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PARTIAL REGULARITY FOR FLOWS OF H-SURFACES

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Abstract

This article studies regularity of weak solutions to the heat equation for H–surfaces. Under the assumption that the function H is Lipschitz and depends only on the first two components, the solution has regularity on its domain, except for a set of measure zero. Moreover, if the solution satisfies certain energy inequality, this set is finite.

§1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain with boundary $\partial\Omega$, and H be a Lipschitz function on $\mathbb{R}^3$. A map $u \in C^2(\Omega, \mathbb{R}^3)$ satisfying

$$-\Delta u = 2H(u)u_{x_1} \wedge u_{x_2}, \tag{1.1}$$

is called a H-surface (parametrized by Ω). It is well known that if $u = (u^1, u^2, u^3)$ is a conformal representation of a surface S, i.e,

$$|u_{x_1}|^2 - |u_{x_2}|^2 = u_{x_1} \cdot u_{x_2} = 0\,,$$

then the mean curvature of S at the point u is $H(u)$; see [S3]. The existence of surfaces with constant mean curvature (i.e. H is constant) under various boundary conditions has been studied by Hildebrandt [Hs], Wente [W], Struwe [S1] [S2] [S3], and Brezis–Coron [Br]. The regularity of weak solutions to (1.1) has been established for constant H in [W], and for H depending only on two variables, or

$$\sup_{p\in\mathbb{R}^3} |H(p)| + \sup_{p\in\mathbb{R}^3} (1 + |p|)|DH(p)| < \infty \tag{1.2}$$

in Heinz [He], Tomi [T], and Bethuel-Ghidaglia [BG]. Bethuel [B] proved that weak solutions to (1.1) are $C^{2,\alpha}$ for any bounded Lipschitz function H.

The heat flow of an H-surface is

$$\partial_t u - \Delta u = 2H(u)u_{x_1} \wedge u_{x_2}, \quad \text{in } \Omega \times R_+\,. \tag{1.3}$$

Since (1.3) describes an evolution process of (1.1), there are results on the existence and regularity of solutions that apply under special conditions on the H-functions; see for example [R] [S2]. It is then a natural question to look at the regularity

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problem of (1.3) for more general H-functions. However (1.3) is a nonlinear parabolic system with borderline nonlinearity, which makes the regularity problem difficult to attack. In this note we consider the partial regularity for weak solutions of (1.3).

We say that $u : \Omega \times R_+ \to \mathbb{R}^3$ is a weak solution of (1.3) if $\partial_t u$ and Du are in $L^2_{\text{loc}}(R_+, L^2(\Omega))$ and u satisfies (1.3) in the sense of distributions.

For H constant, Struwe [S2] has studied (1.3) under free boundary conditions

$$u(x,t) \in \mathcal{S}, \quad \partial_\nu u(x,t) \perp T_{u(x,t)}\mathcal{S}, \tag{1.4}$$

a.e. for $(x,t) \in \partial\Omega \times R_+$, where $\mathcal{S}$ is a smooth surface in $\mathbb{R}^3$. He proved that (1.3)-(1.4) has a unique solution u in

$$\cap_{T<\bar{T}}\{u \in C^0([0,T], H^1(\Omega, \mathbb{R}^3)) : |D^2 u|, |\partial_t u| \in L^2(\Omega \times [0,T])\},$$

which is regular on $B^2 \times (0, \bar{T})$, where $\bar{T} > 0$ is determined by

$$\lim_{T\to\bar{T}} \sup_{(x,t)\in B^2\times(0,T)} \int_{B_R(x)\cap B^2} |Du|^2 \geq \bar{\epsilon}, \tag{1.5}$$

for all $R > 0$, and $\bar{\epsilon}$ depends only on $\mathcal{S}$ and H.

Rey [R] has established the existence of global regular solutions to (1.1) under the Dirichlet boundary conditions

$$u(x,0) = \phi(x),\ x \in \partial\Omega;\ u(x,t) = \phi(x),\ (x,t) \in \partial\Omega \times (0,\infty), \tag{1.6}$$

provided that $\phi \in H^1 \cap L^\infty(\Omega, \mathbb{R}^3)$ and

$$\|\phi\|_{L^\infty(\Omega)} \|H\|_{L^\infty(\mathbb{R}^3)} < 1. \tag{1.7}$$

Note that the nonlinear term occurring in (1.3) is of the same order as that appearing in the equation of harmonic maps from surfaces; see for example [S3]. In general, (1.3) alone does not provide control of $\|Du(\cdot,t)\|_{L^2(\Omega)}$ with respect to t. But, under the assumption (1.7), Rey [R] was able to control $\int_\Omega |Du|^2(\cdot,t)$. Based on this, Rey [R] first obtained the short time existence of a unique regular solution to (1.3) and (1.6), whose life span, $\bar{T}$, is given by (1.5). To show $\bar{T} = \infty$, Rey [R] observed (1.1) does not admit nontrivial entire solution under the assumption (1.7).

For harmonic maps, Freire [F] proved the partial regularity of weak flows of harmonic maps from surfaces to general Riemannian manifolds, whose energy does not increase with respect to t, by showing it must coincide with Struwe's solutions. However, there are serious difference between heat flows of a harmonic map and (1.3). For example, it is not clear whether smooth solutions to (1.3) satisfy the usual energy inequality

$$\int_\Omega |Du|^2(\cdot,t) \leq \int_\Omega |Du|^2(\cdot,s),\ 0 \leq s \leq t < \infty\,. \tag{1.8}$$

However, returning to the partial regularity issue of (1.3), we still prove the following result.

