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Existence principles for inclusions of Hammerstein type involving noncompact acyclic multivalued maps *

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Abstract

We apply Mönch type fixed point theorems for acyclic multivalued maps to the solvability of inclusions of Hammerstein type in Banach spaces. Our approach makes possible to unify and improve the existence theories for nonlinear evolution problems and abstract integral inclusions of Volterra and Fredholm type.

1 Introduction

In [17], the following two fixed point theorems of Mönch type for multivalued maps with convex and compact values were proved:

Theorem 1.1 *Let D be a closed, convex subset of a Banach space X and $N : D \to 2^D \setminus \{\emptyset\}$ be a map with convex values. Assume* graph(N) *is closed, N maps compact sets into relatively compact sets and that for some $x_0 \in D$ one has*

$$\left.\begin{array}{r} M \subset D, \quad M = \operatorname{conv}(\{x_0\} \cup N(M)) \\ \text{and } \overline{M} = \overline{C} \text{ with } C \text{ a countable subset of } M \end{array}\right\} \Longrightarrow \overline{M} \text{ is compact.}$$

Then there exists $x \in D$ with $x \in N(x)$.

Theorem 1.2 *Let K be a closed, convex subset of a Banach space X, U be a relatively open subset of K, and $N : \overline{U} \to 2^K \setminus \{\emptyset\}$ a map with convex values. Assume* graph(N) *is closed, N maps compact sets into relatively compact sets and that for some $x_0 \in U$, the following two conditions are satisfied:*

$$\left.\begin{array}{r} M \subset \overline{U}, \quad M \subset \operatorname{conv}(\{x_0\} \cup N(M)) \\ \text{and } \overline{M} = \overline{C} \text{ with } C \text{ a countable subset of } M \end{array}\right\} \Longrightarrow \overline{M} \text{ is compact;}$$

$$x \notin (1-\lambda)x_0 + \lambda N(x) \text{ for all } x \in \overline{U} \setminus U,\ \lambda \in]0,1[.$$

Then there exists $x \in \overline{U}$ with $x \in N(x)$.

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In [17] and [18], some applications of Theorems 1.1 and 1.2 are presented for Hammerstein integral inclusions of the form

$$u(t) \in \int_0^T k(t,s)g(s,u(s))ds \quad \text{a.e. on } [0,T]. \tag{1.1}$$

Here k is a real single-valued function, while g is a set-valued map with convex, compact values in a Banach space E. Equation (1.1) can be written in the operator form

$$u \in SG(u) \tag{1.2}$$

where G is the Nemitsky multivalued operator associated to g, and S is the linear integral operator of kernel k.

The aim of this paper is to present a unified existence theory for inclusions of type (1.2) with linear and nonlinear operators S. Such inclusions arise naturally in the theory of evolution inclusions of the form

$$u'(t) \in f(t,u(t)) + g(t,u(t))$$

subject to initial conditions. They also arise in the theory of boundary-value problems for second order differential inclusions of the form

$$u''(t) \in f(t,u(t)) + g(t,u(t)).$$

In both cases S is the solution operator assigning to each function w the solution (assuming its existence and uniqueness) of the corresponding problem for

$$u'(t) \in f(t,u(t)) + w(t),$$

respectively

$$u''(t) \in f(t,u(t)) + w(t).$$

If S is nonlinear, we can not assume that the map $N := SG$ has convex values and so Theorems 1.1 and 1.2 do not apply. This was the motivation in [19] to give extensions of Theorems 1.1-1.2 for multivalued maps with non-convex values. These extensions are based on the Eilenberg-Montgomery fixed point theorem [9] and generalize previous results obtained by Fitzpatrick and Petryshyn [10] for condensing set-valued maps. Our approach to (1.2) and several hypotheses are inspired from [5, 6]. Notice in [5] and [6] it is assumed that the nonlinear operator S can be compared (in some sense explained latter) with a Volterra linear integral operator. This assumption together with a suitable compactness property of g guarantees that N is condensing with respect to a specific measure of non-compactness in the space of continuous functions on $[0,T]$. In the present paper, the hypotheses on S and g are more general, so that N have not to be condensing, but just to satisfy a Mönch type compactness condition. Moreover, in this paper we discuss not only continuous solutions but also L^p-solutions, and this is done by a common existence theory. Our results improve and extend those in [5, 6, 17, 18]. They also extend a lot of classical results on perturbed evolution problems and abstract integral inclusions of both Volterra and Fredholm type.

2 Preliminaries

First we recall some definitions. Let $H_* = \{H_n\}_{n\geq 0}$ denote the Čech homology functor with compact carriers and coefficients in the field of rational numbers $\mathbb{Q}$ (see Gorniewicz [12]). A nonempty metric space X is said to be *acyclic* if

$$H_n(X) = \begin{cases} \mathbb{Q} & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases}$$

i.e., X has the same homology as a single point space. A metric space X is said to be *contractible* if there is a homotopy $h : X \times [0,1] \to X$ such that $h(x,0) = x$ and $h(x,1) = x_0$ for every $x \in X$ and with $x_0 \in X$ given.

The space X is an *absolute retract* (AR for short) if for every metric space Z and closed set $A \subset Z$, every continuous map $f : A \to X$ has a continuous extension $\widehat{f} : Z \to X$. We say that X is an *absolute neighborhood retract* (ANR for short) if the above f has a continuous extension to some neighborhood of A.

It is well known that AR's spaces as well as contractible spaces are acyclic. So are R_δ-sets, i.e. compact metric spaces X for which there exists a decreasing sequence $(A_n)_{n\geq 1}$ of compact absolute retracts such that $X = \bigcap_{n\geq 1} A_n$. Also, every convex subset of a normed space is contractible and every compact and convex subset of a normed space is an ANR and is acyclic.

If X is a Hausdorff topological space, we let

$$P_f(X) = \{A \subset X : A \text{ is nonempty, closed}\},$$
$$P_k(X) = \{A \subset X : A \text{ is nonempty, compact}\}.$$

If X is a metric space we define

$$P_a(X) = \{A \subset X : A \text{ is nonempty, acyclic}\},$$
$$P_{ka}(X) = \{A \subset X : A \text{ is nonempty, compact, acyclic}\}.$$

If X is a closed, convex subset of a normed space $(E, |\cdot|)$, then we define

$$P_c(X) = \{A \subset X : A \text{ is nonempty, convex}\},$$
$$P_{kc}(X) = \{A \subset X : A \text{ is nonempty, compact,convex}\},$$

and for any nonempty subset $A \subset E$ we let $|A| = \sup\{|x| : x \in A\}$, By $\mathrm{conv}(A)$ we mean the convex hull of A.

Now we state the Eilenberg-Montgomery fixed point theorem [9].

Theorem 2.1 *Let Ξ be acyclic and ANR, Θ a compact metric space, $\Phi : \Xi \to P_a(\Theta)$ an upper semi-continuous map and $\Gamma : \Theta \to \Xi$ a continuous single-valued map. Then the map $\Gamma\Phi : \Xi \to 2^{\Xi}$ has a fixed point.*

An extension of this theorem for condensing (noncompact) acyclic maps is due to Fitzpatrick-Petryshyn [10]. Next we recall a well-known result of set-valued analysis (see [16], Proposition 1.2.17, Proposition 1.2.23 and Corollary 1.2.20).

