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NONLINEAR PARABOLIC-ELLIPTIC SYSTEM IN MUSIELAK-ORLICZ-SOBOLEV SPACES

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Abstract. The existence of a capacity solution to the thermistor problem in the context of inhomogeneous Musielak-Orlicz-Sobolev spaces is analyzed. This is a coupled parabolic-elliptic system of nonlinear PDEs whose unknowns are the temperature inside a semiconductor material, u, and the electric potential, φ. We study the general case where the nonlinear elliptic operator in the parabolic equation is of the form $Au = -\operatorname{div} a(x,t,u,\nabla u)$, A being a Leray-Lions operator defined on $W_0^{1,x}L_M(Q_T)$, where M is a generalized N-function.

1. Introduction

In the previous decade, there has been an increasing interest in the study of various mathematical problems in modular spaces. These problems have many consideration in applications [8, 21, 23] and have resulted in a renewal interest in Lebesgue and Sobolev spaces with variable exponent, or the general Musielak-Orlicz spaces, the origins of which can be traced back to the work of Orlicz in the 1930s. In the 1950s, this study was carried on by Nakano [19] who made the first systematic study of spaces with variable exponent. Later on, Polish and Czechoslovak mathematicians investigated the modular function spaces (see, for instance, Musielak [18], Kovacik and Rakosnik [16]). The study of variational problems where the function a satisfies a nonpolynomial growth conditions instead of having the usual p-structure arouses much interest with the development of applications to electro-rheological fluids as an important class of non-Newtonian fluids (sometimes referred to as smart fluids). The electro-rheological fluids are characterized by their ability to drastically change the mechanical properties under the influence of an external electromagnetic field. A mathematical model of electro-rheological fluids was proposed by Rajagopal and Ruzicka (we refer to [20, 21] for more details). Another important application is related to image processing [22] where this kind of diffusion operator is used to underline the borders of the distorted image and to eliminate the noise.

From a mathematical standpoint, it is a hard task to show the existence of classical solutions, i.e., solutions which are continuously differentiable as many times

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as the order of the differential equations under consideration. However, the concept of weak solution is not enough to give a formulation to all problems and may not provide existence or stability properties. This is the case when we are dealing with nonuniformly elliptic problems, as in the problem

$$\begin{gathered}\frac{\partial u}{\partial t}-\operatorname{div}\Big(\frac{m\big(x,|\nabla u|\big)}{|\nabla u|}\nabla u\Big)=\rho(u)|\nabla\varphi|^2\quad\text{in } Q_T=\Omega\times(0,T),\\ \operatorname{div}(\rho(u)\nabla\varphi)=0 \text{ in } Q_T, u=0,\quad \varphi=\varphi_0\quad\text{on } \partial\Omega\times(0,T),\\ u(x,0)=u_0(x)\quad\text{in } \Omega,\end{gathered}\tag{1.1}$$

where $\Omega\subset\mathbb{R}^d$, $d\geq 2$, is an open and bounded set and $\rho\in C(\mathbb{R})\cap L^\infty(\mathbb{R})$ is such that $\rho(s)>0$ for all $s\in\mathbb{R}$. In this situation, one readily realizes that the search of weak solutions to problem (1.1) are not well suited. Indeed, $\rho(s)$ may converge to zero as $|s|$ tends to infinity and as a result, if u is unbounded in $\Omega\times(0,T)$, the elliptic equation becomes degenerate at points where u is infinity and, therefore, no a priori estimates for $\nabla\varphi$ will be available and thus, φ may not belong to a Sobolev space. Instead of φ, we may consider the function $\Phi=\rho(u)|\nabla\varphi|^2$ as a whole and then show that it belongs to $L^2(\Omega)^d$. This means that a new formulation of the original system is possible and the solution to this new formulation will be called capacity solution. This concept was first introduced in the 1990s by Xu in [24] in the analysis of a modified version of the thermistor problem where the monotone mapping $a=a(\nabla u)$ is a Leray-Lions operator from $L^2(H^1)$ to $L^2(H^{-1})$. The same author applied this concept to more general settings where weaker assumptions [25] or mixed boundary conditions [26] are considered. Later, González Montesinos and Ortegón Gallego [14] showed the existence of a capacity solution to problem (1.2) where a is a Leray-Lions operator from $L^p(W^{1,p})$ into $L^{p'}(W^{-1,p'})$, $p\geq 2$, $1/p+1/p'=1$. In a recent paper, the existence of a capacity solution in the context of Orlicz-Sobolev spaces has been established by Moussa, Ortegón Gallego and Rhoudaf [17]. The analysis developed in the present paper is a generalization to that given in [17]. Our framework is the Musielak-Orlicz-Sobolev spaces.

This paper deals with the existence of a capacity solution to a coupled system of parabolic-elliptic equations, whose unknowns are the temperature inside a semiconductor material, u, and the electric potential, φ, namely

$$\begin{gathered}\frac{\partial u}{\partial t}-\operatorname{div}a(x,t,u,\nabla u)=\rho(u)|\nabla\varphi|^2\quad\text{in } Q_T=\Omega\times(0,T),\\ \operatorname{div}(\rho(u)\nabla\varphi)=0\quad\text{in } Q_T,\\ \varphi=\varphi_0\quad\text{on } \partial\Omega\times(0,T),\\ u(x,0)=u_0(x)\quad\text{in } \Omega,\\ u=0\quad\text{on } \partial\Omega\times(0,T),\end{gathered}\tag{1.2}$$

where $\Omega\subset\mathbb{R}^d$, $d\geq 2$, is the space region occupied by the semiconductor, $T>0$ is the final time of observation, $Au=-\operatorname{div}a(x,t,u,\nabla u)$ is a Leray-Lions operator defined on $W_0^{1,x}L_M(\Omega)$, M is a generalized N-function, and the functions φ_0 and u_0 are given. The functional spaces to deal with these problems are Musielak-Orlicz-Sobolev spaces. In general, Orlicz-Sobolev spaces are neither reflexive nor separable.

Problem (1.2) may be regarded as a generalization of the so-called thermistor problem arising in electromagnetism [4, 13, 14].

Our analysis makes extensively use of the notion of modular convergence in Musielak-Orlicz spaces. The fundamental studies in this direction are due to Gossez for the case of elliptic equations [11, 12]. The considerations of the problem with an x-dependent modular function formulated in Musielak-Orlicz-Sobolev spaces are due to Benkirane et al. [7] where the authors formulate an approximation theorem with respect to the modular topology. A particular case of Musielak-Orlicz spaces with an x-dependent modular function are the variable exponent spaces $L^{p(x)}(\Omega)$ for which $M(x,t) = |t|^{p(x)}$ [5]. Other possible choices are

$$M(x,t) = |t|^{p(x)} \log(1+|t|),$$
$$M(x,t) = |t| \log(1+|t|)(\log(\tau_0+|t|))^{p(x)}, \text{ for some } \tau_0 \geq 1,$$
$$M(x,t) = \exp\left(|t|^{p(x)}\right) - 1.$$

The reader is referred to [5] for an extensive analysis on the theory of quasilinear of parabolic (and hyperbolic) equations related to some variable exponent spaces, including the $L^{p(x)}(\Omega)$ spaces, and to [9] for a comprehensive summary on these generalized modular spaces.

The main goal of this article is to prove the existence of a capacity solution of (1.2) in the sense of Definition 4.2 for a generalized N-function, M, along with the lack of reflexivity in this setting combined with the nonuniformly elliptic character of the second differential equation.

This work is organized as follows. In Section 2 we recall some well-known properties and results on Musielak-Orlicz-Sobolev spaces. Section 3 is devoted to specify the assumptions on data. In Section 4 we give the definition of a capacity solution of (1.2). Finally, in Section 5 we present the existence result and develop its proof.

2. Preliminaries

In this section we list some definitions and facts about Musielak-Orlicz-Sobolev spaces. Standard reference is [18]. We also include the definition of inhomogeneous Musielak-Orlicz-Sobolev spaces and some preliminaries lemmas to be used later on this paper.

Musielak-Orlicz spaces. Let Ω be a bounded domain in $\mathbb{R}^d$, $d \in \mathbb{N}$.

Definition 2.1. Let $M\colon \Omega \times \mathbb{R} \mapsto \mathbb{R}$ satisfying the following conditions:

(i) For a.a. $x \in \Omega$, $M(x,\cdot)$ is an N-function, that is, convex and even in $\mathbb{R}$, increasing in $\mathbb{R}^+$, $M(x,0)=0$, $M(x,t)>0$ for all $t>0$, $M(x,t)/t \to 0$ as $t \to 0$, $M(x,t)/t \to \infty$ as $t \to \infty$).

(ii) For all $t \in \mathbb{R}$, $M(\cdot,t)$ is a measurable function.

A function $M(x,t)$ which satisfies the conditions (i) and (ii) is called a *Musielak-Orlicz function*, a generalized N-function or a generalized modular function.

From now on, $M\colon \Omega \times \mathbb{R} \mapsto \mathbb{R}$ will stand for a general Musielak-Orlicz function. Notice that

$$\operatorname{ess\,inf}_{x\in\Omega} \frac{M(x,t)}{t} \to \infty \quad \text{as } t \to \infty. \tag{2.1}$$

Indeed, by the definition of $\operatorname{ess\,inf}_{x\in\Omega} M(x,t)$ we have for all $\epsilon > 0$ there exist a measurable $\Omega_\epsilon \subset \Omega$, $\operatorname{meas}(\Omega_\epsilon) > 0$ such that

$$M(y,t) \leq \operatorname{ess\,inf}_{x\in\Omega} M(x,t) + \epsilon, \quad \text{for all } y \in \Omega_\epsilon,$$

dividing by t we obtain

$$\frac{M(y,t)}{t} \le \operatorname{ess\,inf}_{x\in\Omega} \frac{M(x,t)}{t} + \frac{\epsilon}{t}, \quad \text{for all } y \in \Omega_\epsilon,$$

and letting $t \to \infty$, using (i), we obtain (2.1).

In some situations, the growth order with respect to t of two given Musielak-Orlicz functions M and P are comparable. This concept is detailed in the next definition.

Definition 2.2. Let $M, P\colon \Omega \times \mathbb{R} \mapsto \mathbb{R}$ be Musielak-Orlicz functions.

- Assume that there exist two constants $\epsilon > 0$ and $t_0 \ge 0$ such that for a. a. $x \in \Omega$ one has
$$P(x,t) \le M(x,\epsilon t) \text{ for all } t \ge t_0,$$
then we write $P \prec M$ and we say that M dominates P globally if $t_0 = 0$ and near infinity if $t_0 > 0$.
- We say that P grows essentially less rapidly than M at $t = 0$ (respectively, near infinity), and we write $P \ll M$, if for every positive constant k we have
$$\lim_{t\to 0} \sup_{x\in\Omega} \frac{P(x,kt)}{M(x,t)} = 0 \text{ (respectively, } \lim_{t\to\infty} \sup_{x\in\Omega} \frac{P(x,kt)}{M(x,t)} = 0).$$

We will also use the following notation: $M_x(t) = M(x,t)$, for a. a. $x \in \Omega$ and all $t \in \mathbb{R}$, and we associate its inverse function with respect to $t \ge 0$, denoted by M_x^{-1}, that is,

$$M_x^{-1}(M(x,t)) = M(x, M_x^{-1}(t)) = t, \text{ for all } t \ge 0.$$

Remark 2.3. It is easy to check that $P \ll M$ near infinity if and only if

$$\lim_{t\to\infty} \frac{M^{-1}(x,kt)}{P^{-1}(x,t)} = 0 \quad \text{uniformly for } x \in \Omega \setminus \Omega_0$$

for some null subset $\Omega_0 \subset \Omega$.

We define the functional

$$\varrho_{M,\Omega}(u) = \int_\Omega M(x,u(x))\,\mathrm{d}x,$$

for any Lebesgue measurable function $u\colon \Omega \mapsto \mathbb{R}$ is a Lebesgue measurable function.

The set

$$\mathcal{L}_M(\Omega) = \{u\colon \Omega \mapsto \mathbb{R} \text{ measurable such that } \varrho_{M,\Omega}(u) < \infty\}$$

is called the Musielak-Orlicz class related to M in Ω or simply the Musielak-Orlicz class.

The Musielak-Orlicz space $L_M(\Omega)$ is the vector space generated by $\mathcal{L}_M(\Omega)$, that is, $L_M(\Omega)$ is the smallest linear space containing the set $\mathcal{L}_M(\Omega)$. Equivalently,

$$L_M(\Omega) = \{u\colon \Omega \mapsto \mathbb{R} \text{ measurable such that } \varrho_{M,\Omega}(u/\alpha) < \infty, \text{ for some } \alpha > 0\}.$$

For a Musielak-Orlicz function M, we introduce its complementary function, denoted by $\bar{M}$, as

$$\bar{M}(x,s) = \sup_{t\ge 0}\{st - M(x,t)\},$$

that is $\bar{M}(x,s)$ is the complementary to $M(x,t)$ in the sense of Young with respect to the variable s. It turns out that $\bar{M}$ is another Musielak-Orlicz function and the following Young-Fenchel inequality holds

$$|ts| \le M(x,t) + \bar{M}(x,s) \quad \text{for all } t,s \in \mathbb{R} \text{ and a. a. } x \in \Omega. \tag{2.2}$$

In the space $L_M(\Omega)$ we define the following two norms:

$$\|u\|_{M,\Omega} = \inf\big\{\lambda > 0 : \int_\Omega M(x, u(x)/\lambda)\,\mathrm{d}x \le 1\big\},$$

which is called the Luxemburg norm, and the so-called Orlicz norm, namely

$$\|u\|_{(M),\Omega} = \sup_{\varrho_{\bar{M},\Omega}(v)\le 1} \int_\Omega u(x)v(x)\,\mathrm{d}x.$$

where the supremum is taken over all $v \in E_{\bar{M}(\Omega)}$ such that $\varrho_{\bar{M},\Omega}(v) \le 1$. An important inequality in $L_M(\Omega)$ is the following:

$$\int_\Omega M(x,u(x))\,\mathrm{d}x \le \|u\|_{(M),\Omega} \quad \text{for all } u \in L_M(\Omega) \text{ such that } \|u\|_{(M),\Omega} \le 1, \tag{2.3}$$

from we readily deduce

$$\int_\Omega M\Big(x, \frac{u(x)}{\|u\|_{(M),\Omega}}\Big)\,\mathrm{d}x \le 1 \quad \text{for all } u \in L_M(\Omega) \setminus \{0\}. \tag{2.4}$$

It can be shown that the norm $\|\cdot\|_{(M),\Omega}$ is equivalent to the Luxemburg norm $\|\cdot\|_{M,\Omega}$. Indeed,

$$\|u\|_{M,\Omega} \le \|u\|_{(M),\Omega} \le 2\|u\|_{M,\Omega} \quad \text{for all } u \in L_M(\Omega). \tag{2.5}$$

Also, Hölder's inequality holds

$$\int_\Omega |u(x)v(x)|\,\mathrm{d}x \le \|u\|_{M,\Omega}\|v\|_{(\bar{M}),\Omega} \quad \text{for all } u \in L_M(\Omega) \text{ and } v \in L_{\bar{M}}(\Omega),$$

in particular, if Ω has finite measure, Hölder's inequality yields the continuous inclusion $L_M(\Omega) \subset L^1(\Omega)$.

Strong convergence in $L_M(\Omega)$ is rather strict. For most purposes, a mild concept of convergence will be enough, namely, that of modular convergence. The closure in $L_M(\Omega)$ of the bounded measurable functions with compact support in $\bar{\Omega}$ is denoted by $E_M(\Omega)$. The space $E_M(\Omega)$ is the largest linear space such that $E_M(\Omega) \subset \mathcal{L}_M(\Omega) \subset L_M(\Omega)$, where the inclusion is in general strict.

Definition 2.4. We say that a sequence $(u_n) \subset L_M(\Omega)$ is modular convergent to $u \in L_M(\Omega)$ if there exists a constant $\lambda > 0$ such that

$$\lim_{n\to\infty} \varrho_{M,\Omega}((u_n - u)/\lambda) = 0.$$

Musielak-Orlicz-Sobolev spaces. For any fixed nonnegative integer m we define

$$W^m L_M(\Omega) = \{u \in L_M(\Omega) : D^\alpha u \in L_M(\Omega) \text{ for all } \alpha, |\alpha| \le m\}$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{Z}$, $\alpha_j \ge 0$, $j = 1, \dots, d$, with $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$ and $D^\alpha u$ denote the distributional derivative of multiindex α. The space $W^m L_M(\Omega)$ is called the Musielak-Orlicz-Sobolev space (of order m).

Let $u \in W^m L_M(\Omega)$, we define

$$\varrho^{(m)}_{M,\Omega}(u) = \sum_{|\alpha|\le m} \varrho_{M,\Omega}(D^\alpha u),$$

$$\|u\|_{M,\Omega}^{(m)} = \inf\{\lambda > 0 : \varrho_{M,\Omega}^{(m)}(u/\lambda) \le 1\}, \quad \|u\|_{m,M,\Omega} = \sum_{|\alpha|\le m} \|D^\alpha u\|_{M,\Omega}.$$

The functional $\varrho_{M,\Omega}^{(m)}$ is convex in $W^m L_M(\Omega)$, whereas the functionals $\|\cdot\|_{M,\Omega}^{(m)}$ and $\|\cdot\|_{m,M,\Omega}$ are equivalent norms on $W^m L_M(\Omega)$. The pair $(W^m L_M(\Omega), \|\cdot\|_{M,\Omega}^{(m)})$ is a Banach space if there exists a constant $c > 0$ such that

$$\operatorname{ess\,inf}_{x\in\Omega} M(x,1) \ge c. \tag{2.6}$$

From this point on we will assume that (2.6) holds. The space $W^m L_M(\Omega)$ is identified to a subspace of the product $\prod_{|\alpha|\le m} L_M(\Omega) = \prod L_M$, this subspace is $\sigma(\prod L_M, \prod E_{\bar M})$ closed.