Theorem 1. *Assume that $H(p) = H(p^1, p^2) : \mathbb{R}^3 \to \mathbb{R}$, depending only on the first two variables, is bounded and Lipschitz continuous. Let $u \in H^1(\Omega \times R_+, \mathbb{R}^3)$ be a weak solution of (1.3). Then there exists a closed subset $\Sigma = \cup_{t>0}\Sigma_t \subset \Omega \times R_+$, with $\Sigma_t \subset \Omega \times \{t\}$ finite for almost all $t > 0$, such that $u \in C^{2,\alpha}(\Omega \times R_+ \setminus \Sigma, \mathbb{R}^3)$. In particular, Σ has zero Lebesgue measure.*

We believe that the singular set Σ in the above theorem should have Hausdorff dimension with respect to the parabolic metric in $\mathbb{R}^3$ at most 2.

Under the additional assumption (1.8), we confirm, in Remark 6 below, that the singular set Σ in the theorem is finite. It is then very interesting to ask when the singular set Σ in Theorem 1 is finite without (1.8). It is also interesting to ask whether the above theorem is true for any bounded Lipschitz function H. Uniqueness results for (1.3) under Dirichlet conditions are shown by Chen [Ch], in a preprint recently received by the author.

§2. Proof of main theorem

The goal of this section is to prove the theorem stated above. The proof relies on the techniques of Hardy space, Helein's arguments [Hf], and local versions of uniqueness results.

It follows from the assumption of Theorem 1 that $H(u) = H(u^1, u^2)$. First we observe that, for $v \in H^1(\mathbb{R}^2, \mathbb{R}^3)$,

$$H(v^1, v^2)(v^1_{x_1} v^2_{x_2} - v^1_{x_2} v^2_{x_1}) = g_{x_1} v^2_{x_2} - g_{x_2} v^2_{x_1} \in \mathcal{H}^1(\mathbb{R}^2), \tag{2.0}$$

where $g = \int_0^{v^1} H(s, v^2)\, ds$, and $\mathcal{H}^1(\mathbb{R}^2)$ denotes the Hardy space. See [Co] or [BG] for details. Moreover, one has the following norm estimate, see also Proposition 5.3 of [BG].

Lemma 1. *Assume that $H(p) = H(p^1, p^2) \in L^\infty(\mathbb{R}^3)$. For $v \in H^1(\mathbb{R}^2, \mathbb{R}^3)$, we have*

$$\|H(v^1, v^2)(v^1_{x_1} v^2_{x_2} - v^1_{x_2} v^2_{x_1})\|_{\mathcal{H}^1(\mathbb{R}^2)} \le C\|H\|_{L^\infty} \|Dv\|^2_{L^2(\mathbb{R}^2)}. \tag{2.1}$$

Proof. It is given at page 461 of [BG]. For completeness, we sketch it here. First recall that $f \in \mathcal{H}^1(\mathbb{R}^2)$ if

$$f^*(x) := \sup_{r>0} |r^{-2} \int_{\mathbb{R}^2} f(y)\rho(\frac{x-y}{r})\, dy| \in L^1(\mathbb{R}^2),$$

here $\rho \in C_0^\infty(\mathbb{R}^2)$, supp $\rho \subset B(0,1)$, $\rho \ge 0$ and $\int \rho = 1$. Denote $f = H(v^1, v^2)(v^1_{x_1} v^2_{x_2} - v^1_{x_2} v^2_{x_1})$. Concerning f^*, we take $x \in \mathbb{R}^2$, $r > 0$ and set

$$g(y) = \int_\lambda^{v^1(y)} H(s, v^2(y))\, ds, \quad \lambda = (\pi r^2)^{-1} \int_{B(x,r)} v^1(z)\, dz.$$

Then $f = g_{x_1} v^2_{x_2} - g_{x_2} v^2_{x_1}$ and

$$r^{-2} \int_{\mathbb{R}^2} f(y)\rho(\frac{x-y}{r})\, dy = r^{-3} \int_{B(x,r)} (R_1 v^2_{x_2} - R_2 v^2_{x_1}) g\, dy\,,$$

where $R_i = \frac{\partial\rho}{\partial x_i}(\frac{x-y}{r})$ for $i=1,2$. Since $|g(y)| \le \|H\|_{L^\infty}|v^1(y)-\lambda|$, we have

$$|r^{-2}\int_{\mathbb{R}^2} f(y)\rho(\frac{x-y}{r})\,dy| \le C\|H\|_{L^\infty} r^{-3}\int_{B(x,r)} |v^1(y)-\lambda||Dv^2|\,dy.$$

Then we proceed exactly as in [Co] and [BG] to show that

$$\int_{\mathbb{R}^2} f^*(x)\,dx \le C\|H\|_{L^\infty}\|Dv^1\|_{L^2}\|Dv^2\|_{L^2}\,.$$

Which concludes the present proof. □

Let $P_r(x,t) = \{(y,s)\in\mathbb{R}^2\times R_+ |\ |y-x|\le r,\ t-r^2\le s\le t\}$ for $(x,t)\in\mathbb{R}^2\times R_+$ and $r>0$. The following Lemma is the key to the proof of our theorem.