Theorem 2.2 *Let X, Y be Hausdorff topological spaces.*

(a) Let $N : X \to P_f(Y)$, If N is upper semicontinuous, then graph(N) *is closed in $X \times Y$, Conversely, if* graph(N) *is closed and $\overline{N(X)}$ is compact, then N is upper semicontinuous.*

(b) Let $N : X \to P_k(Y)$ be upper semicontinuous. Then $N(A)$ is compact for each compact $A \subset X$.

Throughout this paper E will be a real Banach space with norm $|\cdot|$. A function $u : [a, b] \to E$ is said to be *strongly measurable* on $[a, b]$ if there exists a sequence of finitely-valued functions u_n with

$$u_n(t) \to u(t) \quad \text{as } n \to \infty, \quad \text{a.e. on } [a, b].$$

By $\int_a^b u(t)dt$ we mean the Bochner integral of u, assuming its existence. Recall that a strongly measurable function u is Bochner integrable if and only if $|u|$ is Lebesgue integrable.

For any real $p \in [1, \infty[$, we consider the space $L^p([a, b]; E)$ of all strongly measurable functions $u : [a, b] \to E$ such that $|u|^p$ is Lebesgue integrable on $[a, b]$. Then $L^p([a, b]; E)$ is a Banach space under the norm

$$|u|_p = (\int_a^b |u(s)|^p ds)^{1/p}.$$

Also for $p = \infty$, we let $L^\infty([a, b]; E)$ be the space of all strongly measurable function $u : [a, b] \to E$ which are essentially bounded, i.e.

$$\operatorname*{ess\,sup}_{t\in[a,b]} |u(t)| := \inf\{c \geq 0 : \ |u(t)| \leq c \quad \text{a.e. on } [a, b]\} < \infty.$$

$L^\infty([a, b]; E)$ is a Banach space under the norm $|u|_\infty = \operatorname{ess\,sup}_{t\in[a,b]} |u(t)|$. When $E = \mathbb{R}$ the space $L^p([a, b]; \mathbb{R})$ is simply denoted by $L^p[a, b]$. By $|u|_\infty$ we also denote the max-norm on the space $C([a, b]; E)$ of all continuous functions $u : [a, b] \to E$.

For a function $u : [a, b] \to E$, we define the *translation* by h $(0 < h < b-a)$, to be the function $\tau_h u : [a, b-h] \to E$, given by $\tau_h u(t) = u(t + h)$. We now state a compactness criterion for a subset of vector-valued functions. For the proof see for example [13], Theorems 1.2.5 and 1.2.8.

Theorem 2.3 *Let $p \in [1, \infty]$. Let $M \subset L^p([a, b]; E)$ be countable and suppose there exists some $\nu \in L^p[a, b]$ with $|u(t)| \leq \nu(t)$ a.e. on $[a, b]$ for all $u \in M$. Assume $M \subset C([a, b]; E)$ if $p = \infty$. Then M is relatively compact in $L^p([a, b]; E)$ if and only if*

(i) $\sup_{u\in M} |\tau_h u - u|_{L^p([a,b-h];E)} \to 0$ *as $h \to 0$;*

(ii) $M(t)$ is relatively compact in E for a.a. $t \in [a, b]$.

Next we state a weak compactness criterion in $L^p([a,b];E)$ which follows from the results of Diestel-Ruess-Schachermayer [8].

Theorem 2.4 *Let $p \in [1,\infty[$. Let $M \subset L^p([a,b];E)$ be countable and suppose there exists some $\nu \in L^p[a,b]$ with $|u(t)| \le \nu(t)$ a.e. on $[a,b]$ for all $u \in M$. If $M(t)$ is relatively compact in E for a.a. $t \in [a,b]$, then M is weakly relatively compact in $L^p([a,b];E)$.*

Finally we introduce the following definition. A map $\psi : [a,b] \times D \to 2^Y \setminus \{\emptyset\}$, where $D \subset X$ and $(X, |\cdot|_X)$ $(Y, |\cdot|_Y)$ are two Banach spaces, is said to be (q,p)-Carathéodory ($1 \le q \le \infty$, $1 \le p \le \infty$) if

(C1) $\psi(.,x)$ is strongly measurable for each $x \in D$;

(C2) $\psi(t,.)$ is upper semicontinuous for a.a. $t \in [a,b]$;

(C3) (a) if $1 \le p < \infty$, there exists $c \in L^q([a,b];\mathbb{R}_+)$ and $d \in \mathbb{R}_+$ such that $|\psi(t,x)|_Y \le c(t) + d|x|_X^p$ a.e. on $[a,b]$, for all $x \in D$.

(b) if $p = \infty$, for each $\rho > 0$ there exists $c_\rho \in L^q([a,b];\mathbb{R}_+)$ such that $|\psi(t,x)|_Y \le c_\rho(t)$ a.e. on $[a,b]$, for all $x \in D$ with $|x|_X \le \rho$.

A map ψ which satisfies (C1)-(C2) is said to be a Carathéodory function.

3 Fixed point theorems

In this section, we present the extensions of Theorems 1.1 and 1.2 to set-valued maps with acyclic values, which were established in [19]. For the reader convenience, we also reproduce here their proofs.

Theorem 3.1 *Let D be a closed, convex subset of a Banach space X, Y a metric space, $N : D \to P_a(Y)$ and $J : Y \to D$ continuous. Assume* graph (N) *is closed, N maps compact sets into relatively compact sets and that for some $x_0 \in D$ one has*

$$\left.\begin{array}{c} M \subset D, \quad M = \operatorname{conv}(\{x_0\} \cup JN(M)) \\ \textit{and } \overline{M} = \overline{C} \textit{ with } C \textit{ a countable subset of } M \end{array}\right\} \Longrightarrow \overline{M} \textit{ is compact.} \tag{3.1}$$

Then there exists $x \in D$ with $x \in JN(x)$.

Proof Since J is continuous, the map JN also has a closed graph and maps compact sets into relatively compact sets.

Following the steps (a) and (b) of the proof of Theorem 3.1 in [17], we find a convex set $M \subset D$ with $x_0 \in M$, $M = \operatorname{conv}(\{x_0\} \cup JN(M))$ and $K := \overline{M}$ compact. Next, instead of steps (c)-(d) of the above mentioned proof, we follow:
(c*) Proof of inclusion $JN(K) \subset K$. Let $\varepsilon > 0$ be fixed. According to Theorem 2.2, JN is upper semicontinuous. Consequently, for each $x \in M$ there exists an

open neighborhood V_x of x such that $JN(y) \subset JN(x) + B_\varepsilon(0)$ for all $y \in V_x$. Since for $x \in M$, one has $JN(x) \subset K$, it follows that $JN(y) \subset K_\varepsilon := K + B_\varepsilon(0)$ for every $y \in V_x$. Now M being dense in K, it results that $\{V_x : x \in M\}$ is a cover of K. Consequently, $JN(K) \subset K_\varepsilon$. Hence $JN(K) \subset \bigcap_{\varepsilon>0} K_\varepsilon = K$.
(d*) Application of the Eilenberg-Montgomery theorem. Since every compact and convex subset of a Banach space is an ANR and is acyclic, we may apply Theorem 2.1 to: $\Xi := K$, $\Theta := N(K)$, $\Phi = N$ and $\Gamma = J$. $\square$

Remark 3.1 (a) Under the assumptions of Theorem 3.1, $N : D \to P_{ka}(Y)$.
(b) According to Theorem 2.2, Theorem 3.1 is true under the following assumptions: $N : D \to P_{ka}(Y)$ and N is upper semicontinuous.