Let $W_0^m L_M(\Omega)$ be the $\sigma(\prod L_M, \prod E_{\bar M})$ closure of $\mathcal{D}(\Omega)$ in $W^m L_M(\Omega)$. Let $W^m E_M(\Omega)$ be the space of functions u such that u and its distribution derivatives up to order m lie in $E_M(\Omega)$, and $W_0^m E_M(\Omega)$ is the (norm) closure of $\mathcal{D}(\Omega)$ in $W^m L_M(\Omega)$.

Lemma 2.5 (Poincaré's inequality [2])**.** *Let Ω be a bounded Lipchitz-continuous domain of $\mathbb{R}^d$. Then there exists a constant $C_0 = C_0(\Omega, M) > 0$ such that*

$$\|u\|_{M,\Omega} \le C_0 \|\nabla u\|_{M,\Omega}, \quad \textit{for all } u \in W_0^1 L_M(\Omega). \tag{2.7}$$

Remark 2.6. Let M be a Musielak-Orlicz function and $u \in W_0^1 L_M(\Omega)$. Assume that, for some constant $C \ge 0$, one has $\int_\Omega M(x, \nabla u)\,dx \le C$. Then we also have $\|u\|_{1,M,\Omega} \le C'$ where $C' = (C_0 + 1)\max(C, 1)$. Indeed, since $\|u\|_{1,M,\Omega} = \|u\|_{M,\Omega} + \|\nabla u\|_{M,\Omega}$, by using (2.7), we obtain

$$\|u\|_{1,M,\Omega} \le C_0 \|\nabla u\|_{M,\Omega} + \|\nabla u\|_{M,\Omega} \le (C_0 + 1)\|\nabla u\|_{M,\Omega}.$$

Now, if $C \ge 1$, according to the convexity of $M(x, \cdot)$, it yields

$$\int_\Omega M\Big(x, \frac{\nabla u}{C}\Big)\,dx \le \frac{1}{C}\int_\Omega M(x, \nabla u)\,dx \le \frac{C}{C} = 1,$$

this means that $C \in \{\lambda > 0, \int_\Omega M(x, \nabla u/\lambda)\,dx \le 1\}$, hence $\|\nabla u\|_{M,\Omega} \le C$. On the other hand, if $C < 1$, then $\int_\Omega M(x, \nabla u)\,dx \le C < 1$, which yields $\|\nabla u\|_{M,\Omega} \le 1$.

Since we are going to work with two generalized N-functions, say P and M, such that $P \ll M$, we will consider the following assumptions for both complementary functions $\bar P$ and $\bar M$:

$$\lim_{|\xi|\to\infty} \operatorname{ess\,inf}_{x\in\Omega} \frac{\bar M(x,\xi)}{|\xi|} = \infty, \tag{2.8}$$

$$\lim_{|\xi|\to\infty} \operatorname{ess\,inf}_{x\in\Omega} \frac{\bar P(x,\xi)}{|\xi|} = \infty. \tag{2.9}$$

Remark 2.7. From [15, Remark 2.1] we have that the assumptions (2.8) and (2.9) provide the following:

$$\sup_{\xi\in B(0,R)} \operatorname{ess\,sup}_{x\in\Omega} M(x,\xi) < \infty \text{ for all } 0 < R < +\infty, \tag{2.10}$$

$$\sup_{\xi\in B(0,R)} \operatorname{ess\,sup}_{x\in\Omega} P(x,\xi) < \infty \text{ for all } 0 < R < +\infty. \tag{2.11}$$

Definition 2.8. We say that a sequence $(u_n) \subset W^1 L_M(\Omega)$ converges to $u \in W^1 L_M(\Omega)$ for the modular convergence in $W^1 L_M(\Omega)$ if, for some $h > 0$,

$$\lim_{n\to\infty} \bar\varrho_{M,\Omega}^{(1)}((u_n - u)/h) = 0.$$

The following spaces of distributions will also be used:

$$W^{-1}L_{\bar{M}}(\Omega) = \big\{f \in \mathcal{D}'(\Omega) : f = \sum_{|\alpha|\le 1} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ for some } f_\alpha \in L_{\bar{M}}(\Omega)\big\},$$
$$W^{-1}E_{\bar{M}}(\Omega) = \big\{f \in \mathcal{D}'(\Omega) : f = \sum_{|\alpha|\le 1} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ for some } f_\alpha \in E_{\bar{M}}(\Omega)\big\}.$$

Lemma 2.9. *If $P \ll M$ and $u_n \to u$ for the modular convergence in $L_M(\Omega)$, then $u_n \to u$ strongly in $E_P(\Omega)$. In particular, $L_M(\Omega) \subset E_P(\Omega)$ and $L_{\bar{P}}(\Omega) \subset E_{\bar{M}}(\Omega)$ with continuous injections.*

Proof. Let $\epsilon > 0$ be given. Let $\lambda > 0$ be such that

$$\int_\Omega M\Big(x, \frac{u_n - u}{\lambda}\Big) \to 0, \quad \text{as } n \to \infty.$$

Then, there exists $h \in L^1(\Omega)$ such that

$$M\Big(x, \frac{u_n - u}{\lambda}\Big) \le h \quad \text{and} \quad u_n \to u \text{ a. e. in } \Omega$$

for a subsequence still denoted (u_n). Since $P \ll M$, then for all $r > 0$ there exists $t_0 > 0$ such that

$$\frac{P(x, rt)}{M(x,t)} \le 1, \qquad \text{a. e. in } \Omega \text{ and for all } t \ge t_0.$$

For $r = \frac{\lambda}{\epsilon}$ and $t = \frac{t'}{\lambda}$, we obtain

$$\frac{P(x, \frac{t'}{\epsilon})}{M(x, \frac{t'}{\lambda})} \le 1, \quad \text{when } t' \ge t_0\lambda.$$

Then

$$\begin{aligned} P\Big(x, \frac{u_n - u}{\epsilon}\Big) &\le M\Big(x, \frac{u_n - u}{\lambda}\Big) + \sup_{t' \in B(0, t_0\lambda)} \operatorname{ess\,sup}_{x\in\Omega} P(x, t'/\epsilon) \\ &\le h + \sup_{t' \in B(0,t_0)} \operatorname{ess\,sup}_{x\in\Omega} P(x, t'/\epsilon) \text{ for a. a. } x \in \Omega. \end{aligned}$$

Since $h + \sup_{t' \in B(0,t_0\lambda)} \operatorname{ess\,sup}_{x\in\Omega} P(x, \frac{t'}{\epsilon}) \in L^1(\Omega)$ (from Remark 2.7), it yields, by the Lebesgue dominated convergence theorem,

$$P\Big(x, \frac{u_n - u}{\epsilon}\Big) \to 0 \text{ in } L^1(\Omega),$$

hence, for n big enough, we have $\|u_n - u\|_{P,\Omega} \le \epsilon$. That is, $u_n \to u$ in $L_P(\Omega)$.

The continuous injection $L_M(\Omega) \subset E_P(\Omega)$ is trivial since the convergence in $L_M(\Omega)$ implies the modular convergence in this space. On the other hand, since $P \ll M$ is equivalent to $\bar{M} \ll \bar{P}$, this yields the continuous injection $L_{\bar{P}}(\Omega) \subset E_{\bar{M}}(\Omega)$. $\square$

Lemma 2.10 ([17, Lemma 2.2])**.** *Let $(w_n) \subset L_M(\Omega)$, $w \in L_M(\Omega)$, $(v_n) \subset L_{\bar{M}}(\Omega)$ and $v \in L_{\bar{M}}(\Omega)$. If $w_n \to w$ in $L_M(\Omega)$ for the modular convergence and $v_n \to v$ in $L_{\bar{M}}(\Omega)$ for the modular convergence, then*

$$\lim_{n\to\infty} \int_\Omega w_n v \,\mathrm{d}x = \int_\Omega wv \,\mathrm{d}x \quad \textit{and} \quad \lim_{n\to\infty} \int_\Omega w_n v_n \,\mathrm{d}x = \int_\Omega wv \,\mathrm{d}x.$$

Lemma 2.11 ([3, 6]). *Let Ω be a bounded and Lipschitz-continuous domain in $\mathbb{R}^d$ and let M and $\bar{M}$ be two complementary Musielak-Orlicz functions which satisfy the following conditions:*

(i) *There exists a constant $A > 0$ such that for all $x, y \in \Omega$ with $|x-y| \le \frac{1}{2}$ one has*
$$\frac{M(x,t)}{M(y,t)} \le t^{-A/\log|x-y|} \quad \text{for all } t \ge 1. \tag{2.12}$$
(ii) *There exists a constant $C > 0$ such that*
$$\bar{M}(x,1) \le C \quad \text{a. e. in } \Omega. \tag{2.13}$$

Then the space $\mathcal{D}(\Omega)$ is dense in $L_M(\Omega)$ with respect to the modular convergence, $\mathcal{D}(\Omega)$ is dense in $W_0^1L_M(\Omega)$ for the modular convergence and $\mathcal{D}(\bar{\Omega})$ is dense in $W^1L_M(\Omega)$ for the modular convergence.

Remark 2.12. By taking $t = 1$ in (2.12) it yields that $M(x,1) = \text{constant}$ for a. a. $x \in \Omega$. In particular, the condition (2.6) is obviously satisfied and also
$$\int_\Omega M(x,1)\,\mathrm{d}x < \infty.$$

Remark 2.13 ([6]). Let $p\colon \Omega \mapsto (1,\infty)$ be a measurable function such that there exists a constant $A > 0$ such that for all points $x, y \in \Omega$ with $|x-y| < 1/2$, one has the inequality
$$|p(x) - p(y)| \le -\frac{A}{\log|x-y|}.$$
Then the following Musielak-Orlicz functions satisfy the assumption (2.12):

(1) $M(x,t) = t^{p(x)}$;
(2) $M(x,t) = t^{p(x)}\log(1+t)$;
(3) $M(x,t) = t\log(1+t)(\log(e-1+t))^{p(x)}$.

Inhomogeneous Musielak-Orlicz-Sobolev spaces. Let Ω be a bounded and open subset of $\mathbb{R}^d$ and let $Q_T = \Omega \times (0,T)$ with some given $T > 0$. Let M be a Musielak function. For each $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^d$, $\alpha_j \ge 0$, $j = 1, \dots, d$, we denote by D_x^α the distributional derivative on Q_T of multiindex α with respect to the variable $x \in \mathbb{R}^d$. The inhomogeneous Musielak-Orlicz-Sobolev spaces of order one are defined as follows:
$$\begin{gathered} W^{1,x}L_M(Q_T) = \{u \in L_M(Q_T) : D_x^\alpha u \in L_M(Q_T) \text{ for all } \alpha,\ |\alpha| \le 1\}, \\ W^{1,x}E_M(Q_T) = \{u \in E_M(Q_T) : D_x^\alpha u \in E_M(Q_T) \text{ for all } \alpha,\ |\alpha| \le 1\} \end{gathered}$$
This last space is a subspace of the first one, and both are Banach spaces under the norm
$$\|u\| = \sum_{|\alpha|\le 1} \|D_x^\alpha u\|_{M,Q_T}.$$
These spaces are considered as subspaces of the product space $\Pi L_M(Q_T)$ which has $(d+1)$ copies. We also consider the weak-$*$ topologies $\sigma(\Pi L_M(Q_T), \Pi E_{\overline{M}}(Q_T))$ and $\sigma(\Pi L_M(Q_T), \Pi L_{\overline{M}}(Q_T))$. If $u \in W^{1,x}L_M(Q_T)$ then the function $t :\to u(t)$ is defined on $(0,T)$ with values in $W^1L_M(\Omega)$. If, further, $u \in W^{1,x}E_M(Q_T)$ then this function is a $W^1E_M(\Omega)$-valued and is strongly measurable. The space $W^{1,x}L_M(Q_T)$ is not in general separable. If $u \in W^{1,x}L_M(Q_T)$, we cannot conclude that the function $u(t)$ is measurable on $(0,T)$. However, the scalar function

$t \to \|u(t)\|_{M,\Omega}$ is in $L^1(0,T)$. The space $W_0^{1,x}E_M(Q_T)$ is defined as the (norm) closure in $W^{1,x}E_M(Q_T)$ of $\mathcal{D}(Q)$. We can easily show as in [4] that when Ω is a Lipschitz-continuous domain then each element u of the closure of $\mathcal{D}(Q_T)$ with respect of the weak-$*$ topology $\sigma(\Pi L_M, \Pi E_{\bar{M}})$ is limit, in $W^{1,x}L_M(Q_T)$, of some subsequence $(u_n) \subset \mathcal{D}(Q_T)$ for the modular convergence; i. e., there exists $\lambda > 0$ such that for all α with $|\alpha| \le 1$

$$\int_{Q_T} M\Big(x, \frac{D_x^\alpha u_n - D_x^\alpha u}{\lambda}\Big)\,\mathrm{d}x\,\mathrm{d}t \to 0 \ as\ n \ \to \infty,$$

and, in particular, this implies that (u_n) converges to u in $W^{1,x}L_M(Q_T)$ for the weak-$*$ topology $\sigma(\Pi L_M, \Pi L_{\bar{M}})$. Consequently

$$\overline{\mathcal{D}(Q_T)}^{\sigma(\Pi L_M, \Pi L_{\bar{M}})} = \overline{\mathcal{D}(Q_T)}^{\sigma(\Pi L_M, \Pi E_{\bar{M}})}.$$

This space will be denoted by $W_0^{1,x}L_M(Q_T)$. Furthermore,

$$W_0^{1,x}E_M(Q_T) = W_0^{1,x}L_M(Q_T) \cap \Pi E_{\bar{M}}(Q_T).$$

Poincaré's inequality also holds in $W_0^{1,x}L_M(Q_T)$, i. e. there exists a constant $C > 0$ such that for all $u \in W_0^{1,x}L_M(Q_T)$ one has

$$\sum_{|\alpha|\le 1} \|D_x^\alpha u\|_{M,Q_T} \le C \sum_{|\alpha|=1} \|D_x^\alpha u\|_{M,Q_T}. \tag{2.14}$$

The dual space of $W_0^{1,x}E_M(Q_T)$ will be denoted by $W^{-1,x}L_{\bar{M}}(Q_T)$, and it can be shown that

$$W^{-1,x}L_{\bar{M}}(Q_T) = \Big\{f = \sum_{|\alpha|\le 1} D_x^\alpha f_\alpha : f_\alpha \in L_{\bar{M}}(Q_T),\ \text{for all } \alpha\Big\}.$$

This space will be equipped with the usual quotient norm

$$\|f\| = \inf \sum_{|\alpha|\le 1} \|D_x^\alpha f_\alpha\|_{\bar{M},Q_T}$$

where the infimum is taken over all possible functions $f_\alpha \in L_{\bar{M}}(Q_T)$ from which the decomposition $f = \sum_{|\alpha|\le 1} D_x^\alpha f_\alpha$ holds.

We also denote by $W^{-1,x}E_{\bar{M}}(Q_T)$ the subspace of $W^{-1,x}L_{\bar{M}}(Q_T)$ consisting of those linear forms which are $\sigma(\Pi L_M, \Pi E_{\bar{M}})$-continuous. It can be shown that

$$W^{-1,x}E_{\bar{M}}(Q_T) = \big\{f = \sum_{|\alpha|\le 1} D_x^\alpha f_\alpha : f_\alpha \in E_{\bar{M}}(Q_T)\big\}.$$

The following Lemma will be needed later on this paper.

Lemma 2.14. *Let P be a Musielak function such that* (2.9) *is satisfied. Assume that $s^2 \le P(x,s)$, for all a. a. $x \in \Omega$ and all $s \in \mathbb{R}$. Then the following continuous inclusions hold:*

$$L_P(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L_{\bar{P}}(\Omega).$$

In particular, $W_0^1L_P(\Omega) \hookrightarrow H_0^1(\Omega)$ and $H^{-1}(\Omega) \hookrightarrow W^{-1}L_{\bar{P}}(\Omega)$. Furthermore, if M is a Musielak function verifying (2.8) *and such that $P \ll M$, then the same continuous inclusions hold for M; that is,*

$$L_M(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L_{\bar{M}}(\Omega),$$

and also $W_0^1L_M(\Omega) \hookrightarrow H_0^1(\Omega)$ and $H^{-1}(\Omega) \hookrightarrow W^{-1}L_{\bar{M}}(\Omega)$.

Proof. From the estimate on P we have

$$\int_\Omega v^2 \,\mathrm{d}x \le \int_\Omega P(x,v)\,\mathrm{d}x, \quad \text{for all } v \in \mathcal{L}_P(\Omega). \tag{2.15}$$

Taking $v = u/\|u\|_{(P)}$ with $u \neq 0$ in (2.15) and using (2.4) it yields

$$\|u\|_{L^2(\Omega)} \le \|u\|_{(P)} \quad \text{for all } u \in L_P(\Omega),$$

and the first assertions of this Lemma are readily deduced.

