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EXISTENCE RESULT FOR A SEMILINEAR PARAMETRIC PROBLEM WITH GRUSHIN TYPE OPERATOR

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Abstract. Using a varational method, we prove an existence result depending on a parameter, for a semilinear system in potential form with Grushin type operator.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$, with smooth boundary $\partial\Omega$ and $\{0\} \in \Omega$. We shall be concerned with the existence of solutions of the Dirichlet problem

$$\begin{gathered} L_{\alpha,\beta}U = \lambda \nabla F \quad \text{in } \Omega, \\ U = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where

$$U = (u, v), \quad L_{\alpha,\beta} = \begin{pmatrix} -G_\alpha & 0 \\ 0 & -G_\beta \end{pmatrix}, \quad G_s = \Delta_x + |x|^{2s}\Delta_y \quad \text{for } s \geqslant 0,$$

$$\Delta_x = \sum_{i=1}^{N_1} \frac{\partial^2}{\partial x_i^2}, \quad \Delta_y = \sum_{j=1}^{N_2} \frac{\partial^2}{\partial y_j^2},$$

$F = F(x, y, u, v)$ is potential function, $\nabla F = (\frac{\partial F}{\partial u}, \frac{\partial F}{\partial v})$, $\alpha \geqslant 0$, $\beta \geqslant 0$ and λ is a positive parameter. Denoting $N(s) = N_1 + (s + 1)N_2$, we assume that $N_1, N_2 \geqslant 1$ and $N(\alpha), N(\beta) > 2$.

For $s \geqslant 0, G_s$ is a Grushin type operator [9]. Its properties (such as degeneracy, hypoellipticity) were considered in [9],[15]. A semilinear problem with G_s in scalar case was studied in [14]. Thuy and Tri [14] pointed out the critical Sobolev exponent and proved the existence theorem for subcritical case. Many authors investigated the existence of solutions for scalar cases or potential system cases with Laplace and p-Laplacian operator (see [2, 3, 4, 5, 7, 8, 10, 13] and references therein). On the other hand, existence result for systems in Hamiltonian form with G_s was obtained in [6] and [11]. Our main goal in this paper is using the Moutain Pass scheme and Ekeland's variational principle as in [1], [7] and [13] to find the weak solutions for

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system (1.1) in the suitable Sobolev space when $\lambda \in (0, \lambda^*)$ and observe on the behaviour of that solutions as $\lambda \to 0$. In particular, we consider the system (1.1) with some classes of homogeneous and nonhomogeneous nonlinearities. To state our main result, we need some definitions and notations.

1.1. **Definition 1.** By $S_1^p(\Omega), 1 \leqslant p < +\infty$, we denote the set of all pair $(u, v) \in L^p(\Omega) \times L^p(\Omega)$ such that $\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i}, |x|^\alpha \frac{\partial u}{\partial y_j}, |x|^\beta \frac{\partial v}{\partial y_j} \in L^p(\Omega)$ for all $i = 1, \dots, N_1$, and $j = 1, \dots, N_2$.

For the norm in $S_1^p(\Omega)$, we take

$$\|(u,v)\|_{S_1^p(\Omega)} = \Big[\int_\Omega (|u|^p + |\nabla_x u|^p + |x|^{p\alpha} |\nabla_y u|^p + |v|^p + |\nabla_x v|^p + |x|^{p\beta} |\nabla_y v|^p) dx dy\Big]^{1/p}$$

where

$$\nabla_x = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{N_1}}), \nabla_y = (\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_{N_2}}).$$

For $p = 2$, the inner product in $S_1^2(\Omega)$ is defined by

$$\begin{aligned} &\langle (u, v), (\varphi, \psi) \rangle \\ &= \int_\Omega (u\varphi + \nabla_x u \nabla_x \varphi + |x|^{2\alpha} \nabla_y u \nabla_y \varphi + v\psi + \nabla_x v \nabla_x \psi + |x|^{2\beta} \nabla_y v \nabla_y \psi) dx\, dy. \end{aligned}$$

The space $S_{1,0}^p(\Omega)$ is defined as closure of $C_0^1(\Omega) \times C_0^1(\Omega)$ in space $S_1^p(\Omega)$. By standard arguments, one can prove that $S_1^p(\Omega)$ and $S_{1,0}^p(\Omega)$ are Banach spaces, $S_1^2(\Omega)$ and $S_{1,0}^2(\Omega)$ are Hilbert spaces.

The following Sobolev embedding inequality was proved in [14].

$$\Big(\int_\Omega |u|^q dx\, dy\Big)^{1/q} \leqslant C\Big[\int_\Omega (|\nabla_x u|^2 + |x|^{2s} |\nabla_y u|^2) dx\, dy\Big]^{1/2}, \tag{1.2}$$

where $q = \frac{2N(s)}{N(s)-2} - \tau, C > 0, s \geqslant 0$ and $N(s) = N_1 + (s+1)N_2$, provided small positive number τ. Furthermore, the number $\frac{N(s)+2}{N(s)-2}$ is critical Sobolev exponent for the embedding in (1.2).

