Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 227, pp. 1–42.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

DIFFERENTIAL EQUATIONS WITH A DIFFERENCE QUOTIENT

BRIAN STREET

ABSTRACT. The purpose of this paper is to study a class of ill-posed differential equations. In some settings, these differential equations exhibit uniqueness but not existence, while in others they exhibit existence but not uniqueness. An example of such a differential equation is, for a polynomial P and continuous functions $f(t,x) : [0,1] \times [0,1] \to \mathbb{R}$,

$$\frac{\partial}{\partial t} f(t,x) = \frac{P(f(t,x)) - P(f(t,0))}{x}, \quad x > 0.$$

These differential equations are related to inverse problems.

1. Introduction

The purpose of this paper is to study a family of ill-posed differential equations. In some instances, these equations exhibit existence, but not uniqueness. In other instances, they exhibit uniqueness, but not existence. The questions studied here can be seen as a family of forward and inverse problems, which in special cases become well-known examples from the literature. This is discussed more below and detailed in Section 3.

In this introduction, we informally state the main results, and present their relationship to inverse problems. However, before we enter into the results in full generality, to help the reader understand our somewhat technical results, we give some very simple special cases, where some of the basic ideas already appear in a simple form:

Example 1.1 (Existence without uniqueness)**.** Fix $\epsilon_1, \epsilon_0 > 0$. We consider the differential equation, defined for functions $f(t,x) \in C([0,\epsilon_1] \times [0,\epsilon_0])$ by

$$\frac{\partial}{\partial t} f(t,x) = \frac{f(t,0) - f(t,x)}{x}, \quad x > 0. \tag{1.1}$$

We claim that (1.1) has existence: i.e., given $f_0(x) \in C([0,\epsilon_0])$, there exists a solution $f(t,x)$ to (1.1) with $f(0,x) = f_0(x)$. Indeed, given $a(t) \in C([0,\epsilon_1])$ with $a(0) = f_0(0)$ set

$$f(t,x) = \begin{cases} e^{-t/x} f_0(x) + \frac{1}{x}\int_0^t e^{(s-t)/x} a(s)\, ds, & x > 0, \\ a(t), & x = 0. \end{cases} \tag{1.2}$$

2010 *Mathematics Subject Classification.* 34K29, 34K09.
Key words and phrases. Ill-posed; differential equation with difference quotient; existence without uniqueness; uniqueness without existence; inverse problem.

Submitted January 7, 2017. Published September 21, 2017.

It is immediate to verify that $f(t,x) \in C([0,\epsilon_1] \times [0,\epsilon_0])$ and satisfies (1.1). Furthermore, this is the unique solution, $f(t,x)$, to (1.1) with $f(0,x) = f_0(x)$ *and* $f(t,0) = a(t)$.[1] Thus, to uniquely determine the solution to (1.1) one needs to give both $f(0,x)$ *and* $f(t,0)$. We call this existence without uniqueness, since there are many solutions corresponding to any initial condition $f(0,x)$-one for each choice of $a(t)$.

Example 1.2 (Uniqueness without existence)**.** Fix $\epsilon_1, \epsilon_0 > 0$. We consider the differential equation, defined for functions $f(t,x) \in C([0,\epsilon_1] \times [0,\epsilon_0])$ by

$$\frac{\partial}{\partial t} f(t,x) = \frac{f(t,x) - f(t,0)}{x}, \quad x > 0. \tag{1.3}$$

We claim that (1.3) has uniqueness: i.e., if $f(t,x), g(t,x) \in C([0,\epsilon_1] \times [0,\epsilon_0])$ both satisfy (1.3) and $f(0,x) = g(0,x)$, for all x, then $f(t,x) = g(t,x)$, for all t,x. Indeed, suppose $f(t,x)$ satisfies (1.3). Then, by reversing time, treating $f(\epsilon_1,x)$ as our initial condition, and treating $a(t) := f(t,0)$ as a given function, we may solve the differential equation (1.3), for $x > 0$, to see

$$f(0,x) = e^{-\epsilon_1/x} f(\epsilon_1,x) + \frac{1}{x}\int_0^{\epsilon_1} e^{-u/x} a(u)\,du, \quad x > 0. \tag{1.4}$$

From (1.4) uniqueness follows. Indeed, if $f(t,x)$ and $g(t,x)$ are two solutions to (1.3) with $f(0,x) = g(0,x)$ for all x, then (1.4) shows

$$\frac{1}{x}\int_0^{\epsilon_1} e^{-u/x} f(u,0)\,du = \frac{1}{x}\int_0^{\epsilon_1} e^{-u/x} g(u,0)\,du + O(e^{-\epsilon_1/x}).$$

It then follows (see Corollary 8.4) that $f(t,0) = g(t,0)$ for all t. With $f(t,0) = g(t,0)$ in hand, (1.3) is a standard ODE for $x > 0$ and it follows that $f(t,x) = g(t,x)$ for all t,x. This proves uniqueness. Furthermore, (1.4) shows that (1.3) does not have existence: not every initial condition gives rise to a solution. In fact, every initial condition that does give rise to a solution must be of the form given by (1.4), for some continuous functions $a(t)$ and $f(\epsilon_1,x)$. I.e., the initial condition must be of Laplace transform type, modulo an appropriate error. Furthermore, it is easy to see that for such an initial condition, there exists a solution. Hence, we have exactly characterized the initial conditions which give rise to a solution to (1.3).

The goal of this paper is to extend the above ideas to a nonlinear setting. Consider the following simplified example.

Example 1.3. Let $P(y) = \sum_{j=1}^D c_j y^j$ be a polynomial without constant term. Consider the differential equation, defined for functions $f(t,x) \in C([0,\epsilon_1] \times [0,\epsilon_0])$, given by

$$\frac{\partial}{\partial t} f(t,x) = \frac{P(f(t,x)) - P(f(t,0))}{x}, \quad x > 0. \tag{1.5}$$

- (Uniqueness without existence) If we restrict our attention to solutions $f(t,x)$ with $P'(f(t,0)) > 0$ for all t and we insist that $f(t,0) \in C^2([0,\epsilon_1])$, then (1.5) has uniqueness (but not existence). I.e., if $f(t,x), g(t,x) \in C([0,\epsilon_1] \times [0,\epsilon_0])$ are two solutions to (1.5) with $f(0,x) = g(0,x)$ for all x, $P'(f(t,0)), P'(g(t,0)) > 0$ for all t, and $f(t,0), g(t,0) \in C^2([0,\epsilon_1])$, then $f(t,x) = g(t,x)$ for all t,x. However, not every initial condition gives rise to

[1] Uniqueness is immediate here, since for $x > 0$, if $f(t,0)$ is assumed to be $a(t)$, then (1.1) is a standard ODE and standard uniqueness theorems apply.

a solution. See Section 2.2. This generalizes[2] Example 1.2 where $P(y) = y$ and therefore $P'(y) \equiv 1 > 0$.

- (Existence without uniqueness) Given $f_0(x) \in C([0,\epsilon_0])$ and $a(t) \in C^2([0,\epsilon_1])$ with $a(0) = f_0(0)$ and $P'(a(t)) < 0$ for all t, there exists $\delta > 0$ and a unique solution $f(t,x) \in C([0,\epsilon_1]\times[0,\delta])$ to (1.5) satisfying $f(0,x) = f_0(x)$ *and* $f(t,0) = a(t)$. See Section 2.1. This generalizes Example 1.1 where $P(y) = -y$ and therefore $P'(y) \equiv -1 < 0$.

In short, if one has $P'(f(t,0)) > 0$ for all t, one has uniqueness but not existence, and if one has $P'(f(t,0)) < 0$ for all t, one has existence but not uniqueness.

We now turn to the more general setting of our main results. Fix $m \in \mathbb{N}$ and $\epsilon_0, \epsilon_1 > 0$. For $t \in [0,\epsilon_1]$, $x \in [0,\epsilon_0]$, and $y, z \in \mathbb{R}^m$, let $P(t,x,y,z)$ be a polynomial in y given by

$$P(t,x,y,z) = \sum_{j=1}^{m} \sum_{|\alpha|\le D} c_{\alpha,j}(t,x,z) y^\alpha e_j,$$

where $e_j \in \mathbb{R}^m$ denotes the jth standard basis element. For $f(t,x) \in C([0,\epsilon_1] \times [0,\epsilon_0];\mathbb{R}^m)$ we consider the differential equation

$$\frac{\partial}{\partial t} f(t,x) = \frac{P(t,x,f(t,x),f(t,0)) - P(t,0,f(t,0),f(t,0))}{x}, \quad x > 0. \tag{1.6}$$

We state our assumptions more rigorously in Section 2, but we assume:

- $c_{\alpha,j}(t,x,z) = \frac{1}{x}\int_0^\infty e^{-w/x} b_{\alpha,j}(t,w,z)\,dw$, where the $b_{\alpha,j}(t,w,z)$ have a certain prescribed level of smoothness.
- We consider only solutions $f(t,x) \in C([0,\epsilon_1]\times[0,\epsilon_0];\mathbb{R}^m)$ such that $f(t,0) \in C^2([0,\epsilon_1];\mathbb{R}^m)$.
- For $y \in \mathbb{R}^m$, set $\mathcal{M}_y(t) := d_y P(t,0,y,y)$, so that $\mathcal{M}_y(t)$ is an $m\times m$ matrix. We consider only solutions $f(t,x)$ such that there exists an invertible matrix $R(t)$ which is C^1 in t and such that $R(t)\mathcal{M}_{f(t,0)}(t)R(t)^{-1}$ is a diagonal matrix. When $m = 1$, this is automatic.

Under the above assumptions, we prove the following:

- (Uniqueness without existence) Under the above hypotheses, if $\mathcal{M}_{f(t,0)}(t)$ is assumed to have all strictly positive eigenvalues, then (1.6) has uniqueness, but not existence. I.e., if $f(t,x), g(t,x) \in C([0,\epsilon_1] \times [0,\epsilon_0];\mathbb{R}^m)$ are solutions to (1.6) which satisfy all of the above hypothesis and such that the eigenvalues of $\mathcal{M}_{f(t,0)}(t)$ and $\mathcal{M}_{g(t,0)}(t)$ are strictly positive, for all t, then if $f(0,x) = g(0,x)$ for all x, we have $f(t,x) = g(t,x)$ for all t, x. Furthermore, in this situation we prove stability estimates. Finally, in analogy to Example 1.2, we will see that only certain initial conditions give rise to solutions. See Section 2.2.
- (Existence without uniqueness) Suppose $f_0(x) \in C([0,\epsilon_0];\mathbb{R}^m)$ and $A(t) \in C^2([0,\epsilon_1];\mathbb{R}^m)$ are given such that $f_0(0) = A(0)$ and $\mathcal{M}_{A(t)}(t)$ has all strictly negative eigenvalues. Suppose further that there exists an invertible matrix $R(t)$, which is C^1 in t such that $R(t)\mathcal{M}_{A(t)}R(t)^{-1}$ is a diagonal matrix. Then we show that there exists $\delta > 0$ and a unique function

[2]Since we insisted $f(t,0) \in C^2$, this is not strictly a generalization of Example 1.2, however it does generalize the basic ideas of Example 1.2. A similar remark holds for the next part where we discuss existence without uniqueness.

$f(t,x) \in C([0,\epsilon_1] \times [0,\delta]; \mathbb{R}^m)$ such that $f(0,x) = f_0(x)$, $f(t,0) = A(t)$, and $f(t,x)$ solves (1.6). See Section 2.1.

The main idea is the following. If $f(t,x)$ were assumed to be of Laplace transform type, $f(t,x) = \frac{1}{x}\int_0^\infty e^{-w/x} A(t,w)\,dw$, then (1.6) can be restated as a partial differential equation on $A(t,w)$–and this partial differential equation is much easier to study. As exemplified in Examples 1.1 and 1.2, not every solution is of Laplace transform type. However, we will show (under the above discussed hypotheses) that every solution is of Laplace transform type modulo an error which can be controlled. Once this is done, the above results follow.

1.1. **Motivation and relation to inverse problems.** It is likely that the methods of this paper are the most interesting aspect. The differential equations in this paper seem to not fall under any current methods (the equations are too unstable), and the methods in this paper are largely new. Moreover, as we will see, special cases of the above appear in some inverse problems. Furthermore, there are other (harder) inverse problems where differential equations similar to (but more complicated than) the ones studied in this paper appear. For example, we will see in Section 9.2 that the anisotropic version of the famous Calderón problem involves a "non-commutative" version of some of these differential equations. We hope that the ideas in this paper might shed light on such questions–and, indeed, one of our motivation for these results is as a simpler model case for full anisotropic version of the Calderón problem.

We briefly outline the relationship between these results and inverse problems; these ideas are discussed in greater detail in Section 3. We begin by explaining that the results in this paper can be thought of as a class of forward and inverse problems. For simplicity, consider the setting in Example 1.3, with $\epsilon_0 = \epsilon_1 = 1$. Thus, we are given a polynomial without constant term, $P(y) = \sum_{j=1}^D c_j y^j$. We consider the differential equation, for functions $f(t,x)$, given by

$$\frac{\partial}{\partial t} f(t,x) = \frac{P(f(t,x)) - P(f(t,0))}{x}, \quad x > 0. \tag{1.7}$$

Forward Problem: Given a function $f_0(x) \in C([0,1])$ and $a(t) \in C^2([0,1])$ with $P'(a(t)) < 0$, for all t and $f_0(0) = a(0)$, the results below imply that there exists $\delta > 0$ and a unique solution $f(t,x) \in C([0,1] \times [0,\delta])$ to (1.7) with $f(0,x) = f_0(x)$ and $f(t,0) = a(t)$.

The forward problem is the map $(f_0(\cdot), a(\cdot)) \mapsto f(1,\cdot)$.

Inverse Problem: The inverse problem is given $f(1,\cdot)$, as above, to find $f_0(\cdot)$ and $a(\cdot)$.

To see how the inverse problem relates to the main results of the paper, let $f(t,x)$ be the solution as above. Set $g(t,x) = f(1-t,x)$. If $Q(y) = -P(y)$, then $g(t,0) = a(1-t)$ and $g(t,x)$ satisfies

$$\frac{\partial}{\partial t} g(t,x) = \frac{Q(g(t,x)) - Q(g(t,0))}{x}, \quad x > 0. \tag{1.8}$$

Also, $Q'(g(t,0)) > 0$, for all t. The main results of this paper imply (1.8) has uniqueness in this setting: $g(0,x) \in C([0,\delta])$ uniquely determines $g(t,x) \in C([0,1] \times [0,\delta])$. Since $g(t,x) = f(1-t,x)$, $f(1,x) \in C([0,\delta])$ uniquely determines both $f_0\big|_{[0,\delta]}$ and $a(t)$. Thus, the inverse problem has uniqueness. In short, the map

$(f_0\big|_{[0,\delta]}(\cdot), a(\cdot)) \mapsto f(1,\cdot)$ is injective (though it is far from surjective as we explain below).

