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AVERAGING METHOD FOR ORDINARY DIFFERENTIAL INCLUSIONS WITH MAXIMA

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Abstract. We consider ordinary differential inclusions with maxima perturbed by a small parameter and give justification of the method of averaging for this type of inclusions.

1. Introduction

It is known that equations and inclusions with maxima arise naturally when solving practical and phenomenon problems, in particular, in the study of systems with automatic regulation and automatic control. Some works on these equations and inclusions are [1, 5, 10, 11, 21, 22, 28].

Differential equations and inclusions with maxima displaying nonlinear oscillations are ubiquitous in the scientific literature. The method of averaging is one of the main tool to analyze these oscillatory equations and inclusions. This method was used for ordinary and functional differential equations without maxima in [16, 18, 19, 20]. This method was also applied to ordinary differential equations with maxima in [13, 23, 24, 26] and in the monograph [1, Chap. 7]. It was extended to fuzzy differential equations with maxima in [14] and to set valued differential equations with Hukuhara derivative and maxima in [12], where both the right-hand sides and the solutions are set valued.

For ordinary differential inclusions (without maxima), many authors have contributed to the development of the averaging method in [3, 9, 15, 17, 25, 27] and the references therein. However, to our knowledge this method has not been extended to ordinary differential inclusions with maxima.

In the present work, we consider ordinary differential inclusions with maxima perturbed by a small parameter and establish an averaging result under weak regularity assumptions. More precisely, we consider the initial-value problem

$$
\begin{gathered}
\dot{x} \in \varepsilon F\Big(t, x(t), \max_{s\in S(t)} x(s)\Big), \quad t \geq 0 \\
x(0) = x_0
\end{gathered}
\tag{1.1}
$$

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where $\varepsilon > 0$ is a small parameter, F and S are multifunctions, with $S(t) \subset [0, t]$ for $t \geq 0$, and

$$\max_{s\in S(t)} x(s) := \Big(\max_{s\in S(t)} x_1(s), \dots, \max_{s\in S(t)} x_n(s)\Big).$$

The structure of this article is as follows. In Section 2 we provide an existence result and a Filippov-Pliś type result for ordinary differential inclusions with maxima. In Section 3 we present our main result: Theorem 3.1. We state and prove some preliminary results in Section 4 and then give the proof of Theorem 3.1. The technical tools used in this article are standard, however their exposition in the framework of problem (1.1) is new.

We complete this section with some definitions and notation. Throughout this paper we denote by $\mathbb{R}^n$ the real n-dimensional space. The set of nonnegative real numbers is denoted by $\mathbb{R}_+$. For $X \subseteq \mathbb{R}$ and $Y = \mathbb{R}_+$ or $\mathbb{R}^n$, the set of (locally) Lebesgue integrable functions $\delta : X \to Y$ is denoted by $\mathcal{L}^1_{(\mathrm{loc})}(X, Y)$. In $\mathbb{R}^n$ we use the notation $\langle\cdot,\cdot\rangle$ and $|\cdot|$ for the usual inner product and Euclidean norm, respectively. The set of all nonempty compact (nonempty compact and convex, respectively) subsets of $\mathbb{R}^n$ is denoted $\mathrm{Comp}(\mathbb{R}^n)$ ($\mathrm{Conv}(\mathbb{R}^n)$, respectively). The distance from $\alpha \in \mathbb{R}^n$ to $C \in \mathrm{Comp}(\mathbb{R}^n)$ is given by $d(\alpha, C) = \inf\big\{|\alpha - c|,\ c \in C\big\}$ and the Hausdorff distance between $A, B \in \mathrm{Comp}(\mathbb{R}^n)$ is defined as

$$H(A, B) = \max\Big(\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\Big).$$

Endowed with the Hausdorff distance, $\mathrm{Comp}(\mathbb{R}^n)$ is a complete separable metric space. The support function of $A \in \mathrm{Comp}(\mathbb{R}^n)$ is $\sigma(b, A) = \sup\{\langle b, a\rangle,\ a \in A\}$ for $b \in \mathbb{R}^n$. Notice that for $A \in \mathrm{Conv}(\mathbb{R}^n)$, $\sigma(\cdot, A)$ uniquely determines A.

The definition of the one-sided Lipschitz condition for multifunctions [7], adapted to the multifunction F in problem (1.1), reads as follows.