Lemma 2. *Assume $H(p) = H(p^1,p^2)\in L^\infty(\mathbb{R}^3)$. There exists $\epsilon_0>0$ such that if $u\in H^1(P_1(0,1),\mathbb{R}^3)$ is a weak solution to (1.3) and $\sup_{(0,1]}\int_{B_1}|Du|^2\le\epsilon_0^2$, then $Du\in L^2((0,1],L^4(B_{3/4}))$. In particular, $D^2u\in L^2((0,1],L^{4/3}(B_{1/2}))$.*

Proof. *Let $\bar u\in L^2((0,1],H^1(\mathbb{R}^2,\mathbb{R}^3))$ be such that $\bar u = u$ on B_1 and $\int_{\mathbb{R}^2}|D\bar u|^2\le C\int_{B_1}|Du|^2$ for $t\in(0,1)$. Define $v,w\in L^2((0,1],H^1(B_1))$ by*

$$\begin{aligned} \Delta v &= \partial_t u^3,\ \text{in } B_1,\\ v &= u^3-(u^3)_1(t),\ \text{on } \partial B_1\,, \end{aligned} \tag{2.3}$$

where $(u^3)_1(t) = \frac{1}{|B_1|}\int_{B_1}u^3(x,t)\,dx$, and

$$\begin{aligned} -\Delta w &= H(\bar u^1,\bar u^2)(\bar u^1_{x_1}\bar u^2_{x_2}-\bar u^1_{x_2}\bar u^2_{x_1}),\ \text{in } B_1,\\ w &= 0,\ \text{on } \partial B_1. \end{aligned} \tag{2.4}$$

Then we have

$$u^3-(u^3)_1(t) = v+w,\ \text{in } P_1(0,1). \tag{2.5}$$

For v, one can apply interior $W^{2,2}$ estimates to get, for $t\in(0,1)$,

$$\begin{aligned} \int_{B_{3/4}}|D^2v|^2 &\le C\int_{B_1}|v|^2+|\partial_t u|^2\\ &\le C\int_{B_1}|u^3-(u^3)_1(t)|^2+|w|^2+|\partial_t u|^2\\ &\le C\int_{B_1}|Du|^2+|Dw|^2+|\partial_t u|^2. \end{aligned} \tag{2.6}$$

Here we have used the Poincaré inequality and (2.5).

For w, we can apply Lemma 1 and the results of [Co] to conclude that $w\in W^{2,1}(B_1)$ and hence $Dw\in L^{2,1}(B_1)$. Here $L^{2,1}$ denotes the Lorentz space which is defined as follows: For $1\le q\le\infty$,

$$L^{2,q}(B_1) = \{f: B_1\to R \text{ measurable },\ \|f\|_{L^{2,q}(B_1)}<\infty\},$$

where $\|f\|_{L^{2,q}(B_1)}$ is defined by

$$\|f\|_{L^{2,q}(B_1)} = \begin{cases} (\int_0^\infty [t^{1/2} f^*(t)]^q \frac{1}{t}\, dt)^{1/q}, & \text{if } 1 \le q < \infty\ ; \\ \sup_{t>0} t^{1/2} f^*(t), & \text{if } q = \infty. \end{cases}$$

Here $f^*(t) := \inf\{s > 0 : |\{x \in B_1 : |f(x)| > s\}| \le t\}$ is the the rearrangement of f. Moreover, for $t \in (0,1)$, multiplying (2.4) by w and integrating over B_1, we have

$$\begin{aligned} \int_{B_1} |Dw|^2 &= \int_{\mathbb{R}^2} H(\bar u^1, \bar u^2)(\bar u^1_{x_1} \bar u^2_{x_2} - \bar u^1_{x_2} \bar u^2_{x_1}) w \\ &\le C \|H(\bar u^1, \bar u^2)(\bar u^1_{x_1} \bar u^2_{x_2} - \bar u^1_{x_2} \bar u^2_{x_1})\|_{\mathcal{H}^1(\mathbb{R}^2)} \|w\|_{\mathrm{BMO}(\mathbb{R}^2)} \\ &\le C \|H\|_{L^\infty} \|Du\|^2_{L^2(B_1)} \|Dw\|_{L^2(B_1)}. \end{aligned}$$

Here we extend w to $\mathbb{R}^2$ by letting it to be zero outside B_1, and $\|w\|_{\mathrm{BMO}(\mathbb{R}^2)}$ denotes the BMO norm of w, which is given by

$$\|w\|_{\mathrm{BMO}(\mathbb{R}^2)} = \sup_{x \in \mathbb{R}^2, r>0} r^{-2} \int_{B(x,r)} |w - w_{x,r}|, \qquad w_{x,r} = \frac{1}{|B(x,r)|} \int_{B(x,r)} w\,.$$

Here we have also used the duality between $\mathcal{H}^1(\mathbb{R}^2)$ and $\mathrm{BMO}(\mathbb{R}^2)$ (see for example [S]) and the Poincaré inequality. Therefore, we have

$$\|Dw\|_{L^2(B_1)} \le C \|H\|_{L^\infty} \|Du\|^2_{L^2(B_1)}, \tag{2.7}$$

and

$$\|Dw\|_{L^{2,1}(B_{3/4})} \le C \|H\|_{L^\infty} \|Du\|^2_{L^2(B_1)}. \tag{2.8}$$

