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WEIGHTED PSEUDO ALMOST PERIODIC SOLUTIONS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. By means of the exponential dichotomy method and the properties of the weighted pseudo almost periodic functions, sufficient conditions are obtained for the existence of weighted pseudo almost periodic solutions of functional differential equations with finite delay.

1. INTRODUCTION

The theory of almost periodic function was created during 1924-1926 mainly by Harald Bohr. who gave a strong impetus to the development of harmonic analysis on groups. In particular, the theory of almost periodic equations has been developed in connection with problems of differential equations, stability theory, dynamical systems, and so on (see, e.g., [9]). The existence of almost periodic type solutions is amongst the most attractive topics, which arises in qualitative theory of differential equations due to their significance and applications in physics, mathematical biology, control theory and others. As one of the cases, the existence of pseudo almost periodic solutions to various differential equations has been investigated in many papers (see, e.g.,[1, 2, 3, 4] and references therein). In [5], Diagnana introduced some new classes of functions called weighted pseudo almost periodic functions. Those new functions implement Zhang's [10] pseudo almost periodic functions. After that, in [4], the author investigated the basic properties of weighted pseudo periodic functions and obtained conditions of existence of the weighted pseudo almost periodic solutions for abstract differential equations.

In the present paper, motivated by [1, 3], we are concerned with weighted pseudo almost periodic solutions of delay differential equations. We obtain sufficient conditions for the existence of weighted pseudo almost periodic solution of functional differential equations with finite delay.

2. PRELIMINARIES

For the reader's convenience, we recall some concepts of almost periodicity and weighted pseudo almost periodicity, as well as some basic facts of functional differential equations.

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Throughout the paper, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{X}$ stand for the sets of real, complex numbers, and a Banach space with a norm $\|\cdot\|$, respectively. Let $(BC(\mathbb{R},\mathbb{X}),\|\cdot\|)$ denote the collection of all $\mathbb{X}$-valued bounded continuous functions equipped with the sup norm $\|\phi\|_\infty := \sup_{t\in\mathbb{R}}\|\phi(t)\|$ for each $\phi \in BC(\mathbb{R},\mathbb{X})$.

Definition 1.1 ([9]). A continuous function $f:\mathbb{R}\to\mathbb{X}$ is called almost periodic if it has a relative dense set of ϵ-almost periods for each $\epsilon>0$, that is, if there is a number $l=l(\epsilon)>0$ such that each interval $(a,a+l)\subset\mathbb{R}$ contains at least one number $\tau=\tau(\epsilon)$ satisfying

$$\|f(t+\tau)-f(t)\|<\epsilon$$

for all $t\in\mathbb{R}$. Denote by $AP(\mathbb{R},\mathbb{X})$ the set of all such functions.

Set

$$PAP_0(\mathbb{R},\mathbb{X}) := \{\varphi\in BC(\mathbb{R},\mathbb{X}) : \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\|\varphi(t)\|dt = 0\}.$$

Definition 1.2 ([10]). A function $f:\mathbb{R}\to\mathbb{X}$ is called pseudo almost periodic, if

$$f = g+\varphi,$$

where $g\in AP(\mathbb{R},\mathbb{X})$ and $\varphi\in PAP_0(\mathbb{R},\mathbb{X})$. Denote by $PAP(\mathbb{R},\mathbb{X})$ the set of all such functions.

Let $\mathbb{U}$ be the collection of functions (weights) $\rho:\mathbb{R}\to(0,\infty)$, which are locally integrable over $\mathbb{R}$ such that $\rho>0$ almost everywhere. Set

$$\mu(T,\rho) := \int_{-T}^{T}\rho(t)dt,$$
$$\mathbb{U}_\infty := \{\rho\in\mathbb{U} : \lim_{T\to\infty}\mu(T,\rho)=\infty\},$$
$$\mathbb{U}_B := \{\rho\in\mathbb{U}_\infty : \rho \text{ is bounded with } \inf_{t\in\mathbb{R}}\rho(t)>0\}.$$

Obviously, $\mathbb{U}_B\subset\mathbb{U}_\infty\subset\mathbb{U}$, with strict inclusions.