We go further than just proving existence and uniqueness, though. We have:

- In the forward problem, we do the following (see Section 2.1):
  - Beyond just proving existence, we show that every solution $f(t,x)$ must be of Laplace transform type, modulo an appropriate error, for every $t>0$. This is despite the fact that the initial condition, $f(0,x) = f_0(x)$, can be any continuous function with $P'(f_0(0)) < 0$.
  - We reduce the problem to a more stable PDE, so that solutions can be more easily studied.
- In the inverse problem, we do the following (see Section 2.2.1):
  - We characterize the initial conditions $g(0,x)$ which give rise to solutions to (1.8). In other words, we characterize the image of the map $(f_0(\cdot), a(\cdot)) \mapsto f(1,\cdot)$. We see that all such functions are of Laplace transform type, modulo an appropriate error.
  - We give a procedure to reconstruct $a(t)$ and $f_0\big|_{[0,\delta]}$ from $f(1,\cdot)$. This is necessarily unstable, but we reduce the instability to the instability of the Laplace transform, which is well understood.
  - We prove a kind of stability for the inverse problem. Namely if one has two solutions $g_1(t,x)$ and $g_2(t,x)$ to (1.8) such that $g_1(0,x) - g_2(0,x)$ vanishes sufficiently quickly as $x \downarrow 0$, then $g_1(t,0) = g_2(t,0)$ on a neighborhood of 0 (the size of the neighborhood depends on how quickly $g_1(0,x) - g_2(0,x)$ vanishes in a way which is made precise). In other words if one only knows $f(1,x)$ modulo functions which vanish sufficiently quickly at 0, one can still reconstruct $a(t)$ on a neighborhood of $t = 1$, in a way which we make quantitative.

Some special cases of the main results in this paper can be interpreted as some standard inverse problems in the following way:

- When $P(y) = -y$, we saw in Examples 1.1 and 1.2 that the forward problem is essentially taking the Laplace transform, and the inverse problem is essentially taking the inverse Laplace transform. See Section 3.1 for more details on this. As a consequence, the results in this paper can be interpreted as nonlinear analogs of the Laplace transform.
- In our main results, we allow the coefficients of the polynomial to be functions of x. We will see in Section 3.2 that the special case of $P(x,y) = -y - x^2y^2$ is closely related to Simon's approach [22] to the theorem of Borg [4] and Marčenko [18] that the principal m-function for a finite interval or half-line Schrödinger operator determines the potential.
- In our main results, we allow f to be vector valued, and also allow the coefficients to depend on $f(t,0)$. By doing this, we see in Section 9.1 that the translation invariant version of the anisotropic version of Calderón's inverse problem can be seen in this framework.

Thus, the results in this paper can be viewed as a family of inverse problems which generalize and unify the above examples, and for which we have good results on uniqueness, characterization of solutions, a reconstruction procedure, and stability estimates.

Furthermore, as argued in Section 9.2, a non-commutative analog[3] of these equations arise in the full anisotropic version of the Calderón problem. Thus, a special case of results in this paper can be seen as a simplified model case for the full Calderón problem. Moreover, by replacing functions in our results with pseudodifferential operators, one gives rise to an entire family of conjectures which generalize the Calderón problem.

1.2. **Selected notation.**

- All functions take their values in real vector spaces or spaces of real matrices. Other than in Section 8, there are no complex numbers in this paper.
- Let $\epsilon_1, \epsilon_2 > 0$. For $n_1, n_2 \in \mathbb{N}$, we write $b(t,w) \in C^{n_1,n_2}([0,\epsilon_1]\times[0,\epsilon_2])$ if for $0 \le j \le n_1$, $0 \le k \le n_2$, $\frac{\partial^j}{\partial t^j}\frac{\partial^k}{\partial w^k} b(t,w) \in C([0,\epsilon_1]\times[0,\epsilon_2])$. If $U \subseteq \mathbb{R}^m$ is open, and $n_3 \in \mathbb{N}$, we write $c(t,w,z) \in C^{n_1,n_2,n_3}([0,\epsilon_1]\times[0,\epsilon_2]\times U)$ if for $0 \le j \le n_1$, $0 \le k \le n_2$, and $0 \le |\alpha| \le n_3$, we have $\frac{\partial^j}{\partial t^j}\frac{\partial^k}{\partial w^k}\frac{\partial^\alpha}{\partial z^\alpha} c(t,w,z) \in C([0,\epsilon_1]\times[0,\epsilon_2]\times U)$. We define the norms
$$\|b\|_{C^{n_1,n_2}} := \sum_{j=0}^{n_1}\sum_{k=0}^{n_2} \sup_{t,w} \Big|\frac{\partial^j}{\partial t^j}\frac{\partial^k}{\partial w^k} b(t,w)\Big|,$$
$$\|c\|_{C^{n_1,n_2,n_3}} := \sum_{j=0}^{n_1}\sum_{k=0}^{n_2}\sum_{|\alpha|\le n_3} \sup_{t,w,z} \Big|\frac{\partial^j}{\partial t^j}\frac{\partial^k}{\partial w^k}\frac{\partial^\alpha}{\partial z^\alpha} c(t,w,z)\Big|.$$
- If $V \subseteq \mathbb{R}^n$ is open, and $U \subseteq \mathbb{R}^m$, we write $C^j(V;U)$ to be the usual space of C^j functions on V taking values in U. We use the norm
$$\|g\|_{C^j(V;U)} := \sum_{|\alpha|\le j} \sup_{z\in V} \big|\frac{\partial^\alpha}{\partial z^\alpha} g(z)\big|.$$
- We write $\mathbb{M}^{m\times n}$ to be the space of $m \times n$ real matrices. We write GL_m to be the space of $m\times m$ real, invertible matrices.
- For $a(w), b(w) \in C([0,\epsilon_2])$ we write
$$(a\tilde{*}b)(w) := \int_0^w a(w-r)b(r)\,dr \in C([0,\epsilon_2]). \tag{1.9}$$
Note that $\tilde{*}$ is commutative and associative.
- If $A(w) \in C([0,\epsilon_2];\mathbb{R}^m)$ and $\alpha = (\alpha_1,\dots,\alpha_m) \in \mathbb{N}^m$ is a multi-index, we write
$$\tilde{*}^\alpha A = \underbrace{A_1\tilde{*}\cdots\tilde{*}A_1}_{\alpha_1 \text{ terms}}\tilde{*}\cdots\tilde{*}\underbrace{A_j\tilde{*}\cdots\tilde{*}A_j}_{\alpha_j \text{ terms}}\tilde{*}\cdots\tilde{*}\underbrace{A_m\tilde{*}\cdots\tilde{*}A_m}_{\alpha_m \text{ terms}}.$$
and with a slight abuse of notation, if $|\alpha| = 0$ and $b(w)$ is another function, we write $b\tilde{*}(\tilde{*}^\alpha A) = b$.
- If $A(t,w)$ is a function of t and w, we write $\dot{A} = \frac{\partial}{\partial t}A$ and $A' = \frac{\partial}{\partial w}A$.
- For $\lambda_1,\dots,\lambda_m \in \mathbb{R}$, we write $\mathrm{diag}(\lambda_1,\dots,\lambda_m)$ to denote the $m\times m$ diagonal matrix with diagonal entries $\lambda_1,\dots,\lambda_m$.
- We write $A \lesssim B$ to mean $A \le CB$, where C depends only on certain parameters. It will always be clear from context what C depends on.
- We write $a \wedge b$ to mean $\min\{a,b\}$.

[3]Achieved by replacing functions with pseudodifferential operators: here the frequency plays the role that x^{-1} plays in our main results.

2. Statement of results

Fix $m \in \mathbb{N}$, $\epsilon_0, \epsilon_1, \epsilon_2 \in (0,\infty)$, $U \subseteq \mathbb{R}^m$ open, and $D \in \mathbb{N}$. For $j \in \{1,\dots,m\}$, $\alpha \in \mathbb{N}^m$ a multi-index with $|\alpha| \le D$, let

$$b_{\alpha,j}(t,w,z) \in C^{0,3,0}([0,\epsilon_1]\times[0,\epsilon_2]\times U),$$

with $b_{\alpha,j}(t,0,z)$, $\left(\frac{\partial}{\partial w} b_{\alpha,j}\right)(t,0,z) \in C^1([0,\epsilon_1]\times U)$. Define $c_{\alpha,j}(t,x,z) \in C([0,\epsilon_1]\times[0,\epsilon_0]\times U)$ by

$$c_{\alpha,j}(t,x,z) := \frac{1}{x}\int_0^{\epsilon_2} e^{-w/x} b_{\alpha,j}(t,w,z)\,dw.$$

We assume there is a $C_0 < \infty$ with

$$\|b_{\alpha,j}\|_{C^{0,3,0}([0,\epsilon_1]\times[0,\epsilon_2]\times U)}, \|b_{\alpha,j}\|_{C^1([0,\epsilon_1]\times U)}, \|b_{\alpha,j}\|_{C^1([0,\epsilon_1]\times U)} \le C_0.$$

Example 2.1. Because $\frac{1}{x}\int_0^{\epsilon_2} e^{-w/x}\frac{w^l}{l!}\,dw = x^l + e^{-\epsilon_2/x}G(x)$, with $G \in C([0,\infty))$, any polynomial in x can be written in the form covered by the $c_{\alpha,j}$, modulo error terms of the form $e^{-\epsilon_2/x}G(x)$, $G \in C([0,\infty))$. The results below are invariant under such error terms, so polynomials in x can be considered as a special case of the $c_{\alpha,j}$.

Define $P(t,x,y,z) := (P_1(t,x,y,z),\dots,P_m(t,x,y,z))$, where for $y \in \mathbb{R}^m$,

$$P_j(t,x,y,z) = \sum_{|\alpha|\le D} c_{\alpha,j}(t,x,z) y^\alpha.$$

Let $V \subseteq \mathbb{R}^m$ be an open set with $U \subseteq V$. Let $G(t,x,y,z) \in C([0,\epsilon_1]\times[0,\epsilon_0]\times V\times U;\mathbb{R}^m)$ be such that for every $\gamma \in (0,\epsilon_2)$, $G(t,x,y,z) = e^{-\gamma/x}G_\gamma(t,x,y,z)$, where $G_\gamma(t,x,y,z) \in C([0,\epsilon_1]\times[0,\epsilon_0]\times V\times U;\mathbb{R}^m)$ satisfies for any compact sets $K_1 \Subset U$, $K_2 \Subset V$,

$$\sup_{\substack{t\in[0,\epsilon_1], x\in[0,\epsilon_0], z\in K_1\\ y_1,y_2\in K_2, y_1\ne y_2}} \frac{|G_\gamma(t,x,y_1,z) - G_\gamma(t,x,y_2,z)|}{|y_1-y_2|} < \infty.$$

We will be considering the differential equation, defined for $f(t,x) \in C([0,\epsilon_1]\times[0,\epsilon_0];V)$ with $f(t,0) \in C([0,\epsilon_1];U)$,

$$\begin{aligned}\frac{\partial}{\partial t} f(t,x) = &\frac{P(t,x,f(t,x),f(t,0)) - P(t,0,f(t,0),f(t,0))}{x}\\ &+ G(t,x,f(t,x),f(t,0)), \quad x>0.\end{aligned} \tag{2.1}$$

Corresponding to $P(t,x,y,z)$, for $\delta \in (0,\epsilon_2]$ and $A \in C^1([0,\delta];\mathbb{R}^m)$, we define

$$\widehat{P}(t,A(\cdot),z)(w) = \Big(\widehat{P}_1(t,A(\cdot),z)(w),\dots,\widehat{P}_m(t,A(\cdot),z)(w)\Big)$$

by

$$\widehat{P}_j(t,A(\cdot),z)(w) = \sum_{|\alpha|\le D} \frac{\partial^{|\alpha|+1}}{\partial w^{|\alpha|+1}} \left(b_{\alpha,j}(t,\cdot,z)\tilde{*}(\tilde{*}^\alpha A)\right)(w).$$

2.1. **Existence without uniqueness.**

Theorem 2.2. *Suppose $f_0(x) \in C([0,\epsilon_0];V)$ and $A_0(t) \in C^2([0,\epsilon_1];U)$ are given, with $f_0(0) = A_0(0)$. Set $\mathcal{M}(t) := -d_y P(t, 0, A_0(t), A_0(t))$.*[4] *We suppose that there exists $R(t) \in C^1([0,\epsilon_1]; \mathrm{GL}_m)$ with*

$$R(t)\mathcal{M}(t)R(t)^{-1} = \operatorname{diag}(\lambda_1(t), \dots, \lambda_m(t)),$$

where $\lambda_j(t) > 0$ for all j, t. Then, there exists $\delta_0 > 0$ and a unique solution $f(t,x) \in C([0,\epsilon_1] \times [0,\delta_0]; \mathbb{R}^m)$ to (2.1), satisfying $f(0,x) = f_0(x)$ and $f(t,0) = A_0(t)$.

Remark 2.3. As in the introduction, we call this existence without uniqueness because one has to specify both $f(0,x)$ and $f(t,0)$ (as opposed to just $f(0,x)$).

Beyond proving existence, we can show that the solution given in Theorem 2.2 is of Laplace transform type, modulo an appropriate error, as shown in the next theorem.

Theorem 2.4. *Assume the same assumptions as in Theorem 2.2, and let $f(t,x)$ be the unique solution guaranteed by Theorem 2.2. Take $c_0, C_1, C_2, C_3, C_4 > 0$ such that $\min_{t,j} \lambda_j(t) \geq c_0 > 0$, $\|R\|_{C^1} \leq C_1$, $\|R^{-1}\|_{C^1} \leq C_2$, $\|\mathcal{M}^{-1}\|_{C^1} \leq C_3$, $\|A_0\|_{C^2} \leq C_4$. Then, there exists $\delta = \delta(m, D, c_0, C_0, C_1, C_2, C_3, C_4) > 0$ and $A(t,w) \in C^{0,2}([0,\epsilon_1] \times [0, \delta \wedge \epsilon_2]; \mathbb{R}^m)$ such that*

$$\frac{\partial}{\partial t} A(t,w) = \widehat{P}(t, A(t,\cdot), A(t,0))(w), \quad A(t,0) = A_0(t),$$

and such that if $\lambda_0(t) = \min_j \lambda_j(t)$, then for all $\gamma \in [0,1)$,

$$f(t,x) = \frac{1}{x} \int_0^{\delta \wedge \epsilon_2} e^{-w/x} A(t,w)\, dw + O\big(e^{-\gamma(\delta\wedge\epsilon_2)/x} + e^{-(\gamma/x)\int_0^t \lambda_0(s)\, ds}\big), \tag{2.2}$$

for $x \in (0,\delta_0]$, where the implicit constant in the O in (2.2) does not depend on $(t,x) \in [0,\epsilon_1] \times (0,\delta_0]$. Furthermore, the representation (2.2) is unique in the following sense. Fix $t_0 \in [0,\epsilon_1]$. Suppose there exists $0 < \delta' < \delta \wedge \epsilon_2 \wedge \big(\int_0^{t_0} \lambda_0(s)\, ds\big)$ and $B \in C([0,\delta'];\mathbb{R}^m)$ with

$$f(t_0,x) = \frac{1}{x}\int_0^{\delta'} e^{-w/x} B(w)\, dw + O\big(e^{-\delta'/x}\big), \quad \text{as } x \downarrow 0.$$

Then, $A(t_0,w) = B(w)$, for all $w \in [0,\delta']$.

2.2. **Uniqueness without Existence.**

In addition to the above assumptions, for the next result we assume for every compact set $K \Subset U$,

$$\begin{gathered}
\sup_{\substack{t\in[0,\epsilon_1], w\in[0,\epsilon_2]\\ z_1,z_2\in K, z_1\neq z_2}} \frac{|b_{\alpha,j}(t,w,z_1) - b_{\alpha,j}(t,w,z_2)|}{|z_1 - z_2|} < \infty,\\
\sup_{\substack{t\in[0,\epsilon_1], w\in[0,\epsilon_2]\\ z_1,z_2\in K, z_1\neq z_2}} \frac{|\frac{\partial}{\partial w} b_{\alpha,j}(t,w,z_1) - \frac{\partial}{\partial w} b_{\alpha,j}(t,w,z_2)|}{|z_1 - z_2|} < \infty.
\end{gathered} \tag{2.3}$$

[4] Notice the minus sign in the definition of $\mathcal{M}(t)$. This is in contrast to the notation in the introduction, which lacked the minus sign.