Denoting by

$$L^{p,q}(\Omega) = \{(u, v) : u \in L^p(\Omega), v \in L^q(\Omega)\},$$

and endowing this space with the norm

$$\|(u,v)\|_{L^{p,q}(\Omega)} = \Big[\int_\Omega |u|^p dx\, dy\Big]^{1/p} + \Big[\int_\Omega |v|^q dx\, dy\Big]^{1/q},$$

we have the conclusion in view of (1.2) that $S_{1,0}^2(\Omega) \subset L^{\frac{2N(\alpha)}{N(\alpha)-2} - \tau_1, \frac{2N(\beta)}{N(\beta)-2} - \tau_2}(\Omega)$ and this embedding is a compact mapping for all small positive numbers τ_1 and τ_2 (see [14]).

1.2. **Definition 2.** A pair $(u, v) \in S_{1,0}^2(\Omega)$ is called a weak solution of system (1.1) if

$$\begin{aligned} \int_\Omega (\nabla_x u \nabla_x \varphi + |x|^{2\alpha} \nabla_y u \nabla_y \varphi) dx\, dy &= \lambda \int_\Omega \frac{\partial F}{\partial u} \varphi dx\, dy, \\ \int_\Omega (\nabla_x v \nabla_x \psi + |x|^{2\beta} \nabla_y v \nabla_y \psi) dx\, dy &= \lambda \int_\Omega \frac{\partial F}{\partial v} \psi dx\, dy \end{aligned} \tag{1.3}$$

for every $\varphi, \psi \in C_0^\infty(\Omega)$.

Since the system (1.1) is in the gradient form, we intend to get its solutions as the critical points of the functional

$$
\begin{aligned}
I_\lambda(u,v) =& \frac{1}{2}\int_\Omega (|\nabla_x u|^2 + |x|^{2\alpha}|\nabla_y u|^2 + |\nabla_x v|^2 + |x|^{2\beta}|\nabla_y v|^2)dx\,dy \\
& - \lambda\int_\Omega F(x,y,u,v)dx\,dy,
\end{aligned}
\tag{1.4}
$$

defined on reflexive Banach space $S^2_{1,0}(\Omega)$ with Fréchet derivative given by

$$
\begin{aligned}
\langle I'_\lambda(u,v),(\varphi,\psi)\rangle =& \int_\Omega (\nabla_x u\nabla_x\varphi + |x|^{2\alpha}\nabla_y u\nabla_y\varphi + \nabla_x v\nabla_x\psi + |x|^{2\beta}\nabla_y v\nabla_y\psi)dx\,dy \\
& - \lambda\int_\Omega (\frac{\partial F}{\partial u}\varphi + \frac{\partial F}{\partial v}\psi)dx\,dy.
\end{aligned}
\tag{1.5}
$$

Let $f = \frac{\partial F}{\partial u}$ and $g = \frac{\partial F}{\partial v}$ be two Carathéodory functions satisfying the following conditions:

(H1) There exist positive constants C_i, for $i = 1,\dots,6$ such that

$$
\begin{aligned}
|f(x,y,s,t)| &\leqslant C_1 + C_2|s|^{r_1} + C_3|t|^{r_2}, \\
|g(x,y,s,t)| &\leqslant C_4 + C_5|s|^{r_3} + C_6|t|^{r_4}
\end{aligned}
\tag{1.6}
$$

for a.e. $(x,y)\in\Omega$ and for all $s,t\in\mathbb{R}$, where

$$
0 < r_1, r_2 < \frac{N(\alpha)+2}{N(\alpha)-2}; \quad 0 < r_3, r_4 < \frac{N(\beta)+2}{N(\beta)-2}. \tag{1.7}
$$

(H2) $F(x,y,0,0) = 0$ and there are two positive constants $\mu > 2$ and $M > 0$ such that

$$
0 < \mu F(x,y,u,v) \leqslant uf(x,y,u,v) + vg(x,y,u,v) \tag{1.8}
$$

for a.e. $(x,y)\in\Omega$ and for all $u,v\in\mathbb{R}$ satisfying $|u|,|v| \geqslant M > 0$.

(H3) For a.e. $(x,y)\in\Omega$,

$$
\lim_{|u|+|v|\to\infty} \frac{F(x,y,u,v)}{|u|^2+|v|^2} = +\infty. \tag{1.9}
$$

The inequalities in (1.6) and (1.7) express the subcritical character of the system (1.1) and guarantee the well-definiteness of the functional I_λ. It's now to state our main result.

Theorem 1.1. *Under hypotheses (H1), (H2) and (H3), there exists a positive constant λ^* such that for any $\lambda\in(0,\lambda^*)$, the functional I_λ has a nontrivial critical point (u_λ, v_λ) satisfying $\|(u_\lambda,v_\lambda)\|_{S^2_{1,0}(\Omega)} \to +\infty$ as $\lambda\to 0$.*

Our work is organized as follows. In section 2, we prove some lemmas to establish the analysis framework for the proof of the main theorem in section 3. We shall make a note that, the operator $L_{\alpha,\beta}$ in system (1.1) has some extensions preserved our proofs. In the last section, we are interested in some cases of nonlinearity of the system (1.1).

2. Preliminaries

We first recall standard definitions and notations. Let X be a reflexive Banach space endowed with a norm $\|.\|$. Let $\langle .,.\rangle$ denote the duality pairing between X and its dual X^*. We denote the weak convergence in X by "$\rightharpoonup$" and the strong convergence by "$\to$".

Let $I \in C^1(X,\mathbb{R})$. We say I satisfies the *Palais-Smale condition*, denoted by (PS) condition, if every Palais-Smale sequence (a sequence $\{x_n\} \subset X$ such that $\{I(x_n)\}$ is bounded and $I^{'}(x_n) \to 0$ in dual space X^*) is relatively compact.

