounded near infinity. This fact will be fundamentally a consequence of the hypothesis (H1).*

Since X_2 is nonnegative, the function w_2 is convex and we have the following two possibilities. Either:

(a) $\dot w_2(s) \leq 0$, *for all s, or*

(b) *there is an s_1, such that $\dot w_2(s) > 0$ for all $s \geq s_1$.*

If (a) holds then w_2 is bounded. If we assume that w_1 is not bounded, then $\dot w_1(s) \geq 0$ for all large s, then since w_1 is convex we get, $w_1(s) \geq cs$ for some constant c positive. By integrating (2.11), it follows that

$$\begin{aligned} \dot w_2(s) &\geq \dot w_2(s_1) + \textstyle\int_{s_1}^{s} X_2\, w_1^q(t)\, dt \\ &\geq \dot w_2(s_1) + c \textstyle\int_{s_1}^{s} X_2\, t^q\, dt. \end{aligned}$$

Hence from (H1), $\dot w_2(s)$ goes to infinity as $s \to \infty$, which contradicts (a). Thus we conclude that w_1 is bounded if w_2 is bounded.

Now, if (b) holds, arguing as in case (a) we also have $\dot w_1(s) > 0$ for large s, and $\dot w_i(s)$ goes to infinity as $s \to \infty$, for $i = 1, 2$. Therefore, we can assume that w_1 and w_2 are both unbounded.

Now, multiplying the first inequality in (2.11) by $\dot w_2$ and the second one by $\dot w_1$ and then adding both expressions, we get

$$\frac{d}{ds}(\dot w_1 \dot w_2) \geq X_1 \frac{d}{ds}\left(\frac{w_2^{p+1}}{p+1}\right) + X_2 \frac{d}{ds}\left(\frac{w_1^{q+1}}{q+1}\right) \tag{2.12}$$

for all $s \geq \tilde s$, for some $\tilde s$. Integrating (2.12) from $\tilde s$ to s we have

$$\dot w_1 \dot w_2(s) \geq \int_{\tilde s}^{s} X_1 \frac{d}{ds}\left(\frac{w_2^{p+1}}{p+1}\right) + \int_{\tilde s}^{s} X_2 \frac{d}{ds}\left(\frac{w_1^{q+1}}{q+1}\right). \tag{2.13}$$

Moreover, since X_1 and X_2 are non-increasing functions for large s, from (2.13) we get

$$\dot w_1 \dot w_2(s) \geq X_1(s)\left(\frac{w_2^{p+1}}{p+1}(s) - \frac{w_2^{p+1}}{p+1}(\tilde s)\right) + X_2(s)\left(\frac{w_1^{q+1}}{q+1}(s) - \frac{w_1^{q+1}}{q+1}(\tilde s)\right). \tag{2.14}$$

If s is large enough, $s \geq s_2$ for some s_2, we can take $w_i(s) \geq \frac{1}{2}\, w_i(\tilde{s})$, for $i = 1, 2$, and we obtain

$$\dot{w}_1(s)\dot{w}_2(s) \geq c\left(X_1(s)\; w_2^{p+1}(s) + X_2(s)\; w_1^{q+1}(s)\right), \tag{2.15}$$

for all $s \geq s_2$. Here c is a positive constant.

Now, we use the following relation between the geometric and arithmetic means

$$a_1{}^{p_1} a_2{}^{p_2} \leq \left(\frac{p_1 a_1 + p_2 a_2}{p_1 + p_2}\right)^{p_1+p_2} \tag{2.16}$$

where a_1, a_2, p_1, and p_2 are positive numbers. We can choose p_1 and p_2 as follows

$$\frac{p_1}{p_1+p_2} = \frac{\alpha_1}{p+1}, \quad \frac{p_2}{p_1+p_2} = \frac{\alpha_2}{q+1}.$$

Then if we apply (2.16) into (2.15) with a_1 and a_2 defined by

$$\frac{p_1 a_1}{p_1+p_2} = X_1 w_2{}^{p+1}, \quad \frac{p_2 a_2}{p_1+p_2} = X_2 w_1{}^{q+1}$$

we get

$$\frac{\dot{w}_1 \dot{w}_2(s)}{w_1^{\alpha_2} w_2{}^{\alpha_1}} \geq c X_1^{\alpha_1/(p+1)} X_2^{\alpha_2/(q+1)}.$$

Hence,

$$\frac{(\dot{w}_1 \dot{w}_2)^{1/2}}{w_1{}^{\alpha_2/2} w_2{}^{\alpha_1/2}} \geq c X_1^{\alpha_1/(2(p+1))} X_2^{\alpha_2/(2(q+1))}$$

which in turn implies

$$\frac{\dot{w}_1}{w_1{}^{\alpha_2}} + \frac{\dot{w}_2}{w_2{}^{\alpha_1}} \geq c X_1^{\alpha_1/(2(p+1))} X_2^{\alpha_2/(2(q+1))}$$

for all $s \geq s_2$. Then integrating from $s \geq s_2$ to ∞ we get

$$\int_{w_1(s)}^{\infty} \frac{dt}{t^{\alpha_2}} + \int_{w_2(s)}^{\infty} \frac{dt}{t^{\alpha_1}} \geq c \int_s^{\infty} X_1^{\alpha_1/(2(p+1))} X_2^{\alpha_2/(2(q+1))}\, dt,$$

which because of (H2), and since $\lim_{s\to\infty} w_i(s)/s = +\infty$, for $i = 1, 2$, gives us a contradiction.