Theorem 2.5. *Suppose* $f_1(t,x), f_2(t,x) \in C([0,\epsilon_1]\times[0,\epsilon_0];V)$ *satisfy* $f_j(t,0) \in C^2([0,\epsilon_1];U)$*, both satisfy* (2.1)*, and* $f_1(0,x) = f_2(0,x)$*, for all* $x \in [0,\epsilon_0]$*. Set* $\mathcal{M}_k(t) := d_yP(t,0,f_k(t,0),f_k(t,0))$*. We suppose that there exists* $R_k(t) \in C^1([0,\epsilon_1];\mathrm{GL}_m)$ *with*

$$R_k(t)\mathcal{M}_k(t)R_k(t)^{-1} = \mathrm{diag}(\lambda_1^k(t),\dots,\lambda_m^k(t)),$$

where $\lambda_j^k(t) > 0$*, for all* $j \in \{1,\dots,m\}$*,* $t \in [0,\epsilon_1]$*. Then* $f_1(t,x) = f_2(t,x)$*, for all* $t \in [0,\epsilon_1]$*,* $x \in [0,\epsilon_0]$*.*

Theorem 2.5 shows uniqueness, but we will show more. We will further investigate the following questions:

- Stability: If $f_1(0,x) - f_2(0,x)$ vanishes sufficiently quickly at 0, and under the hypotheses of Theorem 2.5, we will prove that $f_1(t,0)$ and $f_2(t,0)$ agree for small t, and we will make this quantitative. See Theorem 2.10.
- Reconstruction: Given the initial condition $f(0,x)$ for (2.1), and under the hypotheses of Theorem 2.5, we will show how to reconstruct the solution $f(t,x)$, for all t. This is an unstable process, but we will reduce the instability to that of inverting the Laplace transform, which is well understood. See Remark 2.9.
- Characterization: We will show that if $f(t,x)$ is a solution to (2.1), and under the hypotheses of Theorem 2.5, then $f(t,x)$ must be of Laplace transform type, modulo an appropriate error term. In particular, only initial conditions $f(0,x)$ which are of Laplace transform type modulo an appropriate error give rise to solutions. See Theorem 2.6 and Remark 2.7.

We now turn to making these ideas more precise.

2.2.1. *Stability, Reconstruction, and characterization.* For our first result, we take P as in the start of this section, but we drop the assumption (2.3).

Theorem 2.6 (Charaterization)**.** *Suppose* $f(t,x) \in C([0,\epsilon_1]\times[0,\epsilon_0];\mathbb{R}^m)$ *is such that for all* $\gamma \in [0,\epsilon_2)$*,*

$$\frac{\partial}{\partial t}f(t,x) = \frac{P(t,x,f(t,x),f(t,0)) - P(t,0,f(t,0),f(t,0))}{x} + O(e^{-\gamma/x}), \quad x \in [0,\epsilon_0),$$

where the implicit constant in O *is independent of* t,x*. We suppose*

- $f(t,0) \in C^2([0,\epsilon_1];U)$*.*
- *Set* $\mathcal{M}(t) := d_yP(t,0,f(t,0),f(t,0))$*. We suppose there exists* $R(t) \in C^1([0,\epsilon_1];\mathrm{GL}_m)$ *with*

$$R(t)\mathcal{M}(t)R(t)^{-1} = \mathrm{diag}(\lambda_1(t),\dots,\lambda_m(t)),$$

where $\lambda_j(t) > 0$*, for all* j,t*.*

Take $c_0, C_1, C_2, C_3, C_4 > 0$ *such that* $\min_{t,j}\lambda_j(t) \ge c_0 > 0$*,* $\|R\|_{C^1} \le C_1$*,* $\|R^{-1}\|_{C^1} \le C_2$*,* $\|\mathcal{M}^{-1}\|_{C^1} \le C_3$*,* $\|f(\cdot,0)\|_{C^2} \le C_4$*. Then, there exist* $\delta = \delta(m,D,c_0,C_0,C_1,C_2,C_3,C_4) > 0$ *and* $A(t,w) \in C^{0,2}([0,\epsilon_1]\times[0,\delta\wedge\epsilon_2];\mathbb{R}^m)$ *such that*

$$\frac{\partial}{\partial t}A(t,w) = \widehat{P}(t,A(t,\cdot),A(t,0)), \quad A(t,0) = f(t,0), \tag{2.4}$$

and such that if $\lambda_0(t) = \min_j \lambda_j(t)$*, then for all* $\gamma \in (0,1)$*,*

$$f(t,x) = \frac{1}{x}\int_0^{\delta\wedge\epsilon_2} e^{-w/x}A(t,w)\,dw + O\Big(e^{-\gamma(\delta\wedge\epsilon_2)/x} + e^{-(\gamma/x)\int_0^{\epsilon_1-t}\lambda_0(s)\,ds}\Big), \tag{2.5}$$

where the implicit constant in O is independent of t, x. Furthermore, the representation in (2.5) *of $f(t,x)$ is unique in the following sense. Fix $t_0 \in [0,\epsilon_0]$. Suppose there exists $0 < \delta' < \delta \wedge \epsilon_2 \wedge \int_0^{\epsilon_1 - t_0} \lambda(s)\, ds$ and $B \in C([0,\delta'];\mathbb{R}^m)$ with*

$$f(t_0,x) = \frac{1}{x}\int_0^{\delta'} e^{-w/x} B(w)\, dw + O\big(e^{-\delta'/x}\big), \quad \textit{as } x \downarrow 0.$$

Then, $A(t_0,w) = B(w)$, for all $w \in [0,\delta']$.

Remark 2.7. By taking $t = 0$ in (2.5), we see that $f(0,x)$ is of Laplace transform type, modulo an error: for all $\gamma \in (0,1)$,

$$f(0,x) = \frac{1}{x}\int_0^{\delta\wedge\epsilon_2} e^{-w/x} A(0,w)\, dw + O\Big(e^{-\gamma(\delta\wedge\epsilon_2)/x} + e^{-(\gamma/x)\int_0^{\epsilon_1}\lambda_0(s)\, ds}\Big).$$

Thus, under the hypotheses of Theorem 2.6, the only initial conditions that give rise to a solution are of Laplace transform type, modulo an appropriate error. Furthermore, by taking $t_0 = 0$ in the last conclusion of Theorem 2.6, we see that $f(0,x)$ uniquely determines $A(0,w)$.

For the remainder of the results in this section, we assume (2.3).

Proposition 2.8. *The differential equation* (2.4) *has uniqueness in the following sense. Let $\delta' > 0$ and $A(t,w), B(t,w) \in C^{0,2}([0,\epsilon_1]\times[0,\delta'];\mathbb{R}^m)$ satisfy*

$$\frac{\partial}{\partial t}A(t,w) = \widehat{P}(t,A(t,\cdot),A(t,0))(w), \quad \frac{\partial}{\partial t}B(t,w) = \widehat{P}(t,B(t,\cdot),B(t,0))(w), \tag{2.6}$$

and $A(0,w) = B(0,w)$ for $w \in [0,\delta']$. Set $A_0(t) = A(t,0)$, and suppose $A_0(t) \in C^2([0,\epsilon_2];\mathbb{R}^m)$ and set $\mathcal{M}(t) = d_y P(t,0,A_0(t),A_0(t))$. Suppose there exists $R(t) \in C^1([0,\epsilon_1];\mathrm{GL}_m)$ with

$$R(t)\mathcal{M}(t)R(t)^{-1} = \mathrm{diag}(\lambda_1(t),\dots,\lambda_m(t)),$$

where $\lambda_j(t) > 0$ for all j,t. Set $\gamma_0(t) := \max_j \int_0^t \lambda_j(s)\, ds$, and

$$\delta_0 := \begin{cases} \gamma_0^{-1}(\delta'), & \textit{if } \gamma_0(\epsilon_1) \ge \delta', \\ \epsilon_1, & \textit{else.} \end{cases}$$

Then, $A(t,0) = B(t,0)$ for $t \in [0,\delta_0]$.

Remark 2.9 (Reconstruction)**.** Proposition 2.8 leads us to the reconstruction procedure, which is as follows:

(i) Given a solution $f(t,x)$ to (2.1), satisfying the assumptions of Theorem 2.5, we use Theorem 2.6 to see that $f(t,x)$ can be written in the form (2.5). In particular, as discussed in Remark 2.7, $f(0,x)$ uniquely determines $A(0,w)$. Extracting $A(0,w)$ from $f(0,x)$ involves taking an inverse Laplace transform, and this step therefore inherits any instability inherent in the inverse Laplace transform.

(ii) With $A(0,w)$ in hand, and with the knowledge that $A(t,w)$ satisfies (2.4), Proposition 2.8 shows that $A(0,w)$ uniquely determines $A(t,0) = f(t,0)$ for $0 \le t \le \delta'$, for some δ'.

(iii) With $f(t,0)$ in hand, for $x > 0$ (2.1) is a standard ODE, and so uniquely determines $f(t,x)$ for $0 \le t \le \delta'$.

(iv) Iterating his procedure gives $f(t,x)$, for all t.

The above procedure reduces the reconstruction of $f(t,x)$ from $f(0,x)$ to the reconstruction of $A(t,w)$ from $A(0,w)$. As we will see in the proof of Proposition 2.8, the differential equation satisfied by A is much more stable than that satisfied by f. In particular, we will be able to prove Proposition 2.8 by a straightforward application of Grönwall's inequality.

Theorem 2.10 (Stability). *Suppose $f_1(t,x), f_2(t,x) \in C([0,\epsilon_1]\times[0,\epsilon_0];\mathbb{R}^m)$ satisfy, for $k=1,2$, for all $\gamma\in(0,\epsilon_2)$,*

$$\frac{\partial}{\partial t} f_k(t,x) = \frac{P(t,x,f_k(t,x),f_k(t,0)) - P(t,0,f_k(t,0),f_k(t,0))}{x} + O\big(e^{-\gamma/x}\big),$$

for $x\in(0,\epsilon_0]$, where the implicit constant in O may depend on γ, but not on t or x. Suppose, further, for some $r>0$ and all $s\in[0,r)$,

$$f_1(0,x) = f_2(0,x) + O\big(e^{-s/x}\big). \tag{2.7}$$

We assume the following for $k=1,2$:

- *$f_k(t,0)\in C^2([0,\epsilon_1];U)$.*
- *Set $\mathcal{M}_k(t) := d_y P(t,0,f_k(t,0),f_k(t,0))$. We suppose that there exists $R_k(t)$ in $C^1([0,\epsilon_1];\mathrm{GL}_m)$ with*
$$R_k(t)\mathcal{M}_k(t)R_k(t)^{-1} = \operatorname{diag}(\lambda_1^k(t),\dots,\lambda_m^k(t)),$$
where $\lambda_j^k(t)>0$ for all j,t.

Take $c_0, C_1, C_2, C_3, C_4 > 0$ such that for $k=1,2$, $\min_{t,j}\lambda_j^k(t)\ge c_0>0$, $\|R_k\|_{C^1}\le C_1$, $\|R_k^{-1}\|_{C^1}\le C_2$, $\|\mathcal{M}_k^{-1}\|_{C^1}\le C_3$, $\|f_k(\cdot,0)\|_{C^2}\le C_4$. Set

$$\gamma_0(t) := \max_j \int_0^t \lambda_j^1(s)\,ds, \quad \lambda_0^k(t) = \min_j \lambda_j^k(t).$$

Then there exists $\delta=\delta(m,D,c_0,C_0,C_1,C_2,C_3,C_4)>0$ such that the following holds. Define

$$\delta' = \delta\wedge\epsilon_2\wedge\int_0^{\epsilon_1}\lambda_0^1(s)\,ds\wedge\int_0^{\epsilon_1}\lambda_0^2(s)\,ds>0,$$

and set

$$\delta_0 := \begin{cases}\gamma_0^{-1}(r\wedge\delta'), & \text{if } \gamma_0(\epsilon_1)\ge r\wedge\delta',\\ \epsilon_1, & \text{otherwise.}\end{cases}$$

Then, $f_1(t,0)=f_2(t,0)$ for $t\in[0,\delta_0]$.

3. Forward problems, inverse problems, and past work

The results in this paper can be seen as studying a class of nonlinear forward and inverse problems. Indeed, suppose we have the same setup as described at the start of Section 2.

Forward Problem. Given $f_0(x)\in C([0,\epsilon_1];V)$ and $A_0(t)\in C^2([0,\epsilon_1];U)$ with $f_0(0)=A_0(0)$. Let $\mathcal{M}(t)$ be as in Theorem 2.2. Suppose there exists $R(t)\in C^1([0,\epsilon_1];\mathrm{GL}_m)$ with $R(t)\mathcal{M}(t)R(t)^{-1}=\operatorname{diag}(\lambda_1(t),\dots,\lambda_m(t))$, and $\lambda_j(t)>0$, for all t. Let $f(t,x)$ be the solution to (2.1) described in Theorem 2.2, with $f(0,x)=f_0(x)$, $f(t,0)=A_0(t)$. The forward problem is the map:

$$(f_0,A_0)\mapsto f(\epsilon_1,\cdot).$$

Inverse Problem: The inverse problem is, given $f(\epsilon_1,\cdot)$ as described above, find f_0 and A_0. Note that if $f(t,x)$ is the function described above, $\tilde{f}(t,x)=f(\epsilon_1-t,x)$ satisfies all the hypotheses of Theorem 2.5 (here we assume (2.3)). We have the following:

- The map $(f_0,A_0)\mapsto f(\epsilon_1,\cdot)$ is injective–Theorem 2.5.
- The map $(f_0,A_0)\mapsto f(\epsilon_1,\cdot)$ is not surjective. In fact, the only functions in the image of are Laplace transform type, modulo an appropriate error term–Theorem 2.4.
- The inverse map $f(\epsilon_1,\cdot)\mapsto(f_0,A_0)$ is unstable, but we do have some stability results. Indeed, if one only knows $f(\epsilon_1,x)$ up to error terms of the form $O(e^{-r/x})$, then $f(\epsilon_1,\cdot)$ determines $A_0(t)$ for $t\in[\delta_0-\epsilon_1,\epsilon_1]$, where δ_0 is described in Theorem 2.10.
- We have a procedure to reconstruct $A_0(t)$ and $f_0(x)$ from $f(\epsilon_1,x)$–Remark 2.9.

The above class of inverse problems has, as special cases, some already well understood inverse problems. We next describe two of these. For these problems, we reverse time in the above discussion since we are focusing on the inverse problem. In addition, the results in this paper are related to the famous Calderón problem, and we describe this connection in Section 9.

3.1. **Laplace Transform.** As see in Examples 1.1 and 1.2 the Laplace transform is closely related to the case $P(t,x,y,z)=y$ studied in this paper. In fact, the following proposition makes this even more explicit. For $a\in L^\infty([0,\infty))$ define the Laplace transform:

$$\mathcal{L}(a)(x)=\frac{1}{x}\int_0^\infty e^{-w/x}a(w)\,dw.$$

Proposition 3.1. *Let $a\in C([0,\infty))\cap L^\infty([0,\infty))$. For each $x>0$ there is a unique solution to the differential equation*

$$\frac{\partial}{\partial t}f(t,x)=\frac{f(t,x)-a(t)}{x}, \tag{3.1}$$

such that $\sup_{t\ge 0}|f(t,x)|<\infty$. For $t_0,t\ge 0$ define $a_{t_0}(t)=a(t_0+t)$. This solution $f(t,x)$ is given by $f(t,x)=\mathcal{L}(a_t)(x)$. Furthermore, $f(t,x)$ extends to a continuous function $f\in C([0,\infty)\times[0,\infty))$ by setting $f(t,0)=a(t)$.