Putting

$$\begin{aligned}\Phi(u,v) &= \frac{1}{2}\int_\Omega (|\nabla_x u|^2 + |x|^{2\alpha}|\nabla_y u|^2 + |\nabla_x v|^2 + |x|^{2\beta}|\nabla_y v|^2)dx\,dy,\\ \Psi(u,v) &= \int_\Omega F(x,y,u,v)dx\,dy,\end{aligned}$$

we can write

$$I_\lambda(u,v) = \Phi(u,v) - \lambda\Psi(u,v). \tag{2.1}$$

Lemma 2.1. *Suppose f and g are continuous functions satisfying (H2). Then every Palais-Smale sequence of I_λ is bounded.*

Proof. Let $\{(u_n,v_n)\}$ be a Palais-Smale sequence of I_λ, that is,

$$\Phi(u_n,v_n) - \lambda\Psi(u_n,v_n) \to c \tag{2.2}$$

and

$$|\langle\Phi^{'}(u_n,v_n),(\xi,\eta)\rangle - \lambda\langle\Psi^{'}(u_n,v_n),(\xi,\eta)\rangle| \leqslant \epsilon_n\|(\xi,\eta)\|_{S^2_{1,0}(\Omega)}, \tag{2.3}$$

for all $(\xi,\eta) \in S^2_{1,0}(\Omega)$, where $\epsilon_n \to 0$ as $n \to \infty$. From (2.2), we have

$$\mu\Phi(u_n,v_n) - \lambda\mu\Psi(u_n,v_n) \leqslant \mu c + 1. \tag{2.4}$$

Subtracting (2.3), with $(\xi,\eta) = (u_n,v_n)$, yields

$$\begin{aligned}&(\mu-2)\Phi(u_n,v_n) - \lambda[\mu\Psi(u_n,v_n) - \langle\Psi^{'}(u_n,v_n),(u_n,v_n)\rangle]\\ &\leqslant \mu c + 1 + \epsilon_n\|(u_n,v_n)\|_{S^2_{1,0}(\Omega)}.\end{aligned} \tag{2.5}$$

Assumption (H2) ensures that

$$\mu\Psi(u_n,v_n) - \langle\Psi^{'}(u_n,v_n),(u_n,v_n)\rangle \leqslant 0. \tag{2.6}$$

Therefore, (2.5) implies

$$\frac{\mu-2}{2}\|(u_n,v_n)\|^2_{S^2_{1,0}(\Omega)} - \epsilon_n\|(u_n,v_n)\|_{S^2_{1,0}(\Omega)} \leqslant \mu c + 1.$$

Consequently, $\{(u_n,v_n)\}$ is bounded in $S^2_{1,0}(\Omega)$. □

Lemma 2.2. *Let assumption (H1) hold and $(u_n,v_n) \rightharpoonup (u,v)$ in $S^2_{1,0}(\Omega)$. Then*

$$\lim_{n\to\infty}\langle\Psi^{'}(u_n,v_n),(u_n-u,v_n-v)\rangle = 0.$$

Proof. It suffices to prove that

$$\lim_{n\to\infty}\int_\Omega[f(x,y,u_n,v_n)(u_n-u) + g(x,y,u_n,v_n)(v_n-v)]dx\,dy = 0. \tag{2.7}$$

We first show that there exists constant $M_1 > 0$ such that

$$\int_\Omega |f(x,y,u_n,v_n)|^p dx\,dy < M_1, \quad \text{for} \quad p = \frac{2N(\alpha)}{N(\alpha)+2} + \tau, \tag{2.8}$$

if τ is positive and sufficiently small. By assumption (H1) and the fact that $pr_1, pr_2 \leqslant \frac{2N(\alpha)}{N(\alpha)-2} - \delta(\tau)$, we have

$$\begin{aligned}\int_\Omega |f(x,y,u_n,v_n)|^p dx\,dy &\leqslant \int_\Omega (C_1 + C_2|u_n|^{pr_1} + C_3|v_n|^{pr_2}) dx\,dy \\ &\leqslant C(1 + \|(u_n,v_n)\|_{S^2_{1,0}(\Omega)}^{\frac{2N(\alpha)}{N(\alpha)-2}-\delta(\tau)}),\end{aligned}$$

where $C > 0, \delta(\tau)$ is positive and $\delta(\tau) \to 0$ as $\tau \to 0$. Since $\{(u_n,v_n)\}$ is a bounded sequence, (2.8) follows. Similarly,

$$\int_\Omega |g(x,y,u_n,v_n)|^q dx\,dy < M_2, \quad \text{for } q = \frac{2N(\beta)}{N(\beta)+2} + \tau, \tag{2.9}$$

where $M_2 > 0$, τ is positive and sufficiently small.