Now we adapt the method, developed by Hélein [Hf] and [BG] in the context of harmonic maps from surfaces, to estimate u as follows. Denote $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2})$ and $\frac{\partial}{\partial \bar z} = \frac{1}{2}(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2})$. Hence we have $\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar z}$ and $\frac{\partial}{\partial y} = \frac{1}{i}(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar z})$. For $k = 1, 2$, if we denote $M^k = \frac{\partial u^k}{\partial z}$. Then it follows from (2.5) that (1.3) can be written as

$$\begin{aligned} 4 \frac{\partial}{\partial \bar z} \begin{pmatrix} M^1 \\ M^2 \end{pmatrix} =& 2H(u^1, u^2) \begin{pmatrix} w_{x_1} u^2_{x_2} - w_{x_2} u^2_{x_1} \\ w_{x_1} u^1_{x_2} - w_{x_2} u^1_{x_1} \end{pmatrix} \\ &+ 2H(u^1, u^2) \begin{pmatrix} v_{x_1} u^2_{x_2} - v_{x_2} u^2_{x_1} \\ v_{x_1} u^1_{x_2} - v_{x_2} u^1_{x_1} \end{pmatrix} + \begin{pmatrix} \partial_t u^1 \\ \partial_t u^2 \end{pmatrix} \\ =& I + II + III. \end{aligned}$$

By direct computation, we see that

$$\begin{aligned} I &= 4iH(u^1, u^2) \begin{pmatrix} \frac{\partial w}{\partial z} \frac{\partial u^2}{\partial \bar z} - \frac{\partial w}{\partial \bar z} \frac{\partial u^2}{\partial z} \\ \frac{\partial w}{\partial z} \frac{\partial u^1}{\partial \bar z} - \frac{\partial w}{\partial \bar z} \frac{\partial u^1}{\partial z} \end{pmatrix} \\ &= \mathrm{Re}\, [8iH(u^1, u^2) \begin{pmatrix} 0 & -\frac{\partial w}{\partial \bar z} \\ \frac{\partial w}{\partial \bar z} & 0 \end{pmatrix} \begin{pmatrix} M^1 \\ M^2 \end{pmatrix}]. \end{aligned}$$

Hence we obtain

$$\frac{\partial}{\partial \bar z} \begin{pmatrix} M^1 \\ M^2 \end{pmatrix} = \mathrm{Re}[\alpha \begin{pmatrix} M^1 \\ M^2 \end{pmatrix}] + F + G, \quad \text{in } P_1(0,1). \tag{2.9}$$

Here "Re" denotes the real part of complex numbers, $\alpha = 2iH(u^1,u^2)\begin{pmatrix} 0 & -\frac{\partial w}{\partial \bar z} \\ \frac{\partial w}{\partial \bar z} & 0 \end{pmatrix}$, $F = 2H(u^1,u^2)\begin{pmatrix} v_{x_1}u^2_{x_2} - v_{x_2}u^2_{x_1} \\ v_{x_1}u^1_{x_2} - v_{x_2}u^1_{x_1} \end{pmatrix}$, and $G = \begin{pmatrix} \partial_t u^1 \\ \partial_t u^2 \end{pmatrix}$.

For $t \in (0,1)$, define T by

$$Tf = P * (\alpha \mathrm{Re}\, f), \tag{2.10}$$

where $P(z) = 1/(\pi z)$ is the fundamental solution of $\bar\partial$ in $\mathbb{R}^2$. From (2.8), we have

$$\|\alpha\|_{L^{2,1}(\mathbb{R}^2)} \le C\|H\|_{L^\infty}\int_{B_1}|Du|^2\,. \tag{2.11}$$

Since $P \in L^{2,\infty}(\mathbb{R}^2)$, $T : L^\infty(\mathbb{R}^2) \to L^\infty(\mathbb{R}^2)$ is bounded and

$$\|T\| \le C\|P\|_{L^{2,\infty}}\|\alpha\|_{L^{2,1}(\mathbb{R}^2)} \le C\int_{B_1}|Du|^2\,. \tag{2.12}$$

Therefore, if we choose ϵ_0 so small (e.g., $\epsilon_0 \le (2C)^{-1/2}$) then $I + T : L^\infty \to L^\infty$ is invertible. Hence for $k = 1, 2$ there exist $\nu_k \in L^\infty(\mathbb{R}^2)$ such that

$$(I+T)\nu_k = e_k, \tag{2.13}$$

$$\|\nu_k - e_k\|_{L^\infty(\mathbb{R}^2)} \le C|e_k|(1 - C\epsilon_0^2)^{-1}\epsilon_0. \tag{2.14}$$

Here $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Taking $\frac{\partial}{\partial \bar z}$ of (2.13), we get

$$\frac{\partial \nu_k}{\partial \bar z} = \alpha\,\mathrm{Re}\,\nu_k. \tag{2.15}$$

This combines with (2.9) to yield, for $k = 1, 2$,

$$\begin{aligned}
\mathrm{Re}[\frac{\partial}{\partial \bar z}({\nu_k}^T\begin{pmatrix} M^1 \\ M^2 \end{pmatrix})] &= \mathrm{Re}((\frac{\partial \nu_k}{\partial \bar z})^T\begin{pmatrix} M^1 \\ M^2 \end{pmatrix} + {\nu_k}^T\frac{\partial}{\partial \bar z}\begin{pmatrix} M^1 \\ M^2 \end{pmatrix}) \\
&= \mathrm{Re}[(\alpha \mathrm{Re}\nu_k)^T\begin{pmatrix} M^1 \\ M^2 \end{pmatrix}] + (\mathrm{Re}\nu_k)^T(\mathrm{Re}(\alpha\begin{pmatrix} M^1 \\ M^2 \end{pmatrix}) + F + G) \\
&= (\mathrm{Re}\nu_k)^T(F+G).
\end{aligned} \tag{2.16}$$