Proof. If we set

$$f(t,x)=\mathcal{L}(a_t)(x)=\frac{1}{x}\int_0^\infty e^{-s/x}a(t+s)\,ds=\frac{1}{x}\int_t^\infty e^{(t-s)/x}a(s)\,ds,$$

then it is clear that f satisfies (3.1), $\sup_{t\ge 0}|f(t,x)|<\infty$, and that f extends to a continuous function $f\in C([0,\infty)\times[0,\infty))$ by setting $f(t,0)=a(t)$.

Suppose $g(t,x)$ is another solution to (3.1) such that $\sup_{t\ge 0}|g(t,x)|<\infty$. Let $h=f-g$. Then $h(t,x)$ satisfies $\frac{\partial}{\partial t}h(t,x)=h(t,x)/x$, $\sup_{t\ge 0}|h(t,x)|<\infty$. This implies that $h(t,x)=e^{t/x}h(0,x)$ and we conclude $h(0,x)=0=h(t,x)$, for all t. Thus $f(t,x)=g(t,x)$, proving uniqueness. $\square$

In light of Proposition 3.1 one may define $\mathcal{L}(a)$ (at least for $a\in C([0,\infty))\cap L^\infty([0,\infty))$) in another way: there is a unique $f(t,x)\in C([0,\infty)\times[0,\infty))$ with

$\sup_{t\geq 0} |f(t,x)| < \infty$ and satisfying

$$\frac{\partial}{\partial t} f(t,x) = \frac{f(t,x) - f(t,0)}{x}, \quad f(t,0) = a(t).$$

$\mathcal{L}(a)(x)$ is then defined to be $\mathcal{L}(a)(x) = f(0,x)$. Thus, the well known fact that $a \mapsto \mathcal{L}(a)$ is injective follows from uniqueness for the differential equation

$$\frac{\partial}{\partial t} f(t,x) = \frac{f(t,x) - f(t,0)}{x}.$$

Example 3.2. The above discussion leads naturally to the following "nonlinear inverse Laplace transform". Indeed, let $P(y)$ be a polynomial in $y \in \mathbb{R}$. Let $f_1(t,x), f_2(t,x) \in C([0,\epsilon_1] \times [0,\epsilon_0])$ satisfy, for $j = 1,2$,

$$\frac{\partial}{\partial t} f_j(t,x) = \frac{P(f_j(t,x)) - P(f_j(t,0))}{x}, \quad x \in (0,\epsilon_0].$$

Suppose:

- $f_1(0,x) = f_2(0,x)$, for all $x \in [0,\epsilon_0]$.
- $f_j(t,0) \in C^2([0,\epsilon_1])$, $j = 1,2$.
- $P'(f_j(t,0)) > 0$, for $t \in [0,\epsilon_1]$, $j = 1,2$.

Then, by Theorem 2.5, $f_1(t,x) = f_2(t,x)$ for $(t,x) \in [0,\epsilon_1] \times [0,\epsilon_0]$. When $P(y) = y$, this amounts to the inverse Laplace transform as discussed above.

3.2. **Inverse spectral theory.** In this section, we describe the results due to Simon in the influential work [22], where he gave a new approach to the theorem of Borg-Marčenko that the principal m-function for a finite interval or half-line Schrödinger operator determines the potential. As we will show, this is closely related to the special case $P(t,x,y,z) = x^2y^2 + y$ of the results studied in this paper. We will contrast our theorems and methods with those of Simon.

Let $q \in L^1_{\mathrm{loc}}([0,\infty))$ with $\sup_{y>0} \int_y^{y+1} q(t) \vee 0 \, dt < \infty$, and consider the Schrödinger operator $-\frac{d^2}{dt^2} + q(t)$. For each $z \in \mathbb{C} \setminus [\beta,\infty)$ (with $-\beta$ sufficiently large), there is a unique solution (up to multiplication by a constant) $u(\cdot,z) \in L^2([0,\infty))$ of $-\ddot{u} + qu = zu$. The principal m-function is defined by

$$m(t,z) = \frac{\dot{u}(t,z)}{u(t,z)}.$$

It is a theorem of Borg [4] and Marčenko [18] that $m(0,z)$ uniquely determines q–Simon [22] saw this as an instance of uniqueness for a generalized differential equation, which we now explain in the framework of this paper.

Indeed, it is easy to see that m satisfies the Riccati equation

$$\dot{m}(t,z) = q(t) - z - m(t,z)^2, \tag{3.2}$$

and well-known that m has the asymptotics $m(t,-\kappa^2) = -\kappa - \frac{q(t)}{\kappa} + o(\kappa^{-1})$, as $\kappa \uparrow \infty$. Thus, $q(t)$ can be obtained from $m(t,\cdot)$ and (3.2) is a differential equation involving only m. Thus, if the equation (3.2) has uniqueness, then $m(0,z)$ uniquely determines $q(t)$.

However, one does not need to full power of uniqueness for (3.2). In fact, one needs only know uniqueness under the additional assumption that $m(t,z)$ is a principal m-function: i.e., if $m_1(t,z)$ and $m_2(t,z)$ both satisfy (3.2) with $m_1(0,\cdot) = m_2(0,\cdot)$ and are both principal m-functions, then $m_1(t,z) = m_2(t,z)$, for all t,z. Simon proceeds via this weaker statement.

At this point, we rephrase these ideas into the language used in this paper. For $x \geq 0$, $y \in \mathbb{R}$ define $P(x,y) = x^2y^2 + y$. Note that P is of the form covered in this paper (Example 2.1) and $d_y P(0,y) = 1$. Given a principal m-function as above, define for $x \geq 0$ small,

$$f(t,x) := \begin{cases} -\frac{1}{x}\left(m(t, -(2x)^{-2}) + (2x)^{-1}\right), & \text{if } x > 0, \\ q(t), & \text{if } x = 0. \end{cases} \tag{3.3}$$

It is easy to see from the above discussion that f satisfies

$$\frac{\partial}{\partial t} f(t,x) = \frac{P(x, f(t,x)) - P(0, f(t,0))}{x}, \quad x > 0. \tag{3.4}$$

Furthermore, if q is continuous then f is continuous as well. Thus to show $m(t,z)$ uniquely determines $q(t)$ it suffices to show that (3.4) has uniqueness.

In this context, our results and the results of [22] are closely related but have a few differences:

- As discussed above, [22] only considers solutions to (3.2) which are principal m-functions. This forces $f(t,\cdot)$ in (3.3) to be exactly of Laplace transform type, for all t. As we have seen, not all solutions to (3.4) are exactly of Laplace transform type. In this way, our results are stronger than [22] in that we prove uniqueness when the initial condition is not necessarily of Laplace transform type–we do not even require any sort of analyticity.[5]
- We require $q \in C^2$, while [22] requires no additional regularity on q.
- The constant δ in Theorems 2.6 and 2.10 is taken to be ∞ in [22].
- Our results work for much more general polynomials than P.

The reason for the differences above is that, once m is assumed to be a principal m-function, one is able to use many theorems regarding Schrödinger equations to deduce the stronger results, which we did not obtain in our more general setting.

That we assumed $q \in C^2$ is likely not essential. For the specific case discussed in this section, our methods do yield results for q with lower regularity than C^2, though we chose to not pursue this. Moreover, even for the more general setting of our main results, it seems likely that a more detailed study of the partial differential equations which arise in this paper would lead to lower regularity requirements, though this would require some new ideas. That δ is assumed small in Theorems 2.6 and 2.10 seems much more essential–this has to do with the fact that the equations studied in this paper are non-linear in nature, unlike the results in [22] which rested on the underlying linear theory of the Schrödinger equation.

Remark 3.3. Many works followed [22], some of which dealt with m taking values in square matrices; e.g., [8]. All of the above discussion applies to these cases as well.

4. Convolution

In this section, we record several results on the commutative and associative operation $\tilde{*}$ defined in (1.9). In Section 4.2 we distill the consequences of these results into the form in which they will be used in the rest of the paper–and the reader may wish to skip straight to those results on a first reading. For this section, fix some $\epsilon > 0$.

[5] We learn a posteriori, in Theorem 2.6, that the initial condition must be of Laplace transform type modulo an error, but this is not assumed.

Lemma 4.1. *Let $a \in C([0,\epsilon])$, $b \in C^1([0,\epsilon])$. Then $\frac{\partial}{\partial w}(a\tilde{*}b)(w) = a(w)b(0) + (a\tilde{*}b')(w)$. In particular, if $b(0)=0$, then $\frac{\partial}{\partial w}(a\tilde{*}b)(w) = (a\tilde{*}b')(w)$.*

The proof of the above lemma is immediate from the definitions.

Lemma 4.2. *Let $l \geq -1$ and let $a \in C([0,\epsilon])$, $b \in C^{l+1}([0,\epsilon])$. Suppose for $0 \leq j \leq l-1$, $\frac{\partial^j}{\partial w^j} b(0) = 0$. Then $a\tilde{*}b \in C^{l+1}([0,\epsilon])$ and for $0 \leq j \leq l$, $\frac{\partial^j}{\partial w^j}(a\tilde{*}b)(0) = 0$. Furthermore, if $a \in C^1([0,\epsilon])$, then $a\tilde{*}b \in C^{l+2}([0,\epsilon])$.*

Proof. By repeated applications of Lemma 4.1, for $0 \leq j \leq l$, $\frac{\partial^j}{\partial w^j}(a\tilde{*}b) = a\tilde{*}\frac{\partial^j}{\partial w^j}b$, and this expression clearly vanishes at 0. Applying Lemma 4.1 again, we see $\frac{\partial^{l+1}}{\partial w^{l+1}}(a\tilde{*}b) = \frac{\partial}{\partial w}(a\tilde{*}\frac{\partial^l}{\partial w^l}b) = a(w)\frac{\partial^l b}{\partial w^l}(0) + (a\tilde{*}\frac{\partial^{l+1}}{\partial w^{l+1}}b)$. This expression is continuous, so $a\tilde{*}b \in C^{l+1}$. Furthermore, if $a \in C^1$, it follows from one more application of Lemma 4.1 that $\frac{\partial^{l+2}}{\partial w^{l+2}}(a\tilde{*}b) = \frac{\partial}{\partial w}\Big(a(w)\frac{\partial^l b}{\partial w^l}(0) + (a\tilde{*}\frac{\partial^{l+1}}{\partial w^{l+1}}b)\Big)$ is continuous, and therefore $a\tilde{*}b \in C^{l+2}$. □

For the next few results, suppose $a_1,\dots,a_L \in C^1([0,\epsilon])$ are given. For $J = \{j_1,\dots,j_k\} \subseteq \{1,\dots,L\}$, we define

$$\tilde{*}_{j\in J}a = a_{j_1}\tilde{*}\cdots\tilde{*}a_{j_k}.$$

With an abuse of notation, for $b \in C([0,\epsilon])$, we define $b\tilde{*}\big(\tilde{*}_{j\in\emptyset}a\big) = b$.

Lemma 4.3. *For each $n \in \{1,\dots,L\}$, $a_1\tilde{*}\cdots\tilde{*}a_n \in C^n([0,\epsilon])$ and if $0 \leq j \leq n-2$, $\frac{\partial^j}{\partial w^j}\left(a_1\tilde{*}\cdots\tilde{*}a_n\right)(0) = 0$.*

Proof. For $n=1$, the result is trivial. We prove the result by induction on n, the base case being $n=2$ which follows from Lemma 4.1. We assume the result for $n-1$ and prove it for n. By the inductive hypothesis, $a_1\tilde{*}\cdots\tilde{*}a_{n-1} \in C^{n-1}$ and vanishes to order $n-3$ at 0. From here, the result follows from Lemma 4.2 with $l = n-2$. □

Define

$$I_L(a_1,\dots,a_L) := \sum_{J\subsetneq\{1,\dots,L\}} \Big(\prod_{j\in J} a_j(0)\Big)\Big(\tilde{*}_{k\in J^c}a_k'\Big),$$

and let $I_0 = 0$.

Lemma 4.4.

$$\frac{\partial^{L-1}}{\partial w^{L-1}}(a_1\tilde{*}\cdots\tilde{*}a_L) = \Big(\prod_{j=1}^{L-1} a_j(0)\Big)a_L + I_{L-1}(a_1,\dots,a_{L-1})\tilde{*}a_L\,, \tag{4.1}$$

$$\frac{\partial^L}{\partial w^L}(a_1\tilde{*}\cdots\tilde{*}a_L) = I_L(a_1,\dots,a_L). \tag{4.2}$$

Proof. We prove the result by induction on by induction on L. The base case, $L=1$, is trivial. We assume (4.1) and (4.2) for $L-1$ and prove them for L. We have, using repeated applications of Lemmas 4.1 and 4.3,

$$\begin{aligned}
&\frac{\partial^{L-1}}{\partial w^{L-1}}(a_1\tilde{*}\cdots\tilde{*}a_L) = \frac{\partial}{\partial w}\Big(\Big(\frac{\partial^{L-2}}{\partial w^{L-2}}(a_1\tilde{*}\cdots\tilde{*}a_{L-1}\Big)\tilde{*}a_L\Big) \\
&= \Big(\frac{\partial^{L-2}}{\partial w^{L-2}}(a_1\tilde{*}\cdots\tilde{*}a_{L-1})\Big)(0)a_L + \Big(\frac{\partial^{L-1}}{\partial w^{L-1}}(a_1\tilde{*}\cdots\tilde{*}a_{L-1}\Big)\tilde{*}a_L
\end{aligned}$$

Using our inductive hypothesis for (4.1) and the fact that $(b\tilde{*}c)(0) = 0$ for any b, c,

$$\Big(\frac{\partial^{L-2}}{\partial w^{L-2}}(a_1\tilde{*}\cdots\tilde{*}a_{L-1})\Big)(0)a_L = \Big[\prod_{j=1}^{L-1} a_j(0)\Big]a_L,$$

and using our inductive hypothesis for (4.2),

$$\Big(\frac{\partial^{L-1}}{\partial w^{L-1}}(a_1\tilde{*}\cdots\tilde{*}a_{L-1})\Big)\tilde{*}a_L = I_{L-1}(a_1,\dots,a_{L-1})\tilde{*}a_L.$$

Combining the above equations yields (4.1). Taking $\frac{\partial}{\partial w}$ of (4.1) and applying Lemma 4.1, (4.2) follows, completing the proof. $\square$

Corollary 4.5. *Let $A \in C^1([0,\epsilon];\mathbb{R}^m)$, $b \in C^1([0,\epsilon])$. Then, for a multi-index $\alpha \in \mathbb{N}^m$,*

$$\begin{aligned}
&\frac{\partial^{|\alpha|+1}}{\partial w^{|\alpha|+1}}\,(b\tilde{*}(\tilde{*}^{\alpha}A)) \\
&= \sum_{\substack{\beta\leq\alpha\\|\beta|<|\alpha|}} \binom{\alpha}{\beta} b(0)A(0)^{\beta}\big(\tilde{*}^{\alpha-\beta}A'\big) + \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}A(0)^{\beta}\Big(b'\tilde{*}\big(\tilde{*}^{\alpha-\beta}A'\big)\Big).
\end{aligned}$$

The above corollary follows immediately from Lemma 4.4.