For $\rho\in\mathbb{U}_\infty$, define

$$PAP_0(\mathbb{R},\mathbb{X},\rho) := \{f\in BC(\mathbb{R},\mathbb{X}) : \lim_{T\to\infty}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\|f(t)\|\rho(t)dt = 0\}.$$

Definition 1.3 ([4]). Let $\rho\in\mathbb{U}_\infty$. A function $f\in BC(\mathbb{R},\mathbb{X})$ is called weighted pseudo almost periodic or ρ-pseudo almost periodic if it can be expressed as $f = g+\varphi$, where $g\in AP(\mathbb{R},\mathbb{X})$ and $\varphi\in PAP_0(\mathbb{R},\mathbb{X},\rho)$. The collection of such functions will be denoted by $PAP(\mathbb{R},\mathbb{X},\rho)$

Let $\rho_1,\rho_2\in\mathbb{U}_\infty$. One says that ρ_1 is equivalent to ρ_2, denoting this as $\rho_1\prec\rho_2$, if $\frac{\rho_1}{\rho_2}\in\mathbb{U}_B$.

Remark 2.1. (i) Let $\rho_1,\rho_2,\rho_3\in\mathbb{U}_\infty$. It's clear that $\rho_1\prec\rho_1$ (reflexivity); if $\rho_1\prec\rho_2$, then $\rho_2\prec\rho_1$ (symmetry); and if $\rho_1\prec\rho_2$ and $\rho_2\prec\rho_3$, then $\rho_1\prec\rho_3$ (transitivity). So, $\prec$ is a binary equivalence relation on $\mathbb{U}_\infty$. Thus the equivalence class of a given weighted $\rho\in\mathbb{U}_\infty$ will then be denoted by

$$cl(\rho) = \{\varpi\in\mathbb{U}_\infty : \rho\in\varpi\}.$$

It is then clear that $\mathbb{U}_\infty = \bigcup_{\rho\in\mathbb{U}_\infty} cl(\rho)$.

(ii) Let $\rho\in\mathbb{U}_\infty$. If $\rho_1,\rho_2\in cl(\rho)$, then $PAP(\mathbb{R},\mathbb{X},\rho_1) = PAP(\mathbb{R},\mathbb{X},\rho_2)$. In particular, if $\rho\in\mathbb{U}_B$, then $PAP(\mathbb{R},\mathbb{X},\rho) = PAP(\mathbb{R},\mathbb{X},cl(1)) = PAP(\mathbb{R},\mathbb{X})$.

3. Autonomous Linear Equation

We are concerned with the linear delay differential equations

$$\frac{dx(t)}{dt} = Lx_t + f(t) \tag{3.1}$$

and

$$\frac{dx(t)}{dt} = Lx_t. \tag{3.2}$$

Let r be a fixed positive number,and $C = C([-r, 0], \mathbb{R}^n)$ be the Banach space of continuous functions on $[-r, 0]$ with norm $\|\varphi\| = \sup\{|\varphi(\theta)| : -r \le \theta \le 0\}$. If x is a continuous map of $[a - r, b]$ into $\mathbb{R}^n$, then $x_t \in C$ is given, for each $a \le t \le b$, by

$$x_t(\theta) = x(t + \theta), -r \le \theta \le 0.$$

Let L be a continuous linear operator from $C = C([-r, 0], \mathbb{R}^n)$ into $\mathbb{R}^n$. The hypothesis on L implies that there exists an $n \times n$ matrix $\eta(\theta) : -r \le \theta \le 0$, whose elements are of bounded variation, and

$$L(\phi) = \int_{-r}^{0} d[\eta(\theta)]\phi(\theta). \tag{3.3}$$

If ϕ is any given function in C and $x_t(\sigma, \phi)$ is the unique solution of (3.2) with initial data at σ, the solution operator $T(t)$ is given by

$$T(t)\phi = x_t(\sigma, \phi), t \ge \sigma.$$

It is well-known that the solution operator $(T(t))_{t\ge\sigma}$ is a C_0-semigroup with infinitesimal generator

$$D(A) = \{\phi \in C : \frac{d\phi}{d\theta} \in C \text{ and } \frac{d\phi}{d\theta}(0) = L(\phi) \text{ and } A\phi = \frac{d\phi}{d\theta}\}.$$