Here the superscript "T" means the transpose, and we have used that $\alpha^T + \alpha = 0$. One can further rewrite (2.16) as

$$\sum_{k,l,s=1}^{2}\frac{\partial}{\partial x_k}(a^{rs}_{kl}\frac{\partial u^s}{\partial x_l}) = (\mathrm{Re}\,\nu^r)^T(F+G), \tag{2.17}$$

for $r = 1, 2$, where a^{rs}_{kl} are linear combinations of the ν^k's such that

$$\sup_{P_1(0,1)}|a^{rs}_{kl} - \delta^{rs}_{kl}| \le C\sup_{(0,1)}\sum_{r=1}^{2}\|T\nu^r\|_{L^\infty(B_1)} \le C\epsilon_0\,. \tag{2.18}$$

Hence, for small ϵ_0, (a^{rs}_{kl}) is uniformly elliptic. Let $U = (u^1, u^2)^T$, $A = (a^{rs}_{kl})$, and $Id = (\delta^{rs}_{kl})$. Then (2.17) becomes

$$-\Delta U = \bar F + \bar G + \operatorname{div}\left((A - Id)DU\right), \tag{2.19}$$

where $\bar F = \begin{pmatrix} (\mathrm{Re}\nu_1)^T F \\ (\mathrm{Re}\nu_2)^T F \end{pmatrix}$ and $\bar G = \begin{pmatrix} (\mathrm{Re}\nu_1)^T G \\ (\mathrm{Re}\nu_2)^T G \end{pmatrix}$.

It follows that $\bar G \in L^2((0,1], L^2(B_1))$ and

$$\int_{P_1(0,1)} |\bar G|^2 \le C \int_{P_1(0,1)} |\partial_t u|^2 . \tag{2.20}$$

Also note that $|\bar F| \le C|Dv||Du|$. Moreover, for $t \in (0,1)$, by (2.6), (2.7) and the Sobolev inequality, we have

$$\begin{aligned} \|Dv\|_{L^4(B_{3/4})} &\le C\|Dv\|^{1/2}_{L^2(B_1)}(\|Dv\|^{1/2}_{L^2(B_1)} + \|D^2 v\|^{1/2}_{L^2(B_{3/4})}) \\ &\le C(1 + \|\partial_t u\|^{1/2}_{L^2(B_1)}) . \end{aligned} \tag{2.21}$$

Hence $Dv \in L^4((0,1], L^4(B_{3/4}))$. Since $Du \in L^\infty((0,1], L^2(B_1))$, we can apply Hölder inequality to conclude that $\bar F \in L^4((0,1], L^{4/3}(B_{3/4}))$. In fact,

$$\|\bar F\|_{L^4((0,1],L^{4/3}(B_{3/4}))} \le C\|Dv\|_{L^4((0,1],L^4(B_{3/4}))}\|Du\|_{L^\infty((0,1],L^2(B_1))}. \tag{2.22}$$

For $t \in (0,1)$, we now estimate the L^4 norm of DU in $B_{1/2}$ as follows. Let $\eta \in C^\infty_0(B_{3/4})$ be such that $\eta = 1$ on $B_{1/2}$ and $|D\eta| \le 4$. From (2.19), we have

$$-\Delta(\eta U) = \eta\bar F + \eta\bar G + D\eta \cdot A \cdot DU + \operatorname{div}(AD\eta \cdot U) + \operatorname{div}\left((A - Id)D(\eta U)\right). \tag{2.23}$$

By Theorem 6.1 [Si], for $t \in (0,1)$,

$$\|D(\eta U)\|_{L^4(B_1)} \le C \sup_{\phi\in\mathcal{A}} \int_{B_1} D(\eta U)\cdot D\phi ,$$

where $\mathcal{A} = \{\phi \in W^{1,4/3}_0(B_1) | \|\phi\|_{W^{1,4/3}(B_1)} \le 1\}$. On the other hand, multiplying (2.23) by $\phi \in \mathcal{A}$, we have

$$\begin{aligned} &\int_{B_1} D(\eta U)\cdot D\phi \\ &= \int_{B_1} \eta\bar F\phi + \eta\bar G\phi + D\eta\cdot A\cdot DU\cdot\phi - \int_{B_1} A\cdot D(\eta U)\cdot D\phi \\ &\quad - \int_{B_1} (A - Id)D(\eta U)\cdot D\phi \\ &\le \|\eta\bar F\|_{L^{4/3}(B_1)}\|\phi\|_{L^4(B_1)} + \|\eta\bar G\|_{L^2(B_1)}\|\phi\|_{L^2(B_1)} + C\|A\|_{L^\infty}\|DU\|_{L^2(B_1)}\|\phi\|_{L^2(B_1)} \\ &\quad + C\|A\|_{L^\infty}\|U\|_{L^4(B_1)}\|D\phi\|_{L^{4/3}(B_1)} + \|A - Id\|_{L^\infty}\|D(\eta U)\|_{L^4(B_1)}\|\phi\|_{L^{4/3}(B_1)} \\ &\le C(\|\bar F\|_{L^{4/3}(B_{3/4})} + \|\bar G\|_{L^2(B_1)} + \|Du\|_{L^2(B_1)} + \|u\|_{L^4(B_1)}) + C\epsilon_0\|D(\eta U)\|_{L^4(B_1)} . \end{aligned} \tag{2.24}$$