Definition 1.1. A multifunction $F : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathrm{Conv}(\mathbb{R}^n)$ is said to be one-sided Lipschitz (OSL) (with respect to (x, y)), if there exists $\lambda \in \mathbb{R}$ such that for every $t \in \mathbb{R}_+$, $x_1, y_1, x_2, y_2 \in \mathbb{R}^n$ and all $z_1 \in F(t, x_1, y_1)$, there exists $z_2 \in F(t, x_2, y_2)$ such that

$$\langle z_2 - z_1, x_2 - x_1\rangle \leq \lambda\left(|x_2 - x_1|^2 + |x_2 - x_1||y_2 - y_1|\right).$$

This is equivalently expressed by the support function

$$\sigma(x_2 - x_1, F(t, x_1, y_1)) - \sigma(x_2 - x_1, F(t, x_2, y_2)) \leq \lambda\left(|x_2 - x_1|^2 + |x_2 - x_1||y_2 - y_1|\right)$$

for every $t \in \mathbb{R}_+$ and $x_1, y_1, x_2, y_2 \in \mathbb{R}^n$.

Note that the constant λ in Definition 1.1 can take negative values. As in the case of Lipschitz condition, λ is called the OSL constant. It is well known that the OSL condition generalizes the Lipschitz condition with respect to the Hausdorff metric. Note however that it does not imply continuity.

2. Existence and Filippov-Pliś type results

First we recall that a function x is called solution of an ordinary differential equation (resp. inclusion) with a maximum if x is absolutely continuous on some interval and satisfies the differential equation (resp. inclusion) almost everywhere on this interval.

By an application of Schauder's fixed point theorem [29, Chap.2], one can easily prove the following result on existence of solutions of ordinary differential equations with maxima.

Lemma 2.1. *Let $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function. Suppose that f is uniformly bounded by some locally Lebesgue integrable function. Let $S : \mathbb{R}_+ \to \mathrm{Comp}(\mathbb{R})$ be a continuous multifunction, with $S(t) \subset [0, t]$ for $t \geq 0$. Let $x_0 \in \mathbb{R}^n$ and $L > 0$. Then the initial-value problem associated with an ordinary differential equation with a maximum*

$$\begin{gathered} \dot{x} = f\Big(t, x(t), \max_{s\in S(t)} x(s)\Big), \quad t \in [0, L] \\ x(0) = x_0 \end{gathered} \tag{2.1}$$

admits at least one solution defined on $[0, L]$.

By use of the Michael's selection theorem [6, Chap.2] and Lemma 2.1, it is not hard to prove the following result on existence of solutions of ordinary differential inclusions with maxima.

Lemma 2.2. *Let $F : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathrm{Conv}(\mathbb{R}^n)$ be a continuous multifunction. Suppose that F is uniformly bounded by some locally Lebesgue integrable function. Let $S : \mathbb{R}_+ \to \mathrm{Comp}(\mathbb{R})$ be a continuous multifunction, with $S(t) \subset [0, t]$ for $t \geq 0$. Let $x_0 \in \mathbb{R}^n$ and $L > 0$. Then the initial-value problem, associated with an ordinary differential inclusion with a maximum*

$$\begin{gathered} \dot{x}(t) \in F\Big(t, x(t), \max_{s\in S(t)} x(s)\Big), \quad t \in [0, L] \\ x(0) = x_0 \end{gathered} \tag{2.2}$$

admits at least one solution defined on $[0, L]$.

We need the following lemma which is a Filippov-Pliś type result for ordinary differential inclusions with maxima. Its proof follows the same pattern as in [8] where a similar result is obtained in the without maxima case.

Lemma 2.3. *Let $F : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathrm{Conv}(\mathbb{R}^n)$ and $S : \mathbb{R}_+ \to \mathrm{Comp}(\mathbb{R})$ be multifunctions that satisfy the following conditions:*

- *F is continuous.*
- *F is uniformly bounded by some locally Lebesgue integrable function, i.e., there exists $m \in \mathcal{L}^1_{\rm loc}(\mathbb{R}_+, \mathbb{R}_+)$ such that*
$$H(F(t, x, y), 0) \leq m(t), \quad \forall t \in \mathbb{R}_+,\ \forall x, y \in \mathbb{R}^n.$$
- *F is OSL with constant $\lambda \in \mathbb{R}$.*
- *S is continuous, with $S(t) \subset [0, t]$ for $t \geq 0$.*

Let $L > 0$ and $\delta \in \mathcal{L}^1([0, L], \mathbb{R}_+)$. If $x_1 : [0, L] \to \mathbb{R}^n$ is an absolutely continuous function satisfying

$$d\Big(\dot{x_1}(t), F(t, x_1(t), \max_{s\in S(t)} x_1(s))\Big) \leq \delta(t), \quad \forall t \in [0, L]$$

then, for each $x_0 \in \mathbb{R}$, there exists a solution x of problem (2.2) such that, for $t \in [0, L]$,

$$|x_1(t) - x(t)| \leq \Big(|x_1(0) - x_0| + \int_0^L \delta(t)dt\Big) \exp(2\lambda^+ t), \tag{2.3}$$

where $\lambda^+ = \max\{\lambda, 0\}$.