Furthermore the spectrum $\sigma(A)$ of A is given by

$$\sigma(A) = \{\lambda \in \mathbb{C} : \det \Delta(\lambda) = 0\},$$

where

$$\Delta(\lambda) = \lambda I - \int_{-r}^{0} e^{\lambda\theta} d\eta(\theta).$$

Lemma 1 ([8]). Let $\Lambda = \{\lambda \in \sigma(A) : Re\lambda > 0\}$ and suppose C is decomposed by Λ as

$$C = P_\Lambda \oplus Q_\Lambda,$$

where the definition of P_Λ and Q_Λ one can refer [8, p.212]. Then there exit positive constants k and c such that

$$\|T(t)\phi^{P_\Lambda}\| \le ke^{ct}, t \le 0,$$
$$\|T(t)\phi^{Q_\Lambda}\| \le ke^{-ct}, t \ge 0.$$

Lemma 2 ([8]). Assume that $\sigma(A) \bigcap i\mathbb{R} = \emptyset$, then for all bounded functions f, (3.1) has one and only one bounded solution.

The solution of (3.1) with initial value ϕ at σ is

$$x_t = T(t - \sigma)\phi + \int_{\sigma}^{t} d[K(t, s)]f(s),$$

where

$$K(t, s)(\theta) = \int_{\sigma}^{s} X(t + \theta - \alpha)d\alpha, t \ge \sigma,$$

and $X(\cdot)$ denotes the fundamental matrix solution of system (3.2).

Assume that C is decomposed by Λ as $P \oplus Q$, then the solution is given by
$$x_t = x_t^P + x_t^Q,$$
where
$$x_t^P = T(t-\sigma)\phi^P + \int_\sigma^t T(t-s)X_0^P f(s)ds, t \geq \sigma,$$
$$x_t^Q = T(t-\sigma)\phi^Q + \int_\sigma^t d[K(t,s)^Q]f(s)ds, t \geq \sigma.$$
We know from [8] that there exist $\Psi, \Upsilon \in Q$ such that
$$K(t,s)^Q = \begin{cases} \int_{t-s-r}^{t-r} T(\beta)\Psi d\beta & \text{if } t-s \geq r \\ \int_0^{t-r} T(\beta)\Psi d\beta + \Upsilon & \text{if } t-s \leq r. \end{cases} \tag{3.4}$$
For more details on this facts, can be found in [7] and [8]. Now, we are in a position to present our main result.

Theorem 3.1. *Assume that $\sigma(A) \bigcap i\mathbb{R} = \emptyset$, if f is ρ-pseudo almost periodic and*
$$\sup_{T>0}\{\int_{-T}^{T} e^{-c(T+t)}\rho(t)dt\} < \infty, \tag{H}$$
then (3.1) *has one and only one bounded solution which is also $cl(\rho)$-pseudo almost periodic.*