Now let $P \ll M$. For $\varepsilon \in (0,1)$ there exists t_0 that

$$P(x,t) \le M(x,\varepsilon t) \quad \text{for all } t \ge t_0 \text{ and a. a. } x \in \Omega. \tag{2.16}$$

Then, taking $v \in \mathcal{L}_M(\Omega)$ and using Remark 2.7, we deduce that for some constant $C_1 = C_1(t_0)$,

$$\begin{aligned}
\int_\Omega v^2 \,\mathrm{d}x &\le \int_{\{|v|<t_0\}} P(x,v)\,\mathrm{d}x + \int_{\{|v|\ge t_0\}} P(x,v)\,\mathrm{d}x \\
&\le C_1 + \int_\Omega M(x,\varepsilon v)\,\mathrm{d}x \\
&\le C_1 + \varepsilon \int_\Omega M(x,v)\,\mathrm{d}x.
\end{aligned}$$

Making $v = u/\|u\|_{(M),Q_T}$, $u \neq 0$, in this last inequality and using (2.4) we finally deduce

$$\|u\|_{L^2(\Omega)} \le C_3 \|u\|_{(M),Q_T} \quad \text{for all } u \in L_M(\Omega),$$

where $C_3 = (C_1 + \varepsilon)^{1/2}$. □

Remark 2.15. Under the assumptions of Lemma 2.14, we have

$$L^2(0,T;H^{-1}(\Omega)) \hookrightarrow W^{-1,x}L_{\bar{P}}(Q_T) \hookrightarrow W^{-1,x}E_{\bar{M}}(Q_T).$$

Indeed, let $f \in L^2(0,T;H^{-1}(\Omega))$. Then, for some $f_\alpha \in L^2(Q_T)$, $f = \sum_{|\alpha|\le 1} \nabla_x^\alpha f_\alpha$. But according to Lemma 2.9 $L^2(Q_T) \subset L_{\bar{P}}(Q_T) \subset E_{\bar{M}}(Q_T)$ and thus

$$f \in W^{-1,x}L_{\bar{P}}(Q_T) \hookrightarrow W^{-1,x}E_{\bar{M}}(Q_T).$$

We will use truncations in the definition of our approximate problems. To do so, for $K > 0$, we introduce the truncation at height K, denoted by $T_K \colon \mathbb{R} \mapsto \mathbb{R}$, as

$$T_K(s) = \min(K, \max(s,-K)) = \begin{cases} s & \text{if } |s| \le K, \\ Ks/|s| & \text{if } |s| > K, \end{cases} \tag{2.17}$$

3. Compactness results

In the sequel, we will use the following results which concern mollification with respect to time and space variables and some trace results. Also, unless stated the contrary, $\Omega \subset \mathbb{R}^d$ is a bounded and open set with a Lipschitz-continuous boundary, and M is Musielak function. We put $Q_T = \Omega \times (0,T)$. For a function $u \in L^1(Q_T)$ we introduce the function $\tilde{u} \in L^1(\Omega \times \mathbb{R})$ as $\tilde{u}(x,s) = u(x,s)\chi_{(0,T)}$ and define, for all $\mu > 0$, $t \in [0,T]$ and a.e. $x \in \Omega$, the function u_μ given as follows

$$u_\mu(x,t) = \mu \int_{-\infty}^t \tilde{u}(x,s)\exp(\mu(s-t))\,\mathrm{d}s. \tag{3.1}$$

Lemma 3.1 ([1])**.** *The following assertions hold:*

(1) *Let* $u \in L_M(Q_T)$. *Then* $u_\mu \in C([0,T]; L_M(\Omega))$ *and* $u_\mu \to u$ *as* $\mu \to +\infty$ *in* $L_M(Q_T)$ *for the modular convergence.*
(2) *Let* $u \in W^{1,x}L_M(Q_T)$. *Then* $u_\mu \in C([0,T]; W^1L_M(\Omega))$ *and* $u_\mu \to u$ *as* $\mu \to +\infty$ *in* $W^{1,x}L_M(Q_T)$ *for the modular convergence.*
(3) *Let* $u \in E_M(Q_T)$ *(respectively,* $u \in W^{1,x}E_M(Q_T)$*). Then* $u_\mu \to u$ *as* $\mu \to +\infty$ *strongly in* $E_M(Q_T)$ *(respectively, strongly in* $W^{1,x}E_M(Q_T)$*).*
(4) *Let* $u \in W^{1,x}L_M(Q_T)$ *then* $\frac{\partial u_\mu}{\partial t} = \mu(u - u_\mu) \in W^{1,x}L_M(Q_T)$.
(5) *Let* $(u_n) \subset W^{1,x}L_M(Q_T)$ *and* $u \in W^{1,x}L_M(Q_T)$ *such that* $u_n \to u$ *strongly in* $W^{1,x}L_M(Q_T)$ *(respectively, for the modular convergence). Then, for all* $\mu > 0$, $(u_n)_\mu \to u_\mu$ *strongly in* $W^{1,x}L_M(Q_T)$ *(respectively, for the modular convergence).*

Lemma 3.2. *The following embedding holds with continuous injection*

$$E_M(Q_T) \subset L^1(0,T; E_M(\Omega)) \tag{3.2}$$

Proof. Since $M(x,t)$ is convex with respect to t, then for every $\lambda \geq 1$, $t \in [0,T]$ and a. a. $x \in \Omega$ we have

$$\alpha M(x,t) \leq M(x,\lambda t) \text{ and } \lambda M(x,t/\lambda) \leq M(x,t). \tag{3.3}$$

Let $u \in E_M(Q_T) \setminus \{0\}$. Owing to the definition of the space $E_M(Q_T)$, we have $\int_{Q_T} M(x,\lambda u(x,t))\,\mathrm{d}x\,\mathrm{d}t < \infty$ for every $\lambda \geq 0$. Hence, $\int_\Omega M(x,\lambda u(x,t))\,\mathrm{d}x < \infty$ for a. a. $t \in [0,T]$ and for all $\lambda \geq 0$. Therefore the function $u(\cdot,t) \in E_M(\Omega)$ for a. a. $t \in [0,T]$. In particular,

$$\int_\Omega M\Big(x, \frac{u(x,t)}{\|u(\cdot,t)\|_{M,\Omega}}\Big)\,\mathrm{d}x = 1 \quad \text{for a. a. } t \in [0,T].$$

Then, having in mind (3.3),

$$\begin{aligned}
&\int_0^T \|u\|_{M,\Omega}\,\mathrm{d}t \\
&= \int_{\{\|u(\cdot,t)\|_{M,\Omega}<1\}} \|u(\cdot,t)\|_{M,\Omega}\,\mathrm{d}t + \int_{\{\|u(\cdot,t)\|_M \geq 1\}} \|u(\cdot,t)\|_{M,\Omega}\,\mathrm{d}t \\
&\leq T + \int_{\{\|u(\cdot,t)\|_{M,\Omega}\geq 1\}} \|u(\cdot,t)\|_{M,\Omega} \int_\Omega M\Big(x, \frac{u(x,t)}{\|u(\cdot,t)\|_{M,\Omega}}\Big)\,\mathrm{d}x\mathrm{d}t \\
&\leq T + \int_{\{\|u(\cdot,t)\|_{M,\Omega}\geq 1\}} \int_\Omega \|u(\cdot,t)\|_{M,\Omega} M\Big(x, \frac{u(x,t)}{\|u(\cdot,t)\|_{M,\Omega}}\Big)\,\mathrm{d}x\mathrm{d}t \\
&\leq T + \int_{\{\|u(\cdot,t)\|_{M,\Omega}\geq 1\}} \int_\Omega M(x,u(x,t))\,\mathrm{d}x\mathrm{d}t \\
&\leq T + \int_{Q_T} M(x,u(x,t))\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

By taking $u/\|u\|_{M,Q_T}$ instead of u into the first and last terms of this inequality, using (2.4) and (2.5), it follows that $\int_0^T \|u\|_{M,\Omega}\,\mathrm{d}t \leq 2(T+1)\|u\|_{M,Q_T}$. □

A straightforward consequence of Lemma 3.2 is given in the next result.

Lemma 3.3. *The following embeddings hold with continuous injections*

$$W^1E_M(Q_T) \subset L^1\big(0,T; W^1E_M(\Omega)\big), \tag{3.4}$$

$$W^{-1}E_{\bar{M}}(Q_T) \subset L^1\big(0,T; W^{-1}E_{\bar{M}}(\Omega)\big). \tag{3.5}$$

The proof of the next three lemmas are straightforward adaptations of the ones given in [10, Lemmas 2, 5 and Theorem2].

Lemma 3.4. *Let Y be a Banach space such that $L^1(\Omega) \subset Y$ with continuous embedding. If $\mathcal{F}$ is bounded in $W_0^{1,x}L_M(Q_T)$ and relatively compact in $L^1(0,T;Y)$ then $\mathcal{F}$ is relatively compact in $L^1(Q_T)$ and in $E_P(Q_T)$ for all $P \ll M$.*

Lemma 3.5. *Let Ω be a bounded open subset of $\mathbb{R}^d$ with the segment property. Consider the Banach space*

$$W = \Big\{u \in W_0^{1,x}L_M(Q_T) : \frac{\partial u}{\partial t} \in W^{-1,x}L_{\bar{M}}(Q_T) + L^1(Q_T)\Big\}.$$

Then the embedding $W \subset C([0,T];L^1(\Omega))$ holds and is continuous.

Lemma 3.6. *If $\mathcal{F}$ is bounded in $W_0^{1,x}L_M(Q_T)$ and $\big\{\frac{\partial f}{\partial t} : f \in \mathcal{F}\big\}$ is bounded in $W^{-1,x}L_{\bar{M}}(Q_T)$ then $\mathcal{F}$ is relatively compact in $L^1(Q_T)$.*

The existence result given in Theorem 3.7 will be useful in our analysis. It is related to a second-order partial differential operator

$$\mathbf{A}\colon D(\mathbf{A}) \subset W^{1,x}L_M(Q_T) \mapsto W^{-1,x}L_{\bar{M}}(Q_T)$$

in divergence form $\mathbf{A}(u) = -\operatorname{div}\mathbf{a}(x,t,\nabla u)$, where

$$\mathbf{a}\colon \Omega \times (0,T) \times \mathbb{R}^d \mapsto \mathbb{R}^d \text{ is a Carathéodory function} \tag{3.6}$$

and for almost every $(x,t) \in Q_T$ and for all $\xi, \xi' \in \mathbb{R}^d$, $\xi \neq \xi'$, one has

$$|\mathbf{a}(x,t,\xi)| \le \beta(c_1(x,t) + \bar{M}_x^{-1}M(x,k_1|\xi|), \tag{3.7}$$

$$(\mathbf{a}(x,t,\xi) - \mathbf{a}(x,t,\xi'))(\xi - \xi') > 0, \tag{3.8}$$

$$\mathbf{a}(x,t,\xi)\xi \ge \alpha M(x,|\xi|). \tag{3.9}$$

For a function $f \in W^{-1,x}L_{\bar{M}}(Q_T)$ and a function $u_0 \in L^2(\Omega)$ we consider the parabolic problem given by

$$\begin{gathered}\frac{\partial u}{\partial t} - \operatorname{div}\mathbf{a}(x,t,\nabla u) = f \quad \text{in } Q_T,\\ u(x,0) = u_0(x) \text{ in } \Omega,\\ u = 0 \text{ on } \partial\Omega \times (0,T).\end{gathered} \tag{3.10}$$

Theorem 3.7 ([1])**.** *Under assumptions* (3.6)-(3.9) *there exists at least one weak solution to problem* (3.10)*, $u \in D(\mathbf{A}) \cap W_0^{1,x}L_M(Q_T) \cap C([0,T];L^2(\Omega))$ such that $\mathbf{a}(x,t,\nabla u) \in W^{-1,x}L_{\bar{M}}(Q_T)$ and for all $v \in W_0^{1,x}L_M(Q_T)$ with $\frac{\partial v}{\partial t} \in W^{-1,x}L_{\bar{M}}(Q_T)$ and for all $\tau \in [0,T]$ one has*

$$\begin{aligned}&-\langle \frac{\partial v}{\partial t}, u\rangle_{Q_\tau} + \int_\Omega u(x,\tau)v(x,\tau)\,\mathrm{d}x + \int_0^\tau \int_\Omega \mathbf{a}(x,t,\nabla u)\nabla v\,\mathrm{d}x\mathrm{d}t\\ &= \langle f, v\rangle_{Q_\tau} + \int_\Omega u_0(x)v(x,0)\,\mathrm{d}x,\end{aligned}$$

where the $\langle\cdot,\cdot\rangle_{Q_\tau}$ stands for the duality pairing between the spaces $W^{-1,x}L_{\bar{M}}(Q_\tau)$ and $W_0^{1,x}L_M(Q_\tau)$. Moreover, for all $\tau \in [0,T]$, the following energy identity holds

$$\frac{1}{2}\int_\Omega |u(x,\tau)|^2\,\mathrm{d}x + \int_0^\tau \int_\Omega \mathbf{a}(x,t,\nabla u)\nabla u\,\mathrm{d}x\mathrm{d}t = \langle f, u\rangle_{Q_\tau} + \frac{1}{2}\int_\Omega |u_0(x)|^2\,\mathrm{d}x.$$

4. Notion of capacity solution

In this section, we give the definition of a capacity solution for problem (1.2) in the context of the Musielak-Orlicz-Sobolev spaces.

Let Ω be an open subset of $\mathbb{R}^d$ and let M be a Musielak-Orlicz function satisfying the conditions of Lemma 2.11. We first consider the Banach space

$$\mathbf{W} = \Big\{ v \in W_0^{1,x} L_M(Q_T) : \frac{\partial v}{\partial t} \in W^{-1,x} L_{\bar{M}}(Q_T) \Big\}$$

provided with its standard norm

$$\|v\|_{\mathbf{W}} = \|v\|_{W^{1,x} L_M(Q_T)} + \|\frac{\partial v}{\partial t}\|_{W^{-1,x} L_{\bar{M}}(Q_T)}.$$

Throughout this paper $\langle \cdot, \cdot \rangle$ stands for the duality pairing between the spaces $W^{1,x} L_M(Q_T) \cap L^2(Q_T)$ and $W^{-1,x} L_{\bar{M}}(Q_T) + L^2(Q_T)$ or between $W_0^{1,x} L_M(Q_T)$ and $W^{-1,x} L_{\bar{M}}(Q_T)$, and we assume the following conditions:

$$P \ll M \text{ and } t^2 \le P(x,t) \text{ for a. a. } x \in \Omega \text{ and all } t \in \mathbb{R}, \tag{4.1}$$

and their respective complementary functions, $\bar{M}$ and $\bar{P}$, satisfy (2.8) and (2.9), respectively. We consider a second order partial differential operator

$$A \colon D(A) \subset W^{1,x} L_M(Q_T) \mapsto W^{-1,x} L_{\bar{M}}(Q_T)$$

in divergence form $Au = -\operatorname{div} a(x,t,u,\nabla u)$ where $a \colon \Omega \times (0,T) \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}^d$ is a Carathéodory function (that is, $a = a(x,t,s,\xi)$ is measurable in (x,t) for any value of (s,ξ) and continuous with respect to the arguments (s,ξ) for a. a. $(x,t) \in \Omega \times (0,T)$) satisfying the following assumptions, for a. a. $(x,t) \in Q_T$, all $s \in \mathbb{R}$, and all $\xi, \xi' \in \mathbb{R}^d$, $\xi \neq \xi'$,

$$|a(x,t,s,\xi)| \le \zeta(c(x,t) + \bar{M}_x^{-1}(P(x,k|s|)) + \bar{M}_x^{-1}(M(x,k|\xi|)), \tag{4.2}$$

$$[a(x,t,s,\xi) - a(x,t,s,\xi')][\xi - \xi'] \ge \alpha(M(x,|\xi - \xi'|) + M(x,|s|)), \tag{4.3}$$

$$|a(x,t,s_1,\xi) - a(x,t,s_2,\xi)| \le \zeta\Big[e(x,t) + |s_1| + |s_2| + P^{-1}(x,kM(|\xi|))\Big], \tag{4.4}$$

$$a(x,t,s,0) = 0, \tag{4.5}$$

with $c(x,t) \in E_{\bar{M}}(Q_T)$, $e \in E_P(Q_T)$ and α, ζ, $k > 0$ are given real numbers.

$$\rho \in C(\mathbb{R}) \text{ and there exists } \bar{\rho} \in \mathbb{R} \text{ such that } 0 < \rho(s) \le \bar{\rho}, \text{ for all } s \in \mathbb{R}, \tag{4.6}$$

$$\varphi_0 \in L^2(0,T;H^1(\Omega)) \cap L^\infty(Q_T), \tag{4.7}$$

$$u_0 \in L^2(\Omega). \tag{4.8}$$

Remark 4.1. Notice that from (4.3) and (4.5) we obtain the elliptic condition

$$a(x,t,s,\xi)\xi \ge \alpha M(x,|\xi|), \text{ for a. a. } (x,t) \in Q_T, \text{ and all } (s,\xi) \in \mathbb{R} \times \mathbb{R}^d. \tag{4.9}$$

The concept of capacity solution now follows.