Proof. For $t \in [0, L]$ and $\alpha, \beta \in \mathbb{R}^n$, we define the set

$$\begin{aligned} G(t,\alpha,\beta) = \Big\{ & x \in F(t,\alpha,\beta) : \langle \dot{x}_1(t) - x, x_1(t) - \alpha \rangle \\ & \le \lambda |x_1(t) - \alpha|^2 + |x_1(t) - \alpha| \big(\lambda | \max_{s\in S(t)} x_1(s) - \beta| + \delta(t)\big) \Big\}. \end{aligned}$$

We first prove that $G(t,\alpha,\beta)$ is nonempty for every $t \in [0, L]$ and all $\alpha, \beta \in \mathbb{R}^n$.

Let $w \in F(t, x_1(t), \max_{s\in S(t)} x_1(s))$ be such that

$$|\dot{x_1}(t) - w| = d\Big(\dot{x}_1(t), F(t, x_1(t), \max_{s\in S(t)} x_1(s))\Big) \le \delta(t).$$

From assumption (H3) it follows that there exists $x \in F(t,\alpha,\beta)$ such that

$$\langle w - x, x_1(t) - \alpha \rangle \le \lambda \Big(|x_1(t) - \alpha|^2 + |x_1(t) - \alpha| \big| \max_{s\in S(t)} x_1(s) - \beta \big| \Big).$$

Therefore,

$$\begin{aligned} & \langle \dot{x}_1(t) - x, x_1(t) - \alpha \rangle \\ & \le \langle w - x, x_1(t) - \alpha \rangle + |\dot{x}_1(t) - w| |x_1(t) - \alpha| \\ & \le \lambda |x_1(t) - \alpha|^2 + |x_1(t) - \alpha| \Big(\lambda \big| \max_{s\in S(t)} x_1(s) - \beta \big| + \delta(t) \Big), \end{aligned}$$

i.e., $G(t,\alpha,\beta) \neq \emptyset$. Obviously, G is compact and convex valued and is continuous. Furthermore $G(t,\alpha,\beta) \subset F(t,\alpha,\beta)$. Therefore, by Lemma 2.2, there exists a solution x of problem

$$\begin{gathered} \dot{x}(t) \in G\Big(t, x(t), \max_{s\in S(t)} x(s)\Big), \quad t \in [0, L] \\ x(0) = x_0 \end{gathered} \tag{2.4}$$

such that, for $t \in [0, L]$,

$$\begin{aligned} & \langle \dot{x}_1(t) - \dot{x}(t), x_1(t) - x(t) \rangle \\ & \le \lambda |x_1(t) - x(t)|^2 + |x_1(t) - x(t)| \Big(\lambda \big| \max_{s\in S(t)} x_1(s) - \max_{s\in S(t)} x(s) \big| + \delta(t) \Big) \\ & \le \lambda |x_1(t) - x(t)|^2 + |x_1(t) - x(t)| \Big(\lambda \max_{s\in S(t)} |x_1(s) - x(s)| + \delta(t) \Big). \end{aligned} \tag{2.5}$$

Let $r(t) = |x_1(t) - x(t)|$, $t \in [0, L]$. The function r is absolutely continuous. At every $t \in [0, L]$ for which r is differentiable, by (2.5), we have the inequality

$$r(t)\dot{r}(t) = \frac{1}{2}\frac{d}{dt} r^2(t) \le \lambda r(t) \Big(r(t) + \max_{s\in S(t)} r(s) \Big) + r(t)\delta(t). \tag{2.6}$$

Define the set $T = \{t \in [0, L] : r(t) = 0\}$ and let T_0 be the set of the points of density of T. It is known that $\mathrm{meas}(T_0) = \mathrm{meas}(T)$, where meas is the measure of Lebesgue. If $t \notin T$, then, from (2.6) we deduce

$$\dot{r}(t) \le \lambda^+ \Big(r(t) + \max_{s\in S(t)} r(s) \Big) + \delta(t). \tag{2.7}$$

If $t \in T_0$ and if $\dot{r}(t)$ exists, then $\dot{r}(t) = 0$. Hence, (2.7) is satisfied for almost all $t \in [0, L]$. Therefore, one obtains that: $r(t) \le \overline{r}(t)$, for $t \in [0, L]$, where $\overline{r}$ is the solution of

$$\dot{\overline{r}}(t) = \lambda^+\Big(\overline{r}(t) + \max_{s\in S(t)} \overline{r}(s)\Big) + \delta(t), \quad t \in [0, L]$$
$$\overline{r}(0) = r(0).$$

Taking into account that

$$\overline{r}(t) \le r(0) + \int_0^t \Big(2\lambda^+\overline{r}(\tau) + \delta(\tau)\Big)d\tau,$$

by the Gronwall Lemma [2, Chap.1] we deduce the desired boundedness in (2.3). □

3. Averaging result

Let $F : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathrm{Conv}(\mathbb{R}^n)$ and $S : \mathbb{R}_+ \to \mathrm{Comp}(\mathbb{R})$ be multifunctions, with $S(t) \subset [0, t]$ for all $t \ge 0$. Let $\varepsilon > 0$ be a small parameter. We are interested in the limiting behavior of the trajectories of the initial-value problem

$$\begin{gathered} \dot{x} \in \varepsilon F\Big(t, x(t), \max_{s\in S(t)} x(s)\Big), \quad t \ge 0 \\ x(0) = x_0 \end{gathered} \tag{3.1}$$

on intervals of time $[0, L/\varepsilon]$, $L > 0$, as the perturbation parameter ε tends to zero. For this purpose we make use of the averaging method.