Here we have used the fact that for any $\phi \in \mathcal{A}$ $\|\phi\|_{L^2}$ and $\|\phi\|_{L^4}$ are bounded, and (2.18). Hence for small ϵ_0, if we take the supremum of the left hand side of (2.24)we have

$$\|DU\|_{L^4(B_{1/2})} \le C(\|\bar{F}\|_{L^{4/3}(B_{3/4})} + \|\bar{G}\|_{L^2(B_1)} + \|Du\|_{L^2(B_1)}).$$

In particular, $DU \in L^2((0,1], L^4(B_{1/2}))$ so that $2H(u^1,u^2)(u^1_{x_1}u^2_{x_2} - u^1_{x_2}u^2_{x_1}) \in L^2((0,1], L^{4/3}(B_{1/2}))$. The linear theory implies $D^2u^3 \in L^2((0,1], L^{4/3}(B_{1/2}))$ and the Sobolev embedding theorem implies $Du^3 \in L^2((0,1], L^4(B_{1/2}))$. Applying linear theory again, we know that $D^2u^i \in L^2((0,1], L^{4/3}(B_{1/2}))$ for $i = 1, 2$. The proof is now complete. □

To obtain the regularity of weak solutions to (1.3) under the small energy assumption, we need the following lemma. For its proof, we refer the reader to Lemma 3.10 in Struwe [S3], whose proof is identical to the one of this lemma.

Lemma 3. *There exist $\epsilon_0 > 0$, and $0 < \alpha_0 < 1$ such that if $u \in H^1(P_1(0,1), \mathbb{R}^3)$ is a weak solution to (1.3) satisfying $D^2u \in L^2(P_1(0,1))$, $\sup_{(0,1]}\int_{B_1}|Du|^2 \le \epsilon_0^2$, then $u \in C^{\alpha_0}(P_{1/2}(0,1), \mathbb{R}^3)$. Moreover, $u \in C^{2,\alpha_0}(P_{1/2}(0,1), \mathbb{R}^3)$ provided that $H \in W^{1,\infty}(\mathbb{R}^3)$.*

Although Lemma 2 gives us higher regularity of second order derivatives of weak solutions u of (1.3) (e.g., $D^2u \in L^2((0,1], L^{4/3}(B_{1/2})$), it is not sufficient for us to apply Lemma 3 yet. From the linear theory, in order to apply Lemma 3 we need $Du \in L^4(P_{1/2}(0,1))$. To achieve this, we need the following uniqueness Lemma. First, for $1 < p < \infty$, define $I^p((0,1], W^{2,\frac{4}{3}}(B_{1/2}))$ by

$$I^p((0,1], W^{2,\frac{4}{3}}(B_{1/2})) = \{v \in L^p((0,1], W^{2,\frac{4}{3}}(B_{1/2})) |\ \partial_t v \in L^p((0,1], L^{4/3}(B_{1/2}))\}$$

Lemma 4. *There exists $\epsilon_0 > 0$ such that if $Du \in L^2((0,1], L^2(B_{1/2}))$, $\sup_{(0,1]}\int_{B_1}|Du|^2 \le \epsilon_0^2$, and $g \in L^4((0,1], L^{4/3}(B_{1/2}))$, then for $p = 2, 4$ there exists a unique $w \in I^p((0,1], W^{2,\frac{4}{3}}(B_{1/2}))$ such that*

$$\begin{aligned} &\partial_t w - \Delta w = 2H(u)u_x \wedge w_y + g, \text{ in } B_{1/2} \times (0,1), \\ &w(x,\cdot) = 0, \text{ on } \partial B_{1/2}, \\ &w(\cdot,0) = 0, \text{ in } B_{1/2}. \end{aligned} \tag{2.26}$$

Proof. The argument is based on the contraction principle and linear theory. Here we consider only the case $p = 4$. By Theorem 9.3 in Grisvard [Gr], for each $v \in L^4((0,1], W^{1,4}(B_{1/2}))$ there exists a unique $\Phi(v)$ in $I^4((0,1], W^{2,\frac{4}{3}}(B_{1/2}))$ such that

$$\begin{aligned} &\partial_t \Phi - \Delta \Phi = 2H(u)u_x \wedge v_y + g, \text{ in } B_{1/2} \times (0,1), \\ &\Phi(x,\cdot) = 0, \text{ on } \partial B_{1/2}, \\ &\Phi(\cdot,0) = 0, \text{ in } B_{1/2}. \end{aligned} \tag{2.27}$$

Moreover, by Sobolev embedding inequality, we see that Φ defines a mapping from $v \in L^4((0,1], W^{1,4}(B_{1/2}))$ to itself, and by standard $W^{2,\frac{4}{3}}$ estimates for (2.27),

$$\begin{aligned} \|\Phi(v)\|_{L^4((0,1],W^{1,4}(B_{1/2}))} &\le C\|\Phi(v)\|_{L^4((0,1],W^{2,\frac{4}{3}}(B_{1/2}))} \\ &\le C\epsilon_0\|v\|_{L^4((0,1],W^{1,4}(B_{1/2}))} + C\|g\|_{L^4((0,1],L^{4/3}(B_{1/2}))}. \end{aligned} \tag{2.28}$$