Definition 4.2. A triplet (u,φ,Φ) is called a capacity solution of (1.2) if the following conditions are fulfilled:

1. $u \in \mathbf{W}$, $a(x,t,u,\nabla u) \in L_{\bar{M}}(Q_T)^d, \varphi \in L^\infty(Q_T)$ and $\Phi \in L^2(Q_T)^d$.
2. (u,φ,Φ) satisfies the system of partial differential equations

$$\frac{\partial u}{\partial t} - \operatorname{div} a(x,t,u,\nabla u) = \operatorname{div}(\varphi\Phi) \text{ in } Q_T, \tag{4.10}$$

$$\operatorname{div} \Phi = 0 \text{ in } Q_T, \tag{4.11}$$

(3) For every $S \in C_0^1(\mathbb{R})$ (functions of $C^1(\mathbb{R})$ with compact support), one has $S(u)\varphi - S(0)\varphi_0 \in L^2(0,T;H_0^1(\Omega))$, and
$$S(u)\Phi = \rho(u)[\nabla(S(u)\varphi) - \varphi\nabla S(u)], \tag{4.12}$$
(4) $u(\cdot,0) = u_0$ in Ω.

Notice that, thanks to Lemma 3.5 and the regularity of u, we obtain in particular $u \in C([0,T];L^1(\Omega))$ and thus the initial condition in (4) makes sense at least in $L^1(\Omega)$.

Remark 4.3. The notion of capacity solution involves a triplet (u,φ,Φ) whereas the original problem (1.2) refers only to two unknowns, u and φ. Evidently, the vector function Φ is, in some way, related to u and φ. For instance, if we were allowed to take $S = 1$ in (4.12), we would readily obtain $\Phi = \rho(u)\nabla\varphi$. But the choice $S = 1$ is not possible since it does not belong to the space $C_0^1(\mathbb{R})$. To circumvent this situation, consider, for any $m > 0$, a function $S_m \in C_0^1(\mathbb{R})$ such that $S_m(s) = 1$ in $\{|s| \le m\}$. Using S_m in (4.12) and multiplying this expression by $\chi_{\{|u|\le m\}}$ we obtain
$$\chi_{\{|u|\le m\}}\Phi = \chi_{\{|u|\le m\}}\rho(u)\nabla(S_m(u)\varphi), \quad \text{for all } m > 0.$$
This last expression provides a meaning, al least in a pointwise sense, to $\nabla\varphi$ so that $\Phi = \rho(u)\nabla\varphi$ almost everywhere in Q_T.

5. An existence result

This section is devoted to establish the main theorem of this paper:

Theorem 5.1. *Under the assumptions* (2.8), (2.9), (2.12), (2.13) *and* (4.2)-(4.8), *the system* (1.2) *admits a capacity solution in the sense of Definition 4.2.*

To prove this theorem, we need first to show the existence of a weak solution to a similar problem but with stronger assumptions; namely, there exists $c \in E_{\bar{M}}(Q_T)$, and two real numbers $\zeta > 0$ and $k \ge 0$, such that for almost all $(x,t) \in Q_T$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^d$, we have
$$|a(x,t,s,\xi)| \le \zeta[c(x,t) + \bar{M}_x^{-1}(M(x,k|\xi|))], \tag{5.1}$$
and
$$\begin{gathered} \rho \in C(\mathbb{R}) \text{ and there exist } \rho_1 \text{ and } \rho_2 \in \mathbb{R} \text{ such that} \\ 0 < \rho_1 \le \rho(s) \le \rho_2, \text{ for all } s \in \mathbb{R}. \end{gathered} \tag{5.2}$$

Theorem 5.2. *Assume* (2.8), (2.9), (4.3)-(4.5), (4.7), (4.8), (5.1) *and* (5.2) *are satified. Then, there exists a weak solution* (u,φ) *to problem* (1.2)*; that is,*
$$\begin{gathered} u \in W_0^{1,x}L_M(Q_T) \cap C([0,T];L^2(\Omega)), \quad a(x,t,u,\nabla u) \in L_{\bar{M}}(Q_T)^d, \\ \varphi - \varphi_0 \in L^\infty(0,T;H_0^1(\Omega)) \cap L^\infty(Q_T), \\ u(\cdot,0) = u_0 \textit{ in } \Omega, \\ \int_0^t \big\langle \frac{\partial u}{\partial t}, \phi \big\rangle + \int_0^t \int_\Omega a(x,t,u,\nabla u)\nabla\phi = -\int_0^t \int_\Omega \rho(u)\varphi\nabla\varphi\nabla\phi, \\ \textit{for all } \phi \in W_0^{1,x}L_M(Q_T), \textit{ for all } t \in [0,T], \\ \int_\Omega \rho(u)\nabla\varphi\nabla\psi = 0, \quad \textit{for all } \psi \in H_0^1(\Omega), \textit{ a.e. } t \in (0,T). \end{gathered}$$

Proof. To show the existence of a weak solution, Schauder's fixed point theorem will be applied together with the existence and uniqueness result of a weak solution to a parabolic equation.

For every $\omega \in E_P(Q_T)$ and almost everywhere $t \in (0,T)$, we consider the elliptic problem

$$\begin{gathered} \operatorname{div}(\rho(\omega)\nabla\varphi) = 0 \quad \text{in } \Omega \times (0,T), \\ \varphi = \varphi_0 \quad \text{on } \partial\Omega \times (0,T). \end{gathered} \tag{5.3}$$

Thanks to Lax-Milgram's theorem, (5.3) has an unique solution $\varphi(t) \in H^1(\Omega)$, for almost all $t \in (0,T)$. In fact, φ is measurable in t with values in $H^1(\Omega)$ [4]. In that case, it is $\varphi \in L^\infty(0,T;H^1(\Omega))$. Indeed, by the maximum principle we have

$$\|\varphi\|_{L^\infty(Q_T)} \le \|\varphi_0\|_{L^\infty(Q_T)}. \tag{5.4}$$

Using $\varphi - \varphi_0 \in H_0^1(\Omega)$ as a test function in (5.3) we obtain

$$\int_\Omega \rho(\omega)\nabla\varphi\nabla(\varphi - \varphi_0) = 0,$$

hence

$$\rho_1 \int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x \le \int_\Omega \rho(\omega)|\nabla\varphi||\nabla\varphi_0| \,\mathrm{d}x \le \rho_2 \int_\Omega |\nabla\varphi||\nabla\varphi_0| \,\mathrm{d}x.$$

By the Cauchy-Schwarz inequality, we obtain

$$\int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x \le C(\rho_1,\rho_2,\varphi_0) = C, \quad \text{a.e. } t \in (0,T). \tag{5.5}$$

Note that the right-hand side in the original parabolic equation is $\rho(u)|\nabla\varphi|^2 \in L^1(\Omega \times (0,T))$. Thanks to the elliptic equation, this term also belongs to the space $L^2(0,T;H^{-1}(\Omega))$. Indeed, let $\phi \in \mathcal{D}(\Omega)$ and take $\xi = \phi\varphi$ as a test function in (5.3). We have, for a.e. $t \in [0,T]$,

$$\int_\Omega \rho(\omega)\nabla\varphi\nabla(\phi\varphi) \,\mathrm{d}x = 0,$$

that is

$$\int_\Omega \rho(\omega)|\nabla\varphi|^2\phi \,\mathrm{d}x = -\int_\Omega \rho(\omega)\varphi\nabla\varphi\nabla\phi \,\mathrm{d}x = \langle \operatorname{div}(\rho(\omega)\varphi\nabla\varphi), \phi\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}.$$

This means that

$$\rho(\omega)|\nabla\varphi|^2 = \operatorname{div}(\rho(\omega)\varphi\nabla\varphi) \text{ in } \mathcal{D}'(\Omega) \text{ and a.e. in } [0,T]. \tag{5.6}$$

Since $\rho(\omega)\varphi\nabla\varphi \in L^2(Q_T)^d$ we finally deduce the regularity

$$\operatorname{div}(\rho(\omega)\varphi\nabla\varphi) \in L^2(0,T;H^{-1}(\Omega)).$$

The identity (5.6) is one of the keys that allows us to solve the classical thermistor problem and the introduction of the notion of a capacity solution as well.

Now we introduce the parabolic problem

$$\begin{gathered} \frac{\partial u}{\partial t} - \operatorname{div} a(x,t,\omega,\nabla u) = \operatorname{div}(\rho(\omega)\varphi\nabla\varphi) \quad \text{in } Q_T, \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0 \quad \text{in } \Omega. \end{gathered} \tag{5.7}$$

The variational formulation of the parabolic equation is given as follows.

$$
\begin{gathered}
u\in W_0^{1,x}L_M(Q_T)\cap C([0,T];L^2(\Omega)),\quad a(x,t,\omega,\nabla u)\in L_{\bar M}(Q_T)^d,\\
\int_0^t\\
langle\frac{\partial u}{\partial t},\phi\rangle+\int_0^t\int_\Omega a(x,t,\omega,\nabla u)\nabla\phi=-\int_0^t\int_\Omega \rho(\omega)\varphi\nabla\varphi\nabla\phi,\\
\text{for all } \phi\in W_0^{1,x}L_M(Q_T),\quad \text{for all } t\in[0,T],\\
u(\cdot,0)=u_0\quad \text{in } \Omega.
\end{gathered}\tag{5.8}
$$

Note that $\operatorname{div}(\rho(\omega)\varphi\nabla\varphi)\in L^2(0,T;H^{-1}(\Omega))\hookrightarrow W^{-1,x}E_{\bar M}(Q_T)$ due to (5.3), (5.4), (5.5), Lemma 2.14 and Remark 2.15.

By Theorem 3.7, we have the existence of a solution to the problem (5.8). Now, we show that $|\nabla u|\in\mathcal{L}_M(Q_T)$, and the estimates

$$
\int_0^T\int_\Omega M(x,|\nabla u|)\,\mathrm{d}x\,\mathrm{d}t\le C(u_0,\varphi_0,\alpha,T,\rho_2)=C_0,\tag{5.9}
$$

$$
\|a(x,t,\omega,\nabla u)\|_{\bar M,Q_T}\le C_1,\tag{5.10}
$$

where C_1 only depends on data, but not on ω. Indeed, let $\lambda>0$ such that $|\nabla u|/\lambda\in\mathcal{L}_M(Q_T)$. Since $\varphi\in L^2(0,T;H^1(\Omega))\subset W^{1,x}L_{\bar M}(Q_T)$, there exists $\mu>0$ such that $\frac{2}{\alpha\mu}\rho_2\|\varphi_0\|_{L^\infty(Q_T)}|\nabla\varphi|\in\mathcal{L}_{\bar M}(Q_T)$. By taking $\phi=u$ as a test function in (5.7), from (4.2), (4.5), (5.2), (5.4) and Young's inequality, we obtain

$$
\begin{aligned}
&\frac{\alpha}{\lambda\mu}\int_0^T\int_\Omega M(x,|\nabla u|)\,\mathrm{d}x\,\mathrm{d}t\\
&\le\frac{1}{\lambda\mu}\int_0^T\int_\Omega a(x,t,\omega,\nabla u)\nabla u\,\mathrm{d}x\,\mathrm{d}t\\
&\le\frac{1}{2\lambda\mu}\|u_0\|^2_{L^2(\Omega)}+\frac{\alpha\mu}{2}\int_0^T\int_\Omega\bar M(x,\frac{2}{\alpha\mu}\rho_2\|\varphi_0\|_{L^\infty(Q_T)}|\nabla\varphi|)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\frac{\alpha}{2\mu}\int_0^T\int_\Omega M(x,|\nabla u|/\lambda)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

This shows that $|\nabla u|\in\mathcal{L}_M(Q_T)$ and, consequently, estimate (5.9) is derived by just taking $\lambda=1$ in this last inequality. In order to obtain (5.10), first notice that from the last inequality we also have

$$
\int_0^T\int_\Omega a(x,t,\omega,\nabla u)\nabla u\,\mathrm{d}x\,\mathrm{d}t\le\alpha C_0.\tag{5.11}
$$

Then, owing to (4.3), for any $\phi\in W_0^{1,x}E_M(Q_T)$ such that $\|\nabla\phi\|_{M,Q_T}=1/(k+1)$ it yields

$$
0\le\int_0^T\int_\Omega(a(x,t,\omega,\nabla u)-a(x,t,\omega,\nabla\phi))(\nabla u-\nabla\phi)\,\mathrm{d}x\,\mathrm{d}t,
$$

and thus, using (5.11) and Young's inequality,

$$
\int_0^T\int_\Omega a(x,t,\omega,\nabla u)\nabla\phi\,\mathrm{d}x\,\mathrm{d}t
$$

$$
\begin{aligned}
&\le \int_0^T\int_\Omega a(x,t,\omega,\nabla u)\nabla u\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega a(x,t,\omega,\nabla\phi)(\nabla u-\nabla\phi)\,\mathrm{d}x\,\mathrm{d}t\\
&\le \alpha C_0 + \int_0^T\int_\Omega |a(x,t,\omega,\nabla\phi)\nabla u|\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega a(x,t,\omega,\nabla\phi)\nabla\phi\,\mathrm{d}x\,\mathrm{d}t\\
&\le \alpha C_0 + 2\zeta\int_0^T\int_\Omega \Big[\bar M\Big(x,\frac{|a(x,t,\omega,\nabla\phi)|}{2\zeta}\Big) + M(x,|\nabla u|)\Big]\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + 2\zeta\int_0^T\int_\Omega \Big[\bar M\Big(x,\frac{|a(x,t,\omega,\nabla\phi)|}{2\zeta}\Big) + M(x,|\nabla\phi|)\Big]\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
$$

where ζ is the constant appearing in (5.1). Since

$$
\bar M\Big(x,\frac{|a(x,t,\omega,\nabla\phi)|}{2\zeta}\Big) \le \frac{1}{2}(\bar M(x,c(x,t)) + M(x,k|\nabla\phi|)) \quad \text{a. e. in } Q_T,
$$

using (2.3), we have

$$
\int_0^T\int_\Omega \bar M\Big(x,\frac{|a(x,t,\omega,\nabla\phi)|}{2\zeta}\Big)\,\mathrm{d}x\,\mathrm{d}t \le \frac{1}{2}\int_0^T\int_\Omega \bar M(x,c(x,t))\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2} = C_2.
$$

Note that C_2 only depends on data (but not on ω). Therefore, gathering all these estimates, we deduce for all $\phi \in W_0^{1,x}E_M(Q_T)$ such that $\|\nabla\phi\|_{M,Q_T} = 1/(k+1)$

$$
\int_0^T\int_\Omega a(x,t,\omega,\nabla u)\nabla\phi\,\mathrm{d}x\,\mathrm{d}t \le C_1,
$$

which finally yields the estimate (5.10) by considering the dual norm on $L_{\bar M}(Q_T)$.

Also from (4.2), (5.2), (5.4), (5.5) and (5.10) we obtain

$$
\frac{\partial u}{\partial t} \in W^{-1,x}L_{\bar M}(Q_T) \text{ and } \|\frac{\partial u}{\partial t}\|_{W^{-1,x}L_{\bar M}(Q_T)} \le C_3, \tag{5.12}
$$

where, again, C_3 is a constant depending only on data, but not on ω.

We define the operator $G\colon \omega \in E_P(Q_T) \mapsto G(\omega) = u \in \mathbf{W}$, with u being the unique solution to (5.8). From Lemma 3.6, and Lemma 3.4 with $Y = L^1(\Omega)$, we have that $\mathbf{W} \hookrightarrow E_P(Q_T)$ with compact embedding. Consequently, G maps $E_P(Q_T)$ into itself and, due to the estimates (5.9) and (5.12), G is a compact operator. Moreover, from (5.9) we have, for $R > 0$ large enough $G(B_R) \subset B_R$ where $B_R = \{v \in E_P(Q_T) : \|v\|_{L_P(Q_T)} \le R\}$.

To complete the proof, it remains to show that G is a continuous operator. Thus, let $(\omega_n) \subset B_R$ be a sequence such that $\omega_n \to \omega$ strongly in $E_P(Q_T)$ and consider the corresponding functions to ω_n, that is, $u_n = G(\omega_n)$ and φ_n and put $F_n = \rho(\omega_n)\varphi_n\nabla\varphi_n$ and $F = \rho(\omega)\varphi\nabla\varphi$. We have to show that

$$
u_n \to u = G(\omega) \quad \text{strongly in } E_P(Q_T).
$$

Owing to $P \ll M$ and (5.9), we have $\nabla u \in E_P(Q_T)^d$. Since the inclusion $L_P(Q_T) \subset L^2(Q_T)$ is continuous, we also have $\omega_n \to \omega$ strongly in $L^2(Q_T)$ and thus, we may extract a subsequence, still denoted in the same way, such that $\omega_n \to \omega$ a.e. in Q_T. Then, it is an easy task to show that $\varphi_n \to \varphi$ strongly in $L^2(0,T;H^1(Q_T))$ and, consequently, also for another subsequence denoted in the same way, $F_n \to F$ strongly in $L^2(Q_T)$.