Lemma 4.6. *Let $b_1,\dots,b_L,c_1,\dots,c_L \in C([0,\epsilon])$. Then,*

$$b_1\tilde{*}\cdots\tilde{*}b_L - c_1\tilde{*}\cdots\tilde{*}c_L = \sum_{\emptyset\neq J\subseteq\{1,\dots,L\}} (-1)^{|J|+1}\left(\tilde{*}_{l\in J}(b_l-c_l)\right)\tilde{*}\left(\tilde{*}_{l\notin J}b_l\right)$$

The proof of the above lemma is standard, uses only the multilinearity of $\tilde{*}$, and can be proved using a simple induction.

Lemma 4.7. *Suppose $a_1,\dots,a_L \in C^2([0,\epsilon])$. Then,*

$$\begin{aligned}
&\frac{\partial^L}{\partial w^L}(a_1\tilde{*}\cdots\tilde{*}a_L) \\
&= \Big(\prod_{l=1}^{L-1}a_l(0)\Big)a_L' + \Big(\sum_{l=1}^{L-1}\Big(\prod_{\substack{1\leq k\leq L-1\\k\neq l}}a_k(0)\Big)a_l'(0)\Big)a_L \\
&\quad + \Big(\sum_{l=1}^{L-1}\sum_{\substack{J\subsetneq\{1,\dots,L-1\}\\l=\min J^c}}\Big(\prod_{j\in J}a_j(0)\Big)\Big(a_l'(0)\Big(\tilde{*}_{\substack{k\in J^c\\k\neq l}}a_k'\Big) + a_l''\tilde{*}\Big(\tilde{*}_{\substack{k\in J^c\\k\neq l}}a_k'\Big)\Big)\Big)\tilde{*}a_L
\end{aligned}$$

Proof. Using Lemmas 4.1 and 4.4, we have

$$\begin{aligned}
&\frac{\partial^L}{\partial w^L}(a_1\tilde{*}\cdots\tilde{*}a_L) \\
&= \frac{\partial}{\partial w}\Big(\Big(\prod_{j=1}^{L-1}a_j(0)\Big)a_L + I_{L-1}(a_1,\dots,a_{L-1})\tilde{*}a_L\Big) \\
&= \Big(\prod_{j=1}^{L-1}a_j(0)\Big)a_L' + I_{L-1}(a_1,\dots,a_{L-1})(0)a_L + \Big(\frac{\partial}{\partial w}I_{L-1}(a_1,\dots,a_{L-1})\Big)\tilde{*}a_L
\end{aligned}$$

Since $(b\tilde{*}c)(0) = 0$ for any b, c,

$$I_{L-1}(a_1, \dots, a_L)(0) = \sum_{l=1}^{L-1} \Big(\prod_{\substack{1 \le k \le L-1 \\ k \ne l}} a_k(0) \Big) a_l'(0).$$

Using Lemma 4.1,

$$\begin{aligned} &\frac{\partial}{\partial w} I_{L-1}(a_1, \dots, a_{L-1}) \\ &= \sum_{l=1}^{L-1} \sum_{\substack{J \subsetneq \{1,\dots,L-1\} \\ l = \min J^c}} \Big(\prod_{j \in J} a_j(0) \Big) \Big(a_l'(0) \Big(\tilde{*}_{\substack{k \in J^c \\ k \ne l}} a_k' \Big) + a_l'' \tilde{*} \Big(\tilde{*}_{\substack{k \in J^c \\ k \ne l}} a_k' \Big) \Big). \end{aligned}$$

Combining the above equations yields the result. $\square$

Corollary 4.8. *Let $a_1, \dots, a_L \in C^2([0, \epsilon])$.*

$$\frac{\partial^L}{\partial w^L} (a_1 \tilde{*} \cdots \tilde{*} a_L)(w) = \Big(\prod_{l=1}^{L-1} a_l(0) \Big) a_L'(w) + F_1(w), \tag{4.3}$$

$$\frac{\partial^L}{\partial w^L} (a_1 \tilde{*} \cdots \tilde{*} a_L)(w) = F_2(w), \tag{4.4}$$

where

$$|F_1(w)| \lesssim \sup_{0 \le r \le w} |a_L(r)|,$$

$$|F_2(w)| \lesssim \Big(|a_{L-1}(0)| + \sup_{0 \le r \le w} |a_L(r)| \Big) \wedge \Big(|a_L'(w)| + \sup_{0 \le r \le w} |a_L(r)| \Big),$$

where the implicit constants may depend on L, and upper bounds for ϵ and $\|a_j\|_{C^2}$, $1 \le j \le L$.

Proof. The bound for F_1 follows immediately from Lemma 4.7. The bound for F_2 follows from (4.3) and the bound for F_1. $\square$

Lemma 4.9. *Let $a, b \in C^1([0, \epsilon])$. Let $f(x) = \frac{1}{x} \int_0^\epsilon e^{-w/x} a(w)\, dw$ and $g(x) = \frac{1}{x} \int_0^\epsilon e^{-w/x} b(w)\, dw$. Then, there exists $G \in C([0, \infty))$ such that*

$$f(x)g(x) = \frac{1}{x} \int_0^\epsilon e^{-w/x} \frac{\partial}{\partial w} (a \tilde{*} b)(w)\, dw + \frac{1}{x} e^{-\epsilon/x} G(x). \tag{4.5}$$

Also,

$$\frac{f(x) - f(0)}{x} = \frac{1}{x} \int_0^\epsilon e^{-w/x} \frac{\partial a}{\partial w}(w)\, dw - \frac{1}{x} e^{-\epsilon/x} a(\epsilon). \tag{4.6}$$

Proof. A straightforward computation shows

$$\begin{aligned} f(x)g(x) &= \frac{1}{x^2} \int_0^\epsilon e^{-u/x} \int_0^u a(w_1) b(u - w_1)\, dw_1\, du \\ &\quad + \frac{1}{x^2} \int_\epsilon^{2\epsilon} e^{-u/x} \int_{u-\epsilon}^\epsilon a(w_1) b(u - w_1)\, dw_1\, du. \end{aligned}$$

We have

$$\frac{1}{x^2} \int_\epsilon^{2\epsilon} e^{-u/x} \int_{u-\epsilon}^\epsilon a(w_1) b(u - w_1)\, dw_1\, du$$

$$= \frac{1}{x^2} e^{-\epsilon/x} \int_0^\epsilon e^{-u/x} \int_u^\epsilon a(w_1) b(u+\epsilon-w_1)\,dw_1\,du =: \frac{1}{x} e^{-\epsilon/x} G_1(x),$$

where $G_1 \in C([0,\epsilon])$. Also, using that $(a\tilde{*}b)(0)=0$,

$$\begin{aligned}
&\frac{1}{x^2}\int_0^\epsilon e^{-u/x}\int_0^u a(w_1)b(u-w_1)\,dw_1\,du \\
&= -\frac{1}{x}\int_0^\epsilon \Big(\frac{\partial}{\partial u} e^{-u/x}\Big)(a\tilde{*}b)(u)\,du \\
&= -\frac{1}{x} e^{-\epsilon/x}(a\tilde{*}b)(\epsilon) + \frac{1}{x}\int_0^\epsilon e^{-u/x}\frac{\partial}{\partial u}(a\tilde{*}b)(u)\,du.
\end{aligned}$$

Combining the above equations yields (4.5).

We have

$$\begin{aligned}
\frac{1}{x}\int_0^\epsilon e^{-w/x}\frac{\partial a}{\partial w}(w)\,dw &= \frac{1}{x}e^{-\epsilon/x}a(\epsilon) - \frac{1}{x}a(0) + \frac{1}{x^2}\int_0^\epsilon e^{-w/x}a(w)\,dw \\
&= \frac{f(x)-f(0)}{x} + \frac{1}{x}e^{-\epsilon/x}a(\epsilon),
\end{aligned}$$

which proves (4.6). $\square$

Lemma 4.10. *Let $a_1,\dots,a_n \in C^1([0,\epsilon])$. Define $f_j(x) = \frac{1}{x}\int_0^\epsilon e^{-w/x}a_j(w)\,dw$. Then, there are continuous functions $G_1, G_2 \in C([0,\infty))$ such that*

$$\prod_{j=1}^n f_j(x) = \frac{1}{x}\int_0^\epsilon e^{-w/x}\frac{\partial^{n-1}}{\partial w^{n-1}}(a_1\tilde{*}\cdots\tilde{*}a_n)(w)\,dw + \frac{1}{x}e^{-\epsilon/x}G_1(x), \tag{4.7}$$

$$\begin{aligned}
&\frac{\prod_{j=1}^n f_j(x) - \prod_{j=1}^n f_j(0)}{x} \\
&= \frac{1}{x}\int_0^\epsilon e^{-w/x}\frac{\partial^n}{\partial w^n}(a_1\tilde{*}\cdots\tilde{*}a_n)(w)\,dw + \frac{1}{x^2}e^{-\epsilon/x}G_2(x).
\end{aligned} \tag{4.8}$$

Proof. We prove (4.7) by induction on n. $n=1$ is trivial and $n=2$ is contained in Lemma 4.9. We assume the result for $n-1$ and prove it for n. Thus, we assume

$$\prod_{j=1}^{n-1} f_j(x) = \frac{1}{x}\int_0^\epsilon \frac{\partial^{n-2}}{\partial w^{n-2}}(a_1\tilde{*}\cdots\tilde{*}a_{n-1})(w)\,dw + \frac{1}{x}e^{-\epsilon/x}\widetilde{G}_1(x), \tag{4.9}$$

where $\widetilde{G}_1 \in C([0,\infty))$. By Lemma 4.3, $a_1\tilde{*}\cdots\tilde{*}a_{n-1} \in C^{n-1}$ and vanishes to order $n-3$ at 0. Using this, and repeated applications Lemma 4.1, we have

$$\Big(\frac{\partial^{n-2}}{\partial w^{n-2}}(a_1\tilde{*}\cdots\tilde{*}a_{n-1})\Big)\tilde{*}a_n = \frac{\partial^{n-2}}{\partial w^{n-2}}(a_1\tilde{*}\cdots\tilde{*}a_n).$$

Using this and Lemma 4.9 we have, for some $\widetilde{G}_2 \in C([0,\infty))$,

$$\begin{aligned}
&\Big(\frac{1}{x}\int_0^\epsilon \frac{\partial^{n-2}}{\partial w^{n-2}}(a_1\tilde{*}\cdots\tilde{*}a_{n-1})(w)\,dw\Big) f_n(x) \\
&= \frac{1}{x}\int_0^\epsilon e^{-w/x}\frac{\partial}{\partial w}\Big(\frac{\partial^{n-2}}{\partial w^{n-2}}(a_1\tilde{*}\cdots a_{n-1})\tilde{*}a_n\Big)(w)\,dw + \frac{1}{x}e^{-\epsilon/x}\widetilde{G}_2(x) \\
&= \frac{1}{x}\int_0^\epsilon \frac{\partial^{n-1}}{\partial w^{n-1}}(a_1\tilde{*}\cdots\tilde{*}a_n)(w) + \frac{1}{x}e^{-\epsilon/x}\widetilde{G}_2(x).
\end{aligned} \tag{4.10}$$

Combining (4.9) with (4.10), we have

$$\prod_{j=1}^{n} f_j(x) = \frac{1}{x}\int_0^{\epsilon} \frac{\partial^{n-1}}{\partial w^{n-1}}(a_1\tilde{*}\cdots\tilde{*}a_n)(w) + \frac{1}{x}e^{-\epsilon/x}\widetilde{G}_2(x) + \frac{1}{x}e^{-\epsilon/x}\widetilde{G}_1(x)f_n(x),$$

which proves (4.7).

We turn to (4.8). Using (4.6) and (4.7), we have

$$\begin{aligned}&\frac{\prod_{j=1}^{n} f_j(x) - \prod_{j=1}^{n} f_j(0)}{x}\\ &= \frac{1}{x}\int_0^{\epsilon} e^{-w/x}\frac{\partial^{n}}{\partial w^{n}}(a_1\tilde{*}\cdots\tilde{*}a_n)(w)\,dw + \frac{1}{x^2}e^{-\epsilon/x}G_1(x)\\ &\quad - \frac{1}{x}e^{-\epsilon/x}\frac{\partial^{n-1}}{\partial w^{n-1}}\Big|_{w=\epsilon}(a_1\tilde{*}\cdots\tilde{*}a_n)(w).\end{aligned}$$

Since $a_1\tilde{*}\cdots\tilde{*}a_n \in C^n$ (by Lemma 4.3), this completes the proof. □

4.1. **Smoothing.** The operation $\tilde{*}$ has smoothing properties, and this section is devoted to discussing the instances of these smoothing properties which are used in this paper. Fix $m \in \mathbb{N}$, $\epsilon_1, \epsilon_2 > 0$.

Definition 4.11. For $L \geq 0$, $n \geq 1$, and increasing functions $G_1, G_2, G_3 : (0,\infty) \to (0,\infty)$, we say

$$\mathcal{G} : C^{0,L}([0,\epsilon_1]\times[0,\epsilon_2];\mathbb{R}^m) \to C^{0,L}([0,\epsilon_1]\times[0,\epsilon_2];\mathbb{R}^n)$$

is an (L, G_1, G_2, G_3) operation if:

- $\mathcal{G}(A)(t,w)$ depends only on the values of $A(t,r)$ for $r \in [0,w]$. As a result, for $\delta \in (0,\epsilon_2]$, $\mathcal{G}$ defines a map
$$\mathcal{G} : C^{0,L}([0,\epsilon_1]\times[0,\delta];\mathbb{R}^m) \to C^{0,L}([0,\epsilon_1]\times[0,\delta];\mathbb{R}^n).$$
- For $0 \leq k \leq L-1$, there are functions $\mathcal{G}^k : C([0,\epsilon_1];\mathbb{R}^m)^{k+1} \to C([0,\epsilon_1];\mathbb{R}^m)$ such that
$$\begin{aligned}&\mathcal{G}^k : C^{L}([0,\epsilon_1];\mathbb{R}^m)\times C^{L-1}([0,\epsilon_1];\mathbb{R}^m)\times\cdots\times C^{L-1}([0,\epsilon_1];\mathbb{R}^m)\\ &\to C^{L-k-1}([0,\epsilon_1];\mathbb{R}^n),\end{aligned}$$
and
$$\frac{\partial^k}{\partial w^k}\mathcal{G}(A)(t,w)\Big|_{w=0} = \mathcal{G}^k\Big(A(\cdot,0), \frac{\partial A}{\partial w}(\cdot,0), \dots, \frac{\partial^k A}{\partial w^k}(\cdot,0)\Big)(t).$$
- The following holds for all $M \in (0,\infty)$, $\delta \in (0,\epsilon_2]$.
  - for all $A \in C^{0,L}([0,\epsilon_1]\times[0,\delta];\mathbb{R}^m)$ with $\|A\|_{C^{0,L}} \leq M$, $\|\mathcal{G}(A)\|_{C^{0,L}} \leq G_1(M)$.
  - for all $A, B \in C^{0,L}([0,\epsilon_1]\times[0,\delta];\mathbb{R}^m)$ with $\|A\|_{C^{0,L}}, \|B\|_{C^{0,L}} \leq M$,
  $$\|\mathcal{G}(A) - \mathcal{G}(B)\|_{C^{0,L}} \leq G_2(M)\|A-B\|_{C^{0,L}}.$$
  - For $0 \leq k \leq L-1$, and $g_1, \dots, g_k$ with $g_j \in C^{L-j}([0,\epsilon_1];\mathbb{R}^m)$ and $\|g_j\|_{C^{L-j}} \leq M$,
  $$\|\mathcal{G}^k(g_1,\dots,g_k)\|_{C^{L-k-1}} \leq G_3(M).$$

Below we use $\tilde{*}$ to construct several examples, in the case $n = 1$, of $(2, G_1, G_2, G_3)$ operations.