We are now in a position to prove (2.7). Let p' be the conjugate exponent of p, using Hölder inequality, we have

$$\begin{aligned}&\int_\Omega \Big[|f(x,y,u_n,v_n)(u_n-u)| + |g(x,y,u_n,v_n)(v_n-v)|\Big] dx\,dy \\ &\leqslant \Big(\int_\Omega |f(x,y,u_n,v_n)|^p dx\,dy\Big)^{1/p} \Big(\int_\Omega |u_n-u|^{p'} dx\,dy\Big)^{1/p'} \\ &\quad + \Big(\int_\Omega |g(x,y,u_n,v_n)|^q dx\,dy\Big)^{1/q} \Big(\int_\Omega |u_n-u|^{q'} dx\,dy\Big)^{1/q'} \\ &\leqslant M_1^{1/p}\|u_n-u\|_{L^{p'}(\Omega)} + M_2^{1/q}\|v_n-v\|_{L^{q'}(\Omega)} \\ &\leqslant M\|(u_n,v_n)-(u,v)\|_{L^{p',q'}(\Omega)}\end{aligned}$$

where $M > 0$ and

$$p' = \frac{2N(\alpha)+\tau[N(\alpha)+2]}{N(\alpha)-2+\tau[N(\alpha)+2]} = \frac{2N(\alpha)}{N(\alpha)-2} - \delta_1(\tau),$$

$\delta_1(\tau) > 0 \quad$ and $\quad \delta_1(\tau) \to 0 \quad$ as $\quad \tau \to 0$,

$$q' = \frac{2N(\beta)+\tau[N(\beta)+2]}{N(\beta)-2+\tau[N(\beta)+2]} = \frac{2N(\beta)}{N(\beta)-2} - \delta_2(\tau),$$

$\delta_2(\tau) > 0 \quad$ and $\quad \delta_2(\tau) \to 0 \quad$ as $\tau \to 0$.

The compactness of the embedding $S^2_{1,0}(\Omega) \subset L^{\frac{2N(\alpha)}{N(\alpha)-2}-\delta_1(\tau),\frac{2N(\beta)}{N(\beta)-2}-\delta_2(\tau)}(\Omega)$ implies that there exists a subsequence, denoted also by $\{(u_n,v_n)\}$, such that

$$\|(u_n,v_n)-(u,v)\|_{L^{p',q'}(\Omega)} \to 0 \quad \text{as} \quad n \to \infty.$$

The proof is complete. $\square$

Lemma 2.3. *If $(u_n,v_n) \rightharpoonup (u,v)$ in $S^2_{1,0}(\Omega)$ and*

$$\lim_{n\to\infty} \langle \Phi'(u_n,v_n), (u_n-u, v_n-v)\rangle = 0,$$

then $(u_n,v_n) \to (u,v)$ in $S^2_{1,0}(\Omega)$.

Proof. Let $\{(u_n, v_n)\}$ converge weakly to (u, v). Denoting

$$\begin{aligned} J_n = &\int_\Omega [\nabla_x u \nabla_x (u_n - u) + |x|^{2\alpha} \nabla_y u \nabla_y (u_n - u)] dx\, dy \\ &+ \int_\Omega [\nabla_x v \nabla_x (v_n - v) + |x|^{2\beta} \nabla_y v \nabla_y (v_n - v)] dx\, dy, \end{aligned}$$

we have $\lim_{n\to\infty} J_n = 0$. On the other hand,

$$\begin{aligned} &\langle \Phi'(u_n, v_n), (u_n - u, v_n - v)\rangle \\ &= \int_\Omega [|\nabla_x (u_n - u)|^2 + |x|^{2\alpha} |\nabla_y (u_n - u)|^2] dx\, dy \\ &\quad + \int_\Omega [|\nabla_x (v_n - v)|^2 + |x|^{2\beta} |\nabla_y (v_n - v)|^2] dx\, dy + J_n. \end{aligned}$$

Hence $\|(u_n - u, v_n - v)\|_{S^2_{1,0}(\Omega)} \to 0$ as $n \to \infty$. The conclusion of Lemma 2.3 is proved. □

Lemma 2.4. *Let assumptions (H1) and (H2) hold. Then I_λ satisfies the (PS) condition.*

Proof. Suppose that $\{(u_n, v_n)\}$ is a Palais-Smale sequence of I_λ. Lemma 2.1 ensures that $\{(u_n, v_n)\}$ is bounded in $S^2_{1,0}(\Omega)$. Then, there exists a subsequence, denoted also $\{(u_n, v_n)\}$ converging weakly to (u, v) in $S^2_{1,0}(\Omega)$. By the conclusion of Lemma 2.2,

$$\lim_{n\to\infty} \langle \Psi'(u_n, v_n), (u_n - u, v_n - v) = 0,$$

and the fact that

$$\begin{aligned} &\langle I'_\lambda(u_n, v_n), (u_n - u, v_n - v)\rangle \\ &= \langle \Phi'(u_n, v_n), (u_n - u, v_n - v)\rangle - \lambda \langle \Psi'(u_n, v_n), (u_n - u, v_n - v)\rangle, \end{aligned}$$

we obtain

$$\lim_{n\to\infty} \Phi'(u_n, v_n), (u_n - u, v_n - v)\rangle = 0.$$

Lemma 2.3 allows us to conclude that $\{(u_n, v_n)\}$ converge strongly to (u, v) in $S^2_{1,0}(\Omega)$. Hence, the functional I_λ satisfies the (PS) condition. □

3. Proof of the Existence Result

Lemma 3.1. *Assume that the hypotheses of Theorem 1.1 hold. Then there exist positive numbers η_λ and ρ_λ such that $\eta_\lambda \to +\infty$ as $\lambda \to 0$ and,*

$$I_\lambda(u, v) \geqslant \eta_\lambda \quad \text{for all } (u, v) \in S^2_{1,0}(\Omega) \text{ satisfying } \|(u, v)\|_{S^2_{1,0}(\Omega)} \geqslant \rho_\lambda.$$

Moreover, $I_\lambda(tu, tv) \to -\infty$ as $t \to +\infty$ for some $(u, v) \in S^2_{1,0}(\Omega) \backslash \{(0, 0)\}$.