The next result is a version of Theorem 1.2 for set-valued maps with acyclic values.

Theorem 3.2 *Let K be a closed, convex subset of a Banach space X, U a convex, relatively open subset of K, Y a metric space, $N : \overline{U} \to P_a(Y)$ and $J : Y \to K$ continuous. Assume* graph(N) *is closed, N maps compact sets into relatively compact sets and that for some $x_0 \in U$, the following two conditions are satisfied:*

$$\left.\begin{array}{c} M \subset \overline{U}, \quad M \subset \operatorname{conv}(\{x_0\} \cup JN(M)) \\ \textit{and } \overline{M} = \overline{C} \textit{ with } C \textit{ a countable subset of } M \end{array}\right\} \Longrightarrow \overline{M} \textit{ is compact;} \tag{3.2}$$

$$x \notin (1-\lambda)x_0 + \lambda JN(x) \textit{ for all } x \in \overline{U} \setminus U,\ \lambda \in]0,1[. \tag{3.3}$$

Then there exists $x \in \overline{U}$ with $x \in JN(x)$.

Proof. Let $D = \overline{\operatorname{conv}}(\{x_0\} \cup JN(\overline{U}))$. Clearly, $x_0 \in D \subset K$. Let $P : K \to \overline{U}$ be

$$P(x) = \begin{cases} x & \text{for } x \in \overline{U} \\ \overline{x} & \text{for } x \in K \setminus \overline{U} \end{cases}$$

Here $\overline{x} = (1-\lambda)x_0 + \lambda x \in \overline{U} \setminus U$, $\lambda \in]0,1[$. It is easy to see that P is single valued and continuous.

Let $\widetilde{N} : D \to P_a(Y)$, $\widetilde{N}(x) = N(P(x))$. It is easily seen that graph$(\widetilde{N})$ is closed and $\widetilde{N}$ maps compact sets into relatively compact sets. Next we check (3.1) for $J\widetilde{N}$. Let $M \subset D$ be such that $M = \operatorname{conv}(\{x_0\} \cup J\widetilde{N}(M))$ and $\overline{M} = \overline{C}$ for some countable subset C of M. Since

$$\begin{aligned} P(M) &\subset \operatorname{conv}(\{x_0\} \cup M) \subset \operatorname{conv}(\{x_0\} \cup J\widetilde{N}(M)) \\ &= \operatorname{conv}(\{x_0\} \cup JNP(M)), \end{aligned}$$

$\overline{P(M)} = \overline{P(C)}$, $P(C) \subset P(M)$, and $P(C)$ is countable, from (3.2) we deduce that $P(M)$ is relatively compact. Then $J\widetilde{N}(M) = JNP(M)$ is relatively compact and Mazur's lemma implies that $\overline{M}$ is compact. Thus (3.1) holds for $J\widetilde{N}$.

Now we apply Theorem 3.1 to deduce that there exists an $x \in D$ with $x \in J\widetilde{N}(x)$. We claim that $x \in D \cap \overline{U}$. Assume the contrary, that is $x \in$

$D \setminus \overline{U}$. Then $x \in JN(\overline{x})$, where $\overline{x} = (1-\lambda)x_0 + \lambda x \in \overline{U} \setminus U$, $\lambda \in]0,1[$. Then $x = (1/\lambda)\overline{x} + (1 - 1/\lambda)x_0 \in JN(\overline{x})$. Hence $\overline{x} \in (1-\lambda)x_0 + \lambda JN(\overline{x})$, which contradicts (3.3). Thus $x \in D \cap \overline{U}$ and so $x \in JN(x)$. $\square$

4 Inclusions of Hammerstein type

Let $0 < T < \infty$, $I = [0,T]$, $(E, |\cdot|)$ be a real Banach space, $p \in [1,\infty]$ and $q \in [1,\infty[$. Let $r \in]1,\infty]$ be the conjugate exponent of q, that is $1/q + 1/r = 1$.

Consider $g : I \times E \to 2^E$ and the Nemitsky set-valued operator associated to g, p and q: $G : L^p(I;E) \to 2^{L^q(I;E)}$ given by

$$G(u) = \{w \in L^q(I;E) : w(s) \in g(s,u(s)) \text{ a.e. on } I\}.$$

Also consider a single-valued operator

$$S : L^q(I;E) \to L^p(I;E).$$

We discuss here the inclusion of Hammerstein type

$$u \in SG(u), \quad u \in L^p(I;E). \tag{4.1}$$

Theorem 3.2 immediately yields the following existence principle for (4.1).

Theorem 4.1 *Let K be a closed, convex subset of $L^p(I;E)$ $(1 \le p \le \infty)$, U a relatively open subset of K and $u_0 \in U$. Assume*

(H1) $SG : \overline{U} \to P_a(K)$ has closed graph and maps compact sets into relatively compact sets;

(H2) $$\left.\begin{array}{c} M \subset \overline{U}, \quad M \subset \operatorname{conv}(\{u_0\} \cup SG(M)) \\ \overline{M} = \overline{C}, \quad \text{with } C \text{ a countable subset of } M \end{array}\right\} \Longrightarrow \overline{M} \text{ is compact;}$$

(H3) $u \notin (1-\lambda)u_0 + \lambda SG(u)$ for all $\lambda \in]0,1[$ and $u \in \overline{U} \setminus U$.

Then (4.1) has a solution in $\overline{U}$.

For the proof of this theorem, apply Theorem 3.2 to $N = SG$ and J the identity map of K.

Remark 4.1 (solutions in $C(I;E)$) (a) If the values of S are in $C(I;E)$, then any solution of (4.1) in $K \subset L^p(I;E)$ $(1 \le p \le \infty)$ belongs to $C(I;E)$.
(b) The existence theory in $C(I;E)$ appears as a particular case, where $p = \infty$ and $K \subseteq C(I;E)$.

According to Remark 4.1 (a), when S takes values in $C(I;E)$, there is no loss of regularity in t if we work in an L^p space instead of $C(I;E)$. This is, for example, the case of the mild solution operator S associated to the generator of a continuous semigroup. On the other hand, we may image (by topological reasons, or others) that working in an L^p space could be more flexible than working in $C(I;E)$ (especially if E is reflexive and separable).

In what follows, $u_0 = 0$ (so it is assumed that $0 \in K$). For the next result, let $U = B_R$, the open ball $\{u \in K : |u|_p < R\}$. We give sufficient conditions on S and g in order that the assumptions (H1)-(H3) be satisfied. Thus we assume:

(S1) There is a function $k : I^2 \to \mathbb{R}_+$ such that $k(t,.) \in L^r(I)$, the function $t \longmapsto |k(t,.)|_r$ belongs to $L^p(I)$ and

$$|S(w_1)(t) - S(w_2)(t)| \le \int_I k(t,s)|w_1(s) - w_2(s)|ds \qquad (4.2)$$

a.e. on I, for all $w_1, w_2 \in L^q(I;E)$.

(S2) $S : L^q(I;E) \to K$ and for every compact, convex subset C of E, S is sequentially continuous from $L^q_w(I;C)$ to $L^p(I;E)$. Here $L^q_w(I;C)$ stands for the set $L^q(I;C)$ endowed with the weak topology of $L^q(I;E)$.

(g1) $g : I \times E \to P_{kc}(E)$.

(g2) $g(.,x)$ has a strongly measurable selection on I, for each $x \in E$.