On the other hand, since $(\omega_n) \subset L_P(Q_T)$ is bounded, by the estimates obtained above, we deduce, again modulo a subsequence,

$$
u_n \to U \text{ in } E_P(Q_T), \quad \text{for some } U \in E_P(Q_T), \tag{5.13}
$$

$$\nabla u_n \to \nabla U \quad \text{weakly in } L^2(Q_T)^d, \tag{5.14}$$

By subtracting the respective equations of (5.8) for u_n and u, and taking $\phi = u_n - u$ as a test function, for all $t \in [0,T]$, we obtain

$$\begin{aligned}
&\frac{1}{2}\|u_n(t)-u(t)\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega (a(x,s,\omega_n,\nabla u_n) - a(x,s,\omega,\nabla u))\nabla(u_n-u)\,\mathrm{d}x\,\mathrm{d}s \\
&= -\int_0^t \int_\Omega (F_n - F)\nabla(u_n-u)\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

By using (4.3), we obtain

$$\begin{aligned}
&(a(x,s,\omega_n,\nabla u_n) - a(x,s,\omega,\nabla u))\nabla(u_n-u) \\
&\geq \alpha M(x,|\nabla(u_n-u)|) + (a(x,s,\omega_n,\nabla u) - a(x,s,\omega,\nabla u))\nabla(u_n-u).
\end{aligned}$$

Let $h_n = a(x,s,\omega_n,\nabla u) - a(x,s,\omega,\nabla u)$ and $g_n = \nabla(u_n-u)$. Then, $|h_n| \to 0$ a.e. in Q_T. For a given positive number λ_0, to be chosen later, we have

$$\int_0^t \int_\Omega |h_n g_n| = \int_{\{|g_n|\leq\lambda_0\}} |h_n g_n| + \int_{\{|g_n|>\lambda_0\}} |h_n g_n|. \tag{5.15}$$

For the first term of the right hand side of (5.15), we have

$$\int_{\{|g_n|\leq\lambda_0\}} |h_n g_n| \leq \lambda_0 \int_{Q_T} |h_n| = \lambda_0 \int_{\{|h_n|\leq 4\zeta\}} |h_n| + \lambda_0 \int_{\{|h_n|>4\zeta\}} |h_n|.$$

The first of these integrals converges trivially to zero. As for the second one, using the fact that $\frac{|h_n|}{4\zeta} > 1$ on the set $\{|h_n| > 4\zeta\}$ and (2.15), it yields

$$\lambda_0 \int_{\{|h_n|>4\zeta\}} |h_n| \leq 4\zeta\lambda_0 \int_{\{|h_n|>4\zeta\}} \Big(\frac{|h_n|}{4\zeta}\Big)^2 \leq 4\zeta\lambda_0 \int_{Q_T} P\Big(x, \frac{|h_n|}{4\zeta}\Big).$$

Bye (4.4), we deduce

$$P\Big(x, \frac{|h_n|}{4\zeta}\Big) \leq \frac{1}{4}\left(P(x,e) + P(x,\omega_n) + P(x,\omega) + kM(x,|\nabla u|)\right),$$

and since $P(x,\omega_n) \to P(x,\omega)$ strongly in $L^1(Q_T)$, by Lebesgue's dominated theorem we have

$$\lim_{n\to\infty} \int_{Q_T} P\Big(x, \frac{|h_n|}{4\zeta}\Big) = 0,$$

and consequently

$$\lim_{n\to\infty} \int_{\{|g_n|\leq\lambda_0\}} |h_n g_n| = 0.$$

As for the second term of the right-hand side of (5.15), we use Young's inequality and (2.15). It yields,

$$\begin{aligned}
\int_{\{|g_n|>\lambda_0\}} |h_n g_n| &\leq \frac{1}{2\alpha} \int_{Q_T} |h_n|^2 + \frac{\alpha}{2} \int_{\{|g_n|>\lambda_0\}} |g_n|^2 \\
&\leq \frac{(4\zeta)^2}{\alpha} \int_{Q_T} P\Big(x, \frac{|h_n|}{4\zeta}\Big) + \alpha \int_{\{|g_n|>\lambda_0\}} P(x,|g_n|).
\end{aligned}$$

It has been already shown that the first of these terms converges to zero. As for the second one, since $P \ll M$, we can take λ_0 large enough such that $P(x,s) \le M(x,s)$ for $|s| > \lambda_0$, and then,

$$\alpha \int_{\{|g_n|>\lambda_0\}} P(x,|g_n|) \le \alpha \int_0^t \int_\Omega M(x,|g_n|) = \alpha \int_0^t \int_\Omega M(x,|\nabla(u_n - u)|).$$

Consequently, for some sequence $(\epsilon_n) \subset \mathbb{R}$, $\epsilon_n \to 0$, we have the estimate

$$\frac{1}{2}\|u_n(t) - u(t)\|^2_{L^2(\Omega)} \le -\int_0^t \int_\Omega (F_n - F)\nabla(u_n - u)\,\mathrm{d}x\,\mathrm{d}s + \epsilon_n,$$

and integrating this inequality over $[0,T]$, we have

$$\frac{1}{2}\|u_n - u\|^2_{L^2(Q_T)} \le -\int_0^T \int_\Omega (T-t)(F_n - F)\nabla(u_n - u)\,\mathrm{d}x\,\mathrm{d}t + T\epsilon_n. \tag{5.16}$$

The first term of the right hand side in (5.16) converges to zero since $F_n \to F$ strongly in $L^2(Q_T)^d$ and $(T-t)(\nabla u_n - \nabla u)$ is bounded in $L^2(Q_T)^d$. In conclusion, $u_n \to u$ strongly in $L^2(Q_T)$. Since this limit does not depend upon the subsequence one may extract, it is in fact the whole sequence (u_n) which converges to u strongly in $L^2(Q_T)$. On the other hand, in virtue of (5.13), we also have $u_n \to U$ strongly in $L^2(Q_T)$, so that $u = U$ and we can rewrite (5.13) to give $u_n \to u$ strongly in $E_P(Q_T)$. This shows that G is continuous and this ends the proof of Theorem 5.2.

Proof of Theorem 5.1. This stage is the main goal of this work. We start by introducing a sequence of approximate problem and deriving a priori estimates of it and showing two intermediate results, namely the strong convergence in $L^1(Q_T)$ of both ∇u_n and φ_n, where (u_n, φ_n) is a weak solution to the approximate problem of (1.2).

Step 1. For every $n \in \mathbb{N}$, we introduce the following regularization of the data,

$$\rho_n(s) = \rho(s) + \frac{1}{n}, \tag{5.17}$$

$$a_n(x,t,s,\xi) = a(x,t,T_n(s),\xi), \tag{5.18}$$

and consider the approximate system

$$\frac{\partial u_n}{\partial t} - \operatorname{div} a_n(x,t,u_n,\nabla u_n) = \rho_n(u_n)|\nabla\varphi_n|^2 \quad \text{in } Q_T, \tag{5.19}$$

$$\operatorname{div}(\rho_n(u_n)\nabla\varphi_n) = 0 \quad \text{in } Q_T, \tag{5.20}$$

$$u_n = 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{5.21}$$

$$\varphi_n = \varphi_0 \quad \text{on } (0,T)\times\partial\Omega, \tag{5.22}$$

$$u_n(\cdot,0) = u_0 \quad \text{in } \Omega. \tag{5.23}$$

From (4.2) we deduce

$$|a(x,t,T_n(s),\xi)| \le \zeta\left[c(x,t) + \bar{M}_x^{-1}(P(x,k|T_n(s)|)) + \bar{M}_x^{-1}(M(x,k|\xi|))\right],$$

by the Fenchel-Young inequality, we obtain

$$\begin{aligned}|a(x,t,T_n(s),\xi)| &\le \zeta\left[c(x,t) + (P(x,k|T_n(s)|) + M(x,1)) + \bar{M}_x^{-1}(M(x,k|\xi|))\right]\\ &\le \zeta\Big[c_n(x,t) + \bar{M}_x^{-1}(M(x,k|\xi|))\Big],\end{aligned}$$

where $c_n(x,t) = c(x,t) + \sup_{\xi\in B(0,kn)} \operatorname{ess\,sup}_{x\in\Omega} P(x,\xi) + M(x,1)$. Using Remark 2.12, $M(x,1) =$ constant for a. a. $x \in \Omega$. Taking into account that $L^\infty(Q_T) \subset$

$L^2(Q_T) \subset L_{\bar{P}}(Q_T) \subset E_{\bar{M}}(Q_T)$ (Lemma 2.9) and owing to (2.11), it yields that $c_n \in E_{\bar{M}(Q_T)}$.

Also, in view of (4.6), we have =

$$n^{-1} \leq \rho_n(s) \leq \rho_3 + 1 = \rho_4, \quad \text{for all } s \in \mathbb{R}. \tag{5.24}$$

Thus, we can apply Theorem 5.2 to deduce the existence of a weak solution (u_n, φ_n) to system (5.19)-(5.23).

By the maximum principle we have

$$\|\varphi_n\|_{L^\infty(Q_T)} \leq \|\varphi_0\|_{L^\infty(Q_T)}, \tag{5.25}$$

hence there exists a function $\varphi \in L^\infty(Q_T)$ and a subsequence, still denoted in the same way, such that

$$\varphi_n \to \varphi \text{weak-* in } L^\infty(Q_T). \tag{5.26}$$

Now let multiply (5.20) by $\varphi_n - \varphi_0 \in L^2(0, T; H_0^1(\Omega))$ and integrate over Q_T. We obtain

$$\int_0^T \int_\Omega \rho_n(u_n)\nabla\varphi_n\nabla(\varphi_n - \varphi_0)\,\mathrm{d}x\,\mathrm{d}t = 0,$$

hence

$$\int_0^T \int_\Omega \rho_n(u_n)|\nabla\varphi_n|^2\,\mathrm{d}x\,\mathrm{d}t \leq C_1, \quad \text{for all } n \geq 1. \tag{5.27}$$

Where $C_1 = C_1(\bar{\rho}, \|\varphi_0\|_{L^2(0,T;H^1(\Omega))})$. Consequently, the sequence $(\rho_n(u_n)\nabla\varphi_n)$ is bounded in $L^2(Q_T)$. Thus, there exists a function $\Phi \in L^2(Q_T)^d$ and a subsequence, still denoted the same way, such that

$$\rho_n(u_n)\nabla\varphi_n \to \Phi \text{ weakly in } L^2(Q_T)^d. \tag{5.28}$$

This weak limit function $\Phi \in L^2(Q_T)^d$ is in fact the third component of the triplet appearing in the Definition 4.2 of a capacity solution.

Taking u_n as a test function in (5.19), for all $t \in [0, T]$, we obtain

$$\begin{aligned} &\frac{1}{2}\|u_n(t)\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega a(x, t, T_n(u_n), \nabla u_n)\nabla u_n\,\mathrm{d}x\,\mathrm{d}t \\ &= \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} - \int_0^t \int_\Omega \rho_n(u_n)\varphi_n\nabla\varphi_n\nabla u_n\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{5.29}$$

From (4.2), (4.5), (5.25) and (5.24), we obtain

$$\alpha \int_0^t \int_\Omega M(x, |\nabla u_n|) \leq \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega \|\varphi_0\|_{L^\infty(Q_T)}\rho_2\nabla\varphi_n\nabla u_n, \tag{5.30}$$

and by Young's inequality, we may deduce that for all $t \in [0, T]$,

$$\int_0^t \int_\Omega M(x, |\nabla u_n|)\,\mathrm{d}x\,\mathrm{d}t \leq C, \tag{5.31}$$

where C is a positive constant not depending on n. It follows from (2.6) that the sequence (u_n) is bounded in $W_0^{1,x}L_M(Q_T)$. Consequently, there exist a subsequence of (u_n), still denoted in the same way, and a function $u \in W_0^{1,x}L_M(Q_T)$ such that:

$$u_n \rightharpoonup u \text{ in } W_0^{1,x}L_M(Q_T) \text{ for } \sigma(\Pi L_M, \Pi E_{\bar{M}}). \tag{5.32}$$

On the other hand, Let $\phi \in W_0^{1,x}E_M(Q_T)^d$ be an arbitrary function such that $\|\nabla\phi\|_{(M),Q_T} = 1/(k+1)$. In view of the monotonicity of a_n, one easily has

$$\begin{aligned}
&\int_{Q_T} a_n(x,t,u_n,\nabla u_n)\nabla\phi \\
&\le \int_{Q_T} a_n(x,t,u_n,\nabla u_n)\nabla u_n - \int_{Q_T} a_n(x,t,u_n,\nabla\phi)(\nabla u_n - \nabla\phi) \\
&\le C + \int_{Q_T} |a_n(x,t,u_n,\nabla\phi)\nabla u_n| + \int_{Q_T} a_n(x,t,u_n,\nabla\phi)\nabla\phi,
\end{aligned} \tag{5.33}$$

We can show that the two last integrals in (5.33) are bounded with respect to n. Indeed, for the first one, by Young's inequality

$$\int_{Q_T} |a_n(x,t,u_n,\nabla\phi)\nabla u_n| \le 3\zeta \int_{Q_T} \Big[\bar{M}\Big(x, \frac{|a(x,t,T_n(u_n),\nabla\phi)|}{3\zeta}\Big) + M(x,|\nabla u_n|)\Big],$$

using (4.2) we have

$$3\zeta\bar{M}\Big(x, \frac{|a(x,t,T_n(u_n),\nabla\phi)|}{3\zeta}\Big) \le \zeta\left(\bar{M}(x,c(x,t)) + P(x,kT_n(u_n)) + M(x,k\nabla\phi)\right),$$

since (u_n) is bounded in $W_0^{1,x}L_M(Q_T)$, and owing to Poincaré's inequality, there exists $\lambda > 0$ such that $\int_{Q_T} M(x,u_n/\lambda) \le 1$ for all $n \ge 1$. Also, since $P \ll M$, there exists $s_0 > 0$ such that $P(x,ks) \le P(x,ks_0) + M(x,s/\lambda)$ for all $s \in \mathbb{R}$. Consequently, using (2.11) it yields

$$\begin{aligned}
&3\zeta\int_{Q_T} \bar{M}\Big(\frac{|a(x,t,T_n(u_n),\nabla\phi)|}{3\zeta}\Big) \\
&\le \zeta\Big(\int_{Q_T} \bar{M}(x,c(x,t)) + T\int_\Omega P(x,ks_0)\,\mathrm{d}x + \int_{Q_T} M(x,u_n/\lambda) + \int_{Q_T} M(x,k\nabla\phi)\Big) \\
&\le C,
\end{aligned}$$

and thus $\int_{Q_T} |a_n(x,t,u_n,\nabla\phi)\nabla u_n| \le C$, for all $n \ge 1$ and $\phi \in W_0^{1,x}E_M(Q_T)^d$ such that $\|\nabla\phi\|_{(M),Q_T} = 1/(k+1)$. On the other hand, the second integral in (5.33), namely $\int_{Q_T} a_n(x,t,u_n,\nabla\phi)\nabla\phi$ can be dealt in the same way so that it is easy to check that it is also bounded. Gathering all these estimates, and using the dual norm, one easily deduce that

$$(a_n(x,t,u_n,\nabla u_n)) \text{ is bounded in } L_{\bar{M}}(Q_T)^d. \tag{5.34}$$

Thus, up to a subsequence, still denoted in the same way, there exists $\delta \in L_{\bar{M}}(Q_T)^d$ such that

$$a_n(x,t,u_n,\nabla u_n) \rightharpoonup \delta \quad \text{in } L_{\bar{M}}(Q_T)^d \text{ for } \sigma(\Pi L_{\bar{M}}, \Pi E_M). \tag{5.35}$$

Finally, since both sequences $(\operatorname{div} a_n(x,t,u_n,\nabla u_n))$ and $(\operatorname{div}(\rho_n(u_n)\varphi_n\nabla\varphi_n))$ are bounded in the space $W^{-1,x}L_{\bar{M}}(Q_T)$, according to (5.19), we have

$$\Big(\frac{\partial u_n}{\partial t}\Big) \quad \text{is bounded in } W^{-1,x}L_{\bar{M}}(Q_T). \tag{5.36}$$

Consequently, $(u_n) \subset \mathbf{W}$ is bounded and, since the embedding $\mathbf{W} \hookrightarrow E_P(Q_T)$ is compact, for a subsequence, still denoted in the same way, we have

$$u_n \to u \quad \text{strongly in } E_P(Q_T) \text{ and a.e. in } Q_T, \tag{5.37}$$

where $u \in W_0^{1,x}L_M(Q_T)$ is also the limit function appearing in (5.32).

Step 2. Introduction of regularized sequences and the almost everywhere convergence of the gradients.

First we introduce two smooth sequences, namely, $(v_j) \subset \mathcal{D}(Q_T)$ and $(\psi_i) \subset \mathcal{D}(\Omega)$ such that

(1) $v_j \to u$ in $W_0^{1,x}L_M(Q_T)$ for the modular convergence;
(2) $v_j \to u$ and $\nabla v_j \to \nabla u$ and almost everywhere in Q_T;
(3) $\psi_i \to u_0$ strongly in $L^2(\Omega)$;
(4) $\|\psi_i\|_{L^2(\Omega)} \leq 2\|u_0\|_{L^2(\Omega)}$, for all $i \geq 1$.