Proof. Let

$$p = \frac{2N(\alpha)}{N(\alpha) + 2} + \tau, \quad q = \frac{2N(\beta)}{N(\beta) + 2} + \tau.$$

Using the same arguments in the proof of Lemma 2.2, we obtain

$$\begin{aligned} &\int_\Omega |f(x, y, u, v)|^p dx\, dy \leqslant C_p (1 + \|(u, v)\|_{S^2_{1,0}(\Omega)}^{\frac{2N(\alpha)}{N(\alpha)-2} - \gamma_1(\tau)}), \\ &\int_\Omega |g(x, y, u, v)|^q dx\, dy \leqslant C_q (1 + \|(u, v)\|_{S^2_{1,0}(\Omega)}^{\frac{2N(\beta)}{N(\beta)-2} - \gamma_2(\tau)}), \end{aligned} \tag{3.1}$$

where $\gamma_1(\tau), \gamma_2(\tau)$ are positive and $\gamma_1(\tau), \gamma_2(\tau) \to 0$ as $\tau \to 0$. Now, from the inequalities in (3.1), one can estimate

$$\begin{aligned}
&\int_\Omega F(x,y,u,v)dx\,dy \\
&\leqslant C\int_\Omega [uf(x,y,u,v)+vg(x,y,u,v)]dx\,dy \\
&\leqslant C\Big(\int_\Omega |f(x,y,u,v)|^p dx\,dy\Big)^{1/p}\Big(\int_\Omega |u|^{p'}dx\,dy\Big)^{1/p'} \\
&\quad + C\Big(\int_\Omega |g(x,y,u,v)|^q dx\,dy\Big)^{1/q}\Big(\int_\Omega |v|^{q'}dx\,dy\Big)^{1/q'} \\
&\leqslant C\Big(1+\|(u,v)\|_{S^2_{1,0}(\Omega)}^{\frac{2N(\alpha)}{N(\alpha)-2}-\gamma_1(\tau)}\Big)^{1/p}\|u\|_{L^{p'}(\Omega)} \\
&\quad + C\Big(1+\|(u,v)\|_{S^2_{1,0}(\Omega)}^{\frac{2N(\beta)}{N(\beta)-2}-\gamma_2(\tau)}\Big)^{1/q}\|v\|_{L^{q'}(\Omega)} \\
&\leqslant C\Big[\Big(1+\|(u,v)\|_{S^2_{1,0}(\Omega)}^{\frac{2N(\alpha)}{N(\alpha)-2}-\gamma_1(\tau)}\Big)^{1/p}+\Big(1+\|(u,v)\|_{S^2_{1,0}(\Omega)}^{\frac{2N(\beta)}{N(\beta)-2}-\gamma_2(\tau)}\Big)^{1/q}\Big] \\
&\quad \times (\|u\|_{L^{p'}(\Omega)}+\|v\|_{L^{q'}(\Omega)}). \\
&\leqslant C\Big[\Big(1+\|(u,v)\|_{S^2_{1,0}(\Omega)}^{\frac{2N(\alpha)}{N(\alpha)-2}-\gamma_1(\tau)}\Big)^{1/p}+\Big(1+\|(u,v)\|_{S^2_{1,0}(\Omega)}^{\frac{2N(\beta)}{N(\beta)-2}-\gamma_2(\tau)}\Big)^{1/q}\Big]\|(u,v)\|_{S^2_{1,0}(\Omega)},
\end{aligned}$$

for some positive constant C. Note that, the Young inequality gives

$$A^{\frac{1}{p}} \leqslant \frac{1}{q}+\frac{1}{p}A,\ B^{\frac{1}{q}} \leqslant \frac{1}{p}+\frac{1}{q}B \text{ for } A,B>0.$$

From these facts, we have

$$\int_\Omega F(x,y,u,v)dx\,dy \leqslant C_1^*+C_2^*\|(u,v)\|_{S^2_{1,0}(\Omega)}^{\frac{2N(\alpha)}{N(\alpha)-2}+1-\gamma_1(\tau)}+C_3^*\|(u,v)\|_{S^2_{1,0}(\Omega)}^{\frac{2N(\beta)}{N(\beta)-2}+1-\gamma_2(\tau)},$$

for some $C_1^*, C_2^*, C_3^* > 0$. Using the last inequality and taking (2.1) into account, we get

$$\begin{aligned}
I_\lambda(u,v) \geqslant &\frac{1}{2}\|(u,v)\|^2_{S^2_{1,0}(\Omega)} \\
&- \lambda[C_1^*+C_2^*\|(u,v)\|_{S^2_{1,0}(\Omega)}^{\frac{2N(\alpha)}{N(\alpha)-2}+1-\gamma_1(\tau)}+C_3^*\|(u,v)\|_{S^2_{1,0}(\Omega)}^{\frac{2N(\beta)}{N(\beta)-2}+1-\gamma_2(\tau)}].
\end{aligned} \tag{3.2}$$