(g3) $g(t,.)$ is upper semicontinuous for a.a. $t \in I$.

(g4) If $1 \le p < \infty$, then $|g(t,x)| \le a(t) + b|x|^{p/q}$ a.e. on I, for all $x \in E$. If $p = \infty$, then $|g(t,x)| \le a(t)$ a.e. on I, for all $x \in E$ with $|x| \le R$. Here $a \in L^q(I)$ and $b \in \mathbb{R}_+$.

(g5) For every separable closed subspace E_0 of E, there exists a $(q,p/q)$-Carathéodory function $\omega : I \times \mathbb{R}_+ \to \mathbb{R}_+$ such that for almost every $t \in I$,

$$\beta_{E_0}(g(t,M) \cap E_0) \le \omega(t, \beta_{E_0}(M))$$

for every set $M \subset E_0$ satisfying

$$|M| \le |S(0)(t)| + (|a|_q + bR^{p/q})|k(t,.)|_r$$

if $p < \infty$, and respectively

$$|M| \le |S(0)(t)| + |a|_q|k(t,.)|_r$$

if $p = \infty$. In addition $\varphi = 0$ is the unique solution in $L^p(I;\mathbb{R}_+)$ to the inequality

$$\varphi(t) \le \int_I k(t,s)\omega(s,\varphi(s))ds \quad \text{a.e. on } I. \qquad (4.3)$$

Here β_{E_0} is the ball measure of non-compactness on E_0. Recall that for a bounded set $A \subset E_0$, $\beta_{E_0}(A)$ is the infimum of $\varepsilon > 0$ for which A can be covered by finitely many balls of E_0 with radius not greater than ε.

(SG) For every $u \in K$ the set $SG(u)$ is acyclic in K.

Now we can state the main result of this section.

Theorem 4.2 *Assume (S1)-(S2), (g1)-(g5) and (SG) hold. In addition suppose (H3). Then (4.1) has at least one solution u in $K \subset L^p(I;E)$ with $|u|_p \leq R$.*

The proof is based on Theorem 4.1 and the following two lemmas that extend some results in [5].

Lemma 4.3 *Let $S : L^q(I;E) \to L^p(I;E)$ satisfy (S1)-(S2), $q \in [1,\infty[$ and $p \in [1,\infty]$. Let $M \subset L^q(I;\, E)$ be countable with*

$$|u(t)| \leq \nu(t) \tag{4.4}$$

a.e. on I, for all $u \in M$, where $\nu \in L^q(I)$. Let E_0 be a separable closed subspace of E with $u(t) \in E_0$ a.e. on I, for every $u \in M \cup S(M)$. Then the function $\varphi(t) = \beta_{E_0}(M(t))$ belongs to $L^q(I)$ and satisfies

$$\beta_{E_0}(S(M)(t)) \leq \int_I k(t,s)\varphi(s)ds \quad \text{a.e. on } I. \tag{4.5}$$

Proof Let $M = \{u_n : n \in \mathbb{N}\}$. The space E_0 being separable, we may represent it as $\overline{\bigcup_{k\geq 1} E_k}$ where for each k, E_k is a k-dimensional subspace of E_0 with $E_k \subset E_{k+1}$. The fact that φ is measurable follows from the formula of representation of β for separable spaces which yields

$$\varphi(t) = \lim_{k\to\infty} \sup_{n\geq 1} d(u_n(t), E_k). \tag{4.6}$$

Now $\varphi \in L^q(I)$ since $\varphi(t) \leq \nu(t)$ a.e. on I.

Since M is countable, we may suppose that (4.4) hold for all $t \in I$ and $u \in M$. We will prove (4.5) for any fixed $t_0 \in I$. Let $\varepsilon > 0$ and choose $\delta > 0$ such that for every measurable subset Θ of I we have

$$|\Theta| \leq \delta \implies \int_\Theta k(t_0,s)\nu(s)ds < \varepsilon.$$

Here $|\Theta|$ is the Lebesgue measure of Θ. Also choose a constant $\rho > 0$ such that $|\Theta_1| < \delta/3$ for

$$\Theta_1 = \{t \in I : \nu(t) > \rho\}.$$

So we have $d(u_n(t), E_k) \leq |u_n(t)| \leq \rho$ for $t \in I \setminus \Theta_1$ and $n, k \in \mathbb{N}$. Consequently, $d(u_n(t), E_k) = d(u_n(t), \overline{B}_k)$ with $\overline{B}_k = \{x \in E_k : |x| \leq \rho\}$.

From (4.6) and Egorov's Theorem there is a set $\Theta_2 \subset I \setminus \Theta_1$ with $|\Theta_2| < \delta/3$ and an integer k_0 such that

$$\sup_{n\geq 1} d(u_n(t), \overline{B}_k) \leq \varphi(t) + \varepsilon \tag{4.7}$$

for $t \in I \setminus (\Theta_1 \cup \Theta_2)$, $n \geq 1$ and $k \geq k_0$. Since M is a countable set of strongly measurable functions, we may find a set $\Theta_3 \subset I$ with $|\Theta_3| < \delta/3$ and a countable set $\widetilde{M} = \{\widetilde{u}_n : n \geq 1\}$ of finitely-valued functions from I to E with

$$|u_n(t) - \widetilde{u}_n(t)| \leq \varepsilon \tag{4.8}$$

for $t \in I \setminus \Theta_3$ and $n \geq 1$. From (4.7) and (4.8) we obtain

$$d(\widetilde{u}_n(t), \overline{B}_k) \leq \varphi(t) + 2\varepsilon$$

for $n \in \mathbb{N}$, $k \geq k_0$ and $t \in I \setminus \Theta$ with $\Theta = \Theta_1 \cup \Theta_2 \cup \Theta_3$. Then there exists a finitely-valued function $\widehat{u}_{n,k}$ from I to $\overline{B}_k$ with

$$|u_n(t) - \widehat{u}_{n,k}(t)| \leq \varphi(t) + 3\varepsilon$$

for $n \geq 1$, $k \geq k_0$ and $t \in I \setminus \Theta$. We put $\widehat{u}_{n,k}(t) = 0$ for $t \in \Theta$. Note that $|\Theta| \leq \delta$.

For each fixed $k \geq k_0$, Theorem 2.4 guarantees that the sequence $(\widehat{u}_{n,k})_{n\geq 1}$ is relatively compact in $L^q_w(I; \overline{B}_k)$. Then, from (S2) the sequence $(S(\widehat{u}_{n,k}))_{n\geq 1}$ is relatively compact in $L^p(I; E)$. Therefore, for every $t \in I$ the set $(S(\widehat{u}_{n,k}(t)))_{n\geq 1}$ is relatively compact in E. Now using (S1), we obtain

$$\begin{aligned}
&|S(u_n)(t_0) - S(\widehat{u}_{n,k})(t_0)| \\
&\leq \int_I k(t_0, s)|u_n(s) - \widehat{u}_{n,k}(s)|ds \\
&\leq \int_{I\setminus\Theta} k(t_0, s)(\varphi(s) + 3\varepsilon)ds + \int_\Theta k(t_0, s)|u_n(s)|ds \\
&\leq \int_I k(t_0, s)\varphi(s)ds + 3\varepsilon|k(t_0, .)|_1 + \int_\Theta k(t_0, s)\nu(s)ds \\
&\leq \int_I k(t_0, s)\varphi(s)ds + 3\varepsilon|k(t_0, .)|_1 + \varepsilon.
\end{aligned}$$