Lemma 4.12. *Let $\alpha \in \mathbb{N}^m$ be a multi-index with $|\alpha| \geq 2$, and let $b(t,w) \in C([0,\epsilon_1] \times [0,\epsilon_2])$. For $A \in C^{0,2}([0,\epsilon_1]\times[0,\epsilon_2];\mathbb{R}^m)$ set*

$$\mathcal{G}(A)(t,w) := \left(b(t,\cdot)\tilde{*}\left(\tilde{*}^{\alpha} A'(t,\cdot)\right)\right)(w).$$

Then, $\mathcal{G}$ is a $(2, G_1, G_2, G_3)$ operation, where the functions G_1, G_2, and G_3 can be chosen to depend only on α, m, and upper bounds for ϵ_2 and $\|b\|_{C^0}$.

Proof. Let $k_1 = \min\{l : \alpha_l \neq 0\}$ and $k_2 = \min\{l : (\alpha - e_{k_1})_l \neq 0\}$. Using Lemma 4.1, we have

$$\begin{aligned}\frac{\partial}{\partial w}\mathcal{G}(A)(t,w) &= A'_{k_1}(t,0)\left(b(t,\cdot)\tilde{*}\left(\tilde{*}^{\alpha-e_k} A'(t,\cdot)\right)\right)(w)\\ &\quad + \left(b(t,\cdot)\tilde{*}A''_{k_1}(t,\cdot)\tilde{*}\left(\tilde{*}^{\alpha-e_{k_1}} A'(t,\cdot)\right)\right)(w),\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial w^2}\mathcal{G}(A)(t,w) &= A'_{k_1}(t,0)A'_{k_2}(t,0)\left(b(t,\cdot)\tilde{*}\left(\tilde{*}^{\alpha-e_{k_1}-e_{k_2}} A'(t,\cdot)\right)\right)(w)\\ &\quad + A'_{k_1}(t,0)\left(b(t,\cdot)\tilde{*}A''_{k_2}(t,\cdot)\tilde{*}\left(\tilde{*}^{\alpha-e_{k_1}-e_{k_2}} A'(t,\cdot)\right)\right)(w)\\ &\quad + A'_{k_2}(t,0)\left(b(t,\cdot)\tilde{*}A''_{k_1}(t,\cdot)\tilde{*}\left(\tilde{*}^{\alpha-e_{k_1}-e_{k_2}} A'(t,\cdot)\right)\right)(w)\\ &\quad + \left(b(t,\cdot)\tilde{*}A''_{k_1}(t,\cdot)\tilde{*}A''_{k_2}(t,\cdot)\tilde{*}\left(\tilde{*}^{\alpha-e_{k_1}-e_{k_2}} A'(t,\cdot)\right)\right)(w).\end{aligned}$$

For any c_1, c_2, we have $(c_1\tilde{*}c_2)(0) = 0$, we may therefore take $\mathcal{G}^0 = 0$ and $\mathcal{G}^1 = 0$. Using the above formulas, combined with Lemma 4.6, the result follows. □

Lemma 4.13. *Suppose $|\alpha| = 1$ and $b(t,w) \in C^{0,1}([0,\epsilon_1]\times[0,\epsilon_2])$. For $A \in C^{0,2}([0,\epsilon_1]\times[0,\epsilon_2];\mathbb{R}^m)$ set*

$$\mathcal{G}(A)(t,w) := \left(b(t,\cdot)\tilde{*}\left(\tilde{*}^{\alpha} A'(t,\cdot)\right)\right)(w).$$

Then $\mathcal{G}$ is a $(2, G_1, G_2, G_3)$ operation, where the functions G_1, G_2, and G_3 can be chosen to depend only on m and upper bounds for ϵ_2 and $\|b\|_{C^{0,1}}$.

Proof. Without loss of generality we take $\alpha = e_1$, so that $\mathcal{G}(A)(t,w) = (b(t,\cdot)\tilde{*}A'_1(t,\cdot))(w)$. Using Lemma 4.1 we have

$$\begin{gathered}\frac{\partial}{\partial w}\mathcal{G}(A)(t,w) = A'_1(t,0)b(t,w) + (b(t,\cdot)\tilde{*}A''_1(t,\cdot))(w),\\ \frac{\partial^2}{\partial w^2}\mathcal{G}(A)(t,w) = A'_1(t,w)b'(t,w) + b(t,0)A''_1(t,w) + (b'(t,\cdot)\tilde{*}A''_1(t,\cdot))(w).\end{gathered}$$

In particular,

$$\mathcal{G}(A)(t,0) = 0, \quad \frac{\partial}{\partial w}\Big|_{w=0}\mathcal{G}(A)(t,w) = A'_1(t,0)b(t,0).$$

Using the above formulas, the result follows easily. □

Lemma 4.14. *Suppose $|\alpha| \geq 2$ and $b(t) \in C([0,\epsilon_1])$. For $A \in C^{0,2}([0,\epsilon_1]\times[0,\epsilon_2];\mathbb{R}^m)$ set*

$$\mathcal{G}(A)(t,w) := b(t)\left(\tilde{*}^{\alpha} A'(t,\cdot)\right)(w).$$

Then, $\mathcal{G}$ is a $(2, G_1, G_2, G_3)$ operation, where the functions G_1, G_2, and G_3 can be chosen to depend only on m and upper bounds for ϵ_2 and $\|b\|_{C^0}$.

Proof. Let $k_1 = \min\{l : \alpha_l \neq 0\}$ and $k_2 = \min\{l : (\alpha - e_{k_1})_l \neq 0\}$. Using Lemma 4.1 we have

$$\begin{aligned}\frac{\partial}{\partial w}\mathcal{G}(A)(t,w) &= b(t)A'_{k_1}(t,0)\left(\tilde{*}^{\alpha-e_{k_1}}A'(t,\cdot)\right)(w)\\ &\quad + b(t)\left(A''_{k_1}(t,\cdot)\tilde{*}\left(\tilde{*}^{\alpha-e_{k_1}}A'(t,\cdot)\right)\right)(w),\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial w^2}\mathcal{G}(A)(t,w) &= b(t)A'_{k_1}(t,0)A'_{k_2}(t,0)\left(\tilde{*}^{\alpha-e_{k_1}-e_{k_2}}A'(t,\cdot)\right)(w)\\ &\quad + b(t)A'_{k_1}(t,0)\left(A''_{k_2}(t,\cdot)\tilde{*}\left(\tilde{*}^{\alpha-e_{k_1}-e_{k_2}}A'(t,\cdot)\right)\right)(w)\\ &\quad + b(t)A'_{k_2}(t,0)\left(A''_{k_1}(t,\cdot)\tilde{*}\left(\tilde{*}^{\alpha-e_{k_1}-e_{k_2}}A'(t,\cdot)\right)\right)(w)\\ &\quad + b(t)\left(A''_{k_1}(t,\cdot)\tilde{*}A''_{k_2}(t,\cdot)\tilde{*}\left(\tilde{*}^{\alpha-e_{k_1}-e_{k_2}}A'(t,\cdot)\right)\right)(w).\end{aligned}$$

In particular,

$$\mathcal{G}(A)(t,0) = 0, \quad \frac{\partial}{\partial w}\Big|_{w=0}\mathcal{G}(A)(t,w) = \begin{cases} 0, & \text{if } |\alpha| > 2,\\ b(t)A'_{k_1}(t,0)A'_{k_2}(t,0), & \text{if } |\alpha| = 2.\end{cases}$$

Using the above formulas, combined with Lemma 4.6, the result follows easily. □

Lemma 4.15. *Suppose $d \in C^{0,2}([0,\epsilon_1]\times[0,\epsilon_2])$ is such that $d(t,0) \in C^1([0,\epsilon_1])$. For $A \in C^{0,2}([0,\epsilon_1]\times[0,\epsilon_2];\mathbb{R}^m)$ set*

$$\mathcal{G}(A)(t,w) := d(t,w).$$

Then, $\mathcal{G}$ is a $(2, G_1, G_2, G_3)$ operation, where the functions G_1, G_2, and G_3 can be chosen to depend only on upper bounds for $\|d\|_{C^{0,2}}$ and $\|d(\cdot,0)\|_{C^1}$.

The above lemma follows immediately from the definitions.

Lemma 4.16. *Suppose $\mathcal{G} : C^{0,L}([0,\epsilon_1]\times[0,\epsilon_2];\mathbb{R}^m) \to C^{0,L}([0,\epsilon_1]\times[0,\epsilon_2])$ is an (L, G_1, G_2, G_3) operation. Let $\beta \in \mathbb{N}^m$ be a multi-index, and define*

$$\widetilde{\mathcal{G}}(A)(t,w) := A(t,0)^\beta \mathcal{G}(A)(t,w).$$

Then, $\widetilde{\mathcal{G}}$ is an $(L, \widetilde{G}_1, \widetilde{G}_2, \widetilde{G}_3)$ operation, where $\widetilde{G}_1$, $\widetilde{G}_2$, and $\widetilde{G}_3$ can be chosen to depend only on G_1, G_2, G_3, L, and β.

The above lemma follows immediately from the definitions.

4.2. **Polynomials.** For this section, we take all the same notation and assumptions as in the beginning of Section 2. Thus, we have $b_{\alpha,j}$, $c_{\alpha,j}$, $P(t,x,y,z)$, and $\widehat{P}(t, A(\cdot), z)(w)$ as described in that section.

Lemma 4.17. *Let $\delta \in (0,\epsilon_2]$ and $A(t,w) \in C^{0,1}([0,\epsilon_1]\times[0,\delta];\mathbb{R}^m)$ with $A(t,0) \in C([0,\epsilon_1];U)$. Define $f(t,x) \in C([0,\epsilon_1]\times[0,\epsilon_0];\mathbb{R}^m)$ by*

$$f(t,x) = \frac{1}{x}\int_0^\delta e^{-w/x}A(t,w)\,dw.$$

Then

$$\begin{aligned}&\frac{P(t,x,f(t,x),f(t,0)) - P(t,0,f(t,0),f(t,0))}{x}\\ &= \frac{1}{x}\int_0^\delta e^{-w/x}\widehat{P}(t,A(t,\cdot),A(t,0))(w)\,dw + \frac{1}{x^2}e^{-\delta/x}G(t,x),\end{aligned}$$

where $G(t,x) \in C([0,\epsilon_1]\times[0,\epsilon_0];\mathbb{R}^m)$.

The above lemma follows from Lemma 4.10, using the fact that $f(t,0)=A(t,0)$.

Proposition 4.18. *Let $\delta\in(0,\epsilon_2]$. For $A\in C^{0,2}([0,\epsilon_1]\times[0,\delta];\mathbb{R}^m)$ and $A_0(t)\in C^1([0,\epsilon_1];\mathbb{R}^m)$,*
$$\widehat{P}(t,A(t,\cdot),A_0(t))(w)=d_yP(t,0,A(t,0),A_0(t))A'(t,w)+\mathcal{G}_{A_0}(A)(t,w),$$
where $\mathcal{G}_{A_0}:C^{0,2}([0,\epsilon_1]\times[0,\epsilon_2];\mathbb{R}^m)\to C^{0,2}([0,\epsilon_1]\times[0,\epsilon_2];\mathbb{R}^m)$ is a $(2,G_1,G_2,G_3)$ operation[6]*. The functions G_1, G_2, and G_3 can be chosen to depend only on C_0, m, D,*[7] *and upper bounds for ϵ_2 and $\|A_0\|_{C^1}$.*

Proof. By linearity, it suffices to prove the result for P a monomial in y. I.e.,
$$P(t,x,y,z)=c_{\alpha,j}(t,x,z)y^\alpha e_j,$$
for some $j\in\{1,\dots,m\}$, $\alpha\in\mathbb{N}^m$ with $|\alpha|\le D$. In this case,
$$\widehat{P}(t,A(t,\cdot),z)(w)=\frac{\partial^{|\alpha|+1}}{\partial w^{|\alpha|+1}}\left(b_{\alpha,j}(t,\cdot,z)\tilde{*}\left(\tilde{*}^\alpha A(t,\cdot)\right)\right)(w)e_j. \tag{4.11}$$
Using Corollary 4.5 and the fact that $b_{\alpha,j}(t,0,z)=c_{\alpha,j}(t,0,z)$,
$$\begin{aligned}
&\widehat{P}(t,A(t,\cdot),A_0(t))(w)\\
&=\sum_{l=1}^m \alpha_l b_{\alpha,j}(t,0,A_0(t))A(t,0)^{\alpha-e_l}A_l'(t,w)e_j\\
&\quad+\sum_{\substack{\beta\le\alpha\\ |\beta|<|\alpha|-1}}\binom{\alpha}{\beta}b_{\alpha,j}(t,0,A_0(t))A(t,0)^\beta\left(\tilde{*}^{\alpha-\beta}A'(t,\cdot)\right)(w)e_j\\
&\quad+\sum_{\beta\le\alpha}\binom{\alpha}{\beta}A(t,0)^\beta\left(b'_{\alpha,j}(t,\cdot,A_0(t))\tilde{*}\left(\tilde{*}^{\alpha-\beta}A'(t,\cdot)\right)\right)(w)e_j
\end{aligned}\tag{4.12}$$
Note that
$$\sum_{l=1}^m \alpha_l b_{\alpha,j}(t,0,A_0(t))A(t,0)^{\alpha-e_l}A_l'(t,w)e_j=d_yP(t,0,A(t,0),A_0(t))A'(t,w).$$
Thus, it remains to show the final two terms on the right hand side of (4.12) are a $(2,G_1,G_2,G_3)$ operation. This follows from Lemmas 4.12, 4.13, 4.14, 4.15, and 4.16, completing the proof. □

Proposition 4.19. *In addition to the other assumptions of this section, we assume (2.3). Let $\delta\in(0,\epsilon_2]$ and let $A,B\in C^{0,2}([0,\epsilon_1]\times[0,\delta];\mathbb{R}^m)$. Set $g(t,w)=A(t,w)-B(t,w)$. Then*
$$\begin{aligned}
&\widehat{P}(t,A(t,\cdot),A(t,0))(w)-\widehat{P}(t,B(t,\cdot),B(t,0))(w)\\
&=d_yP(t,0,A(t,0),A(t,0))g'(t,w)+F(t,w),
\end{aligned}$$
where there exists a constant C with
$$|F(t,w)|\le C\sup_{0\le r\le w}|g(t,r)|,\quad \forall t,w.$$
Here, C is allowed to depend on any of the ingredients in the proposition, including A and B.

[6]See Definition 4.11 for the definition of a $(2,G_1,G_2,G_3)$ operation.

[7]See Section 2 for the definitions of these various constants.