Moreover, for any $v_1, v_2 \in L^4((0,1], W^{1,4}(B_{1/2}))$, we know that $w = \Phi(v_1) - \Phi(v_2)$ solves (2.27) with v and g replaced by w and 0. Hence, (2.28) implies

$$\|\Phi(v_1) - \Phi(v_2)\|_{L^4((0,1],W^{1,4}(B_{1/2}))} \le C\epsilon_0 \|v_1 - v_2\|_{L^4((0,1],W^{1,4}(B_{1/2}))}. \tag{2.29}$$

The conclusion follows from the contraction principle if we choose ϵ_0 sufficiently small. $\square$

Based on the above Lemma, we can now improve the integrability of Du in the time direction, under the small energy assumptions.

Corollary 5. *Assume $H(p) = H(p^1, p^2) \in W^{1,\infty} \cap L^\infty(\mathbb{R}^3)$. There exists $\epsilon_0 > 0$ such that if $u \in H^1(P_1(0,1), \mathbb{R}^3)$ is a weak solution to (1.3) and $\sup_{(0,1]} \int_{B_1} |Du|^2 \le \epsilon_0^2$, then $u \in C^{2,\alpha}(P_{\frac{1}{4}}(0,1), \mathbb{R}^3)$ for some $\alpha \in (0,1)$.*

Proof. Applying Lemma 2, we know that $Du \in L^2((0,1], W^{1,4}(B_{1/2}))$. Let $w \in H^1(P_{1/2}(0,1), \mathbb{R}^3)$ be a solution to

$$\begin{aligned} \partial_t w - \Delta w &= 0, \text{ in } P_{1/2}(0,1), \\ w &= u, \text{ on } \partial P_{1/2}(0,1)\,, \end{aligned} \tag{2.30}$$

where $\partial P_{1/2}(0,1)$ denotes the parabolic boundary of $P_{1/2}(0,1)$.

Claim 1. $u - w \in L^4((0,1], L^4(B_{1/2}))$. To prove this claim, we first observe, by Sobolev embedding theorem, that

$$\int_{B_{1/2}} |u-w|^4 \le C \int_{B_{1/2}} |u-w|^2 \int_{B_{1/2}} |Du - Dw|^2. \tag{2.31}$$

This implies that $u - w \in L^2((0,1], L^4(B_{1/2}))$. Now multiplying (1.3) and (2.30) by $u - w$, subtracting each other, and integrating over $B_{1/2} \times (0,1)$, we have

$$\begin{aligned} &\sup_{t\in(0,1]} \int_{B_{1/2}} |u-w|^2 + \int_0^1 \int_{B_{1/2}} |Du - Dw|^2 \\ &\quad \le C \int_0^1 \|Du\|_{L^2(B_{1/2})} \|Du\|_{L^4(B_{1/2})} \|u-w\|_{L^4(B_{1/2})} \\ &\quad \le C \|Du\|_{L^\infty((0,1],L^2(B_{1/2}))} \|Du\|_{L^2((0,1],L^4(B_{1/2}))} \|u-w\|_{L^2((0,1],L^4(B_{1/2}))} < \infty. \end{aligned}$$

This implies that $u - w \in L^\infty((0,1], L^2(B_{1/2}))$. Hence (2.31) yields the claim.

Claim 2. $Du \in L^4(P_{1/4}(0,1))$. To prove this claim, we first note, by the linear theory, that $Dw \in L^\infty(P_{\frac{1}{4}}(0,1))$. Hence it suffices to prove $D(u-w) \in L^4(P_{\frac{1}{4}}(0,1))$. To do so, let $\eta \in C^\infty(P_{1/2}(0,1))$ be such that $\eta = 1$ on $P_{1/4}(0,1)$, $\eta = 0$ outside $P_{1/2}(0,1)$, and $|\partial_t \eta| + |D\eta| \le 4$. Then

$$\partial_t(\eta(u-w)) - \Delta(\eta(u-w)) = 2H(u)u_x \wedge (\eta(u-w))_y + g, \tag{2.32}$$

where

$$g = (\partial_t \eta)(u-w) - 2(D\eta)D(u-w) - \Delta\eta(u-w) + 2H(u)u_x \wedge \eta_y(u-w) + 2\eta H(u)u_x \wedge w_y\,.$$

Hence $g \in L^4((0,1], L^{4/3}(B_1))$ and $\|g\|_{L^4((0,1],L^{4/3}(B_1))} \le C$, where C depends on $\|Du\|_{L^\infty((0,1],L^2(B_1))}$ and $\|Du\|_{L^2((0,1],L^4(B_{1/2}))}$. Applying Lemma 4, we conclude that $D(\eta(u-w)) \in L^4(B_{1/2} \times (\frac{1}{4}, 1))$, which proves Claim 2.