First, let us formulate the assumptions on the multifunctions F and S, needed for proving our averaging result.

(H1) $F = F(t, x, y)$ is continuous and the continuity in (x, y) is uniform with respect to t.

(H2) There exist $m \in \mathcal{L}^1_{\rm loc}(\mathbb{R}_+, \mathbb{R}_+)$ and a constant $M > 0$ such that
$$H(F(t, x, y), 0) \le m(t), \quad \forall t \in \mathbb{R}_+,\ \forall x, y \in \mathbb{R}^n$$
with
$$\int_{t_1}^{t_2} m(t)dt \le M(t_2 - t_1), \quad \forall t_1, t_2 \in \mathbb{R}_+,\ t_1 \le t_2.$$

(H3) F is OSL with constant $\lambda \in \mathbb{R}$.

(H4) S is uniformly continuous.

(H5) For all $x, y \in \mathbb{R}^n$, there exists a limit
$$\overline{F}(x, y) := \lim_{T\to+\infty} \frac{1}{T} \int_0^T F(t, x, y)dt, \tag{3.2}$$
i.e.,
$$\lim_{T\to+\infty} H\Big(\overline{F}(x, y), \frac{1}{T}\int_0^T F(t, x, y)dt\Big) = 0.$$

Note that in (3.2) and in what follows the integral of a multifunction G is understood in the Lebesgue-Aumann sense [4], i.e.

$$\int_{t_1}^{t_2} G(t)dt = \Big\{ \int_{t_1}^{t_2} g(t)dt : g \in \mathcal{L}^1([t_1, t_2], \mathbb{R}^n), g(t) \in G(t)\Big\}, \quad \forall t_1, t_2 \in \mathbb{R},\ t_1 \le t_2.$$

Consider now problem (3.1) with the initial-value averaged problem

$$\begin{gathered} \dot{y} \in \varepsilon\overline{F}\Big(y(t), \max_{s\in S(t)} y(s)\Big), \quad t \ge 0 \\ y(0) = x_0. \end{gathered} \tag{3.3}$$

The main result of this article is contained in the following theorem.

Theorem 3.1. *Suppose that* (H1)–(H5) *are fulfilled. Let* $x_0 \in \mathbb{R}^n$. *Then, for every* $L > 0$ *and* $\eta > 0$, *there exists* $\varepsilon_0 = \varepsilon_0(x_0, L, \eta) > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon_0]$, *the following holds:*

- (i) *for any solution* x *of problem* (3.1), *there exists a solution* y *of problem* (3.3) *such that*
$$|x(t) - y(t)| \leq \eta, \quad \forall t \in [0, L/\varepsilon]; \tag{3.4}$$
- (ii) *for any solution* y *of problem* (3.3), *there exists a solution* x *of problem* (3.1) *such that inequality* (3.4) *holds.*

Let $x_0 \in \mathbb{R}^n$. For $L > 0$, denote by $\mathrm{Sol}(\varepsilon F, x_0, L)$ and $\mathrm{Sol}(\varepsilon \overline{F}, x_0, L)$ the solutions sets on $[0, L/\varepsilon]$ of problems (3.1) and (3.3), respectively, and consider the associated reachable sets at time $t \in [0, L/\varepsilon]$ given by
$$R(\varepsilon F, x_0, t) = \{x(t) : x \in \mathrm{Sol}(\varepsilon F, x_0, L)\},$$
$$R(\varepsilon \overline{F}, x_0, t) = \{y(t) : y \in \mathrm{Sol}(\varepsilon \overline{F}, x_0, L)\}.$$
From Theorem 3.1 we obtain the following corollary.

Corollary 3.2. *Suppose that* (H1)–(H5) *are fulfilled. Let* $x_0 \in \mathbb{R}^n$. *For any* $L > 0$, *we have*
$$\lim_{\varepsilon \to 0} \sup \Big\{ H\big(R(\varepsilon F, x_0, t), R(\varepsilon \overline{F}, x_0, t)\big) : t \in [0, L/\varepsilon] \Big\} = 0.$$

Remark 3.3. In Theorem 3.1, solutions of problems (3.1) and (3.3) are defined globally in time. On any interval of time $[0, L/\varepsilon]$, $L > 0$, they are contained in the compact ball in $\mathbb{R}^n$ of radius ML, centered at x_0.