For a fixed positive real number K, we consider the truncation function at height K, T_K, defined in (2.17). Then, for every $K, \mu > 0$ and $i, j \in \mathbb{N}$, we introduce the function $w^i_{\mu,j} \in W_0^{1,x}L_M(Q_T)$ (to simplify the notation, we drop the index K) defined as $w^i_{\mu,j} = T_K(v_j)_\mu + e^{-\mu t}T_K(\psi_i)$, where $T_K(v_j)_\mu$ is the mollification with respect to time of $T_K(v_j)$ given in (3.1). From Lemma 3.1, we know that

$$\frac{\partial w^i_{\mu,j}}{\partial t} = \mu(T_K(v_j) - w^i_{\mu,j}), \quad w^i_{\mu,j}(\cdot, 0) = T_K(\psi_i), \quad |w^i_{\mu,j}| \leq K \text{ a.e in } Q_T, \tag{5.38}$$

$$w^i_{\mu,j} \to w^i_\mu \overset{\text{def}}{=} T_K(u)_\mu + e^{-\mu t}T_K(\psi_i) \text{ in } W_0^{1,x}L_M(Q_T), \tag{5.39}$$

for the modular convergence as $j \to \infty$.

$$T_K(u)_\mu + e^{-\mu t}T_K(\psi_i) \to T_K(u) \quad \text{in } W_0^{1,x}L_M(Q_T), \tag{5.40}$$

for the modular convergence as $\mu \to \infty$. Since we may consider subsequences in (5.38)-(5.40), we will assume without loss of generality that the convergences (5.39) and (5.40) also hold almost everywhere in Q_T.

We will establish the following proposition.

Proposition 5.3. *Let (u_n, φ_n) be a solution of the approximate problem* (5.19)-(5.23)*. Then, for a suitable subsequence, still denoted in the same way, we have*

$$\nabla u_n \to \nabla u \text{ a.e. in } Q_T, \tag{5.41}$$

as n tends to $+\infty$.

Proof. In the sequel and throughout this article, χ^j_s and χ_s will denote, respectively, the characteristic functions of the sets

$$Q^j_s = \{(x,t) \in Q_T : |\nabla T_K(v_j)| \leq s\}, \quad Q_s = \{(x,t) \in Q_T : |\nabla T_K(u)| \leq s\}.$$

We also introduce the primitive of the truncation function T_K vanishing at the origin, S_K, that is,

$$S_K(t) = \int_0^t T_K(s)ds = \begin{cases} t^2/2 & \text{if } |t| \leq K, \\ K|t| - K^2/2 & \text{if } |t| > K. \end{cases} \tag{5.42}$$

It is straightforward to show that $0 \leq S_K(t) \leq K|t|$ for all $t \in \mathbb{R}$.

We will also use the following notation for vanishing sequences: $\epsilon(n)$ means a sequence such that $\lim_{n\to\infty} \epsilon(n) = 0$ or $\limsup_{n\to\infty} \epsilon(n) = 0$; $\epsilon(n, j)$ is a term such that $\lim_{j\to\infty} \lim_{n\to\infty} \epsilon(n, j) = 0$ where any occurrence of lim may be substituted by lim sup. And so on for $\epsilon(n, j, \mu)$, etc.

For any $\mu,\nu>0$ and $i,j,n\geq 1$ we may use the admissible test function $\varphi^{\mu,i}_{n,j,\nu}=T_\nu(u_n-w^i_{\mu,j})$ in (5.19). This leads to

$$\begin{aligned}&\langle\frac{\partial u_n}{\partial t},\varphi^{\mu,i}_{n,j,\nu}\rangle+\int_{Q_T}a_n(x,t,u_n,\nabla u_n)\nabla T_\nu(u_n-w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t\\&=\int_{Q_T}\rho_n(u_n)|\nabla\varphi_n|^2\varphi^{\mu,i}_{n,j,\nu}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}\tag{5.43}$$

By using (5.27), we obtain

$$\langle\frac{\partial u_n}{\partial t},\varphi^{\mu,i}_{n,j,\nu}\rangle+\int_{Q_T}a_n(x,t,u_n,\nabla u_n)\nabla T_\nu(u_n-w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t\leq C_1\nu.\tag{5.44}$$

As far as the parabolic term is concerned, we have

$$\begin{aligned}\langle\frac{\partial u_n}{\partial t},T_\nu(u_n-w^i_{\mu,j})\rangle&=\langle\frac{\partial u_n}{\partial t}-\frac{\partial w^i_{\mu,j}}{\partial t},T_\nu(u_n-w^i_{\mu,j})\rangle\\&\quad+\langle\frac{\partial w^i_{\mu,j}}{\partial t},T_\nu(u_n-w^i_{\mu,j})\rangle.\end{aligned}\tag{5.45}$$

The first term of the right-hand side in (5.45) can be written as

$$\langle\frac{\partial u_n}{\partial t}-\frac{\partial w^i_{\mu,j}}{\partial t},T_\nu(u_n-w^i_{\mu,j})\rangle=\int_\Omega S_\nu(u_n(T)-w^i_{\mu,j}(T))-\int_\Omega S_\nu(u_0-T_K(\psi_i)).$$

Since

$$\begin{aligned}0&\leq\int_\Omega S_\nu(u_0-T_K(\psi_i))\leq\nu\int_\Omega|u_0-T_K(\psi_i)|\\&\leq\nu|\Omega|^{1/2}(\int_\Omega|u_0-T_K(\psi_i)|^2)^{1/2}\\&\leq 3\|u_0\|_{L^2(\Omega)}|\Omega|^{1/2}\nu=C_2\nu,\end{aligned}$$

we deduce that for all $i,j,n\geq 1$ and $\mu,n,K>0$,

$$\langle\frac{\partial u_n}{\partial t}-\frac{\partial w^i_{\mu,j}}{\partial t},T_\nu(u_n-w^i_{\mu,j})\rangle\geq -C_2\nu.\tag{5.46}$$

As for the second term of the right-hand side in (5.45) we have

$$\langle\frac{\partial w^i_{\mu,j}}{\partial t},T_\nu(u_n-w^i_{\mu,j})\rangle=\mu\int_{Q_T}(T_K(v_j)-w^i_{\mu,j})T_\nu(u_n-w^i_{\mu,j}).\tag{5.47}$$

Passing to the limit first in $n\to\infty$, then in $j\to\infty$, it yields

$$\lim_{j\to\infty}\lim_{n\to\infty}\langle\frac{\partial w^i_{\mu,j}}{\partial t},T_\nu(u_n-w^i_{\mu,j})\rangle=\mu\int_{Q_T}(T_K(u)-w^i_\mu)T_\nu(u-w^i_\mu).$$

Owing to (5.38) and (5.39) we have $|w^i_\mu|\leq K$ almost everywhere in Q_T. Also, since $sT_\nu(s)\geq 0$ for all $s\in\mathbb{R}$, we deduce, for all $\mu,\nu,K>0$ and $i\geq 1$,

$$\lim_{j\to\infty}\lim_{n\to\infty}\langle\frac{\partial w^i_{\mu,j}}{\partial t},T_\nu(u_n-w^i_{\mu,j})\rangle\geq 0.\tag{5.48}$$

Gathering (5.45), (5.46) and (5.48) we finally obtain, for all $\mu,\nu,K>0$ and $i\geq 1$, the following estimate for the parabolic term

$$\liminf_{j\to\infty}\liminf_{n\to\infty}\langle\frac{\partial u_n}{\partial t},T_\nu(u_n-w^i_{\mu,j})\rangle\geq -C_2\nu.\tag{5.49}$$

It remains to analyze the diffusion term of (5.43). We have

$$
\begin{aligned}
&\int_{Q_T} a_n(x,t,u_n,\nabla u_n)\nabla T_\nu(u_n-w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t\\
&=\int_{\{|u_n-w^i_{\mu,j}|\le\nu\}} a_n(x,t,u_n,\nabla u_n)\nabla(u_n-w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t\\
&=\int_{\{|u_n|>K\}\cap\{|u_n-w^i_{\mu,j}|\le\nu\}} a_n(x,t,u_n,\nabla u_n)\nabla(u_n-w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_{\{|u_n|\le K\}\cap\{|u_n-w^i_{\mu,j}|\le\nu\}} a_n(x,t,u_n,\nabla u_n)\nabla(u_n-w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t\\
&=\int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}} a_n(x,t,T_K(u_n),\nabla T_K(u_n))(\nabla T_K(u_n)-\nabla w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_{\{|u_n|>K\}\cap\{|u_n-w^i_{\mu,j}|\le\nu\}} a_n(x,t,u_n,\nabla u_n)\nabla u_n\,\mathrm{d}x\,\mathrm{d}t\\
&\quad-\int_{\{|u_n|>K\}\cap\{|u_n-w^i_{\mu,j}|\le\nu\}} a_n(x,t,u_n,\nabla u_n)\nabla w^i_{\mu,j}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

By (4.9) and (4.5) we have

$$
\begin{aligned}
&\int_{\{|u_n|>K\}\cap\{|u_n-w^i_{\mu,j}|\le\nu\}} a_n(x,t,u_n,\nabla u_n)\nabla u_n\,\mathrm{d}x\,\mathrm{d}t\\
&\ge\alpha\int_{\{|u_n|>K\}\cap\{|u_n-w^i_{\mu,j}|\le\nu\}} M(x,|\nabla u_n|)\,\mathrm{d}x\,\mathrm{d}t\ge 0,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\int_{Q_T} a_n(x,t,u_n,\nabla u_n)\nabla T_\nu(u_n-w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t\\
&\ge\int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}} a_n(x,t,T_K(u_n),\nabla T_K(u_n))(\nabla T_K(u_n)-\nabla w^i_{\mu,j})\\
&\quad-\int_{\{|u_n|>K\}\cap\{|u_n-w^i_{\mu,j}|\le\nu\}} a_n(x,t,u_n,\nabla u_n)\nabla w^i_{\mu,j}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{5.50}
$$

On the one hand, let us observe that for any $K>0$, and for n large enough, namely $n>K+\nu\ge K$, we have

$$
a_n(x,t,T_K(u_n),\nabla T_K(u_n))=a(x,t,T_K(u_n),\nabla T_K(u_n)).\tag{5.51}
$$

On the other hand, from (5.38), we have $|w^i_{\mu,j}|\le K$ a.e. in Q_T, then in the set $\{|u_n-w^i_{\mu,j}|\le\nu\}$, we have $|u_n|\le|u_n-w^i_{\mu,j}|+|w^i_{\mu,j}|\le\nu+K$. Then for $n>\nu+K$, we obtain

$$
\begin{aligned}
&\int_{\{|u_n|>K\}\cap\{|u_n-w^i_{\mu,j}|\le\nu\}} a_n(x,t,u_n,\nabla u_n)\nabla w^i_{\mu,j}\,\mathrm{d}x\,\mathrm{d}t\\
&=\int_{\{|u_n|>K\}\cap\{|u_n-w^i_{\mu,j}|\le\nu\}} a(x,t,T_{\nu+K}(u_n),\nabla T_{\nu+K}(u_n))\nabla w^i_{\mu,j}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{5.52}
$$

From (5.51) and (5.52), inequality (5.50) becomes

$$\begin{aligned} &\int_{Q_T} a_n(x,t,u_n,\nabla u_n)\nabla T_\nu(u_n - w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t \\ &\geq \int_{\{|T_K(u_n)-w^i_{\mu,j}|\leq\nu\}} a(x,t,T_K(u_n),\nabla T_K(u_n))(\nabla w^i_{\mu,j} T_K(u_n) - \nabla w^i_{\mu,j}) \\ &\quad - \int_{\{|u_n|>K\}\cap\{|u_n-w^i_{\mu,j}|\leq\nu\}} a(x,t,T_{\nu+K}(u_n),\nabla T_{\nu+K}(u_n))\nabla w^i_{\mu,j}. \end{aligned} \tag{5.53}$$

We put

$$J_1 = \int_{\{|u_n|>K\}\cap\{|u_n-w^i_{\mu,j}|\leq\nu\}} a(x,t,T_{\nu+K}(u_n),\nabla T_{\nu+K}(u_n))\nabla w^i_{\mu,j}\,\mathrm{d}x\,\mathrm{d}t.$$

Since $(a(x,t,T_{K+\nu}(u_n),\nabla T_{K+\nu}(u_n)))$ is bounded in $L_{\bar{M}}(Q_T)^d$, we have

$$a(x,t,T_{K+\nu}(u_n),\nabla T_{K+\nu}(u_n)) \rightharpoonup l_{K+\nu}$$

weakly in $L_{\bar{M}}(Q_T)$ in $\sigma(\Pi L_{\bar{M}}, \Pi E_M)$ as n tends to infinity and since

$$\nabla w^i_{\mu,j}\chi_{\{|u_n|>K\}\cap\{|u_n-w^i_{\mu,j}|\leq\nu\}} \to \nabla w^i_{\mu,j}\chi_{\{|u|>K\}\cap\{|u-w^i_{\mu,j}|\leq\nu\}}$$

strongly in $E_M(Q_T)^d$ as n tends to infinity, we have

$$\begin{aligned} &\int_{\{|u_n|>K\}\cap\{|u_n-w^i_{\mu,j}|\leq\nu\}} a(x,t,T_{\nu+K}(u_n),\nabla T_{\nu+K}(u_n))\nabla w^i_{\mu,j}\,\mathrm{d}x\,\mathrm{d}t \\ &\to \int_{\{|u|>K\}\cap\{|u-w^i_{\mu,j}|\leq\nu\}} l_{K+\nu}\nabla w^i_{\mu,j}\,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

as n approaches infinity.

Using Lemma 2.9 with the convergences (5.39), (5.40), together with the almost everywhere convergence, and letting first j then μ tend to infinity, we obtain (note that the index i disappears in this process)

$$\int_{\{|u|>K\}\cap\{|u-w^i_{\mu,j}|\leq\nu\}} l_{K+\nu}\nabla w^i_{\mu,j} \to \int_{\{|u|>K\}\cap\{|u-T_K(u)|\leq\nu\}} l_{K+\nu}\nabla T_K(u) = 0$$

since $\nabla T_K(u) = 0$ in the set $\{|u| > K\}$. This gives

$$J_1 = \epsilon(n,j,\mu,i). \tag{5.54}$$

Using (5.49), (5.53) and (5.54) in (5.44), we obtain

$$\begin{aligned} &\int_{\{|T_K(u_n)-w^i_{\mu,j}|\leq\nu\}} a(x,t,T_K(u_n),\nabla T_K(u_n))(\nabla T_K(u_n) - \nabla w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t \\ &\leq C\nu + \epsilon(n,j,\mu,i). \end{aligned} \tag{5.55}$$

where $C = (C_1 + C_2)$.

On the other hand, note that

$$\begin{aligned}
&\int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}} a(x,t,T_K(u_n),\nabla T_K(u_n))(\nabla T_K(u_n)-\nabla w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t\\
&=\int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}} a(x,t,T_K(u_n),\nabla T_K(u_n))(\nabla T_K(u_n)-\nabla T_K(v_j)\chi^s_j)\\
&\quad+\int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}} a(x,t,T_K(u_n),\nabla T_K(u_n))(\nabla T_K(v_j)\chi^s_j-\nabla w^i_{\mu,j})\\
&=J_2+J_3.
\end{aligned}\tag{5.56}$$

The integral term J_3 tends to 0 as first n, then j, μ, i and s go to ∞. Indeed, since,

$$a(x,t,T_K(u_n),\nabla T_K(u_n))\rightharpoonup l_K \quad \text{weakly in } L_{\bar M}(Q_T)^d,$$

and since

$$(\nabla T_K(v_j)\chi^s_j-\nabla w^i_{\mu,j})\chi_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}}\to(\nabla T_K(v_j)\chi^s_j-\nabla w^i_{\mu,j})\chi_{\{|T_K(u)-w^i_{\mu,j}|\le\nu\}}$$

strongly in $E_{\bar M}(Q_T)^d$ as $n\to\infty$, we have

$$\lim_{n\to\infty}J_3=\int_{\{|T_K(u)-w^i_{\mu,j}|\le\nu\}} l_K\cdot(\nabla T_K(v_j)\chi^s_j-\nabla w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t.$$

Letting j, μ, i and s, in this order, tend to infinity we readily deduce that

$$J_3=\epsilon(n,j,\mu,i,s).\tag{5.57}$$

Consequently, from (5.55), (5.56) and (5.57), one has

$$\begin{aligned}
J_2&=\int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}} a(x,t,T_K(u_n),\nabla T_K(u_n))(\nabla T_K(u_n)-\nabla T_K(v_j)\chi^s_j)\\
&\le C\nu+\epsilon(n,i,j,\mu,s).
\end{aligned}\tag{5.58}$$

Let M_n be the non-negative expression

$$M_n=(a(x,t,T_K(u_n),\nabla T_K(u_n))-a(x,t,T_K(u_n),\nabla T_K(u)))(\nabla T_K(u_n)-\nabla T_K(u)),$$

then for any $0<\theta<1$, we write

$$I_{n,r}=\int_{Q_r}M_n^\theta\,\mathrm{d}x\,\mathrm{d}t.$$

We have

$$\int_{Q_r}M_n^\theta\,\mathrm{d}x\,\mathrm{d}t=\int_{Q_r}M_n^\theta\chi_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}}+\int_{Q_r}M_n^\theta\chi_{\{|T_K(u_n)-w^i_{\mu,j}|>\nu\}}.\tag{5.59}$$

Using Hölder's inequality the second term of the right-side hand is less than

$$\Big(\int_{Q_r}M_n\,\mathrm{d}x\,\mathrm{d}t\Big)^\theta\Big(\int_{Q_r}\chi_{\{|T_K(u_n)-w^i_{\mu,j}|>\nu\}}\,\mathrm{d}x\,\mathrm{d}t\Big)^{1-\theta}.$$

Note that

$$\begin{aligned}
\int_{Q_r}M_n\,\mathrm{d}x\,\mathrm{d}t&=\int_{Q_r}a(x,t,T_K(u_n),\nabla T_K(u_n))\nabla T_K(u_n)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad-\int_{Q_r}a(x,t,T_K(u_n),\nabla T_K(u_n))\nabla T_K(u)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&+\int_{Q_r} a(x,t,T_K(u_n),\nabla T_K(u))\nabla T_K(u)\,\mathrm{d}x\,\mathrm{d}t\\
&-\int_{Q_r} a(x,t,T_K(u_n),\nabla T_K(u))\nabla T_K(u_n)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Since $(a(x,t,T_K(u_n),\nabla T_K(u_n)))$ is bounded in $L_{\bar{M}}(Q_T)^d$, $(\nabla T_K(u_n))$ is bounded in $L_M(Q_T)^d$ and $(a(x,t,T_K(u_n),\nabla T_K(u)))$ is bounded in $L^\infty(Q_r)$, we have (M_n) is bounded in $L^1(Q_r)$.

