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EXISTENCE OF SOLUTIONS TO p-LAPLACIAN DIFFERENCE EQUATIONS UNDER BARRIER STRIPS CONDITIONS

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ABSTRACT. We study the existence of solutions to the boundary-value problem

$$\Delta(\phi_p(\Delta u(k-1))) = f(k, u(k), \Delta u(k)), \quad k \in \mathbb{T}_{[1,N]},$$
$$\Delta u(0) = A, \quad u(N+1) = B,$$

with barrier strips conditions, where $N > 1$ is a fixed natural number, $\phi_p(s) = |s|^{p-2}s$, $p > 1$.

1. INTRODUCTION

Given $a, b \in \mathbf{Z}$ and $a < b$, we employ $\mathbb{T}_{[a,b]}$ to denote $\{a, a+1, a+2, \dots, b-1, b\}$. In this paper, we are concerned with the following p-Laplacian difference equation

$$\Delta(\phi_p(\Delta u(k-1))) = f(k, u(k), \Delta u(k)), \quad k \in \mathbb{T}_{[1,N]}, \tag{1.1}$$

satisfying the boundary conditions

$$\Delta u(0) = A, u(N+1) = B, \tag{1.2}$$

where $N > 1$ is a fixed natural number, $f : \mathbb{T}_{[1,N]} \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, $\phi_p(s) = |s|^{p-2}s, p > 1, (\phi_p)^{-1} = \phi_q, \frac{1}{p} + \frac{1}{q} = 1$.

In recent years, p-Laplacian discrete boundary-value problems have been investigated in literature [1,2,4]. But, almost all of the works discussed these problems when f satisfies growth restriction at ∞. Now, the question is: Is there still a solution to those problems when f is not restricted at ∞?

In 1994, Kelevedjiev [3] used Leray-Schauder principle to discuss the solutions to the nonlinear differential boundary-value problem

$$x''(t) = f(t, x(t), x'(t)), \quad t \in [0, 1], \tag{1.3}$$
$$x'(0) = A, x(1) = B. \tag{1.4}$$

He established the following results:

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Theorem 1.1. *Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. Suppose there are constants L_i, $i = 1,2,3,4$, such that $L_2 > L_1 \geq A$, $L_3 < L_4 \leq A$,*

$$f(t,x,p) \leq 0, \quad (t,x,p) \in [0,1] \times \mathbb{R} \times [L_1, L_2],$$
$$f(t,x,p) \geq 0, \quad (t,x,p) \in [0,1] \times \mathbb{R} \times [L_3, L_4].$$

Then (1.3)-(1.4) has at least one solution in $C^2[0,1]$, where $C^2[0,1]$ is the set of functions whose second derivative is continuous on $[0,1]$.

Clearly, growth restrictions on f are not imposed in Theorem 1.1. So, we use the Leray-Schauder principle to discuss the existence of solutions to boundary-value problem (1.1)-(1.2) when f is not restricted at ∞.

2. Preliminaries

Let $X := \{u | u : \mathbb{T}_{[0,N+1]} \to \mathbb{R}\}$ be equipped with the norm

$$\|u\|_X = \max_{k \in \mathbb{T}_{[0,N+1]}} |u(k)|,$$

and $Y := \{u | u : \mathbb{T}_{[1,N]} \to \mathbb{R}\}$ with the norm

$$\|u\|_Y = \max_{k \in \mathbb{T}_{[1,N]}} |u(k)|.$$

It is easy to see that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces.

The main result of our work is based on the following special form of Leray-Schauder principle.

Theorem 2.1. *Let $f : \mathbb{T}_{[1,N]} \times \mathbb{R}^2 \to \mathbb{R}$ be continuous, $L : D(L) \subset X \to Y$ be a bijection, and L^{-1} be completely continuous. If there exists a constant M such that an arbitrary solution of the boundary-value problem*

$$Lu(k) = \lambda f(k, u(k), \Delta u(k)), \quad k \in \mathbb{T}_{[1,N]}, \quad \lambda \in [0,1], \quad u \in D(L)$$

satisfies $\|u\|_X < M$, then the boundary-value problem

$$Lu(k) = f(k, u(k), \Delta u(k)), \quad k \in \mathbb{T}_{[1,N]}, \quad u \in D(L)$$

has at least one solution in X.