Proof. By linearity, it suffices to prove the result for P a monomial in y. I.e.,

$$P(t,x,y,z) = c_{\alpha,j}(t,x,z)y^\alpha e_j,$$

for some $j \in \{1,\dots,m\}$ and $\alpha \in \mathbb{N}^m$ with $|\alpha| \le D$. In this case $\widehat{P}$ is given by (4.11). Using Lemma 4.6,

$$\begin{aligned}
&\widehat{P}(t,A(t,\cdot),A(t,0))(w) - \widehat{P}(t,B(t,\cdot),B(t,0))\\
&= \sum_{l=1}^m \alpha_l \frac{\partial^{|\alpha|+1}}{\partial w^{|\alpha|+1}} \left(b_{\alpha,j}(t,\cdot,A(t,0)) \tilde{*} g_l(t,\cdot) \tilde{*} \left(\tilde{*}^{\alpha-e_l} A(t,\cdot)\right)\right)(w)e_j\\
&\quad + \sum_{\substack{\beta\le\alpha\\ |\beta|\ge 2}} (-1)^{|\beta|+1}\binom{\alpha}{\beta} \frac{\partial^{|\alpha|+1}}{\partial w^{|\alpha|+1}} \left(b(t,\cdot,A(t,0)) \tilde{*} \left(\tilde{*}^{\beta} g(t,\cdot)\right) \tilde{*} \left(\tilde{*}^{\alpha-\beta} A(t,\cdot)\right)\right)(w)e_j\\
&\quad + \frac{\partial^{|\alpha|+1}}{\partial w^{|\alpha|+1}} \left(\left(b_{\alpha,j}(t,\cdot,A(t,0)) - b_{\alpha,j}(t,\cdot,B(t,0))\right) \tilde{*} \left(\tilde{*}^{\alpha} B(t,\cdot)\right)\right)(w)e_j\\
&=: (I) + (II) + (III).
\end{aligned}$$

We study the three terms on the right hand side of the above equation separately. Applying (4.3) to each term of the sum in (I), with g_l playing the role of a_L, and using the fact that $b_{\alpha,j}(t,0,z) = c_{\alpha,j}(t,0,z)$,

$$\begin{aligned}
(I) &= \sum_{l=1}^m \alpha_l b_{\alpha,j}(t,0,A(t,0))A(t,0)^{\alpha-e_l} g_l'(t,0)e_j + F_1(t,w)\\
&= d_y P(t,0,A(t,0),A(t,0))g'(t,0) + F_1(t,w),
\end{aligned}$$

where $|F_1(t,w)| \lesssim \sup_{0\le r\le w} |g(t,r)|$. Turning to (II), we note that in each term in the sum defining (II), $|\beta| \ge 2$, and so there are at least two coordinates (counting repetitions) of $g(t,\cdot)$ in the convolution. Applying (4.4) to each term of the sum, with these two coordinates of $g(t,\cdot)$ playing the roles of a_L and a_{L-1}, we see $(II) = F_2(t,w)$ where $|F_2(t,w)| \lesssim \sup_{0\le r\le w} |g(t,r)|$. Finally, for (III), we use that as t varies over $[0,\epsilon_1]$, $A(t,0)$ and $B(t,0)$ range over a compact subset of U. Applying (4.4) with $b_{\alpha,j}(t,\cdot,A(t,0)) - b_{\alpha,j}(t,\cdot,B(t,0))$ playing the role of a_L, and using (2.3), we see $(III) = F_3(t,w)$, where

$$\begin{aligned}
|F_3(t,w)| &\lesssim \sup_{0\le r\le w} |b_{\alpha,j}(t,r,A(t,0)) - b_{\alpha,j}(t,r,B(t,0))|\\
&\quad + |b'_{\alpha,j}(t,w,A(t,0)) - b'_{\alpha,j}(t,w,B(t,0))|\\
&\lesssim |g(t,0)| \lesssim \sup_{0\le r\le w} |g(t,r)|.
\end{aligned}$$

Summing the above three estimates completes the proof. □

5. Ordinary differential equations

In this section, we prove some auxiliary results concerning ODEs which are needed in the remainder of the paper.

5.1. **Chronological Calculus.** Let $m \in \mathbb{N}$ and let $J = [a,b]$ for some $a < b$. Let $M(t) : J \to \mathbb{M}^{m\times m}$ be locally bounded and measurable.

Definition 5.1. For $t \in J$, we define

$$\overleftarrow{\exp}\Big(\int_a^t A(s)\,ds\Big) = E(t),$$

to be the unique solution $E : J \to \mathbb{M}^{m\times m}$ to the differential equation

$$\dot{E}(t) = A(t)E(t), \quad E(a) = I,$$

where I denotes the $m \times m$ identity matrix.

For the rest of this section, fix $\epsilon_0 > 0$.

Proposition 5.2. *Let $\mathcal{M}(t,x) \in C(J \times [0,\epsilon_0]; \mathbb{M}^{m\times m})$ be such that there exists $R(t) \in C^1(J; \mathrm{GL}_m)$ with $R(t)\mathcal{M}(t,0)R(t)^{-1} = \mathrm{diag}(\lambda_1(t),\dots,\lambda_m(t))$, with $\lambda_j(t) > 0$ for all t. Set $\lambda_0(t) = \min_{1\le j\le m} \lambda_j(t)$. Then, for all $\delta \in [0,1)$, $\exists x_0 \in (0,\epsilon_0]$, for all $x \in (0,x_0]$, for all $t \in J$,*

$$\Big\|\overleftarrow{\exp}\Big(-\frac{1}{x}\int_a^t \mathcal{M}(s,x)\,ds\Big)\Big\| \le \|R(t)^{-1}\|\,\|R(a)\| \exp\Big(-\frac{\delta}{x}\int_a^t \lambda_0(s)\,ds\Big).$$

To prove Proposition 5.2, we introduce the following lemma.

Lemma 5.3. *Let $\mathcal{M}(t,x) \in C(J \times [0,\epsilon_0]; \mathbb{M}^{m\times m})$ and set $2\lambda_0(t)$ to be the least eigenvalue of $\mathcal{M}(t,0)^\top + \mathcal{M}(t,0)$. We assume $\lambda_0(t) > 0$, for all $t \in J$. Then, for all $\delta \in [0,1)$, there exists $x_0 \in (0,\epsilon_0]$, such that for all $x \in (0,x_0]$,*

$$\Big\|\overleftarrow{\exp}\Big(-\frac{1}{x}\int_a^t \mathcal{M}(s,x)\,ds\Big)\Big\| \le \exp\Big(-\frac{\delta}{x}\int_a^t \lambda_0(s)\,ds\Big).$$

Proof. Let $\mathcal{N}(t,x) = \mathcal{M}(t,x) - \mathcal{M}(t,0)$ so that $\mathcal{N}(t,x) \in C(J\times[0,\epsilon_0];\mathbb{M}^{m\times m})$ and $\mathcal{N}(t,0) = 0$. Fix $\delta \in [0,1)$ and take $x_0 \in (0,\epsilon_0]$ so small for all $(t,x) \in J \times [0,x_0]$, $\|\mathcal{N}(t,x)\| \le \inf_{s\in J}(1-\delta)\lambda_0(s)$.

Let $\theta_0 \in \mathbb{R}^m$ and set $\theta(t,x) := \overleftarrow{\exp}\Big(-\frac{1}{x}\int_a^t \mathcal{M}(s,x)\,ds\Big)\theta_0$. Then

$$\begin{aligned}
\frac{\partial}{\partial t}|\theta(t,x)|^2 &= -\big\langle \theta(t,x), \big(\frac{1}{x}\mathcal{M}(t,x)^\top + \frac{1}{x}\mathcal{M}(t,x)\big)\theta(t,x)\big\rangle \\
&= -\frac{1}{x}\langle \theta(t,x), \big(\mathcal{M}(t,0)^\top + \mathcal{M}(t,0)\big)\,\theta(t,x)\rangle \\
&\quad - \frac{1}{x}\langle \theta(t,x), \big(\mathcal{N}(t,x)^\top + \mathcal{N}(t,x)\big)\,\theta(t,x)\rangle \\
&\le -\frac{2}{x}\lambda_0(t)|\theta(t,x)|^2 + \frac{2}{x}\|\mathcal{N}(t,x)\||\theta(t,x)| \\
&\le -\frac{2}{x}\delta\lambda_0(t)|\theta(t,x)|^2.
\end{aligned}$$

By Grönwall's inequality, we have

$$|\theta(t,x)|^2 \le |\theta_0|^2 \exp\Big(-\frac{2\delta}{x}\int_a^t \lambda_0(s)\,ds\Big).$$

Taking square roots yields the result. $\square$

Proof of Proposition 5.2. Let $\Lambda(t) = \mathrm{diag}(\lambda_1(t),\dots,\lambda_m(t)) = R(t)\mathcal{M}(t,0)R(t)^{-1}$. For $\theta_0 \in \mathbb{R}^m$, set $\theta(t,x) = \overleftarrow{\exp}\Big(-\frac{1}{x}\int_a^t \mathcal{M}(s,x)\,ds\Big)\theta_0$. Let $\gamma(t,x) = R(t)\theta(t,x)$. γ satisfies

$$\frac{\partial}{\partial t}\gamma(t,x) = -\frac{1}{x}R(t)\mathcal{M}(t,x)R(t)^{-1}\gamma(t,x) + \dot{R}(t)R(t)^{-1}\gamma(t,x)$$

$$= -\frac{1}{x}\widetilde{\mathcal{M}}(t,x)\gamma(t,x),$$

where $\widetilde{\mathcal{M}}(t,x) = \Lambda(t) + R(t)\left(\mathcal{M}(t,x) - \mathcal{M}(t,0)\right)R(t)^{-1} - x\dot{R}(t)R(t)^{-1}$. In particular, note $\mathcal{M}(t,0) = \Lambda(t)$. It follows that $\gamma(t,x) = \overleftarrow{\exp}\Big(-\frac{1}{x}\widetilde{\mathcal{M}}(s,x)\,ds\Big)\gamma(a,x)$.

Fix $\delta \in [0,1)$. By Lemma 5.3, there exists $x_0 \in (0,\epsilon_0]$ (independent of θ_0) such that for $x \in (0,x_0]$,

$$\Big\|\overleftarrow{\exp}\Big(-\frac{1}{x}\int_a^t \widetilde{\mathcal{M}}(s,x)\,ds\Big)\Big\| \le \exp\Big(-\frac{\delta}{x}\int_a^t \lambda_0(s)\,ds\Big).$$

Hence, for $x \in (0,x_0]$,

$$\begin{aligned}
|\theta(t,x)| &\le \|R(t)^{-1}\|\,|\gamma(t,x)| \\
&\le \|R(t)^{-1}\|\exp\Big(-\frac{\delta}{x}\int_a^t \lambda_0(s)\,ds\Big)|\gamma(a,x)| \\
&\le \|R(t)^{-1}\|\|R(a)\|\exp\Big(-\frac{\delta}{x}\int_a^t \lambda_0(s)\,ds\Big)|\theta_0|.
\end{aligned}$$

The result follows. □

5.2. **A basic existence result.** Fix $\epsilon_1, \epsilon_0 > 0$ and let $W \subseteq \mathbb{R}^m$ be an open neighborhood of $0 \in \mathbb{R}^m$. Suppose $\mathcal{M}(t,x,y) \in C([0,\epsilon_1] \times [0,\epsilon_0] \times W; \mathbb{M}^{m\times m})$ be such that for every compact set $K \Subset W$,

$$\sup_{\substack{t\in[0,\epsilon_1],x\in[0,\epsilon_0]\\ y_1,y_2\in K, y_1\neq y_2}} \frac{\|\mathcal{M}(t,x,y_1) - \mathcal{M}(t,x,y_2)\|}{|y_1 - y_2|} < \infty.$$

Let $G(t,x,y) \in C([0,\epsilon_1] \times [0,\epsilon_0] \times W; \mathbb{R}^m)$ be such that for every compact set $K \Subset W$,

$$\sup_{\substack{t\in[0,\epsilon_1],x\in[0,\epsilon_0]\\ y_1,y_2\in K, y_1\neq y_2}} \frac{|G(t,x,y_1) - G(t,x,y_2)|}{|y_1 - y_2|} < \infty.$$

Let $g_0 \in C([0,\epsilon_0]; \mathbb{R}^m)$ have $g_0(0) = 0$. The goal of this section is to study the differential equation

$$\frac{\partial}{\partial t} g(t,w) = -\frac{1}{x}\mathcal{M}(t,x,g(t,x))g(t,x) + G(t,x,g(t,x)), \quad x > 0 \tag{5.1}$$

with the initial condition $g(0,x) = g_0(x)$. The main result is the following.

Proposition 5.4. *Set $\mathcal{M}_0(t) = \mathcal{M}(t,0,0)$. We suppose that there exists $R(t) \in C^1([0,\epsilon_1]; \mathrm{GL}_m)$ such that*

$$R(t)\mathcal{M}_0(t)R(t)^{-1} = \operatorname{diag}(\lambda_1(t),\dots,\lambda_m(t))$$

and $\lambda_j(t) > 0$, for all j,t. Then, there exists $\delta_0 \in (0,\epsilon_0]$ and a function $g(t,x) \in C([0,\epsilon_1] \times [0,\delta_0]; W)$ such that $g(0,x) = g_0(x)$ for all $x \in [0,\delta_0]$, $g(t,0) = 0$ for all $t \in [0,\epsilon_1]$, and g satisfies (5.1).

To prove Proposition 5.4, we need two lemmas. As in Proposition 5.4, set $\mathcal{M}_0(t) = \mathcal{M}(t,0,0)$. For these lemmas, instead of assuming the existence of $R(t)$ as in Proposition 5.4, we let $2\lambda_0(t)$ be the least eigenvalue of $\mathcal{M}_0(t)^\top + \mathcal{M}_0(t)$ and we assume $\lambda_0(t) > 0$, for all $t \in [0,\epsilon_1]$.

Lemma 5.5. *Under the the assumption $\lambda_0(t) > 0$ for all t, the following holds. For all $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_0 \in (0, \delta]$, there exists a unique solution $g^{x_0}(t) \in C^1([0, \epsilon_1]; B^m(\epsilon) \cap W)$ to the differential equation*

$$\frac{\partial}{\partial t} g^{x_0}(t) = -\frac{1}{x_0}\mathcal{M}(t, x_0, g^{x_0}(t))g^{x_0}(t) + G(t, x_0, g^{x_0}(t)), \quad g^{x_0}(0) = g_0(x_0).$$

In the above, $B^m(\epsilon) = \{y \in \mathbb{R}^m : |y| < \epsilon\}$.

Proof. Fix $\epsilon > 0$. Set $\mathcal{N}(t, x, y) := \mathcal{M}(t, x, y) - \mathcal{M}_0(t)$, so that $\mathcal{N}(t, 0, 0) = 0$. Fix $r > 0$ so small $\overline{B^m(r)} \subset W$. Take $\gamma > 0$ so small that if $x, |y| \leq \gamma$, $\sup_{t\in[0,\epsilon_1]} \|\mathcal{N}(t, x, y)\| \leq \frac{1}{2} \inf_{t\in[0,\epsilon_1]} \lambda_0(t)$. Without loss of generality, we assume $\epsilon < r \wedge \gamma$. Let

$$C := \sup_{\substack{t\in[0,\epsilon_1], x\in[0,\epsilon_0] \\ |y|\leq r}} |G(t, x, y)| < \infty.$$

Take $\delta \in (0, \gamma]$ so small that

$$\frac{\epsilon}{\delta} \inf_{t\in[0,\epsilon_1]} \lambda_0(t) > 2C, \quad \sup_{x\in[0,\delta]} |g_0(x)| < \epsilon.$$

Fix $x_0 \in (0, \delta]$. The Picard-Lindelöf theorem shows that the solution $g^{x_0}(t)$ exists and is unique for t in some interval $[0, s]$, where $s \in (0, \epsilon_1]$. We will show that for $t \in [0, s]$, $|g^{x_0}(t)| < \epsilon$. By iterating this process, it follows that we do not have blow up in small time, and can take $s = \epsilon_1$.