It follows that there exists a constant $C_3>0$ such that

$$
\int_{Q_r} M_n^\theta \chi_{\{|T_K(u_n)-w^i_{\mu,j}|>\nu\}}\,\mathrm{d}x\,\mathrm{d}t \le C_3 \operatorname{meas}\{|T_K(u_n)-w^i_{\mu,j}|>\nu\}^{1-\theta}. \tag{5.60}
$$

Using again Hölder's inequality, we have

$$
\begin{aligned}
&\int_{Q_r} M_n^\theta \chi_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}}\,\mathrm{d}x\,\mathrm{d}t\\
&\le \Big(\int_{Q_r} 1\,\mathrm{d}x\,\mathrm{d}t\Big)^{1-\theta}\Big(\int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}\cap Q_r} M_n\,\mathrm{d}x\,\mathrm{d}t\Big)^{\theta}\\
&\le C_4\Big(\int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}\cap Q_r} M_n\,\mathrm{d}x\,\mathrm{d}t\Big)^{\theta}.
\end{aligned} \tag{5.61}
$$

From (5.60) and (5.61), we obtain

$$
\begin{aligned}
I_{n,r} &\le C_3 \operatorname{meas}\{|T_K(u_n)-w^i_{\mu,j}|>\nu\}^{1-\theta}\\
&\quad + C_4\Big(\int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}\cap Q_r} M_n\,\mathrm{d}x\,\mathrm{d}t\Big)^{\theta}.
\end{aligned} \tag{5.62}
$$

On the other hand, for every $s\ge r$ and $r>0$, we have

$$
\begin{aligned}
&\int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}\cap Q_r} M_n\,\mathrm{d}x\,\mathrm{d}t\\
&\le \int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}\cap Q_s} M_n\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}\cap Q_s} \cdot[a(x,t,T_K(u_n),\nabla T_K(u_n))\\
&\quad - a(x,t,T_K(u_n),\nabla T_K(u)\chi_s)]\cdot[\nabla T_K(u_n)-\nabla T_K(u)\chi_s]\,\mathrm{d}x\,\mathrm{d}t\\
&\le \int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}} [a(x,t,T_K(u_n),\nabla T_K(u_n))\\
&\quad - a(x,t,T_K(u_n),\nabla T_K(u)\chi_s)]\cdot[\nabla T_K(u_n)-\nabla T_K(u)\chi_s]\,\mathrm{d}x\,\mathrm{d}t\\
&\le \int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}} [a(x,t,T_K(u_n),\nabla T_K(u_n))\\
&\quad - a(x,t,T_K(u_n),\nabla T_K(v_j)\chi_j^s)]\cdot[\nabla T_K(u_n)-\nabla T_K(v_j)\chi_j^s]\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}} a(x,t,T_K(u_n),\nabla T_K(u_n))\cdot[\nabla T_K(v_j)\chi_j^s-\nabla T_K(u)\chi_s]\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}} \cdot[a(x,t,T_K(u_n),\nabla T_K(v_j)\chi_j^s)
\end{aligned}
$$

$$
\begin{aligned}
&- a(x,t,T_K(u_n),\nabla T_K(u)\chi^s)]\cdot\nabla T_K(u_n)\,\mathrm{d}x\,\mathrm{d}t\\
&-\int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}} a(x,t,T_K(u_n),\nabla T_K(v_j)\chi^s_j)\cdot\nabla T_K(v_j)\chi^s_j\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}} a(x,t,T_K(u_n),\nabla T_K(u)\chi_s)\cdot\nabla T_K(u)\chi_s\,\mathrm{d}x\,\mathrm{d}t\\
&= I_1+I_2+I_3+I_4+I_5.
\end{aligned}
$$

We will take the limit first in n then in j, μ, i and s as they tend to infinity in these last five integrals.

Starting with I_1, we have

$$
\begin{aligned}
I_1 &= \int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}}(a(x,t,T_K(u_n),\nabla T_K(u_n))-a(x,t,T_K(u_n),\nabla T_K(v_j)\chi^s_j))\\
&\quad\cdot(\nabla T_K(u_n)-\nabla T_K(v_j)\chi^s_j)\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}} a(x,t,T_K(u_n),\nabla T_K(u_n))\cdot(\nabla T_K(u_n)-\nabla T_K(v_j)\chi^s_j)\\
&\quad -\int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}} a(x,t,T_K(u_n),\nabla T_K(v_j)\chi^s_j)\cdot(\nabla T_K(u_n)-\nabla T_K(v_j)\chi^s_j)\\
&= J_2-J_3.
\end{aligned}
$$

Since the sequence $(a(x,t,T_K(u_n),\nabla T_K(v_j)\chi^s_j)\chi_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}})_n$ converges to $a(x,t,T_K(u),\nabla T_K(v_j)\chi^s_j)\chi_{\{|T_K(u)-w^i_{\mu,j}|\le\nu\}}$ strongly in $E_{\bar M}(Q_T)^d$ and $(\nabla T_K(u_n))$ converges to $\nabla T_K(u)$ weakly in $L_M(Q_T)^d$ for $\sigma(\Pi L_M,\Pi E_{\bar M})$, we then have

$$
\begin{aligned}
J_3 &= \int_{\{|T_K(u)-w^i_{\mu,j}|\le\nu\}} a(x,t,T_K(u),\nabla T_K(v_j)\chi^s_j)\big(\nabla T_K(u)-\nabla T_K(v_j)\chi^s_j\big)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\epsilon(n).
\end{aligned}
$$

Using the almost everywhere convergence of $w^i_{\mu,j}$ and since $(\nabla T_K(v_j)\chi^s_j)_j$ converges to $\nabla T_K(u)\chi_s$ strongly in $E_M(Q_T)^d$ and $(a(x,t,T_K(u),\nabla T_K(v_j)\chi^s_j))_j$ converges to $a(x,t,T_K(u),\nabla T_K(u)\chi_s)$ strongly in $L_{\bar M}(Q_T)^d$, we deduce

$$
\begin{aligned}
J_3 &= \int_{Q_T} a(x,t,T_K(u),\nabla T_K(u)\chi_s)(\nabla T_K(u)-\nabla T_K(u)\chi_s)\,\mathrm{d}x\,\mathrm{d}t+\epsilon(n,j,\mu,i)\\
&= \epsilon(n,j,\mu,i,s).
\end{aligned}
$$

Gathering all these estimates, taking into account (5.58), we obtain

$$
I_1\le C\nu+\epsilon(n,j,\mu,i,s)=\epsilon(n,j,\mu,i,s,\nu). \tag{5.63}
$$

As for I_2, since $(a(x,t,T_K(u_n),\nabla T_K(u_n)))_n$ converges to l_K weakly in $L_{\bar M}(Q_T)^d$ for $\sigma(\Pi L_{\bar M},\Pi E_M)$ and $((\nabla T_K(v_j)\chi^s_j-\nabla T_K(u)\chi_s)\chi_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}})_n$ converges to $(\nabla T_K(v_j)\chi^s_j-\nabla T_K(u)\chi_s)\chi_{\{|T_K(u)-w^i_{\mu,j}|\le\nu\}}$ strongly in $E_M(Q_T)^d$, we obtain

$$
I_2=\int_{\{|T_K(u)-w^i_{\mu,j}|\le\nu\}} l_K(\nabla T_K(v_j)\chi^s_j-\nabla T_K(u)\chi_s)\,\mathrm{d}x\,\mathrm{d}t+\epsilon(n).
$$

By letting now $j\to\infty$, and using Lebesgue's theorem, we deduce that

$$
I_2=\epsilon(n,j). \tag{5.64}
$$

Similar tools as above yield

$$I_3 = \epsilon(n, j), \tag{5.65}$$

$$I_4 = -\int_{Q_T} a(x, t, T_K(u), \nabla T_K(u)\chi_s)\nabla T_K(u)\chi_s + \epsilon(n, j, \mu, i, s), \tag{5.66}$$

$$I_5 = \int_{Q_T} a(x, t, T_K(u), \nabla T_K(u)\chi_s)\nabla T_K(u)\chi_s + \epsilon(n, j, \mu, i, s). \tag{5.67}$$

Combining (5.62)-(5.67), we obtain

$$I_{n,r} \le C_4\epsilon(n, j, \mu, i, s, \nu)^\theta + C_3 \operatorname{meas}\{|T_K(u_n) - w^i_{\mu,j}| > \nu\}^{1-\theta}. \tag{5.68}$$

Consequently, when we take the limit superior first in n, then in j, μ, i, s and ν in (5.68), we obtain

$$\begin{aligned}\limsup_{n\to\infty} \int_{Q_r} &\Big((a(x, t, T_K(u_n), \nabla T_K(u_n)) - a(x, t, T_K(u_n), \nabla T_K(u)))\\ &\cdot (\nabla T_K(u_n) - \nabla T_K(u))\Big)^\theta \mathrm{d}x\,\mathrm{d}t = 0.\end{aligned}$$

According to (4.9) this last expression implies that

$$\lim_{n\to\infty} \int_{Q_r} M(x, \nabla T_K(u_n) - \nabla T_K(u))^\theta \mathrm{d}x\,\mathrm{d}t = 0.$$

hence, for a subsequence, $\nabla T_K(u_n) \to \nabla T_K(u)$ almost everywhere in Q_r. Since $r > 0$ is arbitrary, we may deduce that, maybe for another subsequence, $\nabla T_K(u_n) \to \nabla T_K(u)$ almost everywhere in Q_T. Finally, since $K > 0$ is arbitrary, it yields, still for a subsequence,

$$\nabla u_n \to \nabla u \quad \text{a.e in } Q_T. \tag{5.69}$$

This completes the proof. □

Remark 5.4. A straightforward consequence of Proposition 5.3 is that, owing to (5.35), $\delta = a(x, t, u, \nabla u)$; that is,

$$a_n(x, t, u_n\nabla u_n) \rightharpoonup a(x, t, u, \nabla u) \quad \text{in } L_{\bar M}(Q_T)^d \text{ for } \sigma(\Pi L_{\bar M}, \Pi E_M). \tag{5.70}$$

Step 3. In this step, we will show that $\varphi_n \to \varphi$ strongly in $L^1(Q_T)$ modulo a subsequence. The strongly convergence of (φ_n) in $L^1(Q_T)$ is based in the next result which generalizes that of González Montesinos and Ortegón Gallego in [14, Lemma 4].

Lemma 5.5. *Let P be a Musielak function such that* (2.9) *is satisfied. Assume that $s^2 \le P(x, s)$, for all a. a. $x \in \Omega$ and all $s \in \mathbb{R}$, and let (u_n) be a bounded sequence in $W^{1,x}L_M(Q_T)$ such that $u_n \to u$ strongly in $E_P(Q_T)$. Then there exists a subsequence $(u_{n(k)}) \subset (u_n)$ such that, for every $\epsilon > 0$, there exists a constant value $\boldsymbol{M} = \boldsymbol{M}(\epsilon)$ and a function $\psi \in L^1(0, T; W^{1,1}(\Omega))$ satisfying the following properties:*

$$0 \le \psi \le 1. \tag{5.71}$$

$$\|\psi - 1\|_{L^1(Q_T)} + \|\nabla\psi\|_{L^1(Q_T)} \le \epsilon. \tag{5.72}$$

$$|u|, |u_{n(k)}| \le \boldsymbol{M} \quad \text{on } \{\psi > 0\} \text{ for all } k \ge 1. \tag{5.73}$$

Proof. According to lemmas 2.9 and 2.14 we deduce the the following continuous inclusions:

$$L_P(Q_T) \hookrightarrow L_{\bar{P}}(Q_T) \hookrightarrow L_{\bar{M}}(Q_T).$$

Since (u_n) is relatively compact in $E_P(Q_T)$, we can extract a subsequence $(u_{n(k)}) \subset (u_n)$ such that

$$\sum_{k=1}^{\infty} \|u_{n(k)} - u\|_{L_{\bar{M}}(Q_T)} \leq 1. \tag{5.74}$$

Fix $K > 0$ to be chosen later big enough and introduce the function γ given by

$$\gamma = (|u| - K)^+ + \sum_{k=1}^{\infty} (|u_{n(k)} - u| - K)^+. \tag{5.75}$$

Then putting $v_k = u_{n(k)} - u$, $k \geq 1$, and $v_0 = u$, we have

$$\begin{aligned}
&\int_{Q_T} (|v_k| - K)^+ + \int_{Q_T} |\nabla(|v_k| - K)^+| \\
&= \int_{\{|v_k|>K\}} (|v_k| - K)^+ \frac{|v_k|}{|v_k|} + \int_{\{|v_k|>K\}} |\nabla(|v_k| - K)^+| \frac{|v_k|}{|v_k|} \\
&\leq \frac{1}{K} (\|v_k\|_{L_M(Q_T)} + \|\nabla v_k\|_{L_M(Q_T)}) \|v_k\|_{L_{\bar{M}}(Q_T)}
\end{aligned}$$

Summing these inequalities, bearing in mind that $(u_{n(k)})$ and (v_k) are bounded in $W^{1,x}L_M(Q_T)$ and (5.75), we deduce

$$\begin{aligned}
&\sum_{k=0}^{\infty} (\|(|v_k| - K)^+\|_{L^1(Q_T)} + \|(|\nabla v_k| - K)^+\|_{L^1(Q_T)}) \\
&\leq \frac{C_0}{K} \sum_{k=0}^{\infty} \|v_k\|_{L_{\bar{M}}(Q_T)} \\
&= \frac{C_0}{K} \Big(\|u\|_{L_{\bar{M}}(Q_T)} + \sum_{k=1}^{\infty} \|u_{n(k)} - u\|_{L_{\bar{M}}(Q_T)} \Big) \\
&\leq \frac{C_0}{K} (\|u\|_{L_{\bar{M}}(Q_T)} + 1) = \frac{C}{K}.
\end{aligned}$$

Hence

$$\|\gamma\|_{L^1(0,T;W^{1,1}(\Omega))} \leq \frac{C}{K}.$$

It is straightforward to check that the function $\psi = (1-\gamma)^+$ satisfies the asserted condition (5.71)-(5.73) for $K \geq C/\epsilon$ and $\mathbf{M} = K + 1$. $\square$

The next two results analyze the behavior of certain subsequences of (φ_n). They will allow us, together with the convergences deduced in the previous steps, to pass to the limit in the approximate problems (5.19)-(5.23) in order to show the existence of a capacity solution to the system (1.2).

Lemma 5.6 ([14]). *Let* (u_n, φ_n) *be a weak solution to the system* (5.19)-(5.23), $u \in E_P(Q_T)$ *and* $\varphi \in L^\infty(Q_T)$ *the limit functions appearing, respectively, in* (5.26) *and* (5.37). *Then, for any function* $S \in C_0^1(\mathbb{R})$, *there exists a subsequence, still denoted in the same way, such that*

$$S(u_n)\varphi_n \rightharpoonup S(u)\varphi \quad \text{weakly in } L^2(0,T;H^1(\Omega)). \tag{5.76}$$

Moreover, if $0 \le S \le 1$, then there exists a constant $C > 0$, independent of S, such that

$$\limsup_{n\to\infty} \int_{Q_T} \rho_n(u_n)|\nabla[S(u_n)\varphi_n - S(u)\varphi]|^2 \le C\|S'\|_\infty(1 + \|S'\|_\infty). \tag{5.77}$$

Lemma 5.7. *There exists a subsequence $(\varphi_{n(k)}) \subset (\varphi_n)$ such that*

$$\lim_{k\to\infty} \int_{Q_T} |\varphi_{n(k)} - \varphi| = 0. \tag{5.78}$$

Proof. The proof of this result is almost identical to that of [14, Lemma 4.8]. For the sake of completeness, we include it here.