Proof. By Lemma 2, (3.1) has one and only one solution which is
$$x_t = x_t^P + x_t^Q.$$
We will show that both
$$\int_{-\infty}^t T(t-s)X_0^P f(s)ds \quad \text{and} \quad \int_{-\infty}^t d[K(t,s)^Q]f(s)$$
are ρ-pseudo almost periodic.Assume that $f = g + \phi$, where g is its almost periodic component and ϕ satisfies
$$\lim_{T\to\infty} \frac{1}{\mu(T,\rho)} \int_{-T}^T \|\phi(t)\|\rho(t)dt = 0.$$
We will show that
$$\int_{-\infty}^t T(t-s)X_0^P g(s)ds \quad \text{and} \quad \int_{-\infty}^t d[K(t,s)^Q]g(s)ds$$
are almost periodic. Let $\alpha = (\alpha_n)$ be a real sequence. By almost periodicity of f, there exists a subsequence of α noted by α' and a continuous function $h(t)$ such that
$$h(t) = \lim_{n\to\infty} g(t+\alpha') \quad \text{uniformly in } \mathbb{R}.$$
So,
$$\int_{-\infty}^t T(t-s)X_0^P g(s+\alpha_n')ds \to \int_{-\infty}^t T(t-s)X_0^P h(s)ds$$
uniformly in $\mathbb{R}$, and
$$\int_{-\infty}^t d[K(t,s+\alpha_n')^Q]g(s+\alpha_n') \to \int_{-\infty}^t d[K(t,s)^Q]h(s)$$

uniformly in $\mathbb{R}$. Thus
$$\int_{-\infty}^{t} T(t-s)X_0^P g(s)ds$$
and
$$\int_{-\infty}^{t} d[K(t,s)^Q]g(s)$$
are almost periodic. It remains to show that
$$\lim_{T\to\infty}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\|\int_{-\infty}^{t} T(t-s)X_0^P\phi(s)ds\|\rho(t)dt=0, \tag{3.5}$$
and
$$\lim_{T\to\infty}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\|\int_{-\infty}^{t} d[K(t,s)^Q]\phi(s)\|\rho(t)dt=0. \tag{3.6}$$
Let us put
$$I(t)=\int_{-\infty}^{t} T(t-s)X_0^P\phi(s)ds$$
and
$$J(t)=\int_{-\infty}^{t} d[K(t,s)^Q]\phi(s).$$
Because of (3.4), there exists a positive constant M such that
$$\begin{aligned}
&\lim_{T\to\infty}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\|J(t)\|\rho(t)dt\\
&\le \lim_{T\to\infty}\frac{M}{\mu(T,\rho)}\int_{-T}^{T}\int_{-\infty}^{t}\exp(-c(t-s))\|\phi(s)\|\rho(t)dsdt\\
&= \lim_{T\to\infty}\frac{M}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)\int_{-\infty}^{-T}\exp(-c(t-s))\|\phi(s)\|dsdt\\
&\quad + \lim_{T\to\infty}\frac{M}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)\int_{-T}^{t}\exp(-c(t-s))\|\phi(s)\|dsdt
\end{aligned} \tag{3.7}$$
Note that
$$\begin{aligned}
J_1 &= \lim_{T\to\infty}\frac{M}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)\int_{-T}^{t}\exp(-c(t-s))\|\phi(s)\|dsdt\\
&= \lim_{T\to\infty}\frac{M}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)dt\int_{-T}^{t}\exp(-c(t-s))\|\phi(s)\|ds\\
&= \lim_{T\to\infty}\frac{M}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)\|\phi(t)\|dt\int_{-T}^{t}\exp(-c(t-s))ds\\
&= \lim_{T\to\infty}\frac{M}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)\|\phi(t)\|dt\{\frac{1}{c}[1-\exp(-c(t+T))]\}.
\end{aligned} \tag{3.8}$$
Since $-T\le t\le T$ and $c>0$, it follows that $\frac{1}{c}[1-\exp(-c(t+T))]$ is bounded. Note that $\lim_{T\to\infty}\frac{M}{\mu(T,\rho)}\int_{-T}^{T}\|\phi(t)\|\rho(t)dt=0$. Then it follows that $J_1=0$. Also,

by (H) we have

$$
\begin{aligned}
J_2 &= \lim_{T\to\infty} \frac{M}{\mu(T,\rho)} \int_{-T}^{T} \rho(t) \int_{-\infty}^{-T} \exp(-c(t-s))\|\phi(s)\| ds dt \\
&= \lim_{T\to\infty} \frac{M}{\mu(T,\rho)} \int_{-T}^{T} \rho(t) e^{-ct} dt \int_{-\infty}^{-T} e^{cs} \|\phi(s)\| ds \\
&\leq \lim_{T\to\infty} \{ \frac{M \sup_{s\in\mathbb{R}} \|\phi(s)\| \int_{-T}^{T} \rho(t) e^{-c(c+T)} dt}{c\mu(T,\rho)} \} \\
&= 0.
\end{aligned} \tag{3.9}
$$