The next result is a particular case of the above lemma, and is the key for proving the main results of this section.

Lemma 2.4 *Let (w_1, w_2) be a nonnegative solution of (2.9). Assume that $p, q > 0$ and $pq > 1$. Moreover, assume that $\delta_1 \leq p+1$ and $\delta_2 \leq q+1$. Then w_1 and w_2 are bounded near infinity.*

Proof. *Let us call* ${\delta_i}^+ = \max\{\delta_i, 0\}$ *for* $i = 1, 2$. *We can take on the above lemma,* $X_i = c_i s^{-{\delta_i}^+}$, *for* $i = 1, 2$. *Then* X_1 *and* X_2 *are non-increasing functions, and* (w_1, w_2) *is a nonnegative solution of*

$$\begin{aligned} \ddot{w}_1 &\geq X_1 w_2^p \\ \ddot{w}_2 &\geq X_2 w_2^q, \end{aligned}$$

for all s *large. We have to prove the validity of the conditions (H1) and (H2) given on the above result.*

(H1): $\int^\infty X_1 s^p = \infty$, *is equivalent with* ${\delta_1}^+ \leq p+1$, *which is satisfied since* $\delta_1 \leq p+1$. *In the same way* $\int^\infty X_2 s^q = \infty$, *since* $\delta_2 \leq q+1$.

(H2): We have to find α_1 *and* α_2 *satisfying condition (H2) on Lemma 2.3. Let us denote* $x = \alpha_1/(p+1)$ *and* $y = \alpha_2/(q+1)$. *The problem of finding* α_1 *and* α_2 *is reduced to find* x, y *which verify the following conditions*

$$x + y = 1, \quad x > \tfrac{1}{p+1}, \quad y > \tfrac{1}{q+1},$$
$$\left(2(q+1) - {\delta_2}^+\right) y \geq {\delta_1}^+ x, \ \text{and} \ \left(2(p+1) - {\delta_1}^+\right) x \geq {\delta_2}^+ y\,.$$

Let

$$a = \frac{{\delta_2}^+}{2(p+1) - {\delta_1}^+ + {\delta_2}^+} \ \text{and}\ b = \frac{2(q+1) - {\delta_2}^+}{{\delta_1}^+ + 2(q+1) - {\delta_2}^+}\,.$$

Then a *and* b *are well defined and* $(a, 1-a)$ *is the intersection of the lines* $x + y = 1$, $\left(2(p+1) - {\delta_1}^+\right) x = {\delta_2}^+ y$ *and* $(b, 1-b)$ *is the intersection of* $x + y = 1$ *with* $\left(2(q+1) - {\delta_2}^+\right) y = {\delta_1}^+ x$.

Now, since $pq > 1$ *and* $\delta_1^+ \leq p+1$, $\delta_2^+ \leq q+1$, *we always have*

$$a < \frac{q}{q+1} \quad \text{and} \quad \frac{1}{p+1} < b.$$

Also $a \leq b$, *so that*

$$A \equiv \max\left\{\frac{1}{p+1}, a\right\} \leq \min\left\{\frac{q}{q+1}, b\right\} \equiv B.$$

If $A \neq B$, *we can choose any* x *such that* $A < x < B$. *On the contrary, if* $A = B$, *it can be proved that* $A = a = b$. *In this case, we choose* $x = a$.

The above systems can have only one component bounded but not the other. This is enough for some of our purposes, as we will see on section 3. The following two lemmas are concerned with the boundedness of at least one of the components of the pair (w_1, w_2).

Lemma 2.5 *Let* (w_1, w_2) *be a positive solutions of (2.9) for some* $p, q > 0$ *and* $pq > 1$. *Let us call* $\bar{\delta}_1 \equiv \delta_1 - p - 1$, $\bar{\delta}_2 \equiv \delta_2 - q - 1$. *Assume that*

$$\bar{\delta}_1 \leq 0 \qquad \text{and} \qquad \gamma_2(\delta_1, \delta_2) \leq 1.$$

Then w_2 *is bounded.*

Proof. *The proof is divided into three cases, depending on the values of* δ_1 *and* δ_2.