Since the conditions of Lemma 5.5 are fulfilled by a suitable subsequence $(u_{n(k)})$, we have for every $\epsilon > 0$ there exists $\mathbf{M} > 0$ and $\psi \in L^1(0,T;W^{1,1}(\Omega))$ such that (5.71)-(5.73) are satisfied. By (5.73), there exists $C_{\mathbf{M}} > 0$ such that

$$\xi_k \overset{\text{def}}{=} \rho_{n(k)}(u_{n(k)}) \ge C_{\mathbf{M}} \quad \text{on } \{\psi > 0\}, \text{ for all } k \ge 1. \tag{5.79}$$

We consider a sequence of regular functions $(S_m) \subset C_0^1(\mathbb{R})$ such that

$$0 \le S_m \le 1, \quad S_m = 1 \quad \text{in } [-\mathbf{M},\mathbf{M}], \text{ for all } k \ge 1, \tag{5.80}$$

$$\|S_m'\|_{L^\infty(\mathbb{R})} \le \frac{1}{m}, \quad \text{for all } m \ge 1. \tag{5.81}$$

From (5.73) and (5.80), we write

$$\int_{Q_T} |\varphi_{n(k)} - \varphi| = \int_{\{\psi>0\}} |S_m(u_{n(k)})\varphi_{n(k)} - S_m(u)\varphi| + \int_{\{\psi=0\}} |\varphi_{n(k)} - \varphi|.$$

Inserting $\pm\psi|S_m(u_{n(k)})\varphi_{n(k)} - S_m(u)\varphi|$ in the first integral above and $-\psi|\varphi_{n(k)} - \varphi| = 0$ in the second one, then owing to (5.25), (5.26), (5.71) and using Poincaré's inequality, we obtain

$$\begin{aligned}
&\int_{Q_T} |\varphi_{n(k)} - \varphi| \\
&= \int_{\{\psi>0\}} \psi|S_m(u_{n(k)})\varphi_{n(k)} - S_m(u)\varphi| \\
&\quad + \int_{\{\psi>0\}} (1-\psi)|S_m(u_{n(k)})\varphi_{n(k)} - S_m(u)\varphi| + \int_{\{\psi=0\}} (1-\psi)|\varphi_{n(k)} - \varphi| \\
&\le C_0 \int_{Q_T} |\nabla(\psi(S_m(u_{n(k)})\varphi_{n(k)} - S_m(u)\varphi))| + 2\|\varphi_0\|_{L^\infty(Q_T)} \int_{Q_T} |1-\psi| \\
&\le 2C_0\|\varphi_0\|_{L^\infty(Q_T)} \int_{Q_T} |\nabla\psi| + C_0 \int_{Q_T} |\nabla(S_m(u_{n(k)}))\varphi_{n(k)} - S_m(u)\varphi)| \\
&\quad + 2\|\varphi_0\|_{L^\infty(Q_T)} \int_{Q_T} |1-\psi|,
\end{aligned}$$

Putting $C^* = 2\|\varphi_0\|_{L^\infty(\Omega)} \max(C_0, 1)$, $K_{\mathbf{M}} = C_0 C_{\mathbf{M}}^{-1/2}|\Omega|^{1/2}T^{1/2}$ and taking into account (5.72) and (5.79), we deduce

$$\begin{aligned}
&\int_{Q_T} |\varphi_{n(k)} - \varphi| \le C^*\epsilon + C_0 \int_{Q_T} \xi_k^{-1/2}\xi_k^{1/2}|\nabla(S_m(u_{n(k)}))\varphi_{n(k)} - S_m(u)\varphi)| \\
&\le C^*\epsilon + K_{\mathbf{M}} \Big(\int_{Q_T} \xi_k|\nabla(S_m(u_{n(k)}))\varphi_{n(k)} - S_m(u)\varphi|^2\Big)^{1/2},
\end{aligned}$$

Owing to (5.77) and (5.81), we obtain

$$\begin{aligned}\limsup_{k\to\infty}\int_{Q_T}|\varphi_{n(k)}-\varphi| &\le C^*\epsilon+K_M\Big(C\|S'_m\|_\infty(1+\|S'_m\|_\infty)\Big)^{1/2}\\ &\le C^*\epsilon+K_M C^{1/2}\\ big[\frac{1}{m}\Big(1+\frac{1}{m}\Big)\big]^{1/2}.\end{aligned}$$

And since $\epsilon>0$ and $m\ge 1$ are arbitrary, we derive the desired result. $\square$

Step 5. Passing to the limit. According to (5.26), (5.28), (5.32), (5.34) and (5.36), it is straightforward that the condition (C_1) of Definition 4.2 is fulfilled. The convergences in Proposition 5.3 and Lemma 5.7 lead us to (C_2) of Definition 4.21, and in order to obtain the condition (C_3), using Proposition 5.3 and Lemma 5.7 again with (5.76), it is sufficient to let k goes to infinity in the expression

$$S(u_{n(k)})\rho_{n(k)}(u_{n(k)})\nabla\varphi_{n(k)}=\rho_{n(k)}(u_{n(k)})[\nabla(S(u_{n(k)})\varphi_{n(k)})-\varphi_{n(k)}\nabla S(u_{n(k)})]$$

Step 6. Regularity of u. Finally, it remains to show the regularity of $u\in C([0,T];L^1(\Omega))$ [1]. To this end, we go back to the expression (5.43) but the integration in time happens in the interval $(0,\tau)$ for any $\tau\in(0,T]$, namely

$$\begin{aligned}\langle\frac{\partial u_n}{\partial t},T_\nu(u_n-w^i_{\mu,j})\rangle_{Q_\tau}&=\int_{Q_\tau}a_n(x,t,u_n,\nabla u_n)(\nabla w^i_{\mu,j}-\nabla u_n)\chi_{\{|u_n-w^i_{\mu,j}|\le\nu\}}\\ &\quad-\int_{Q_\tau}\rho_n(u_n)\varphi_n\nabla\varphi_n\nabla T_\nu(u_n-w^i_{\mu,j}).\end{aligned}\tag{5.82}$$

where $\nu\in(0,1]$, $Q_\tau=(0,\tau)\times\Omega$ and $\langle\cdot,\cdot\rangle_{Q_\tau}$ is the duality product between $W^{-1,x}L_{\bar M}(Q_\tau)$ and $W^{1,x}_0L_M(Q_\tau)$. We will consider the necessary subsequences to assure the almost everywhere convergence in Q_T of $\varphi_n\to\varphi$, $u_n\to u$, $\nabla u_n\to\nabla u$, and also for $(T_\nu(u_n-w^i_{\mu,j}))$, etc.

From (5.70) we readily obtain

$$\begin{aligned}&\lim_{n\to\infty}\int_{Q_\tau}a_n(x,t,u_n,\nabla u_n)\nabla w^i_{\mu,j}\chi_{\{|u_n-w^i_{\mu,j}|\le\nu\}}\\ &=\int_{Q_\tau}a(x,t,u,\nabla u)\nabla w^i_{\mu,j}\chi_{\{|u-w^i_{\mu,j}|\le\nu\}}\end{aligned}$$

Also, by Fatou's lemma we obtain

$$\int_{Q_\tau}a(x,t,u,\nabla u)\nabla u\chi_{\{|u-w^i_{\mu,j}|\le\nu\}}\le\liminf_{n\to\infty}\int_{Q_\tau}a_n(x,t,u_n,\nabla u_n)\nabla u_n\chi_{\{|u_n-w^i_{\mu,j}|\le\nu\}}$$

then, passing to the limit in these two expressions, first in j, then in μ, i and K, we deduce, uniformly in τ, that

$$\int_{Q_\tau}a_n(x,t,u_n,\nabla u_n)(\nabla w^i_{\mu,j}-\nabla u_n)\chi_{\{|u_n-w^i_{\mu,j}|\le\nu\}}\le\epsilon(n,j,\mu,i,K)\tag{5.83}$$

The analysis of the term $\int_{Q_\tau}\rho_n(u_n)\varphi_n\nabla\varphi_n\nabla T_\nu(u_n-w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t$ is more involved. Here the difficulty relies on the fact that the sequence $(\rho_n(u_n)|\nabla\varphi_n|^2)$ does not converge, in general, strongly in $L^1(Q_T)$. To deal with this situation, we are going to make use of the properties already shown for a capacity solution. Indeed, we first notice that $\nabla T_\nu(u_n - w^i_{\mu,j}) = 0$ in the set $\{|u_n| \le K+\nu\} \subset \{|u_n| \le K+1\}$. Then we consider a sequence of functions $S_K \subset C^1_0(\mathbb{R})$ such that

$$0 \le S_K \le 1, \quad S_K = 1 \quad \text{in } [-(K+1), K+1], \text{ for all } K>0,$$
$$\|S'_K\|_{L^\infty(\mathbb{R})} \le \frac{1}{K+1}, \quad \text{for all } K>0.$$

We have

$$\begin{aligned}
&\int_{Q_\tau} \rho_n(u_n)\varphi_n \nabla\varphi_n \nabla T_\nu(u_n - w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_{Q_\tau} \rho_n(u_n)\varphi_n \nabla[S_K(u_n)\varphi_n] \nabla T_\nu(u_n - w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_{Q_\tau} \rho_n(u_n)\varphi_n \nabla[S_K(u_n)\varphi_n - S(u)\varphi] \nabla T_\nu(u_n - w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_{Q_\tau} \rho_n(u_n)\varphi_n \nabla[S_K(u)\varphi] \nabla T_\nu(u_n - w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t = I_1 + I_2.
\end{aligned}$$

According to the almost everywhere convergence of (u_n) and (φ_n) together with (5.25) and (5.32), we readily deduce that

$$\lim_{n\to\infty} I_2 = \int_{Q_\tau} \rho(u)\varphi\nabla[S_K(u)\varphi]\nabla T_\nu(u - w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t,$$

and using the identity (C_3), already shown in the previous step, namely,

$$\rho(u)\nabla[S_K(u)\varphi] = S_K(u)\Phi + \varphi\nabla S_K(u),$$

we can easily obtain the estimate

$$I_2 = \epsilon(n,j,\mu,i,K).$$

As for the term I_1, we use (5.77) to get, for some constant $C>0$,

$$\begin{aligned}
|I_1|^2 &\le \Big(\int_{Q_\tau} \rho_n(u_n)|\nabla[S_K(u_n)\varphi_n - S(u)\varphi]|^2\,\mathrm{d}x\,\mathrm{d}t\Big)\\
&\quad \times \Big(\int_{Q_\tau} \rho_n(u_n)|\varphi_n|^2|\nabla T_\nu(u_n - w^i_{\mu,j})|^2\,\mathrm{d}x\,\mathrm{d}t\Big)\\
&\le \frac{C}{K+1},
\end{aligned}$$

and thus it is also

$$I_1 = \epsilon(n,j,\mu,i,K).$$

Consequently, we obtain, for any fixed $\nu \in (0,1]$ and uniformly in $\tau\in[0,T]$,

$$\int_{Q_\tau} \rho_n(u_n)\varphi_n\nabla\varphi_n\nabla T_\nu(u_n - w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t \le \epsilon(n,j,\mu,i,K), \tag{5.84}$$

Gathering (5.82), (5.83) and (5.84) we obtain the estimate

$$\langle \frac{\partial u_n}{\partial t}, T_\nu(u_n - w^i_{\mu,j})\rangle_{Q_\tau} \le \epsilon(n,j,\mu,i,K). \tag{5.85}$$

Then we write, as in (5.45)-(5.48),

$$\int_\Omega S_\nu(u_n(x,\tau) - w^i_{\mu,j}(x,\tau))\,\mathrm{d}x$$

$$
\begin{aligned}
&= \big\langle \frac{\partial(u_n - w^i_{\mu,j})}{\partial t}, T_\nu(u_n - w^i_{\mu,j})\big\rangle_{Q_\tau} + \int_\Omega S_\nu(u_0 - T_K(\psi_i))\,\mathrm{d}x \\
&= \big\langle \frac{\partial u_n}{\partial t}, T_\nu(u_n - w^i_{\mu,j})\big\rangle_{Q_\tau} - \big\langle \frac{\partial w^i_{\mu,j}}{\partial t}, T_\nu(u_n - w^i_{\mu,j})\big\rangle_{Q_\tau} + \int_\Omega S_\nu(u_0 - T_K(\psi_i))\,\mathrm{d}x.
\end{aligned}
$$

Consequently, owing to (5.48) and (5.85), it yields, for every fixed $\nu \in (0,1]$ and uniformly in $\tau \in [0,T]$,

$$
\int_\Omega S_\nu(u_n(x,\tau) - w^i_{\mu,j}(x,\tau))\,\mathrm{d}x \le \epsilon(n,j,\mu,i,K),
$$

and using the convexity of the function S_ν we may also derive the estimate

$$
\begin{aligned}
&\int_\Omega S_\nu\Big(\frac{1}{2}(u_n(x,\tau) - u_m(x,\tau))\Big)\,\mathrm{d}x \\
&\le \frac{1}{2}\int_\Omega S_\nu(u_n(x,\tau) - w^i_{\mu,j}(x,\tau))\,\mathrm{d}x + \frac{1}{2}\int_\Omega S_\nu(u_m(x,\tau) - w^i_{\mu,j}(x,\tau))\,\mathrm{d}x \\
&\le \epsilon(n,j,\mu,i,K) + \epsilon(m,j,\mu,i,K),
\end{aligned}
$$

and thus, for any fixed $\nu > 0$ and uniformly in $\tau \in [0,T]$, we have

$$
\int_\Omega S_\nu\Big(\frac{1}{2}(u_n(x,\tau) - u_m(x,\tau))\Big)\,\mathrm{d}x \le \epsilon(n) + \epsilon(m). \tag{5.86}
$$

Consequently, using the definition of S_ν and (5.86), for all $\tau \in [0,T]$, we have

$$
\begin{aligned}
&\int_\Omega \frac{1}{2}|u_n(x,\tau) - u_m(x,\tau)|\,\mathrm{d}x \\
&\le \int_{\{|u_n(x,\tau)-u_m(x,\tau)|\le 2\nu\}} \frac{1}{2}|u_n(x,\tau) - u_m(x,\tau)|\,\mathrm{d}x \\
&\quad + \int_{\{|u_n(x,\tau)-u_m(x,\tau)|> 2\nu\}} \frac{1}{2}|u_n(x,\tau) - u_m(x,\tau)|\,\mathrm{d}x \\
&\le |\Omega|\nu + \frac{1}{\nu}\int_{\{|u_n(x,\tau)-u_m(x,\tau)|> 2\nu\}} \frac{\nu}{2}|u_n(x,\tau) - u_m(x,\tau)|\,\mathrm{d}x \\
&= |\Omega|\nu + \frac{1}{\nu}\int_{\{|u_n(x,\tau)-u_m(x,\tau)|> 2\nu\}} \big[S_\nu\Big(\frac{1}{2}|u_n(x,\tau) - u_m(x,\tau)|\Big) + \frac{\nu^2}{2}\big]\,\mathrm{d}x \\
&= \frac{3}{2}|\Omega|\nu + \frac{1}{\nu}(\epsilon(n) + \epsilon(m)).
\end{aligned}
$$

This last estimate shows that (u_n) is a Cauchy sequence in the space $C([0,T];L^1(\Omega))$ and, in particular, its limit u lies in this space. This completes the proof. □

Remark 5.8. The previous result given in Theorem 5.1 gives just the existence of a capacity solution. The uniqueness of the capacity solution is an open problem, even in a Hilbertian context. Other interesting questions on this capacity solution are concerned with the establishment of certain qualitative properties [5] as the derivation of some energy estimate, the analysis of large time behavior or even the occurrence of a blow-up situation.

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Addendum posted on August 14, 2019

After publication, the authors found two errata which do not affect the main result of this article; see explanations below.

Erratum 1. Statement in (2.1), namely

$$\operatorname{ess\,inf}_{x\in\Omega} \frac{M(x,t)}{t} \to \infty \quad \text{as } t \to \infty, \tag{2.1}$$

does not follow from the definition of Musielak-Orlicz functions given in the Definition 2.1. Indeed, one just may consider the case $d = 1$, $\Omega = (0,1)$ and $M(x,t) = xt^2$ in which $\operatorname{ess\,inf}_{x\in\Omega} M(x,t)/t = 0$ for all $t > 0$.

Statement (2.1) is easily deduced under the two conditions given in (4.1), which are assumed in this article, namely

$$P \ll M \text{ near infinity and } t^2 \le P(x,t) \text{ for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R}. \tag{4.1}$$

Erratum 2. The estimate for the term $\int_0^T\int_\Omega M(x,|\nabla u|)\,\mathrm{d}x\,\mathrm{d}t$ appearing on page 17 is not correctly deduced. It should be:

$$\begin{aligned}
&\alpha \int_0^T\int_\Omega M(x,|\nabla u|)\,\mathrm{d}x\,\mathrm{d}t \\
&\le \int_0^T\int_\Omega a(x,t,\nabla u)\nabla u\,\mathrm{d}x\,\mathrm{d}t \\
&\le \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + \frac{1}{2}\rho_2^2\|\varphi_0\|^2_{L^\infty(Q_T)}\int_0^T\int_\Omega |\nabla\varphi|^2 + \frac{1}{2}\int_0^T\int_\Omega |\nabla u|^2 < \infty.
\end{aligned}$$

This shows that $|\nabla u| \in \mathcal{L}_M(Q_T)$. Now, using Lemma 2.14, we have

$$\begin{aligned}
\alpha \int_0^T\int_\Omega M(x,|\nabla u|)\,\mathrm{d}x\,\mathrm{d}t &\le \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + \frac{C}{2}\rho_2^2|\varphi_0\|^2_{L^\infty(Q_T)} + \frac{1}{2}\int_0^T\int_\Omega |\nabla u|^2 \\
&\le C_2 + \frac{1}{2}\Big(C_1 + \varepsilon\int_0^T\int_\Omega M(x,|\nabla u|)\Big).
\end{aligned}$$

Then it suffices to take $\varepsilon = \alpha$ in the above inequality to obtain the estimate

$$\int_0^T\int_\Omega M(x,|\nabla u|)\,\mathrm{d}x\,\mathrm{d}t \le C(u_0,\varphi_0,\alpha,T,\rho_2) = C_0.$$

End of addendum.

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