In problem (3.1), S is a general multifunction which is uniformly continuous. In [24], the authors considered problem (3.1) in single-valued case (differential equations with maxima) with S an interval valued multifunction which is uniformly continuous, that is, $S(t) = [g(t), \gamma(t)]$, where $g, \gamma : \mathbb{R}_+ \to \mathbb{R}_+$ are uniformly continuous functions such that $0 \leq g(t) \leq \gamma(t) \leq t$, for all $t \in \mathbb{R}_+$.

If a multifunction $F = F(t, x, y)$ is continuous in t and satisfies a Lipschitz condition on (x, y) (as assumed in [24]), then assumptions (H1) and (H3) are automatically fulfilled.

In assumption (H5), when the limit (3.2) is uniform with respect to (x, y), then ε_0 in the conclusion of Theorem 3.1 does not depend on the initial condition x_0.

4. Proof of the main result

To prove Theorem 3.1 we need to establish the following two lemmas.

Lemma 4.1. *Let* $F : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathrm{Conv}(\mathbb{R}^n)$ *be a multifunction.*

- (i) *If* F *satisfies assumptions* (H1) *and* (H2), *then its average* $\overline{F}$ *in* (H5) *is uniformly bounded by the constant* M *in* (H2) *and is continuous.*
- (ii) *If* F *satisfies assumption* (H3) *then its average* $\overline{F}$ *in* (H5) *satisfies the OSL condition with constant* λ *in* (H3).

Proof. For the proof of (i) see [3].

(ii) Note that, for $x \in \mathbb{R}^n$ and $A, B \in \mathrm{Conv}(\mathbb{R}^n)$, we have
$$|\sigma(x, A) - \sigma(x, B)| \leq |x| \Big| \sigma\Big(\frac{x}{|x|}, A\Big) - \sigma\Big(\frac{x}{|x|}, B\Big) \Big| \leq |x| H(A, B). \tag{4.1}$$

Now, let $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$. Using inequality (4.1), by assumption (H5) we can easily deduce that, for any $\eta > 0$ there exists $T_0 = T_0(x_1, x_2, y_1, y_2, \eta) > 0$ such that, for all $T \geq T_0$ we have

$$
\begin{aligned}
&\sigma(x_2 - x_1, \overline{F}(x_1, y_1)) - \sigma(x_2 - x_1, \overline{F}(x_2, y_2)) \\
&\leq \Big[\sigma(x_2 - x_1, \overline{F}(x_1, y_1)) - \sigma\Big(x_2 - x_1, \frac{1}{T}\int_0^T F(t, x_1, y_1)dt\Big)\Big] \\
&\quad + \Big[\sigma\Big(x_2 - x_1, \frac{1}{T}\int_0^T F(t, x_1, y_1)dt\Big) - \sigma\Big(x_2 - x_1, \frac{1}{T}\int_0^T F(t, x_2, y_2)dt\Big)\Big] \\
&\quad + \Big[\sigma\Big(x_2 - x_1, \frac{1}{T}\int_0^T F(t, x_2, y_2)dt\Big) - \sigma(x_2 - x_1, \overline{F}(x_2, y_2))\Big] \\
&\leq |x_2 - x_1| H\Big(\overline{F}(x_1, y_1), \frac{1}{T}\int_0^T F(t, x_1, y_1)dt\Big) \\
&\quad + \frac{1}{T}\int_0^T \big[\sigma(x_2 - x_1, F(t, x_1, y_1)) - \sigma(x_2 - x_1, F(t, x_2, y_2))\big]dt \\
&\quad + |x_2 - x_1| H\Big(\frac{1}{T}\int_0^T F(t, x_2, y_2)dt, \overline{F}(x_2, y_2)\Big) \\
&\leq 2|x_2 - x_1|\eta + \lambda\left(|x_2 - x_1|^2 + |x_2 - x_1||y_2 - y_1|\right).
\end{aligned}
$$

Since the value of η is arbitrary, in the limit we obtain that

$$
\sigma(x_2 - x_1, \overline{F}(x_1, y_1)) - \sigma(x_2 - x_1, \overline{F}(x_2, y_2)) \leq \lambda\left(|x_2 - x_1|^2 + |x_2 - x_1||y_2 - y_1|\right),
$$

which completes the proof that $\overline{F}$ is OSL with constant λ. $\square$

Lemma 4.2. *Suppose that* (H1)–(H4) *are fulfilled. Let $x_0 \in \mathbb{R}^n$. Then, for every solution x of* (3.1) *and $L > 0$ there exists a solution $\overline{z} : [0, L/\varepsilon] \to \mathbb{R}^n$ of the discrete problem*