Case 1: $\overline{\delta}_1 \le 0$ *and* $\overline{\delta}_2 \le 0$. *We are in the previous lemma.*

Case 2: *Assume next that* $\overline{\delta}_1 < 0$ *and* $\gamma_2(\delta_1, \delta_2) < 1$. *The condition* $\gamma_2 < 1$ *is equivalent to* $\overline{\delta}_1 q + \overline{\delta}_2 < 0$. *We proceed by contradiction. If* w_2 *is not bounded, then there exists an* s_0 *such that* $\dot{w}_2(s_0) > 0$. *Now, since* w_2 *is convex we get* $w_2(s) \ge cs$ *for all large* s *and for some nonnegative constant* c. *Going back to (2.9) we get*

$$\ddot{w}_1 \ge cs^{-\overline{\delta}_1 - 1}, \tag{2.17}$$

for all s *large enough. Integrating twice from* s_0 *to* s *in the above inequality and using the fact that* $\overline{\delta}_1 < 0$, *we obtain*

$$w_1(s) \ge cs^{-\overline{\delta}_1 + 1}, \tag{2.18}$$

for all s *large. Applying the estimate (2.18) into (2.9), we have the following for* w_2:

$$w_2(s) \ge cs^{-\overline{\delta}_2 - \overline{\delta}_1 q + 1}$$

for all large s. *Iterating the above process, as in [4], we get for* $n \in \mathbb{N}$

$$w_1(s) \ge cs^{p_n}$$

$$w_2(s) \ge cs^{q_n}$$

for s *large, where*

$$\begin{aligned} p_n &= -\delta_1 + 2 + pq_n \\ q_{n+1} &= -\delta_2 + 2 + qp_n \\ q_1 &= 1. \end{aligned}$$

(The constant c *represents any positive value). Due to the condition* $\overline{\delta}_1 q + \overline{\delta}_2 < 0$, *we deduce that the sequences* $\{p_n\}$ *and* $\{q_n\}$ *are strictly increasing. Let us call*

$$P = \lim_{n\to\infty} p_n \quad \text{and} \quad Q = \lim_{n\to\infty} q_n.$$

Then either $P = Q = \infty$ *or*

$$P = -\delta_1 + 2 + pQ \text{ and } Q = -\delta_2 + 2 + qP\,. \tag{2.19}$$

Thus, multiplying the first equation on (2.19) by q *and adding the second one, we get*

$$0 = -\overline{\delta}_1 q - \overline{\delta}_2 + (Q-1)(pq-1),$$

which is a contradiction to $Q > q_1 = 1$ *and* $-\overline{\delta}_1 q - \overline{\delta}_2 > 0$.

Now, if $P = Q = \infty$, then for all p' and q' with, $p' < p$ and $q' < q$, we have

$$\ddot{w}_1 \geq cs^{-\delta_1} {w_2}^p \geq w_2^{p'}$$
$$\ddot{w}_2 \geq cs^{-\delta_2} {w_1}^q \geq w_1^{q'} .$$

Moreover, choosing p' and q' such that $p'q' > 1$, from Lemma 2.3, we deduce that w_1 and w_2 are bounded which is a contradiction.

Case 3: $\gamma_2(\delta_1, \delta_2) = 1$ *and* $\overline{\delta}_1 < 0$. *As in the previous case, we proceed by contradiction. If w_2 is not bounded, we claim that for all $k > 0$*

$$\lim_{s\to\infty} \frac{w_1(s)}{s^k} = \infty \textit{ and } \lim_{s\to\infty} \frac{w_2(s)}{s^k} = \infty .$$

If the claim is true, then arguing as we did at the end of Case 2, we will get a contradiction. Next we will prove the claim. Since we are assuming that w_2 is not bounded, one can prove the following estimate for w_2 near infinity:

$$w_2(s) \geq cs \log s ,$$

so that

$$\lim_{s\to\infty} \frac{w_2(s)}{s} = \infty . \tag{2.20}$$

Also, w_1 and w_2 are increasing functions for large s. Integrating the first inequality on (2.9) from s to $2s$, we get

$$\dot{w}_1(2s) \geq \dot{w}_1(2s) - \dot{w}_1(s) \geq c \int_s^{2s} t^{-\delta_1} {w_2}^p(t)\, dt. \tag{2.21}$$

Hence,

$$\dot{w}_1(2s) \geq c \int_s^{2s} t^{-\delta_1} w_2^p(t)\, dt \geq c w_2^p(s) s^{-\delta_1+1} \tag{2.22}$$

Integrating (2.22) from s to $2s$, and arguing as above, we get

$$w_1(4s) \geq c {w_2}^p(s) s^{-\delta_1+2} \tag{2.23}$$

In the same way, but now starting with the second inequality on (2.9), we get

$$w_2(4s) \geq c w_1^q(s) s^{-\delta_2+2} \tag{2.24}$$

If we use (2.23) in (2.24) we obtain

$$w_2(16s) \geq c {w_2}^{pq}(s) s^{-\overline{\delta}_1 q - \overline{\delta}_2 + 1 - pq}.$$