Define the operator $L : D(L) \subset X \to Y$ by

$$Lu(k) = \Delta(\phi_p(\Delta u(k-1))), \quad u \in D(L),\ k \in \mathbb{T}_{[1,N]},$$

where $D(L) = \{u | u \in X, \Delta u(0) = A, u(N+1) = B\}$.

Lemma 2.2. *Let $h \in Y$. Then the boundary-value problem*

$$\Delta\phi_p(\Delta u(k-1)) = h(k), \quad k \in \mathbb{T}_{[1,N]}, \tag{2.1}$$
$$\Delta u(0) = A, \quad u(N+1) = B \tag{2.2}$$

has a unique solution

$$u(k) = B - \sum_{s=k+1}^{N+1} \Big(\phi_q\Big(\sum_{l=1}^{s-1} h(l) + \phi_p(A)\Big)\Big), \quad k \in \mathbb{T}_{[0,N+1]}.$$

Proof. Summing the equation (2.1) from $s=1$ to $s=k-1$, we obtain

$$\phi_p(\Delta u(k-1)) = \phi_p(A) + \sum_{s=1}^{k-1} h(s).$$

Applying ϕ_q on both sides of the above equation, we obtain

$$\Delta u(k-1) = \phi_q(\phi_p(A) + \sum_{s=1}^{k-1} h(s)).$$

Summing again from $s=k+1$ to $s=N+1$, we have

$$B - u(k) = \sum_{s=k+1}^{N+1} (\phi_q(\sum_{l=1}^{s-1} h(l) + \phi_p(A))),$$
$$u(k) = B - \sum_{s=k+1}^{N+1} (\phi_q(\sum_{l=1}^{s-1} h(l) + \phi_p(A))), \quad k \in \mathbb{T}_{[0,N+1]}.$$

Next, we show that there is only one solution to (1.1)-(1.2). Suppose that u_1, u_2 are solutions. Then

$$\Delta(\phi_p(\Delta u_1(k-1))) = \Delta(\phi_p(\Delta u_2(k-1))), \quad k \in \mathbb{T}_{[1,N]}, \tag{2.3}$$

and $\Delta u_i(0) = A$, $u_i(N+1) = B$, $i = 1, 2$. Now, summing (2.3) from $s=1$ to $s=k-1$, we get

$$\phi_p(\Delta u_1(k-1)) - \phi_p(\Delta u_2(k-1)) = \phi_p(\Delta u_1(0)) - \phi_p(\Delta u_2(0)),$$

furthermore, $\Delta u_i(0) = A, i = 1, 2$,

$$\phi_p(\Delta u_1(k-1)) = \phi_p(\Delta u_2(k-1)),$$

and since ϕ_p is a bijection,

$$\Delta u_1(k-1) = \Delta u_2(k-1).$$

Summing the above equation from $s=k+1$ to $s=N+1$, we have

$$\sum_{s=k+1}^{N+1} \Delta u_1(k-1) = \sum_{s=k+1}^{N+1} \Delta u_2(k-1),$$
$$B - u_1(k) = B - u_2(k),$$

so $u_1(k) = u_2(k)$, $k \in \mathbb{T}_{[1,N]}$, and from the boundary conditions $\Delta u_i(0) = A$, $u_i(N+1) = B$, we have

$$u_1(k) = u_2(k), \quad k \in \mathbb{T}_{[0,N+1]}.$$

□

We remark that from Lemma 2.2, it follows that L is a bijection.