Combining Claim 2 with Lemma 3, we complete the present proof. □

Completion of the proof of Theorem 1.
Define the parabolic metric: $\delta((x,t),(y,s)) = \max\{|x-y|, \sqrt{|t-s|}\}$. For $(x,t) \in \Omega \times R_+$ and $R \in (0, \delta((x,t), \partial(\Omega \times R_+)))$. Define

$$M_R(x,t) = \limsup_{s \uparrow t} \int_{B_R(x)} |Du|^2(x,s)\, dx,$$

for the weak solution u of (1.3). It is easy to see that $M_R(x,t)$ is non-decreasing with respect to R so that $M(x,t) = \lim_{R\downarrow 0} M_R(x,t)$ exists and is upper semi-continuous for any $(x,t) \in \Omega \times R_+$. Let ϵ_1 be the smallest of the constant obtained in lemmas 2, 3, 4, and Corollary 5. For $t > 0$, define $\Sigma_t \subset \Omega \times \{t\}$ by

$$\Sigma_t = \{x \in \Omega : M(x,t) \ge \epsilon_1^2\},$$

and let $\Sigma = \cup_{t>0} \Sigma_t$. Then it is easy to see that Σ is a closed subset of $\Omega \times R_+$.

Claim. $u \in C^{2,\alpha}(\Omega \times R_+ \setminus \Sigma, \mathbb{R}^3)$ for some $\alpha \in (0,1)$. To prove this claim, Let $(x_0,t_0) \in \Omega \times R_+ \setminus \Sigma$. By definition, there exists $r_0 > 0$ such that $M_{r_0}(x_0,t_0) < \epsilon_1^2$. For such r_0, there exists $0 < \delta_0 \le r_0$ such that

$$\int_{B_{r_0}(x_0)} |Du|^2(x,t)\, dx \le \epsilon_1^2, \quad \forall t \in [t_0 - \delta_0^2, t_0].$$

Hence if we define the rescaled mappings $u_{\delta_0} : P_1(0,0) \to \mathbb{R}^3$ by $u_{\delta_0}(x,t) = u(x_0 + \delta_0 x, t_0 + \delta_0^2 t)$ then u_{δ_0} is a weak solution to (1.3) on $P_1(0,0)$ and satisfies $\sup_{(0,1]} \int_{B_1} |Du_{\delta_0}|^2(x,t)\, dx \le \epsilon_1^2$. Hence Corollary 5 implies

$$u_{\delta_0} \in C^{2,\alpha}(P_{\frac{1}{4}}(0,0), \mathbb{R}^3),$$

which is the same as saying that $u \in C^{2,\alpha}(P_{\delta_0/4}(x_0,t_0), \mathbb{R}^3)$. Since (x_0,t_0) is arbitrary in $\Omega \times R_+ \setminus \Sigma$, the claim is proven.

Now we estimate the size Σ_t for a.e. $t > 0$. Since $Du \in L^2_{\text{loc}}(\Omega \times R_+)$, the set

$$A = \{t_0 \in R_+ : \liminf_{t \uparrow t_0} \int_\Omega |Du|^2(x,t)\, dx = +\infty\}$$

has Lebesgue measure, $|A|$, equal to zero. For any $t_1 \in R_+ \setminus A$, we claim that Σ_{t_1} is finite. In fact, let $\{x_1, \cdots, x_N\}$ be a finite subset of Σ_{t_1}. Then we can choose $R_0 > 0$ such that $\{B_{R_0}(x_i)\}_{i=1}^N$ are mutually disjoint and

$$\limsup_{t \uparrow t_1} \int_{B_{R_0}(x_i)} |Du|^2(x,t)\, dx \ge \epsilon_1^2, \ 1 \le i \le N.$$

Therefore,

$$\begin{aligned}\liminf_{t\uparrow t_1}\int_{\Omega\setminus\cup_{i=1}^N B_{R_0}(x_i)}|Du|^2 &\le \liminf_{t\uparrow t_1}\int_\Omega |Du|^2-\sum_{i=1}^N \limsup_{t\uparrow t_1}\int_{B_{R_0}(x_i)}|Du|^2\\ &\le \liminf_{t\uparrow t_1}\int_\Omega |Du|^2 - N\epsilon_1^2.\end{aligned}$$

Hence $N\le \epsilon_1^{-2}\liminf_{t\uparrow t_1}\int_\Omega |Du|^2$, which implies Σ_{t_1} is finite. By Fubini's theorem, we see that Σ has zero Lebesgue measure. □

Remark 6. *Under the condition (1.8), the set* Σ *in Theorem 1 is finite.*

Proof. Let $0<t_1<\cdots<t_N$ be such that there exist $x_1,\cdots,x_N\in\Omega$ so that $\{(x_i,t_i)\}\subset\Sigma$. Then for $1\le i\le N-1$,

$$\begin{aligned}\int_\Omega |Du|^2(\cdot,t_{i+1}) &= \lim_{R\downarrow 0}\int_{\Omega\setminus B_R(x_{i+1})}|Du|^2(\cdot,t_{i+1})\\ &\le \lim_{R\downarrow 0}\liminf_{t\uparrow t_{i+1}}\int_{\Omega\setminus B_R(x_{i+1})}|Du|^2\\ &\le \liminf_{t\uparrow t_{i+1}}\int_\Omega |Du|^2-\lim_{R\downarrow 0}\limsup_{t\uparrow t_{i+1}}\int_{B_R(x_{i+1})}|Du|^2\\ &\le \int_\Omega |Du|^2(\cdot,t_i)-\epsilon_1^2\,.\end{aligned}$$

Hence,

$$\int_\Omega |Du|^2(\cdot,t_N)\le\int_\Omega |Du|^2(\cdot,t_1)-N\epsilon_1^2.$$

This clearly implies the set $\{t\in R_+ : \Sigma\cap\Omega\times\{t\}\ne\emptyset\}$ is finite. Hence Σ is finite. □

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