From the hypothesis $\overline{\delta}_1 q + \overline{\delta}_2 = 0$, we then have

$$w_2(16s) \geq c w_2^{pq}(s) s^{1-pq}. \tag{2.25}$$

We rewrite (2.25) in the form

$$\frac{w_2(16s)}{16s} \geq c\left(\frac{w_2(s)}{s}\right)^{pq}. \tag{2.26}$$

For $n \in \mathbb{N}$*, choose* $s = 2^{4n}$ *in (2.26), and* $x_n = c^{1/(pq-1)} w_2(2^{4n})/2^{4n}$*. Then*

$$x_{n+1} \geq {x_n}^{pq}, \tag{2.27}$$

for all n *large,* $n \geq n_0$*, for some* n_0*. A repeated iteration on (2.27) leads to the estimate*

$$x_{n+1} \geq {x_{n_0}}^{(pq)^{n+1-n_0}},$$

for all $n \geq n_0$*. From (2.20),* $x_n \to \infty$ *as* $n \to \infty$*, then we can take* $n_0 \in \mathbb{N}$ *such that*

$$x_{n_0} > 1\,.$$

Therefore, for all $\beta > 0$ *we obtain*

$$\lim_{n\to\infty} \frac{x_{n+1}}{\left(2^{4(n+1)}\right)^{\beta}} = \infty.$$

Going back to the definition of x_n*, we deduce*

$$\lim_{n\to\infty} \frac{w_2\left(2^{4n}\right)}{\left(2^{4n}\right)^{\beta+1}} = \infty.$$

Next, we prove that $\lim_{s\to\infty} \frac{w_2(s)}{s^{\beta+1}} = \infty$*. Let* s *be sufficiently large and* $n \in \mathbb{N}$ *be such that* $s \in \left[2^{4n}, 2^{4(n+1)}\right]$*. Since* $w_2(s)$ *is nondecreasing, then*

$$\frac{w_2(s)}{s^{\beta+1}} \geq \frac{w_2\left(2^{4n}\right)}{2^{4(n+1)(\beta+1)}},$$

which implies $\lim_{s\to\infty} \frac{w_2(s)}{s^{\beta+1}} = \infty$*, for all* $\beta > 0$ *and the claim follows from (2.23).*

In analogous form, we obtain

Lemma 2.6 *Let* (w_1, w_2) *be a positive solution of (2.9) with* $p, q > 0$ *and* $pq > 1$*. Let us call* $\bar{\delta}_1 \equiv \delta_1 - p - 1$*,* $\quad\bar{\delta}_2 \equiv \delta_2 - q - 1$*. Assume that*

$$\bar{\delta}_2 \leq 0 \qquad \text{and} \qquad \gamma_1(\delta_1, \delta_2) \leq 1.$$

Then w_1 *is bounded.*

In the following lemmas, we prove that for certain values of δ_1, δ_2, p *and* q *in (2.9), if one component of the pair* (w_1, w_2) *is bounded, then the other is bounded, too. This allows extending the regions of boundedness of* w_1 *and* w_2 *obtained in previous lemmas.*

Lemma 2.7 *Let (w_1, w_2) be a positive solution of (2.9) with $p, q > 0$ and $pq > 1$. Assume that w_2 is bounded and*

$$\min\left\{\gamma_1(\delta_1, \delta_2),\ \bar{\delta}_2\right\} \le 1.$$

Then w_1 is bounded.

With respect to the boundedness of w_2, assuming boundedness of w_1, we have

Lemma 2.8 *Let (w_1, w_2) be a positive solution of (2.9) with $p, q > 0$ and $pq > 1$. Assume that w_1 is bounded and*

$$\min\left\{\gamma_2(\delta_1, \delta_2),\ \bar{\delta}_1\right\} \le 1.$$

Then w_2 is bounded.

Proof of Lemma 2.7. *We distinguish two cases, according to whether $\gamma_1 \le 1$ or $\bar{\delta}_2 \le 1$. We assume first*

Case 1: *$\gamma_1 \le 1$. This case is equivalent to $\bar{\delta}_1 + p\bar{\delta}_2 \le 0$. Now, since w_2 is bounded at infinity it must be a non-increasing function for all s large. Suppose by contradiction that w_1 is not bounded near infinity. Then w_1 is increasing for s large enough. Integrating the first inequality on (2.9) from $s/2$ to s it follows that*

$$\dot{w}_1(s) \ge c\left(\int_{s/2}^{s} t^{-\delta_1}\right) w_2{}^p(s) \ge c s^{-\delta_1+1} w_2{}^p(s)\,. \tag{2.28}$$

Integrating once again from $s/2$ to s in (2.28), we obtain

$$w_1(s) \ge c s^{-\delta_1+2} w_2{}^p(s). \tag{2.29}$$

Similarly, but now integrating from s to $2s$ in the second inequality of (2.9), we get

$$w_2(s) \ge c s^{-\delta_2+2} w_1{}^q(s). \tag{2.30}$$

Therefore, by using (2.29) and (2.30), in the first inequality of (2.9) we have the following for w_1

$$\begin{aligned} \ddot{w}_1 &\ge c s^{-\delta_1+p(-\delta_2+2)} w_1{}^{pq} \\ &\equiv c s^{-\gamma} w_1{}^{pq}, \end{aligned} \tag{2.31}$$

where $\gamma = \delta_1 - p(-\delta_2 + 2)$. By the assumption $\overline{\delta}_1 + p\overline{\delta}_2 \le 0$, it follows that $\gamma \le pq + 1$. Thus, by Lemma 2.4 (see also [1]) w_1 must be bounded.