Hence $\{S(\widehat{u}_{n,k})(t_0) : n \geq 1\}$ is a relatively compact γ-net of the set $\{S(u_n)(t_0) : n \geq 1\}$ with

$$\gamma = \int_I k(t_0, s)\varphi(s)ds + 3\varepsilon|k(t_0, .)|_1 + \varepsilon \to \int_I k(t_0, s)\varphi(s)ds$$

as $\varepsilon \to 0$. □

Lemma 4.4 *Assume (S1) and (S2). Let M be a countable subset of $L^q(I; E)$ such that $M(t)$ is relatively compact for a.a. $t \in I$ and there is a function $\nu \in L^q(I)$ with $|u(t)| \leq \nu(t)$ a.e. on I, for every $u \in M$. Then the set $S(M)$ is relatively compact in $L^p(I; E)$. In addition S is continuous from M equipped with the relative weak topology of $L^q(I; E)$ to $L^p(I; E)$ equipped with its strong topology.*

Proof. Let $M = \{u_n : n \geq 1\}$. Let $\varepsilon > 0$. As in the proof of Lemma 4.3, we can find functions $\widehat{u}_{n,k}$ with values in the compact $\overline{B}_k \subset E$ such that

$$|u_n - \widehat{u}_{n,k}|_q \leq \varepsilon$$

for every $n \geq 1$. Then (S1) implies via Hölder's inequality that

$$|S(u_n) - S(\widehat{u}_{n,k})|_p \leq \gamma := \varepsilon||k(t, .)|_r|_p. \tag{4.9}$$

On the other hand, from Theorem 2.4 the set $\{\widehat{u}_{n,k} : n \geq 1\} \subset L^q(I;E)$ is weakly relatively compact in $L^q(I;E)$. Next (S2) guarantees that $\{S(\widehat{u}_{n,k}) : n \geq 1\}$ is relatively compact in $L^p(I;E)$. Hence from (4.9) we see that $\{S(\widehat{u}_{n,k}) : n \geq 1\}$ is a relatively compact γ-net of $S(M)$. Since ε is arbitrary, we conclude that $S(M)$ is relatively compact.

Now suppose that $(w_m)_m$ converges weakly in $L^q(I;E)$ to w and $w_m \in M$. In view of the relative compactness of $S(M)$, we may assume that $(S(w_m))_m$ converges in $L^p(I;E)$ towards some function h_∞. We have to prove

$$h_\infty = S(w). \tag{4.10}$$

For each fixed $\varepsilon > 0$, we have already seen that the proof of Lemma 4.3 again provides a compact set K_ε and a sequence $(w_m^\varepsilon)_m$ of K_ε-valued functions satisfying

$$|w_m - w_m^\varepsilon|_q \leq \varepsilon \tag{4.11}$$

for every $m \geq 1$. The sequence $(w_m^\varepsilon)_m$ being weakly precompact in $L^q(I,E)$, a suitable subsequence $(w_{m_j}^\varepsilon)_j$ must be weakly convergent in $L^q(I,E)$ towards some w_∞^ε. Then the Masur's Theorem and (4.11) provide

$$|w - w_\infty^\varepsilon|_q \leq \varepsilon. \tag{4.12}$$

The triangle inequality yields

$$\begin{aligned} |h_\infty - S(w)|_p \leq & |h_\infty - S(w_{m_j})|_p + |S(w_{m_j}) - S(w_{m_j}^\varepsilon)|_p \\ & + |S(w_{m_j}^\varepsilon) - S(w_\infty^\varepsilon)|_p + |S(w_\infty^\varepsilon) - S(w)|_p. \end{aligned} \tag{4.13}$$

Passing to the limit when j approaches infinity in (4.13) and using Assumption (S2) we obtain

$$|h_\infty - S(w)|_p \leq \limsup_j |S(w_{m_j}) - S(w_{m_j}^\varepsilon)|_p + |S(w_\infty^\varepsilon) - S(w)|_p. \tag{4.14}$$

According to (4.11) and (4.12) we deduce from (4.14) and Assumption (S1)

$$|h_\infty - S(w)|_p \leq 2\varepsilon||k(t,.)|_r|_p.$$

Since ε is arbitrary the proof of Lemma 4.4 is ended. □

Proof of Theorem 4.2 (a) First we show that $G(u) \neq \emptyset$ and so $SG(u) \neq \emptyset$ for every $u \in \overline{B}_R$. Indeed, since g takes nonempty, compact values and satisfies (g2)-(g3), for each strongly measurable function u there exists a strongly measurable selection w of $g(.,u(.))$ (see [7], Proof of Proposition 3.5 (a)). Next, if $u \in L^p([0,T];E)$, (g4) guarantees $w \in L^q([0,T];E)$. Hence $w \in G(u)$.
(b) The values of SG are acyclic according to condition (SG).
(c) The graph of SG is closed. To show this, let $(u_n, v_n) \in \mathrm{graph}(SG)$, $n \geq 1$, with $|u_n - u|_p$, $|v_n - v|_p \to 0$ as $n \to \infty$. Let $v_n = S(w_n)$, $w_n \in L^q([0,T];E)$, $w_n \in G(u_n)$. Since $|u_n - u|_p \to 0$, by Theorem 2.3 we may suppose that for

every $t \in I$, there exists a compact set $C \subset E$ with $\{u_n(t); n \geq 1\} \subset C$. Furthermore, since g satisfies (g3) and has compact values, Theorem 2.2 (b) guarantees that $g(t, C)$ is compact. Consequently, $\{w_n(t) : n \geq 1\}$ is relatively compact in E. If we also take into account (g4) we may apply Theorem 2.4 to conclude that (at least for a subsequence) (w_n) converges weakly in $L^q(I; E)$ to some w. As in [11], since g has convex values and satisfies (g3), we can show that $w \in G(u)$. Furthermore, by using Lemma 4.4 and a suitable subsequence we deduce $S(w_n) \to S(w)$. Thus $v = S(w)$ and so $(u, v) \in \operatorname{graph}(SG)$.

(d) We show that $SG(M)$ is relatively compact for each compact $M \subset \overline{B}_R$. Let $M \subset \overline{B}_R$ be a compact set and (v_n) be any sequence of elements of $SG(M)$. We prove that (v_n) has a convergent subsequence. Let $u_n \in M$ and $w_n \in L^q([0, T]; E)$ with

$$v_n = S(w_n) \text{ and } w_n \in G(u_n).$$

The set M being compact, we may assume that $|u_n - u|_p \to 0$ for some $u \in \overline{B}_R$. As above, there exists a $w \in G(u)$ with $w_n \rightharpoonup w$ weakly in $L^q([0, T]; E)$ (at least for a subsequence) and $S(w_n) \to S(w)$. Hence $v_n \to S(w)$. Thus (H1) is completely verified.