Choosing $(u,v) \in S^2_{1,0}(\Omega)$ such that $\|(u,v)\|_{S^2_{1,0}(\Omega)} = \lambda^{-s}$, with s satisfying

$$0 < s < s^* = \min\Big(\frac{N(\alpha)-2}{N(\alpha)+2}, \frac{N(\beta)-2}{N(\beta)+2}\Big),$$

we have

$$I_\lambda(u,v) \geqslant \lambda^{-2s}\Big[\frac{1}{2}-C_2^*\lambda^{1-s\frac{N(\alpha)+2}{N(\alpha)-2}}-C_3^*\lambda^{1-s\frac{N(\beta)+2}{N(\beta)-2}}\Big]-\lambda C_1^*.$$

Taking $\rho_\lambda = \lambda^{-s}, \eta_\lambda = \frac{1}{2}\lambda^{-2s}$, we conclude that $I_\lambda(u,v) \geqslant \eta_\lambda$ if λ is sufficiently small and $\|(u,v)\|_{S^2_{1,0}(\Omega)} \geqslant \rho_\lambda$.

Our task is now to show that $I_\lambda(tu,tv) \to -\infty$ as $t \to +\infty$, for some $(u,v) \in S^2_{1,0}(\Omega)\backslash\{(0,0)\}$. It follows from (H3) that, given $M > 0$, there exists $K(M) > 0$ such that

$$F(x,y,s,t) \geqslant M(s^2+t^2), \quad \text{for all } s,t \in \mathbb{R} \text{ satisfying } |s|+|t| \geqslant K(M).$$

Let $(u,v) \in S^2_{1,0}(\Omega)$ with $\|(u,v)\|_{S^2_{1,0}(\Omega)} = 1$ and $\int_\Omega (u^2+v^2)dx\,dy = a$. Then

$$I_\lambda(tu,tv) = \frac{1}{2}t^2 - \lambda \int_\Omega F(x,y,tu,tv)dx\,dy$$

and

$$\int_\Omega F(x,y,tu,tv)dx\,dy \geqslant Mt^2 \int_{\Omega\cap\{(x,y)\in\Omega:|u|+|v|\geqslant \frac{K(M)}{t}\}} (u^2+v^2)dx\,dy - b, \tag{3.3}$$

where b is a constant depending on a.

For t sufficient large,

$$\int_{\Omega\cap\{(x,y)\in\Omega:|u|+|v|\geqslant \frac{K(M)}{t}\}} (u^2+v^2)dx\,dy \geqslant \frac{1}{2}a.$$

From this and (3.3), we arrive at the conclusion

$$I_\lambda(tu,tv) \leqslant \frac{1}{2}t^2 - \frac{1}{2}a\lambda Mt^2 + b,$$

for t sufficient large. Choosing $M = \frac{2}{a\lambda}$ leads to

$$I_\lambda(tu,tv) \leqslant b - \frac{1}{2}t^2, \text{ for } t \text{ sufficient large.}$$

Hence, $I_\lambda(tu,tv) \to -\infty$ as $t \to +\infty$. □

Proof of Theorem 1.1. By Lemma 2.4 and Lemma 3.1, we may apply the Mountain Pass Theorem [12]. It follows that there exists $\lambda^* > 0$ such that for all $\lambda \in (0,\lambda^*)$, the functional I_λ has a critical point (u_λ, v_λ) satisfying $I_\lambda(u_\lambda,v_\lambda) > \eta_\lambda > 0$ and $\|(u_\lambda,v_\lambda)\|_{S^2_{1,0}(\Omega)} \geqslant \rho_\lambda = \lambda^{-s} \to +\infty$ as $\lambda \to 0$. □

Remark. (1) The operator G_s can be extended to the more complicated form

$$\Delta_x + |x|^{2s_1}\Delta_y + |x|^{2s_2}\Delta_z,$$

in the domain $\Omega \subset \mathbb{R}^{N_1}\times\mathbb{R}^{N_2}\times\mathbb{R}^{N_3}$. Following [14], the critical exponent for this case is

$$\frac{N_1 + (s_1+1)N_2 + (s_2+1)N_3 + 2}{N_1 + (s_1+1)N_2 + (s_2+1)N_3 - 2}.$$

Generally, G_s has the form

$$\Delta_{\omega_0} + \sum_{i=1}^m |\omega_0|^{2s_i}\Delta_{\omega_i},$$

in the domain $\Omega = \{(\omega_0,\omega_1,\dots,\omega_m)\} \subset \prod_{i=0}^m \mathbb{R}^{N_i}$. The associated critical exponent is given by

$$\frac{N_0 + \sum_{i=1}^m (s_i+1)N_i + 2}{N_0 + \sum_{i=1}^m (s_i+1)N_i - 2}.$$

Putting $\omega = (\omega_0,\omega_1,\dots,\omega_m)$, we can preserve the hypotheses (H1)-(H3) and proceed with the functional

$$\begin{aligned} I_\lambda(u,v) =& \frac{1}{2}\int_\Omega (|\nabla_{\omega_0}u|^2 + \sum_{i=1}^m |\omega_0|^{2s_i}|\nabla_{\omega_i}u|^2)d\omega \\ &+ \frac{1}{2}\int_\Omega (|\nabla_{\omega_0}v|^2 + \sum_{i=1}^m |\omega_0|^{2s_i}|\nabla_{\omega_i}v|^2)d\omega - \lambda\int_\Omega F(\omega,u,v)d\omega. \end{aligned}$$