Case 2: $\overline{\delta}_2 \le 1$. *Assume that w_1 is not bounded, then $w_1 \ge cs$ for s large. As before, from (2.9) it follows that*

$$w_2(s) \ge cs^{-\overline{\delta}_2+1}\,,$$

which in turn implies that $w_2 \to \infty$ as $s \to \infty$ if $\overline{\delta}_2 < 1$. Now, if $\overline{\delta}_2 = 1$ we get the same conclusion by integrating in

$$\ddot{w}_2(s) \ge cs^{-\delta_2} {w_1}^q \ge cs^{-1}.$$

Remark *Theorem 2.1 is a consequence of Lemma2 2.5, 2.6, and 2.7. Similarly, Theorem 2.2 is a consequence of the Lemmas 2.5, 2.6 and 2.8.*

3 Nonexistence in $\mathbb{R}^N$

In this section we consider Ω in (1.1) to be either an exterior domain, for instance $\Omega = \{x : |x| \ge 1\}$, or $\Omega = \mathbb{R}^N$. For exterior domains, we will give bounds near infinity for one or both of the components of the pair (u, v), where (u, v) is a nonnegative solution of (1.1) (Theorem 3.1, Theorem 3.2 and Theorem 3.3). In the whole space we will prove a nonexistence result, Theorem 3.4, for nonnegative nontrivial solutions of (1.1). We remark that Theorem 3.4 is optimal for the system (2.4).

Throughout this section we will assume that a and b are nonnegative functions in $L^\infty_{\rm loc}(\Omega)$. Moreover, there exist three constants α, β and c, with c positive, such that

$$\left.\begin{array}{rcl} a_p(|x|) & \ge & c|x|^{-\alpha} \\ b_q(|x|) & \ge & c|x|^{-\beta} \end{array}\right\} \qquad \text{at infinity,} \tag{3.1}$$

where a_p and b_q are defined in Lemma 2.2.

Theorem 3.1 *Let $(u, v) \in (C(|x| \ge 1))^2$ be a positive solution of*

$$\left.\begin{array}{rcl} \Delta u & \ge & a(x)v^p \\ \Delta v & \ge & b(x)u^q \end{array}\right\} \qquad \text{in} \qquad |x| \ge 1\,, \tag{3.2}$$

where $p \ge 1, q \ge 1$ and $pq > 1$. Assume a and b are nonnegative functions defined in $|x| \ge 1$ and satisfying (3.1) with α, β such that either

(i) $\gamma_1(\alpha,\ \beta) \le 0$, *or*

(ii) $\gamma_2(\alpha,\ \beta) \le 0$ *and* $\beta \le N$.

Then $|x|^{N-2}u$ is bounded.

For the equation (1.6) the conditions (i) *and* (ii) *in Theorem 3.1 are equivalent with* $\alpha \leq 2$.

Before proving Theorem 3.1 let us enunciate the boundedness for v.

Theorem 3.2 *Let* $(u, v) \in (C(|x| \geq 1))^2$ *be a positive solution of (3.2). Let* $p \geq 1, q \geq 1$ *and* $pq > 1$. *Assume* a *and* b *are nonnegative functions defined in* $|x| \geq 1$ *and satisfying (3.1) with* α, β *such that either*

(i) $\gamma_2(\alpha,\ \beta) \leq 0$, *or*

(ii) $\gamma_1(\alpha,\ \beta) \leq 0$ *and* $\alpha \leq N$.

Then $|x|^{N-2}v$ *is bounded.*

Proof of Theorem 3.1. *From (3.1), (3.2) and Lemma 2.2, we have*

$$\begin{array}{rcl} \bar{u}'' + \frac{N-1}{r}\bar{u}' & \geq & cr^{-\alpha}\, \bar{v}^p \\ \bar{v}'' + \frac{N-1}{r}\bar{v}' & \geq & cr^{-\beta}\, \bar{u}^q, \end{array} \tag{3.3}$$

for all r *large enough. Let* $s = r^{N-2}$ *and let*

$$\begin{array}{rcl} w_1(s) & = & s\bar{u}(r) \\ w_2(s) & = & s\bar{v}(r)\,. \end{array}$$

Then w_1 *and* w_2 *satisfy*

$$\begin{array}{rcl} \ddot{w}_1(s) & \geq & cs^{-\delta_1} w_2^p \\ \ddot{w}_2(s) & \geq & cs^{-\delta_2} w_1^q \end{array} \tag{3.4}$$

where

$$\delta_1 = \frac{\alpha - 2}{N-2} + p + 1 \quad and \quad \delta_2 = \frac{\beta - 2}{N-2} + q + 1.$$

It follows from the hypothesis on α, β, p, q *and Theorem 2.1 that* w_1 *is bounded. Thus, from the definition of* w_1, *we get that* $r^{N-2}\bar{u}$ *is bounded. To prove that* $|x|^{N-2}u$ *is also bounded we use the following mean value inequality for subharmonic functions (see [5])*

$$u(x) \leq \frac{1}{|B_{|x|/2}(x)|} \int_{B_{|x|/2}(x)} u(y)\, dy,$$

then

$$u(x) \leq c|x|^{-N} \int_{|x|/2}^{3|x|/2} r^{N-1}\bar{u}(r)\, dr. \tag{3.5}$$

Since $r^{N-2}\bar{u}$ *is bounded for* r *large enough and* u *satisfies (3.5), then the conclusion of the theorem follows.*

Next we apply the previous results to the biharmonic.