(e) Finally, we check (H2). Suppose $M \subset \overline{B}_R$, $M \subset \operatorname{conv}(\{0\} \cup SG(M))$ and $\overline{M} = \overline{C}$ for some countable set $C \subset M$. Since

$$C \subset M \subset \operatorname{conv}(\{0\} \cup SG(M)) \quad \text{and} \quad C \text{ is countable},$$

we can find a countable set $V = \{v_n : n \geq 1\} \subset SG(M)$ with $C \subset \operatorname{conv}(\{0\} \cup V)$. Then, there exists $u_n \in M$ and $w_n \in L^q([0, T]; E)$ with

$$v_n = S(w_n) \quad \text{and} \quad w_n \in G(u_n).$$

From (S2) and (g4) with $v_n \in V$ and $v_0 = S(0)$, we have

$$\begin{aligned} |v_n(t) - v_0(t)| &= |S(w_n)(t) - S(0)(t)| \\ &\leq \int_I k(t,s)|w_n(s)| ds \\ &\leq \int_I k(t,s)(a(s) + b|u_n(s)|^{p/q}) ds \\ &\leq (|a|_q + bR^{p/q})|k(t,.)|_r. \end{aligned}$$

Hence

$$|v_n(t)| \leq |S(0)(t)| + (|a|_q + bR^{p/q})|k(t,.)|_r \text{ a.e. on } I \tag{4.15}$$

for every $n \geq 1$. From $M \subset \overline{C} \subset \overline{\operatorname{conv}}(\{0\} \cup V)$ it follows that (4.15) is also true with any $u \in M$ instead of v_n. Since V and $\{w_n : n \geq 1\}$ are countable sets of strongly measurable functions, we may suppose that their values belong to a separable closed subspace E_0 of E. Clearly, the same is true for $\overline{M} = \overline{C}$. Now Lemma 4.3 guarantees

$$\begin{aligned} \beta_{E_0}(M(t)) &= \beta_{E_0}(C(t)) \leq \beta_{E_0}(V(t)) \\ &= \beta_{E_0}(\{S(w_n)(t) : n \geq 1\}) \\ &\leq \int_I k(t,s)\beta_{E_0}(\{w_n(s) : n \geq 1\}) ds \end{aligned}$$

while (g5) gives

$$\beta_{E_0}(\{w_n(s) :\ n \geq 1\}) \leq \beta_{E_0}(g(s, M(s)) \cap E_0) \leq \omega(s, \beta_{E_0}(M(s))).$$

It follows

$$\beta_{E_0}(C(t)) \leq \int_I k(t,s)\omega(s, \beta_{E_0}(C(s)))ds.$$

Moreover, the function φ given by $\varphi(t) = \beta_{E_0}(C(t))$ belongs to $L^p(I;\mathbb{R}_+)$. Consequently, $\varphi = 0$, and so $\varphi(t) = \beta_{E_0}(M(t)) = 0$ a.e. $t \in [0,T]$. Moreover, according to (4.15) and Assumption (g4) we have

$$|w_n(t)| \leq a(t) + b(|S(0)(t)| + (|a|_q + bR^{p/q})|k(t,.)|_r)^{p/q} := \nu(t)$$

a.e. on I, and $\nu \in L^q(I)$. Let $(v_{n_k})_{k\geq 1}$ be any subsequence of V. Then, as at step (c), $(w_{n_k})_{k\geq 1}$ has a weakly convergent subsequence in $L^q(I;E)$, say to w. Owing to Lemma 4.4 the corresponding subsequence of $(S(w_{n_k}))_{k\geq 1} = (v_{n_k})_{k\geq 1}$ converges to $S(w)$ in $L^p(I;E)$. Hence V is relatively compact. Now Mazur's Lemma guarantees $\overline{\text{conv}}(\{0\} \cup V)$ is compact and so $\overline{C} = \overline{M}$ is compact too. Thus (H2) also holds and Theorem 4.1 applies. $\square$

Remark 4.2 The following condition is sufficient for *(SG)* to hold:

(S3) S is affine, i.e.

$$S(\lambda w_1 + (1-\lambda)w_2) = \lambda S(w_1) + (1-\lambda)S(w_2)$$

for all $w_1, w_2 \in L^q(I;E)$, or for all w_0, w_1, $w_2 \in L^q(I;E)$, the relation $S(w_1) = S(w_2)$ implies

$$S(1_{[0,\lambda]}w_1 + 1_{[\lambda,T]}w_0) = S(1_{[0,\lambda]}w_2 + 1_{[\lambda,T]}w_0)$$

for every $\lambda \in I$. Here $1_{[a,b]}$ is the characteristic function of the interval $[a,b]$.

Indeed, let $u \in K$ and $v_0 \in SG(u)$. Then $v_0 = S(w_0)$ for some $w_0 \in G(u)$. Define $H : [0,1] \times SG(u) \to SG(u)$,

$$H(\lambda, v) = S(1_{[0,(1-\lambda)T]}w + 1_{[(1-\lambda)T,T]}w_0)$$

where $w \in G(u)$ and $v = S(w)$. According to (S3), the definition of $H(\lambda, v)$ does not depend on the choice of w. Clearly,

$$H(0,v) = v \quad \text{and} \quad H(1,v) = v_0.$$

It remains to prove the continuity of H. Let $\lambda_n \to \lambda$ and $v_n \to v$ with $v_n = S(w_n)$ and $w_n \in G(u)$. As at step (c) in the Proof of Theorem 4.2, we show that a subsequence of (w_n) converges weakly in $L^q(I;E)$ to some w, and $w \in G(u)$. Finally, by Lemma 4.4 we obtain

$$S(1_{[0,(1-\lambda_n)T]}w_n + 1_{[(1-\lambda_n)T,T]}w_0) \to S(1_{[0,(1-\lambda)T]}w + 1_{[(1-\lambda)T,T]}w_0)$$

in $L^p(I;E)$. Hence $H(\lambda_n, v_n) \to H(\lambda, v)$. Thus $SG(u)$ is contractible and so acyclic for every $u \in K$.

Note that (S3) holds whenever S is one-to-one. An open problem is to find weaker conditions to guarantee (SG) in order to extend the applicability of Theorems 4.1-4.2. For example, we may think to find conditions such that the values of SG are R_δ-sets. Such conditions are known for particular classes of problems (see [2]).

Remark 4.3 A sufficient condition for (H3) is

$$|S(0)|_p + (|a|_q + bR^{p/q})||k(t,.)|_r|_p \le R \tag{4.16}$$

if $p < \infty$ and respectively,

$$|S(0)|_\infty + |a|_q||k(t,.)|_r|_\infty \le R$$

if $p = \infty$.

Indeed, if $u \in \overline{B}_R$ is any solution of $u \in \lambda SG(u)$ for some $\lambda \in]0,1[$ and $u = \lambda S(w)$ with $w \in G(u)$, then for almost every $t \in [0,T]$, we have

$$\begin{aligned} |u(t)| &= \lambda|S(w)(t)| \le \lambda|S(0)(t)| + \lambda \int_I k(t,s)(a(s) + b|u(s)|^{p/q})ds \\ &\le \lambda|S(0)(t)| + \lambda|k(t,.)|_r|a + b|u|^{p/q}|_q \\ &\le \lambda[|S(0)(t)| + |k(t,.)|_r(|a|_q + b|u|_p^{p/q})]. \end{aligned}$$

This and (4.16) yield

$$\begin{aligned} |u|_p &\le \lambda[|S(0)|_p + ||k(t,.)|_r|_p(|a|_q + b|u|_p^{p/q})] \\ &\le \lambda[|S(0)|_p + ||k(t,.)|_r|_p(|a|_q + bR^{p/q})] < R. \end{aligned}$$

Hence (H3) is satisfied.