Lemma 2.3. *$L^{-1} : Y \to X$ is completely continuous.*

Proof. Since the range of L^{-1} has finite dimension, it is not difficult to check that it is compact; and from the continuity of f and ϕ_q, we can see that L^{-1} is continuous. □

3. Main results

Theorem 3.1. *Let $f : \mathbb{T}_{[1,N]} \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. Suppose there exist constants L_i, $i = 1, 2, 3, 4$ satisfying $L_2 > L_1 \geq A$, $L_3 < L_4 \leq A$, such that*

$$f(k, u, p) \leq 0, \quad (k, u, p) \in \mathbb{T}_{[1,N]} \times \mathbb{R} \times [L_1, L_2], \tag{3.1}$$

$$f(k, u, p) \geq 0, \quad (k, u, p) \in \mathbb{T}_{[1,N]} \times \mathbb{R} \times [L_3, L_4]. \tag{3.2}$$

Then the boundary-value problem (1.1)-(1.2) *has at least one solution in X.*

Proof. Let us define the function $\Phi : \mathbb{R} \to \mathbb{R}$ as follows.

$$\Phi(v) = \begin{cases} L_2, & v > L_2, \\ v, & L_3 \leq v \leq L_2, \\ L_3, & v < L_3. \end{cases}$$

Now, we consider the problem

$$\Delta(\phi_p(\Delta u(k-1))) = f(k, u(k), \Phi(\Delta u(k))), \quad k \in \mathbb{T}_{[1,N]}, u \in D(L). \tag{3.3}$$

Suppose that $u \in D(L)$ is an arbitrary solution to the family of problems

$$\Delta(\phi_p(\Delta u(k-1))) = \lambda f(k, u(k), \Phi(\Delta u(k))), \quad k \in \mathbb{T}_{[1,N]}. \tag{3.4}$$

To apply Theorem 2.1, we need a priori bounds for $\|u\|_X$ independent of $\lambda \in [0, 1]$. First, let us examine $\Delta u(k)$. Now, we prove that the set

$$S_0 = \{k \in \mathbb{T}_{[0,N]} | \Delta u(k) > L_1\}$$

is empty. Suppose it is not empty. Let $k_0 \in S_0$ be fixed. Then $\Delta u(k_0) > L_1$. From the construction of Φ, we know

$$L_1 < \Phi(\Delta u(k_0)) \leq L_2.$$

From (3.1) and $\Delta(\phi_p(\Delta u(k_0 - 1))) \leq 0$, we have

$$|\Delta u(k_0)|^{p-2} \Delta u(k_0) \leq |\Delta u(k_0 - 1)|^{p-2} \Delta u(k_0 - 1). \tag{3.5}$$

Now, we prove $k_0 - 1 \in S_0$. It will be discussed in three cases:
Case 1: $\Delta u(k_0) > 0$. Then from (3.5), we know $L_1 < \Delta u(k_0) \leq \Delta u(k_0 - 1)$;
Case 2: $\Delta u(k_0) = 0$. Then the result is obvious;
Case 3: $\Delta u(k_0) < 0$. Then $\Delta u(k_0 - 1)$ will be discussed under two cases.
Case 3.1: $\Delta u(k_0 - 1) \geq 0$. Then from (3.5), $\Delta u(k_0 - 1) > L_1$;
Case 3.2: $\Delta u(k_0 - 1) < 0$. Then p will be discussed under different situations.
Case 3.2.1: p is an odd number. Then $(-\Delta u(k_0))^{p-2} = -(\Delta u(k_0))^{p-2}$. From (3.5), we know $-(\Delta u(k_0))^{p-1} \leq |\Delta u(k_0 - 1)|^{p-2} \Delta u(k_0 - 1)$. Moreover, $\Delta u(k_0 - 1) < 0$, we have $-(\Delta u(k_0))^{p-1} \leq -(\Delta u(k_0 - 1))^{p-1}$. Since $p - 1$ is an even number and $\Delta u(k_0), \Delta u(k_0 - 1) < 0$, it's not difficult to get