Corollary 3.1 *Let $q > 1$ and $u \in C^2\left(\mathbb{R}^N\right)$ be a positive solution of*

$$\Delta^2 u = b(x)u^q \qquad \textit{in} \qquad \mathbb{R}^N \tag{3.6}$$

Assume that b is a nonnegative function defined in $\mathbb{R}^N$ and satisfying

$$b_q(x) \geq c|x|^{-\beta}, \qquad \textit{for all} \quad |x| \quad \textit{large},$$

with $\beta \leq 2(q+1)$. Then u is a super-harmonic function in $\mathbb{R}^N$.

Proof. *Let us define $v := \Delta u$. Then, the pair (u, v) is a solution for*

$$\left.\begin{array}{rcl} \Delta u & = & v \\ \Delta v & = & b(x)u^q \end{array}\right\} \qquad \textit{in} \qquad \mathbb{R}^N\,. \tag{3.7}$$

Since v is a sub-harmonic function in $\mathbb{R}^N$ we get the following two possibilities for $\bar{v}$, either

(1) *There is a positive r_0 so that $\bar{v}(r) \geq 0$, for all r larger than r_0. Moreover, $\lim_{r\to\infty} r^{N-2}\bar{v}(r) = \infty$, or*

(2) *$\bar{v}(r) \leq 0$, for all $r > 0$.*

Theorem 3.2 and the hypothesis on β imply that case 1 is impossible and then $\bar{v} \leq 0$. Repeating the above argument for the functions $v_y(x) := v(x+y)$ with $y \in \mathbb{R}^N$, we obtain that $\overline{v_y} \leq 0$ for all y. Then the conclusion follows. As a consequence of the two previous theorems we obtain the following, which gives us the boundedness of u and v at the same time.

Corollary 3.2 *Let $(u, v) \in \left(C\left(|x| \geq 1\right)\right)^2$ be a positive solution of (3.2). Let $p \geq 1, q \geq 1$ and $pq > 1$. Assume a and b are nonnegative functions defined in $|x| \geq 1$ satisfying (3.1) with α, β such that either*

(i) $\max\{\gamma_1(\alpha,\beta), \gamma_2(\alpha,\ \beta)\} \leq 0$, *or*

(ii) $\alpha \leq N,\ \beta \leq N$ *and* $\min\{\gamma_1(\alpha,\ \beta), \gamma_2(\alpha,\ \beta)\} \leq 0$.

Then $|x|^{N-2}u$ and $|x|^{N-2}v$ are bounded.

Our main result of this section, in a way, extends those of [12] and [6].

Theorem 3.3 *Let $(u, v) \in \left(C\left(\mathbb{R}^N\right)\right)^2$ be a positive solution of*

$$\left.\begin{array}{rcl} \Delta u & \geq & a(x)v^p \\ \Delta v & \geq & b(x)u^q \end{array}\right\} \qquad \textit{in} \qquad \mathbb{R}^N\,, \tag{3.8}$$

Let $p \geq 1, q \geq 1$ and $pq > 1$. Assume a and b are nonnegative functions defined in $\mathbb{R}^N$ and satisfying (3.1) with α, β such that

$$\min\{\gamma_1(\alpha,\ \beta), \gamma_2(\alpha,\ \beta)\} \leq 0 \tag{3.9}$$

Then $u \equiv 0$ and $v \equiv 0$.

Proof. *The proof follows from Lemma 2.1, Theorem 3.1, and Theorem 3.2.*

Remark. *For the equation (1.6), condition (3.9) in the above theorem is the well known condition* $\alpha \leq 2$ *(see [12] and [6]). If (3.9) in the above theorem is not satisfied, then* γ_1 *and* γ_2 *are both positive. Therefore, we can get a positive radial solution* (u, v) *for the system* (2.4) *in* $\mathbb{R}^N$*, with* $u(r) = l_1 r^{\gamma_1}$ *and* $v(r) = l_2 r^{\gamma_2}$.

4 Removable singularities

Brèsis and Véron ([3]) have proven removable singularities for nonlinear elliptic equations in a ball. In the sequel we give the same type of result but now for a system. To obtain the behavior of solutions to (1.1) at zero, we use the Kelvin transform together with the results in section 3. Let $B_1(0)$ *be the open unit ball centered at zero of* $\mathbb{R}^N$*, with* $N \geq 3$*. Throughout this section the functions* a *and* b *are nonnegative functions in* $L^\infty_{loc}(B_1(0)\backslash\{0\})$ *such that*

$$\left.\begin{array}{rcl} a_p(|x|) & \geq & c|x|^{-\alpha} \\ b_q(|x|) & \geq & c|x|^{-\beta} \end{array}\right\} \qquad \text{for all } x \text{ small,} \tag{4.1}$$

for some positive constant c*, and* a_p *and* b_q *defined in Lemma 2.2.*

Theorem 4.1 *Let* $(u, v) \in \left(C\left(B_1(0)\backslash\{0\}\right)\right)^2$ *be a positive solution of*

$$\left.\begin{array}{rcl} \Delta u & \geq & a(x)v^p \\ \Delta v & \geq & b(x)u^q \end{array}\right\} \qquad in \quad B_1(0)\backslash\{0\} \tag{4.2}$$

where $p \geq 1, q \geq 1$*, and* $pq > 1$*. Assume that* a *and* b *are nonnegative functions satisfying (4.1) with* α, β *such that either*

(i) $\gamma_1(\alpha,\ \beta) \geq 2 - N$*, or*

(ii) $\gamma_2(\alpha,\ \beta) \geq 2 - N$ *and* $q \geq (2-\beta)/(N-2)$.