Corollary 4.5 *Assume $q \le p$ and (S1)-(S2), (g1)-(g4), (SG) and (H3) hold. In addition suppose*

(g5) For every separable closed subspace E_0 of E, there exists a function $\delta \in L^{pq/(p-q)}(I)$ such that for almost every $t \in I$,*

$$\beta_{E_0}(g(t,M) \cap E_0) \le \delta(t)\beta_{E_0}(M) \tag{4.17}$$

for every subset $M \subset E_0$ satisfying

$$|M| \le |S(0)(t)| + (|a|_q + bR^{p/q})|k(t,.)|_r\,,$$

if $p < \infty$, respectively

$$|M| \le |S(0)(t)| + |a|_q|k(t,.)|_r$$

if $p = \infty$, and

$$|\delta|_{pq/(p-q)}||k(t,.)|_r|_p < 1. \tag{4.18}$$

Then (4.1) has at least one solution u in $K \subset L^p(I;E)$ with $|u|_p \le R$. Here $pq/(p-q) = q$ if $p = \infty$ and $pq/(p-q) = \infty$ if $p = q$.

Proof Let $\varphi \in L^p(I;\mathbb{R}_+)$ be a solution of (4.3) with $\omega(t,s) = \delta(t)s$. From (4.17) via Hölder's inequality we obtain

$$\varphi(t) \leq |k(t,.)|_r |\delta|_{pq/(p-q)} |\varphi|_p.$$

It follows

$$|\varphi|_p \leq |\delta|_{pq/(p-q)} ||k(t,.)|_r|_p |\varphi|_p.$$

This together with (4.18) implies $|\varphi|_p = 0$ and so $\varphi = 0$, Thus (g5) also holds and Theorem 4.2 applies. □

We say that (4.1) is in the *Volterra case* if the function k in (S1) satisfies $k(t,s) = 0$ for $t < s$.

Corollary 4.6 *Assume (4.1) is in the Volterra case. In addition suppose that all the assumptions of Corollary 4.5 except (4.18) are satisfied. Then (4.1) has at least one solution $u \in K \subset L^p(I;E)$ with $|u|_p \leq R$.*

Proof In the Volterra case, from (4.3) we obtain

$$\varphi(\tau) \leq \int_0^\tau k(\tau,s)\delta(s)\varphi(s)ds \leq |k(\tau,.)|_r |\delta|_{pq/(p-q)} \Big(\int_0^\tau \varphi(s)^p ds\Big)^{1/p}.$$

Then

$$\int_0^t \varphi(\tau)^p d\tau \leq C \int_0^t (|k(\tau,.)|_r^p \int_0^\tau \varphi(s)^p ds) d\tau.$$

Now Gronwall's inequality implies $\int_0^t \varphi(\tau)^p d\tau = 0$ for all $t \in I$. So $\varphi = 0$. Thus (g5) holds without (4.18). □

Remark 4.4 In particular, if $K = C(I;E)$, $q = 1$, $p = \infty$ and

(g5**) there exists a function $\delta \in L^1(I)$ such that for every bounded subset $M \subset E$ and almost every $t \in I$ one has

$$\beta(g(t,M)) \leq \delta(t)\beta(M),$$

the result in Corollary *4.6* was established in [5] by showing that the operator SG is condensing with respect to a suitable measure of non-compactness on $C(I;E)$ and using the continuation principle for condensing operators.

Corollary 4.7 *Assume (4.1) is in the Volterra case. Let $1 \leq q = p < \infty$ and (S1), (S2), (g1)-(g4) and (SG) hold. Suppose that for the function k in (S1) there exists $r' > r$ such that $k(t,.) \in L^{r'}[0,T]$ for a.a. $t \in I$ and the map $t \mapsto k(t,.)$ belongs to $L^p(I;L^{r'}(I))$. In addition suppose that for every separable closed subspace E_0 of E, there exists a function $\delta \in L^\infty(I)$ such that for almost every $t \in I$,*

$$\beta_{E_0}(g(t,M) \cap E_0) \leq \delta(t)\beta_{E_0}(M) \tag{4.19}$$

for every subset $M \subset E_0$ satisfying

$$|M| \leq |S(0)(t)| + (|a|_p + bR)|k(t,.)|_r \,.$$

Then (4.1) has at least one solution u in K.

Proof We apply Theorem 4.1 to $U = \{u \in K : \|u\| < R\}$, for any $R > |S(0)|_p$ and a suitable equivalent norm $\|\cdot\|$ on $L^p(I;E)$.

According to the proof of Theorem 4.2 and of Corollary 4.6, the assumptions (H1)-(H2) are fulfilled. It remains to guarantee (H3). Let $u \in K$ be any solution of $u \in \lambda SG(u)$ for some $\lambda \in]0,1[$. Then, for any $\theta > 0$, we have

$$|u(t)| \leq \lambda |S(0)(t)| + \lambda \int_0^t k(t,s) e^{\theta s} (|a(s)| + b|u(s)|e^{-\theta s}) ds.$$

Define an equivalent norm on $L^p(I;E)$, by

$$\|u\| = |u(t)e^{-\theta t}|_p.$$

Then, since $1/r' + (r'-r)/(rr') + 1/p = 1$, Hölder's inequality guarantees

$$\begin{aligned} |u(t)| &\leq \lambda |S(0)(t)| + \lambda |k(t,.)|_{r'} (|a|_p + b\,\|u\|) (\int_0^t e^{\theta rr'/(r'-r)s} ds)^{(r'-r)/(rr')} \\ &\leq \lambda |S(0)(t)| + \lambda |k(t,.)|_{r'} (|a|_p + b\,\|u\|) (\frac{r'-r}{\theta rr'})^{(r'-r)/(rr')} e^{\theta t}. \end{aligned}$$

Consequently

$$\|u\| \leq \lambda [|S(0)|_p + (|a|_p + b\,\|u\|)(\frac{r'-r}{\theta rr'})^{(r'-r)/(rr')} ||k(t,.)|_{r'}|_p]. \tag{4.20}$$

Now we choose $\theta > 0$ so large that

$$|S(0)|_p + (|a|_p + bR)(\frac{r'-r}{\theta rr'})^{(r'-r)/(rr')} ||k(t,.)|_{r'}|_p \leq R.$$

Then, since $\lambda < 1$, from (4.20) we have $\|u\| < R$, so (H3) holds. □

5 Examples

The aim of this section is to show the wide field of applications of our abstract existence principles. Roughly speaking, our theory yields existence results for perturbed problems by a multivalued state-depending term, when the unperturbed original problem has a unique solution and the solution operator satisfies (S1), (S2) and the condition of acyclicity. Thus, our theory gives applications whenever a univoque operator S with the above required properties is detected.

First we note that if S is any operator from $L^1(I;E)$ to $C(I;E)$ such that for every compact, convex subset C of E, the operator S is sequentially continuous from $L^1_w(I;C)$ to $C(I;E)$ (this being condition (a2) in [5] and [6]), then S satisfies our condition (S2) for every $p \in [1,\infty]$, $q \in [1,\infty[$ and $K = L^p(I;E)$. Also note that condition (a1) in [5], [6] guarantees (S1). This remark shows that all the examples of an operator S given in [6] can be used in our more general framework. In particular, by applying Corollary 4.6 and Corollary 4.7 we obtain extensions of a lot of classical results on the Cauchy problem for semilinear evolution inclusions (see [14] and [20]). The extension comes from the generality of our compactness condition for the perturbation term g and also, in case of Corollary 4.6, from the localization of solutions in a given ball of $L^p(I;E)$. New existence results for evolution problems with Osgood type perturbations (see [4]) will be presented in a forthcoming paper.