$$
\begin{gathered}
\dot{\overline{z}}(t) \in \varepsilon F\Big(t, \overline{z}(t_i), \max_{s \in S(t_i)} \overline{z}(s)\Big), \quad t \in [t_i, t_{i+1}] \\
\overline{z}(0) = x_0
\end{gathered} \tag{4.2}
$$

where $0 = t_0 < t_1 < \dots < t_p = L/\varepsilon$ with $t_{i+1} = t_i + L/\varepsilon p$, $i = 0, \dots, p-1$, such that, for $t \in [0, L/\varepsilon]$

$$
|\overline{z}(t) - x(t)| \leq \big(L\exp(2\lambda^+ L)\big)\omega_F\Big(\frac{M}{p}(L + \omega_S(L)) + \varepsilon M \omega_S(L)\Big),
$$

where $\lambda^+ = \max\{\lambda, 0\}$ and ω_G is the modulus of continuity of multifunction G.

Remark 4.3. Note that on $[0, L/\varepsilon]$, $L > 0$, solutions of (4.2) are contained in the compact ball in $\mathbb{R}^n$ of radius ML, centered at x_0.

Proof of Lemma 4.2. We present two steps.

Step 1. Let $\overline{z}(0) = x_0$ and suppose that $\overline{z}$ exists on $[0, t_i]$. We prove inductively that it exists on $[t_i, t_{i+1}]$, $i = 0, \dots, p-1$. For a given $t \in [t_i, t_{i+1}]$ and $\alpha, \beta \in \mathbb{R}^n$ consider the map

$$
G(t, \alpha, \beta) = E(t, \alpha, \beta) \cap \varepsilon F\Big(t, \overline{z}(t_i), \max_{s \in S(t_i)} \overline{z}(s)\Big)
$$

where

$$\begin{aligned} E(t,\alpha,\beta) = \Big\{z \in \mathbb{R}^n : \langle \dot{x}(t) - z, x(t)-\alpha\rangle \le \varepsilon\Big[\lambda|x(t)-\alpha|^2 \\ + |x(t)-\alpha|\Big(\lambda\big|\max_{s\in S(t)} x(s) - \beta\big| + \delta(t)\Big)\Big]\Big\} \end{aligned}$$

with

$$\delta(t) = H\Big(F(t,\alpha,\beta), F\Big(t, \overline{z}(t_i), \max_{s\in S(t_i)} \overline{z}(s)\Big)\Big).$$

We obtain the existence of a solution of the initial-value problem

$$\begin{gathered} \dot{\alpha} \in G\Big(t, \alpha(t), \max_{s\in S(t)} \alpha(s)\Big), \quad t \in [t_i, t_{i+1}] \\ \alpha(t_i) = \overline{z}(t_i). \end{gathered} \tag{4.3}$$

We have, $G(t,\alpha,\beta)$ is nonempty for every t, α and β. Indeed, by assumption (H3) (OSL condition) there is $w \in \varepsilon F(t,\alpha,\beta)$ such that

$$\langle \dot{x}(t) - w, x(t) - \alpha\rangle \le \varepsilon\lambda\Big(|x(t)-\alpha|^2 + |x(t)-\alpha|\big|\max_{s\in S(t)} x(s) - \beta\big|\Big).$$

Further, for w we find $z \in \varepsilon F\left(t, \overline{z}(t_i), \max_{s\in S(t_i)} \overline{z}(s)\right)$ such that

$$|w - z| \le \varepsilon H\Big(F(t,\alpha,\beta), F\Big(t, \overline{z}(t_i), \max_{s\in S(t_i)} \overline{z}(s)\Big)\Big) = \varepsilon\delta(t).$$

Then

$$\langle \dot{x}(t) - z, x(t)-\alpha\rangle \le \varepsilon\Big[\lambda|x(t)-\alpha|^2 + |x(t)-\alpha|\Big(\lambda\big|\max_{s\in S(t)} x(s) - \beta\big| + \delta(t)\Big)\Big];$$

that is, $z \in G(t,\alpha,\beta)$.

Now, it is easy to see that G is compact and convex valued, and is continuous. Hence, problem (4.3) has a solution that we denote also by $\overline{z}$. This completes the induction step.