Then u *is bounded near zero.*

Proof. *This result is a consequence of those of section 3; we transform our problem near zero to a problem near infinity. Let* u_1 *and* v_1 *be the Kelvin transform of* u *and* v*, that is*

$$\left.\begin{array}{rcl} u_1(x) & = & |x|^{2-N}\, u(x/|x|^2) \\ v_1(x) & = & |x|^{2-N}\, v(x/|x|^2) \end{array}\right\} \qquad for \quad |x| \geq 1\,,$$

then, (u_1, v_1) *satisfies ([5])*

$$\left.\begin{array}{rcl} \Delta u_1 & \geq & a_1(x)v_1^p \\ \Delta v_1 & \geq & b_1(x)u_1^q \end{array}\right\} \qquad for \quad |x| \geq 1\,, \tag{4.3}$$

where a_1 *and* b_1 *satisfy*

$$\begin{array}{rcccl} a_1(x) & = & |x|^{(N-2)p-(N+2)}a(x/|x|^2) & \geq & c|x|^{-\alpha_1} \\ b_1(x) & = & |x|^{(N-2)q-(N+2)}b(x/|x|^2) & \geq & c|x|^{-\beta_1}, \end{array}$$

and α_1, β_1 *are defined by*

$$\begin{array}{rcl} \alpha_1 & = & N+2-(N-2)p-\alpha \\ \beta_1 & = & N+2-(N-2)q-\beta. \end{array}$$

Then we obtain

$$\gamma_i(\alpha_1,\beta_1) = -\gamma_i(\alpha,\beta) - (N-2), \quad \textit{for i=1,2.}$$

From here, we easily get that α_1, β_1, p *and* q *satisfy the hypotheses of Theorem 3.1, thus* $|x|^{N-2}u_1$ *is bounded at infinity. Therefore* u *is bounded near zero.*

Remark. *If in the previous theorem,* $p = q$, $\alpha = 0 = \beta$ *and* $u = v$, *then we obtain Theorem 1 of [3]. In this case conditions* (i) *and* (ii) *on Theorem 4.1 are equivalent to* $p \geq N/(N-2)$. *Analogously, we get for* v *the following theorem:*

Theorem 4.2 *Let* $(u,v) \in (C(B_1(0)\backslash\{0\}))^2$ *be a positive solution of (4.2), where* $p \geq 1, q \geq 1$, *and* $pq > 1$. *Assume that* a *and* b *are nonnegative functions satisfying (4.1) with* α, β *such that either*

(i) $\gamma_2(\alpha,\ \beta) \geq 2-N$, *or*

(ii) $\gamma_1(\alpha,\ \beta) \geq 2-N$ *and* $p \geq (2-\alpha)/(N-2)$.

Then v *is bounded near zero.*

The intersection of the region of (α,β) *where* u *is bounded with the region where* v *is bounded gives us the main result of the section.*

Theorem 4.3 *Let* $(u,v) \in (C(B_1(0)\backslash\{0\}))^2$ *be a positive solution of* (4.2), *where* $p \geq 1, q \geq 1$, *and* $pq > 1$. *Assume that* a *and* b *are nonnegative functions satisfying* (4.1) *with* α, β *such that either*

(i) $\min\{\gamma_1(\alpha,\ \beta),\ \gamma_2(\alpha,\ \beta)\} \geq 2-N$, *or*

(ii) $\max\{\gamma_1(\alpha,\ \beta),\ \gamma_2(\alpha,\ \beta)\} \geq 2-N$ *and* $p \geq (2-\alpha)/(N-2)$, $q \geq (2-\beta)/(N-2)$.

Then u *and* v *are bounded near zero, and* (u,v) *satisfies (4.2) in* $\mathcal{D}'(B_1(0))$.

As a consequence of the above result we can state the following for the biharmonic case:

Corollary 4.1 *Let $u \in C^2(B_1(0)\backslash\{0\})$ be a positive sub-harmonic solution of*

$$\Delta^2 u = |x|^{-\beta} u^q, \tag{4.4}$$

where $q > 1$. Assume that either

(i) $\beta \geq 4$, *or*

(ii) $N > 4$, $\beta < 4$ *and* $q \geq (N+2-\beta)/(N-2)$.

Then u is bounded near zero. Moreover, u satisfies (4.4) in $\mathcal{D}'(B_1(0))$.

Soranzo [14] has proven removability results for nonnegative super-harmonic solutions of (4.4). We remark that for a radially symmetric nonnegative solution u of (4.4), we get that u is either sub-harmonic or super-harmonic near zero.