(2) Using the same argument used above, we can deal with the system of m unknowns

$$\begin{gathered} LU = \lambda \nabla F \quad \text{in } \Omega, \\ U = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

with $U = (u_1, u_2, \dots, u_m)$ $F = F(x, y, u_1, u_2, \dots, u_m)$ and

$$L = \begin{pmatrix} G_{s_1} & 0 & \dots & 0 \\ 0 & G_{s_2} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & G_{s_m} \end{pmatrix}.$$

For this system, hypotheses (H1), (H2) and (H3) are replaced by

(H1') All components of ∇F are Caratheodory functions satisfying

$$\Big|\frac{\partial F}{\partial u_i}(x, y, U)\Big| \leqslant C_{i0} + \sum_{j=1}^{m} C_{ij} |u_i|^{r_{ij}},$$

$$0 \leqslant r_{ij} < \frac{N(s_i) + 2}{N(s_i) - 2}, \quad i = 1..n,\ j = 1..m,$$

for a.e, $(x, y) \in \Omega$ and for all $U \in \mathbb{R}^n$.

(H2') For a.e. $(x, y) \in \Omega$ and for all $U \in \mathbb{R}^n$ satisfying $|U| \geqslant M$, $F(x, y, 0) = 0$ and $0 < \mu F \leqslant \nabla F.U$, where μ, M are real numbers, $\mu > 2$ and $M > 0$.

(H3') $F(x, y, U)$ is superlinear, i.e. $\lim_{|U|\to\infty} \frac{F(x,y,U)}{|U|^2} = +\infty$. The associated functional is represented by

$$I_\lambda(U) = \frac{1}{2} \int_\Omega \Big[\sum_{i=1}^{m} (|\nabla_x u_i|^2 + |x|^{2s_i} |\nabla_y u_i|^2)\Big] dx\, dy - \lambda \int_\Omega F(x, y, U) dx\, dy.$$

4. Some Special Cases of Nonlinearity

Homogeneous cases. Let $q \in \mathbb{R}$, $q > 1$. The potential function $F(x, y, u, v)$ is called q-homogeneous in (u, v) if $F(x, y, tu, tv) = t^q F(x, y, u, v)$ for a.e. $(x, y) \in \Omega$, for all $t > 0$ and $(u, v) \in \mathbb{R}^2$.

Assume that $F(x, y, u, v) \geqslant 0$ and F is q-homogeneous in (u, v). Furthermore, for fixed $(x, y) \in \Omega, F(x, y, ., .) \in C^1(\mathbb{R}^2)$ and for fixed $(u, v) \in \mathbb{R}^2$, $F(., ., u, v) \in L^\infty(\Omega)$. Then the following properties of $F(x, y, u, v)$ are verified:

(1) For a.e. $(x, y) \in \Omega$ and for all $(u, v) \in \mathbb{R}^2$,

$$m(|u|^q + |v|^q) \leqslant F(x, y, u, v) \leqslant M(|u|^q + |v|^q), \tag{4.1}$$

where

$$M = \operatorname*{esssup}_{(x,y)\in\Omega} \max_{(u,v)\in\mathbb{R}^2} \{F(x, y, u, v) : |u|^q + |v|^q = 1\}, \tag{4.2}$$

$$m = \operatorname*{essinf}_{(x,y)\in\Omega} \min_{(u,v)\in\mathbb{R}^2} \{F(x, y, u, v) : |u|^q + |v|^q = 1\}. \tag{4.3}$$

(2) For all $(u, v) \in \mathbb{R}^2$ and a.e. $(x, y) \in \Omega$,

$$u\frac{\partial F(x, y, u, v)}{\partial u} + v\frac{\partial F(x, y, u, v)}{\partial v} = qF(x, y, u, v). \tag{4.4}$$

(3)

$$\nabla F \text{ is } (q - 1)\text{- homogeneous in } (u, v). \tag{4.5}$$

It is easy to see that, for $q > 2$, the condition (H2) is followed from (4.4), the q-*homogeneity* of F implies the condition (H3). Moreover, we deduce from properties (4.5) and (4.1) that

$$\Big|\frac{\partial F(x,y,u,v)}{\partial u}\Big| \leqslant M_1(|u|^{q-1}+|v|^{q-1}),$$
$$\Big|\frac{\partial F(x,y,u,v)}{\partial v}\Big| \leqslant M_2(|u|^{q-1}+|v|^{q-1}),$$

where M_1, M_2 are positive constants. Then, the condition

$$2 < q < \min\{\frac{2N(\alpha)}{N(\alpha)-2}, \frac{2N(\beta)}{N(\beta)-2}\} \tag{4.6}$$

ensures the validity of (H1).

We now show some examples of $F(x,y,u,v)$, which by their homogeneity, one can obtain the solutions of problem (1.1).

Example 1. Let $F(x,y,u,v) = a(x,y)|u|^k|v|^\ell$ where

$$2 < k+\ell < q^* = \min\{\frac{2N(\alpha)}{N(\alpha)-2}, \frac{2N(\beta)}{N(\beta)-2}\},$$

$a(x,y) \in L^\infty_+(\Omega)$ denoted the set of all nontrivial nonnegative functions in $L^\infty(\Omega)$. Obviously, this class of potential functions satisfies all our conditions and yields the existence result.