Another new feature in this paper, contrary to [5] and [6], is that the theory is achieved in a such way that the case of Fredholm type inclusions be included. For Hammerstein integral inclusions involving linear operators of the form

$$S(w)(t) = \int_I k(t,s)w(s)ds,$$

this was realized in [18]. A source of such operators are the boundary value problems for second order abstract linear ordinary differential equations, when k is the corresponding Green function.

Nonlinear operators of Fredholm type arise from the theory of boundary value problems for second order nonlinear differential equations in abstract spaces. For example, let us consider the following boundary-value problem

$$\begin{gathered} u''(t) \in Au(t) + g(t,u(t)) \quad \text{a.e. on } I \\ u(0) = u(T) = 0. \end{gathered} \tag{5.1}$$

From [3] (Corollary 5.2.1) if follows that if E is a real Hilbert space and A is a maximal monotone set in $E \times E$, then for each $w \in L^2(I;E)$ there exists a unique solution $u \in H^2(I;E)$ of the problem

$$\begin{gathered} u''(t) \in Au(t) + w(t) \quad \text{a.e. on } I \\ u(0) = u(T) = 0. \end{gathered} \tag{5.2}$$

Let us consider the solution operator $S : L^2(I;E) \to C(I;E)$, given by $S(w) = u$, where u is the unique solution of (5.2). Assume $0 \in A0$.

Proposition 5.1 *The above operator S satisfies (S1) and (S2).*

Proof First we show that S satisfies (S1). For this, let w_1, $w_2 \in L^2(I;E)$. Denote $u_i = S(w_i)$, $i = 1,2$. We have $u_i'' = p_i + w_i$, where $p_i(t) \in Au_i(t)$ a.e. on I. Then

$$(u_1 - u_2)''(t) = p_1(t) - p_2(t) + w_1(t) - w_2(t).$$

Multiplying by $u_1(t) - u_2(t)$ and using the monotonicity of A, we obtain

$$\frac{1}{2}(|u_1(t) - u_2(t)|^2)'' - |u_1'(t) - u_2'(t)|^2 \geq (w_1(t) - w_2(t),\, u_1(t) - u_2(t)).$$

Hence $-(|u_1 - u_2|^2)'' \leq -2(w_1 - w_2,\, u_1 - u_2)$ a.e. on I. Consequently

$$|u_1(t) - u_2(t)|^2 \leq -2\int_I G(t,s)(w_1(s) - w_2(s),\, u_1(s) - u_2(s))ds. \tag{5.3}$$

Here G is the Green function of the differential operator $-u''$ corresponding to the boundary conditions $u(0) = u(T) = 0$. It follows

$$|u_1(t) - u_2(t)|^2 \le m|u_1 - u_2|_\infty \int_I |w_1(s) - w_2(s)|ds$$

where $m = 2\max_{(t,s)\in I^2} G(t,s)$. As a result, we obtain

$$|S(w_1)(t) - S(w_2)(t)| \le m \int_I |w_1(s) - w_2(s)|ds. \tag{5.4}$$

Thus (S1) holds.

Next we prove that for each compact, convex subset C of E, S is sequentially continuous from $L^2_w(I;C)$ to $C(I;E)$. This is achieved in three steps:
(1) First we show that graph (S) is closed in $L^2_w(I;E) \times C(I;E)$. For this, let $w_j \to w$ weakly in $L^2(I;E)$ and $S(w_j) \to u$ strongly in $C(I;E)$. Then

$$(w_j - w,\, S(w_j) - S(w)) \to 0 \text{ strongly in } L^1(I).$$

Using (5.3) we find that for each $t \in I$, one has

$$|S(w_j)(t) - S(w)(t)|^2 \to 0 \text{ as } j \to \infty.$$

Hence $S(w) = u$.
(2) For each positive integer n, we let

$$J_n = (J + n^{-1}A)^{-1}, \ \ A_n = n(J - J_n)$$

(J being the identity map of E) and we consider the map $S_n : L^2(I;E) \to C(I;E)$, given by $S_n(w) = u_n$, where u_n is the unique solution to

$$\begin{gathered} u_n''(t) = A_n u_n(t) + w(t) \quad \text{a.e. on } I \\ u_n(0) = u_n(T) = 0. \end{gathered} \tag{5.5}$$

Using the well known machinery on approximate solutions (see [3]), we can prove that for each bounded $M \subset L^2(I;E)$ and every $\varepsilon > 0$, there exists a positive integer $n_0 = n_0(M,\varepsilon)$ such that

$$|S_{n_0}(w) - S(w)|_\infty \le \varepsilon \quad \text{for all } w \in M,$$

that is $S_{n_0}(M)$ is an ε-net for $S(M)$. We omit here the details.
(3) From (5.4) we see that for each n and any bounded $M \subset L^2(I;E)$, the set $S_n(M)$ is bounded in $C(I;E)$. In addition, using

$$u_n(t) = -\int_0^T G(t,s)[A_n u_n(s) + w(s)]ds$$

and the Lipschitz property of A_n, we obtain

$$|u_n(t) - u_n(t')| \le \int_0^T |G(t,s) - G(t',s)|[2n|u_n(s)| + |w(s)|]ds.$$

This implies the equicontinuity of $S_n(M)$.

Now we consider a compact, convex subset C of E and a countable set $M \subset L^2(I;C)$, We claim that $S_n(M)(t)$ is relatively compact in E for every $t \in I$. Indeed, for any $w \in M$, the unique solution $u_n = S_n(w)$ of (5.5) satisfies

$$-u_n'' + nu_n = nJ_n u_n - w \quad \text{a.e. on } I.$$

If we denote by $\widetilde{G}$ the Green function of the operator $-u'' + nu$ corresponding to the boundary conditions $u(0) = u(T) = 0$, then

$$u_n(t) = \int_0^T \widetilde{G}(t,s)[nJ_n u_n(s) - w(s)]ds. \tag{5.6}$$

Using a result by Heinz, really a particular case of Lemma 4.3, the nonexpansivity of J_n and the inclusion $M(s) \subset C$ a.e. on I, from (5.6), we obtain

$$\beta_0(S_n(M)(t)) \leq n \int_0^T \widetilde{G}(t,s)\beta_0(S_n(M)(s))ds. \tag{5.7}$$

Here β_0 is the ball measure of noncompactness corresponding to a suitable separable closed subspace of E. Let

$$\varphi(t) = \beta_0(S_n(M)(t)), \quad v(t) = \int_0^T \widetilde{G}(t,s)\varphi(s)ds.$$

We have

$$-v'' + nv = \varphi, \quad v(0) = v(T) = 0.$$

According to (5.7), $\varphi \leq nv$. Hence $-v'' \leq 0$. This, since $v(0) = v(T) = 0$, implies $v \leq 0$ on I. The function v being nonnegative, it follows $v \equiv 0$. Thus $\beta_0(S_n(M)(t)) = 0$ for all $t \in I$, that is $S_n(M)(t)$ is relatively compact in E. As a result, $S_n(M)$ is relatively compact in $C(I;E)$.

Therefore, we have shown that for each $\varepsilon > 0$, there exists a relatively compact ε-net of $S(M)$. By Hausdorff's Theorem, $S(M)$ is relatively compact in $C(I;E)$. $\square$

Remark 5.1 Proposition 5.1 together with Theorem 4.2 gives new existence results for the problem (5.1) if the multivalued perturbation g satisfies (g1)–(g5) and (SG).

Similar results can be obtained for problems of type (5.1) with some other boundary conditions like those in [1] and [15].

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