5 Nonexistence of singular solutions in $\mathbb{R}^N\backslash\{0\}$.

This section is devoted to nonexistence results of nonnegative solutions (singular or not) for (1.1) in $\mathbb{R}^N\backslash\{0\}$. These results can be obtained as a consequence of those of the previous sections. We give them without proof.

Throughout this section the functions a and b are nonnegative functions in $L^\infty_{loc}(\mathbb{R}^N\backslash\{0\})$. In some of the next results we need also the following properties for a and b

$$\left.\begin{array}{rcl} a_p(|x|) & \geq & c|x|^{-\alpha_0} \\ b_q(|x|) & \geq & c|x|^{-\beta_0} \end{array}\right\} \qquad \text{for all } x \text{ small} \tag{5.1}$$

and

$$\left.\begin{array}{rcl} a_p(|x|) & \geq & c|x|^{-\alpha_\infty} \\ b_q(|x|) & \geq & c|x|^{-\beta_\infty} \end{array}\right\} \qquad \text{for all } x \text{ large enough} \tag{5.2}$$

where a_p and b_q are defined in Lemma 2.2, and c is some positive constant.

Theorem 5.1 *Let $(u,v) \in \left(C\left(\mathbb{R}^N\backslash\{0\}\right)\right)^2$ be a positive solution of*

$$\left.\begin{array}{rcl} \Delta u & \geq & a(x)v^p \\ \Delta v & \geq & b(x)u^q \end{array}\right\} \qquad \text{in} \quad \mathbb{R}^N\backslash\{0\} \tag{5.3}$$

where $p \geq 1, q \geq 1$, with $pq > 1$. Moreover, we assume that a and b satisfy (5.1), with α_0, β_0 satisfying either

(i) $\gamma_1(\alpha_0,\ \beta_0) \geq 2 - N$, *or*

(ii) $\gamma_2(\alpha_0,\ \beta_0) \geq 2 - N$ *and* $q \geq (2-\beta_0)/(N-2)$.

Then the system (5.3) does not possess any positive solution (u,v) with u going to 0 at infinity.

Likewise, we get the following

Theorem 5.2 *Let* $(u,v) \in \left(C\left(\mathbb{R}^N\backslash\{0\}\right)\right)^2$ *be a positive solution of (5.3). Let* $p \geq 1, q \geq 1$*, and* $pq > 1$*. Moreover, we assume that* a *and* b *satisfy (5.1), with* α_0, β_0 *satisfying either*

(i) $\gamma_2(\alpha_0,\ \beta_0) \geq 2-N$*, or*

(ii) $\gamma_1(\alpha_0,\ \beta_0) \geq 2-N$ *and* $p \geq (2-\alpha_0)/(N-2)$.

Then the system (5.3) does not posses any positive solution (u,v) *with* v *going to* 0 *at infinity.*

In [1] Benguria, Lorca, and Yarur prove, among others, the nonexistence of nonnegative singular solutions for the equation (1.6), with decay conditions on $a(x)$ *for* x *near zero and infinity. Our next two results extend those of [1] to the system (5.3).*

Theorem 5.3 *Let* $(u,v) \in \left(C\left(\mathbb{R}^N_0\right)\right)^2$ *be a positive solution of (5.3). Let* $p \geq 1, q \geq 1$ *and* $pq > 1$*. Moreover, we assume that* $a(x)$*,* $b(x)$ *satisfies (5.1) and (5.2). Suppose that* α_∞ *and* β_∞ *are such that either*

(i) $\gamma_1(\alpha_\infty,\ \beta_\infty) \leq 0$*, or*

(ii) $\gamma_2(\alpha_\infty,\ \beta_\infty) \leq 0$ *and* $\beta_\infty \leq N$.

For α_0 *and* β_0 *we assume that either*

$(\mathrm{i})_0$ $\gamma_1(\alpha_0,\ \beta_0) \geq 2-N$*, or*

$(\mathrm{ii})_0$ $\gamma_2(\alpha_0,\ \beta_0) \geq 2-N$*, and* $q \geq (2-\beta_0)/(N-2)$.

Then $u \equiv 0$ *and* $v \equiv 0$.

Theorem 5.4 *Let* $(u,v) \in \left(C\left(\mathbb{R}^N_0\right)\right)^2$ *be a positive solution of (5.3). Let* $p \geq 1$*,* $q \geq 1$ *and* $pq > 1$*. Moreover, we assume that* $a(x)$*,* $b(x)$ *satisfy (5.1) and (5.2). Suppose that* α_∞ *and* β_∞ *are such that either*

(i) $\gamma_2(\alpha_\infty,\ \beta_\infty) \leq 0$*, or*

(ii) $\gamma_1(\alpha_\infty,\ \beta_\infty) \leq 0$ *and* $\alpha_\infty \leq N$.

For α_0 *and* β_0 *we assume that either*

$(\mathrm{i})_0$ $\gamma_2(\alpha_0,\ \beta_0) \geq 2-N$*, or*

$(\mathrm{ii})_0$ $\gamma_1(\alpha_0,\ \beta_0) \geq 2-N$ *and* $p \geq (2-\alpha_0)/(N-2)$.

Then $u \equiv 0$ *and* $v \equiv 0$.

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