Step 2. For $t \in [0, L/\varepsilon]$, we have $t \in [t_i, t_{i+1}]$ for some $i = 0, \dots, p-1$ and:

On the one hand,

$$\begin{gathered} |\overline{z}(t) - \overline{z}(t_i)| \le \int_{t_i}^{t_{i+1}} \varepsilon m(s)ds \le \varepsilon M(t_{i+1} - t_i) \le \frac{LM}{p}, \\ \big|\max_{s\in S(t)} \overline{z}(s) - \max_{s\in S(t_i)} \overline{z}(s)\big| \le \varepsilon M\omega_S\Big(\frac{L}{\varepsilon p}\Big) \le M\Big(\frac{1}{p} + \varepsilon\Big)\omega_S(L) \end{gathered}$$

where ω_S is the modulus of continuity of the multifunction S, and then

$$\begin{aligned} \overline{\delta}(t) &:= H\Big(F\Big(t, \overline{z}(t), \max_{s\in S(t)} \overline{z}(s)\Big), F\Big(t, \overline{z}(t_i), \max_{s\in S(t_i)} \overline{z}(s)\Big)\Big) \\ &\le \omega_F\Big(\frac{M}{p}(L + \omega_S(L)) + \varepsilon M\omega_S(L)\Big), \end{aligned}$$

where ω_F is the modulus of continuity of the multifunction F which is, by assumption (H1), independent of t.

On the other hand

$$\begin{aligned} &\langle \dot{x}(t) - \dot{\overline{z}}, x(t) - \overline{z}\rangle \\ &\le \varepsilon\Big[\lambda|x(t)-\overline{z}|^2 + |x(t)-\overline{z}|\Big(\lambda\big|\max_{s\in S(t)} x(s) - \max_{s\in S(t_i)} \overline{z}(s)\big| + \overline{\delta}(t)\Big)\Big]. \end{aligned}$$

We repeat the arguments following inequality (2.5) to obtain that, for all $t \in [0, L/\varepsilon]$,

$$\begin{aligned}|\overline{z}(t) - x(t)| &\le \Big(\int_0^{L/\varepsilon} \varepsilon\overline{\delta}(t)dt\Big)\exp(2\varepsilon\lambda^+ t)\\ &\le \big(L\exp(2\lambda^+ L)\big)\omega_F\Big(\frac{M}{p}(L + \omega_S(L)) + \varepsilon M\omega_S(L)\Big),\end{aligned}$$

with $\lambda^+ = \max\{\lambda, 0\}$. □

Proof of Theorem 3.1. Let $x_0 \in \mathbb{R}^n$ and x be a solution of (3.1). Let $L > 0$. By Lemma 4.2 there exists a solution $\overline{z} : [0, L/\varepsilon] \to \mathbb{R}^n$ of the discrete problem

$$\begin{gathered}\dot{\overline{z}}(t) \in \varepsilon F\Big(t, \overline{z}(t_i), \max_{s\in S(t_i)} \overline{z}(s)\Big), \quad t \in [t_i, t_{i+1}]\\ \overline{z}(0) = x_0\end{gathered}$$

where $0 = t_0 < t_1 < \dots < t_p = L/\varepsilon$ with $t_{i+1} = t_i + L/\varepsilon p$, $i = 0, \dots, p-1$, such that, for $t \in [0, L/\varepsilon]$

$$|\overline{z}(t) - x(t)| \le \big(L\exp(2\lambda^+ L)\big)\omega_F\Big(\frac{M}{p}(L + \omega_S(L)) + \varepsilon M\omega_S(L)\Big), \tag{4.4}$$

where $\lambda^+ = \max\{\lambda, 0\}$.

Notice that by assumption (H5), for any $\mu > 0$ there exists $\overline{\varepsilon}$ such that for every $\varepsilon \in (0, \overline{\varepsilon}]$ we have

$$H\Big(\frac{\varepsilon p}{L}\int_{t_i}^{t_{i+1}} F\Big(t, \overline{z}(t_i), \max_{s\in S(t_i)} \overline{z}(s)\Big)dt, \overline{F}\Big(\overline{z}(t_i), \max_{s\in S(t_i)} \overline{z}(s)\Big)\Big) \le \mu. \tag{4.5}$$

For $i = 0, \dots, p-1$, let $v_i : [t_i, t_{i+1}] \to \mathbb{R}^n$ be continuous function satisfying: for $t \in [t_i, t_{i+1}]$, $v_i(t) \in F\Big(t, \overline{z}(t_i), \max_{s\in S(t_i)} \overline{z}(s)\Big)$ and $\overline{z}(t) = \overline{z}(t_i) + \varepsilon\int_{t_i}^t v_i(s)ds$. There exists $v^i \in \overline{F}\big(\overline{z}(t_i), \max_{s\in S(t_i)} \overline{z}(s)\big)$ such that, by (4.5),

$$\Big|\frac{\varepsilon p}{L}\int_{t_i}^{t_{i+1}} v_i(t)dt - v^i\Big| = \Big|\frac{\varepsilon p}{L}\int_{t_i}^{t_{i+1}} (v_i(t) - v^i)dt\Big| \le \mu.$$

Then we consider the function $z^1 : [0, L] \to \mathbb{R}^n$ given by

$$z^1(t) = z^1(t_i) + \varepsilon\int_{t_i}^{t_{i+1}} v^i ds, \quad t \in [t_i, t_{i+1}].$$