The proof of Theorem 1 is complete. □

4. Nonautonomous Linear Equation

In this section we consider the linear delay differential equation

$$\frac{dx(t)}{dt} = L(t)x_t. \tag{4.1}$$

We say that linear system (4.1) admits an exponential dichotomy on $\mathbb{R}$, if the solution operators $T(t,s)$ satisfies the following property:

there exist positive constants $k > 1, c > 0$, and a projection operator $P(s) : X \to X, (P(s) = P^2(s)), s \in \mathbb{R}$, such that if $Q(s) = I - P(s)$, then

(i) $T(t,s)P(s) = P(t)T(t,s), t \geq s$;

(ii) The restriction $T(t,s)R(Q(s)), t \geq s$, is an isomorphism of $R(Q(s))$ onto $R(Q(t))$ and we define $T(s,t)$ as the inverse mapping;

(iii) $\|T(t,s)P(s)\| \leq \exp(-c(t-s))$, for $s \leq t$;

(iv) $\|T(t,s)Q(s)\| \leq \exp(-c(t-s))$, for $s \geq t$.

We will apply the exponential dichotomy theory to prove the existence of ρ-pseudo almost periodic solutions for the following delay differential equations (4.2) with ρ-pseudo almost periodic coefficients.

For $(\sigma, \phi) \in \mathbb{R} \times C$, consider the linear system

$$
\begin{gathered}
\frac{dx(t)}{dt} = L(t)x_t + f(t), t \geq \sigma \\
x_\sigma = \phi.
\end{gathered} \tag{4.2}
$$

Theorem 4.1. *Assume that $L(t)$ is almost periodic in t and* (4.1) *has an exponential dichotomy. Then, for any $f \in BC(\mathbb{R},\mathbb{R})$, it follows that* (4.2) *has an unique solution $Sf \in BC(\mathbb{R}, C)$. Moreover, if f is ρ-pseudo almost periodic and condition (H) is satisfied, then $Sf \in PAP(\mathbb{R}, \mathbb{X}, cl(\rho))$.*

Proof. As we know in [8] for all bounded functions f, the unique bounded solution of (4.2) is given by

$$Sf(t) = \int_{-\infty}^{t} d[P(s)K(t,s)]f(s) - \int_{t}^{+\infty} d[Q(s)K(t,s)]f(s).$$

where the function$K(t,s)$ is defined by

$$K(t,s) = \int_{\sigma}^{s} X(t+\theta, \alpha) d\alpha,$$

and $K(t,s)$ is the fundamental matrix solution to system (4.1).

If f is ρ-pseudo almost periodic, then $f(t) = g(t) + \phi(t)$, where $g \in AP(\mathbb{R}, \mathbb{R}^n)$,and $\rho \in PAP(\mathbb{R}, \mathbb{R}^n, \rho)$. By the exponential dichotomy property, we can show similarly to the one given in Theorem 1 that $Sg \in AP(\mathbb{R}, \mathbb{R}^n)$, $S\phi \in PAP(\mathbb{R}, \mathbb{R}^n, cl(\rho))$, respectively. $\square$

Remark 4.2. In the particular case when $\rho = 1$; that is, $PAP(\mathbb{R}, \mathbb{R}^n, cl(\rho)) = PAP(\mathbb{R}, \mathbb{R}^n)$ by Remark 2, we retrieve the pseudo almost periodic situation,sine condition (H) is always achieved in that event. This means that Theorem 1 and Theorem 2 we obtained here are good generalizations of Theorem 3.4 and Proposition 4.2 in [1].

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