More generally, we can apply the polynomial function given by

$$F(x,y,u,v) = \sum_i a_i(x,y)|u|^{k_i}|v|^{\ell_i} \tag{4.7}$$

to nonlinearity of problem (1.1), where $i \in \Theta(\#\Theta < \infty)$, k_i, ℓ_i are nonnegative numbers satisfying $2 < k_i + \ell_i = q < q^*$ and $a_i(x,y) \in L^\infty_+(\Omega)$.

Example 2. Let us denote

$$P_q(x,y,u,v) = \sum_i a_i(x,y)|u|^{k_i}|v|^{\ell_i}, \quad 2 < k_i+\ell_i = q, a_i \in L^\infty_+(\Omega).$$

Our potential function F may be the following functions and some possible combinations of them:

$$F(x,y,u,v) = \sqrt[r]{P_{rq}(x,y,u,v)}, \quad F(x,y,u,v) = \frac{P_{q+r}(x,y,u,v)}{P_r(x,y,u,v)},$$

where r is a positive real number.

Nonhomogeneous cases. In this part, we assume that the nonlinearity in (1.1) has the form

$$F(x,y,u,v) = G(x,y,u,v) + H(x,y,u,v), \tag{4.8}$$

where G and H are nontrivial nonnegative functions such that: $G(x,y,u,v)$ is p-homogeneous with $1 < p \leqslant 2, H(x,y,u,v)$ is q-homogeneous with $2 < q < q^*$ (q^* is denoted in example 1). It's not difficult to check that $F(x,y,u,v)$ satisfies the hypotheses (H1)-(H3), so by Theorem 1.1 one can obtain the solution $(u_{1,\lambda}, v_{1,\lambda})$ of system (1.1) with behaviour that $\|(u_{1,\lambda}, v_{1,\lambda})\|_{S^2_{1,0}(\Omega)} \to +\infty$ as $\lambda \to 0$. Now suppose $1 < p < 2$, we will use the Ekeland's variational principle to get another nontrivial solution named $(u_{2,\lambda}, v_{2,\lambda})$ such that $\|(u_{2,\lambda}, v_{2,\lambda})\|_{S^2_{1,0}(\Omega)} \to 0$ as $\lambda \to 0$.

Firstly, following the arguments in the proof of Lemma 3.1, we have the inequality

$$\begin{aligned} I_\lambda(u,v) \geqslant & \frac{1}{2}\|(u,v)\|^2_{S^2_{1,0}(\Omega)} \\ & - \lambda[C_1^* + C_2^*\|(u,v)\|^{\frac{2N(\alpha)}{N(\alpha)-2}+1-\gamma_1(\tau)}_{S^2_{1,0}(\Omega)} + C_3^*\|(u,v)\|^{\frac{2N(\beta)}{N(\beta)-2}+1-\gamma_2(\tau)}_{S^2_{1,0}(\Omega)}]. \end{aligned} \tag{4.9}$$

Choosing $(u,v) \in S^2_{1,0}(\Omega)$ such that $\|(u,v)\|_{S^2_{1,0}(\Omega)} = \lambda^{r/2}, 0 < r < 1$, we deduce

$$I_\lambda(u,v) \geqslant \frac{1}{2}\lambda^r > 0, \tag{4.10}$$

for λ sufficiently small. On the other hand, under the assumptions on G and H, the following inequalities hold for all $(u,v) \in \mathbb{R}^2, (x,y) \in \Omega$:

$$\begin{gathered} m_G(|u|^p + |v|^p) \leqslant G(x,y,u,v) \leqslant M_G(|u|^p + |v|^p), \\ m_H(|u|^q + |v|^q) \leqslant H(x,y,u,v) \leqslant M_H(|u|^q + |v|^q), \end{gathered} \tag{4.11}$$

where M_G, M_H, m_G and m_H are defined in (4.2) and (4.3). This fact allows us to estimate

$$\begin{aligned} & I_\lambda(tu,tv) \\ & \leqslant \frac{1}{2}t^2\|(u,v)\|_{S^2_{1,0}(\Omega)} - t^p m_G \int_\Omega (|u|^p + |v|^p)dx\,dy - t^q m_H \int_\Omega (|u|^q + |v|^q)dx\,dy, \end{aligned}$$

for all $t > 0$. Taking $(u,v) \in S^2_{1,0}(\Omega)$, $\|(u,v)\|_{S^2_{1,0}(\Omega)} = 1$ we have

$$I_\lambda(tu,tv) \leqslant C_1t^2 - C_2t^p - C_3t^q,$$

where C_1, C_2 and C_3 are positive constants. Since $1 < p < 2$, we can state that

$$I_\lambda(tu,tv) < 0, \tag{4.12}$$

for small positive t.

We now consider the functional I_λ in the ball $B(0,\lambda^r) \subset S^2_{1,0}(\Omega)$. Using Ekeland's variational principle and arguments similar to those used in [1], we have

$$I_0 = \inf_{\bar{B}(0,\lambda^r)} I_\lambda(u,v) < 0$$

and there exists a (PS) sequence $\{(u_n,v_n)\}$ of I_λ in $B(0,\lambda^r)$. Since I_λ satisfies (PS) condition, one can conclude that $\{(u_n,v_n)\}$ converges to a critical point named $(u_{2,\lambda}, v_{2,\lambda}) \in B(0,\lambda^r)$ up to a subsequence.

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