For $t \in [t_i, t_{i+1}]$ we have

$$|z^1(t) - z^1(t_i)| \le \int_{t_i}^{t_{i+1}} M\varepsilon ds \le \frac{ML}{p}.$$

By the definition of z^1 and $\overline{z}$, we have

$$\begin{aligned}|z^1(t_{i+1}) - \overline{z}(t_{i+1})| &\le |z^1(t_i) - \overline{z}(t_i)| + \varepsilon\Big|\int_{t_i}^{t_{i+1}} (v_i(t) - v^i)dt\Big|\\ &\le |z^1(t_i) - \overline{z}(t_i)| + \frac{L\mu}{p} \le \dots\\ &\le p\frac{L\mu}{p} = L\mu.\end{aligned}$$

For $t \in [t_i, t_{i+1}]$ we obtain

$$\begin{aligned} |z^1(t) - \overline{z}(t)| &\le |z^1(t) - z^1(t_i)| + |z^1(t_i) - \overline{z}(t_i)| + |\overline{z}(t_i) - \overline{z}(t)| \\ &\le L\mu + \frac{2ML}{p} \end{aligned} \tag{4.6}$$

and

$$\Big| \max_{s \in S(t_i)} z^1(s) - \max_{s \in S(t_i)} \overline{z}(s) \Big| \le \max_{s \in S(t_i)} \big| z^1(s) - \overline{z}(s) \big| \le L\mu + \frac{2ML}{p}$$

so that

$$H\Big(\overline{F}\Big(\overline{z}(t_i), \max_{s \in S(t_i)} \overline{z}(s)\Big), \overline{F}\Big(z^1(t), \max_{s \in S(t_i)} z^1(s)\Big)\Big) \le \omega_{\overline{F}}\Big(2L\mu + \frac{4ML}{p}\Big),$$

where $\omega_{\overline{F}}$ is the modulus of continuity of the multifunction $\overline{F}$.

Therefore, for $t \in [t_i, t_{i+1}]$, $i = 0, \dots, p-1$,

$$\begin{aligned} &d\Big(\dot{z}^1(t), \varepsilon\overline{F}\Big(z^1(t), \max_{s \in S(t_i)} z^1(s)\Big)\Big) \\ &\le \varepsilon d\Big(v^i, \overline{F}\Big(z^1(t), \max_{s \in S(t_i)} z^1(s)\Big)\Big) \\ &\le \varepsilon H\Big(\overline{F}\Big(\overline{z}(t_i), \max_{s \in S(t_i)} \overline{z}(s)\Big), \overline{F}\Big(z^1(t), \max_{s \in S(t_i)} z^1(s)\Big)\Big) \\ &\le \varepsilon\omega_{\overline{F}}\Big(2L\mu + \frac{4ML}{p}\Big). \end{aligned}$$

Taking into account that $\varepsilon\overline{F}$ is OSL with constant $\varepsilon\lambda$, by Lemma 2.3 there exists a solution y of (3.3), such that, for $t \in [0, L/\varepsilon]$,

$$\begin{aligned} |z^1(t) - y(t)| &\le \big(\exp(2\lambda^+ L)\big) \int_0^{L/\varepsilon} \varepsilon\omega_{\overline{F}}\Big(2L\mu + \frac{4ML}{p}\Big) ds \\ &\le \big(L\exp(2\lambda^+ L)\big)\omega_{\overline{F}}\Big(2L\mu + \frac{4ML}{p}\Big). \end{aligned} \tag{4.7}$$

By inequalities (4.4), (4.6) and (4.7) it follows that, for $t \in [0, L/\varepsilon]$,

$$\begin{aligned} |x(t) - y(t)| &\le |x(t) - \overline{z}(t)| + |\overline{z}(t) - z^1(t)| + |z^1(t) - y(t)| \\ &\le \big(L\exp(2\lambda^+ L)\big)\omega_F\Big(\frac{M}{p}(L + \omega_S(L)) + \varepsilon M\omega_S(L)\Big) + L\mu + \frac{2ML}{p} \\ &\quad + \big(L\exp(2\lambda^+ L)\big)\omega_{\overline{F}}\Big(2L\mu + \frac{4ML}{p}\Big). \end{aligned}$$

Therefore, for any $\eta > 0$, by appropriate choice of μ, sufficiently large p and sufficiently small ε, we get the inequality $|x(t) - y(t)| \le \eta$ for $t \in [0, L/\varepsilon]$. The proof of assertion (i) is now complete.

Adopting the process presented above, we obtain assertion (ii). In this way the proof is complete. □

Remark 4.4. In all the results above, it is not necessary to consider the whole space $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n$. One can restrict the domains of definition of function f in (2.1) and multifunctions F in (2.2) and (3.1) to $\mathbb{R}_+ \times \mathbb{U} \times \mathbb{U}$ for any open subset $\mathbb{U} \subset \mathbb{R}^n$ with additional technical assumptions.

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