$$L_1 < \Delta u(k_0) \leq \Delta u(k_0 - 1);$$

Case 3.2.2: p is an even number. Then we have $(\Delta u(k_0))^{p-1} \leq (\Delta u(k_0 - 1))^{p-1}$, and since $p - 1$ is an odd number, we know that

$$L_1 < \Delta u(k_0) \leq \Delta u(k_0 - 1);$$

so, when $\Delta u(k_0) < 0$, $\Delta u(k_0 - 1) < 0$, there also exists

$$L_1 < \Delta u(k_0) \leq \Delta u(k_0 - 1).$$

From Case 1, Case 2, Case 3, we obtain

$$L_1 < \Delta u(k_0) \le \Delta u(k_0 - 1),$$

so $k_0 - 1 \in S_0$. If we continue the above process, we get

$$\Delta u(0) \ge \Delta u(1) > L_1,$$

which contradicts with $\Delta u(0) = A$, so $S_0 = \emptyset$.

Similarly, we can obtain that the set

$$S_1 = \{k \in \mathbb{T}_{[0,N]} | \Delta u(k) < L_4\}$$

is also empty.

Then for $k \in \mathbb{T}_{[0,N]}$,

$$L_4 \le \Delta u(k) \le L_1, \tag{3.6}$$

i.e.,

$$\max_{k \in \mathbb{T}_{[0,N]}} |\Delta u(k)| \le C, \tag{3.7}$$

where $C = \max\{|L_1|, |L_4|\}$.

On the other hand, for $k \in \mathbb{T}_{[0,N]}$, since $u(N+1) = B$, we can construct $u(k) = -\sum_{s=k}^{N} \Delta u(s) + B$. Thus for $u \in D(L)$, we have

$$\max_{k \in \mathbb{T}_{[0,N+1]}} |u(k)| \le C_1, \tag{3.8}$$

where $C_1 = (N+1) \cdot C + |B|$. From (3.8), we can see that all of the solutions to problems (3.4) satisfy

$$\|u\|_X \le C_1.$$

Then there exists at least one solution $u \in D(L)$ to problem (3.3). And from (3.6), we know that

$$L_3 < L_4 \le \Phi(\Delta u(k)) \le L_1 < L_2, \quad k \in \mathbb{T}_{[1,N]},$$

together with the definition of Φ, the following conclusion

$$\Phi(\Delta u(k)) = \Delta u(k), \quad k \in \mathbb{T}_{[1,N]},$$

can be obtained. Thus u is also a solution to the problem (1.1)-(1.2).

Example. Consider the problem

$$\Delta(\phi_p(\Delta u(k-1))) = (\Delta u(k))^4 - 6(\Delta u(k))^3 + 11(\Delta u(k))^2 - 6\Delta u(k), \quad k \in \mathbb{T}_{[1,N]},$$
$$\Delta u(0) = 2, u(N+1) = B,$$

where $N > 1$ is a fixed natural number, B is an arbitrary number. Let $f(k,u,p) = p^4 - 6p^3 + 11p^2 - 6p$, $L_1 = \frac{5}{2}$, $L_2 = 3$, $L_3 = 1$, $L_4 = \frac{3}{2}$, $A = 2$. We can prove that $f(k,u,p)$ satisfies all conditions of Theorem 3.1, so this problem has at least one solution. $\square$

The next theorem can be proved by similar arguments.

Theorem 3.2. *Let $f : \mathbb{T}_{[1,N]} \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. Suppose there are constants L_i, $i = 1,2,3,4$ with $L_2 > L_1 \ge B$, $L_3 < L_4 \le B$, such that* (3.1), (3.2) *are satisfied. Then the boundary-value problem*

$$\Delta(\phi_p(\Delta u(k-1))) = f(k, u(k), \Delta u(k)), \quad k \in \mathbb{T}_{[1,N]},$$
$$u(0) = A, \quad \Delta u(N) = B$$

has at least one solution in X.

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