Thus, we wish to show that for all $t \in [0, s]$, $|g^{x_0}(t)| < \epsilon$. Suppose, for contradiction, there is $t_0 \in [0, s]$ with $|g^{x_0}(t_0)| \geq \epsilon$. Take the least such t_0. Since $|g^{x_0}(0)| = |g_0(x_0)| < \epsilon$, $t_0 > 0$. Hence, $|g^{x_0}(t_0)| = \epsilon$ and

$$\frac{\partial}{\partial t}\Big|_{t=t_0} |g^{x_0}(t)|^2 \geq 0. \tag{5.2}$$

But, for $t \in [0, t_0]$, $|g^{x_0}(t)| \leq \epsilon < r \wedge \gamma$, and therefore,

$$\begin{aligned}
&\frac{\partial}{\partial t}|g^{x_0}(t)|^2 \\
&= -\frac{1}{x_0}\langle g^{x_0}(t), \left(\mathcal{M}_0(t)^\top + \mathcal{M}_0(t)\right) g^{x_0}(t)\rangle \\
&\quad - \frac{1}{x_0}\langle g^{x_0}(t), \left(\mathcal{N}(t, x_0, g^{x_0}(t))^\top + \mathcal{N}(t, x_0, g^{x_0}(t))\right) g^{x_0}(t)\rangle \\
&\quad + 2\langle g^{x_0}(t), G(t, x_0, g^{x_0}(t))\rangle \\
&\leq -\frac{1}{x_0} 2\lambda_0(t)|g^{x_0}(t)|^2 + 2\frac{1}{x_0}\|\mathcal{N}(t, x_0, g^{x_0}(t))\||g^{x_0}(t)|^2 \\
&\quad + 2|G(t, x_0, g^{x_0}(t)||g^{x_0}(t)| \\
&\leq -\frac{1}{x_0}\lambda_0(t)|g^{x_0}(t)|^2 + 2C|g^{x_0}(t)| \leq -\frac{1}{\delta}\lambda_0(t)|g^{x_0}(t)|^2 + 2C|g^{x_0}(t)|.
\end{aligned}$$

Hence,

$$\frac{\partial}{\partial t}\Big|_{t=t_0} |g^{x_0}(t)|^2 \leq -\frac{\epsilon^2}{\delta}\lambda_0(t_0) + 2C\epsilon < 0,$$

contradicting (5.2) and completing the proof. □

Lemma 5.6. *Under the the assumption $\lambda_0(t) > 0$ for all t, there exists $\delta_0 \in (0, \epsilon_0]$ and a function $g(t, x) \in C([0, \epsilon_1] \times [0, \delta_0]; W)$ such that $g(0, x) = g_0(x)$ for all $x \in [0, \delta_0]$, $g(t, 0) = 0$ for all $t \in [0, \epsilon_1]$, and g satisfies* (5.1).

Proof. Let $\delta_0 > 0$ be the δ guaranteed by Lemma 5.5 with $\epsilon = 1$. For $x \in (0, \delta]$, set $g(t,x) = g^x(t)$, where $g^x(t)$ is the unique solution from Lemma 5.5. Standard theorems from ODEs show $g(t,x) : [0,\epsilon_1] \times (0,\delta_0] \to \mathbb{R}^m$ is continuous. All that remains to show is that $g(t,x)$ extends to a continuous function at $x = 0$ by setting $g(t,0) = 0$. This follows immediately from Lemma 5.5. $\square$

Proof of Proposition 5.4. Set

$$\widetilde{\mathcal{M}}(t,x,y) := -x\dot{R}(t)R(t)^{-1} + R(t)\mathcal{M}(t,x,R(t)^{-1}y)R(t)^{-1},$$

and $\widetilde{G}(t,x,y) = R(t)G(t,x,R(t)^{-1}y)$. Note that $\widetilde{\mathcal{M}}(t,0,0) = \operatorname{diag}(\lambda_1(t),\dots,\lambda_m(t))$. Lemma 5.6 shows that there exists $\delta_0 \in (0,\epsilon_0]$ and and a function $h(t,x) \in C([0,\epsilon_1] \times [0,\delta_0]; W)$ such that $h(0,x) = R(0)g_0(x)$ for all $x \in [0,\delta_0]$, $h(t,0) = 0$ for all $t \in [0,\epsilon_1]$, and h satisfies

$$\frac{\partial}{\partial t}h(t,w) = -\frac{1}{x}\widetilde{\mathcal{M}}(t,x,h(t,x))h(t,x) + \widetilde{G}(t,x,h(t,x)), \quad x > 0.$$

Setting $g(t,x) = R(t)^{-1}h(t,x)$ gives the desired solution, and completes the proof. $\square$

6. Existence

In this section, we prove Theorems 2.2 and 2.4. The key result needed for these, which is also useful for proving Theorem 2.6, is the next proposition. For it, we take all the same notation and assumptions as in the beginning of Section 2. Thus, we have $m \in \mathbb{N}$, $\epsilon_0, \epsilon_1, \epsilon_2 \in (0,\infty)$, $U \subseteq \mathbb{R}^m$ open, $D \in \mathbb{N}$, and $b_{\alpha,j}$, $c_{\alpha,j}$, C_0, P, and $\widehat{P}$ as described in that section.

Proposition 6.1. *Let $A_0(t) \in C^2([0,\epsilon_1];U)$ and $\mathcal{M}(t) := -d_yP(t,0,A_0(t),A_0(t))$. Suppose there exists $R(t) \in C^1([0,\epsilon_1];\mathrm{GL}_m)$ with $R(t)\mathcal{M}(t)R(t)^{-1} =$ $\operatorname{diag}(\lambda_1(t),\dots,\lambda_m(t))$, where $\lambda_j(t) > 0$, for all t,j. Take $c_0, C_1, C_2, C_3, C_4 > 0$ such that $\min_{t,j}\lambda_j(t) \ge c_0 > 0$, $\|R\|_{C^1} \le C_1$, $\|R^{-1}\|_{C^1} \le C_2$, $\|\mathcal{M}^{-1}\|_{C^1} \le C_3$, $\|A_0\|_{C^2} \le C_4$. Then, there exists $\delta = \delta(m,D,c_0,C_0,C_1,C_2,C_3,C_4) > 0$ and $A(t,w) \in C^{0,2}([0,\epsilon_1] \times [0,\delta \wedge \epsilon_2];\mathbb{R}^m)$ such that*

$$\frac{\partial}{\partial t}A(t,w) = \widehat{P}(t,A(t,\cdot),A(t,0))(w), \quad A(t,0) = A_0(t). \tag{6.1}$$

Moreover, if we set

$$f(t,x) = \begin{cases} \frac{1}{x}\int_0^{\delta\wedge\epsilon_2} e^{-w/x}A(t,w)\,dw, & \text{if } x > 0,\\ A_0(t), & \text{if } x = 0,\end{cases} \tag{6.2}$$

then $f(t,x) \in C([0,\epsilon_1] \times [0,\epsilon_0];\mathbb{R}^m)$ and there exists $\widetilde{G}(t,x) \in C([0,\epsilon_1] \times [0,\epsilon_0];\mathbb{R}^m)$ such that

$$\begin{gathered}\frac{\partial}{\partial t}f(t,x) = \frac{P(t,x,f(t,x),f(t,0)) - P(t,0,f(t,0),f(t,0))}{x} \\ + \frac{1}{x^2}e^{-(\delta\wedge\epsilon_2)/x}\widetilde{G}(t,x), \\ f(t,0) = A_0(t).\end{gathered} \tag{6.3}$$

Finally, if $\delta_1 \in [0, \epsilon_2 \wedge \delta)$ *and* $\tilde{f}(t,x) \in C([0,\epsilon_1] \times [0,\epsilon_0]; \mathbb{R}^m)$ *satisfies*

$$\begin{gathered} \frac{\partial}{\partial t}\tilde{f}(t,x) = \frac{P(t,x,\tilde{f}(t,x),\tilde{f}(t,0)) - P(t,0,\tilde{f}(t,0),\tilde{f}(t,0))}{x} + O(e^{-\delta_1/x}), \\ \tilde{f}(t,0) = A_0(t), \end{gathered} \tag{6.4}$$

then if $\lambda_0(t) = \min_{1\leq j\leq m} \lambda_j(t)$*, we have that for all* $\gamma \in [0,1)$*,*

$$f(t,x) = \tilde{f}(t,x) + O\Big(e^{-\delta_1/x} + e^{-\frac{\gamma}{x}\int_0^t \lambda_0(s)\,ds}\Big).$$

In the above, the implicit constants in O *are independent of* $(t,x) \in [0,\epsilon_1] \times [0,\epsilon_0]$*.*

Without loss of generality, we may assume $\epsilon_2 \leq 1$ in Proposition 6.1; and we assume this for the rest of the section. The heart of Proposition 6.1 is an abstract existence result, which we now present.

Proposition 6.2. *Fix* $L \geq 0$*. Suppose* $\mathcal{G} : C^{0,L}([0,\epsilon_1]\times[0,\epsilon_2];\mathbb{R}^m) \to C^{0,L}([0,\epsilon_1]\times[0,\epsilon_2];\mathbb{R}^m)$ *is an* (L, G_1, G_2, G_3) *operation (see Definition 4.11). Let* $\mathcal{M}(t) \in C^{(L-1)\vee 0}([0,\epsilon_1];\mathbb{M}^{m\times m})$ *be such that there exists* $R(t) \in C^1([0,\epsilon_1];\mathrm{GL}_m)$ *satisfying* $R(t)\mathcal{M}(t)R(t)^{-1} = \mathrm{diag}(\lambda_1(t),\dots,\lambda_m(t))$*, where* $\lambda_j(t) > 0$*, for all* j,t*. Fix* $A_0 \in C^L([0,\epsilon_1];\mathbb{R}^m)$ *and take* $c_0, C_1, C_2, C_3, C_4 > 0$ *such that* $\min_{t,j}\lambda_j(t) \geq c_0 > 0$*,* $\|R\|_{C^1} \leq C_1$*,* $\|R^{-1}\|_{C^1} \leq C_2$*,* $\|\mathcal{M}^{-1}\|_{C^{(L-1)\vee 0}} \leq C_3$*,* $\|A_0\|_{C^L} \leq C_4$*. Then, there exists* $\delta = \delta(L, m, G_1, G_2, G_3, c_0, C_1, C_2, C_3, C_4) > 0$ *such that there exists a solution* $A(t,w) \in C^{0,L}([0,\epsilon_1]\times[0,\delta\wedge\epsilon_2];\mathbb{R}^m)$ *to the equation*

$$\frac{\partial}{\partial t}A(t,w) = -\mathcal{M}(t)\frac{\partial}{\partial w}A(t,w) + \mathcal{G}(A)(t,w), \quad A(t,0) = A_0(t). \tag{6.5}$$

We prove Proposition 6.2 by induction on L. We begin with the inductive step, which is contained in the next lemma.

Lemma 6.3. *Let* $L \geq 1$*, and* $\mathcal{G}$*,* A_0*,* $\mathcal{M}$*, and* C_4 *be as in Proposition 6.2. For* $B(t,w) \in C^{0,L-1}([0,\epsilon_1]\times[0,\delta];\mathbb{R}^m)$ *let* $\mathcal{I}(A_0,B) = A_0(t) + \int_0^w B(t,r)\,dr$*, and set* $\widetilde{\mathcal{G}}_{A_0}(B)(t,w) := \frac{\partial}{\partial w}\mathcal{G}(\mathcal{I}(A_0,B))(t,w)$*, and let* $B_0(t) = \mathcal{M}(t)^{-1}\big[-\dot{A}_0(t) + \mathcal{G}^0(A_0)(t)\big] \in C^{L-1}([0,\epsilon_1];\mathbb{R}^m)$ *(here* $\mathcal{G}^0$ *is as in Definition 4.11). Then,* $\widetilde{\mathcal{G}}_{A_0}$ *is an* $(L-1, \widetilde{G}_1, \widetilde{G}_2, \widetilde{G}_3)$ *operation, where* $\widetilde{G}_1$*,* $\widetilde{G}_3$*, and* $\widetilde{G}_3$ *can be chosen to depend only on* G_1*,* G_2*,* G_3*, and* C_4*. Furthermore, consider the differential equation*

$$\frac{\partial}{\partial t}B(t,w) = -\mathcal{M}(t)\frac{\partial}{\partial w}B(t,w) + \widetilde{\mathcal{G}}_{A_0}(B)(t,w), \quad B(t,0) = B_0(t). \tag{6.6}$$

Then, solutions to (6.5) *and* (6.6) *are in bijective correspondence in the following sense:*

- *(i) If* $A(t,w) \in C^{0,L}([0,\epsilon_1]\times[0,\delta];\mathbb{R}^m)$ *is a solution to* (6.5)*, then* $B(t,w) = A'(t,w) \in C^{0,L-1}([0,\epsilon_1]\times[0,\delta];\mathbb{R}^m)$ *is a solution to* (6.6)*.*
- *(ii) If* $B(t,w) \in C^{0,L-1}([0,\epsilon_1]\times[0,\delta];\mathbb{R}^m)$ *is a solution to* (6.6)*, then* $A(t,w) = \mathcal{I}(A_0,B)(t,w) \in C^{0,L}([0,\epsilon_1]\times[0,\delta];\mathbb{R}^m)$ *is a solution to* (6.5)*.*

Proof. That $\widetilde{\mathcal{G}}_{A_0}$ is an $(L-1, G_1, G_2, G_3)$ operation follows immediately from the definitions. Suppose $A(t,w) \in C^{0,L}([0,\epsilon_1]\times[0,\delta];\mathbb{R}^m)$ is a solution to (6.5) and set $B(t,w) = A'(t,w) \in C^{0,L-1}([0,\epsilon_1]\times[0,\delta];\mathbb{R}^m)$. Putting $w = 0$ in (6.5) and solving for $B(t,0)$ shows $B(t,0) = B_0(t)$. Taking $\frac{\partial}{\partial w}$ of (6.5) and writing $A(t,w) = \mathcal{I}(A_0,B)(t,w)$ shows B satisfies $\frac{\partial}{\partial t}B(t,w) = -\mathcal{M}(t)\frac{\partial}{\partial w}B(t,w) + \widetilde{\mathcal{G}}_{A_0}(B)(t,w)$. This proves (i).

Suppose $B(t,w) \in C^{0,L-1}([0,\epsilon_1] \times [0,\delta];\mathbb{R}^m)$ is a solution to (6.6) and set $A(t,w) = \mathcal{I}(A_0,B)(t,w) \in C^{0,L}([0,\epsilon_1] \times [0,\delta];\mathbb{R}^m)$. We wish to show (6.5) holds. Clearly, $A(t,0) = A_0(t)$. At $w = 0$, (6.5) is equivalent to $\dot{A}_0(t) + \mathcal{M}(t)B_0(t) - \mathcal{G}^0(A_0)(t) = 0$, and this follows from the choice of $B_0(t)$. Thus, (6.5) follows if:

$$\frac{\partial}{\partial w}\left[\frac{\partial}{\partial t}A(t,w) + \mathcal{M}(t)\frac{\partial}{\partial w}A(t,w) - \mathcal{G}(A)(t,w)\right] = 0. \tag{6.7}$$

But (6.7) is exactly (6.6), completing the proof. □

In light of Lemma 6.3, it suffices to prove Proposition 6.2 in the case $L = 0$. The next lemma reduces this to the case when $\mathcal{M}(t)$ is diagonal and $R(t) = I$.

Lemma 6.4. *Let $L = 0$, and $\mathcal{G}$, A_0, $\mathcal{M}$, $\lambda_1,\dots,\lambda_m$, and R be as in Proposition 6.2. For $B \in C([0,\epsilon_1] \times [0,\epsilon_2];\mathbb{R}^m)$, set $\widetilde{\mathcal{G}}(B)(t,w) := \dot{R}(t)R(t)^{-1}B(t,w) + R(t)\mathcal{G}(R(\cdot)^{-1}B)(t,w)$. Then, $\widetilde{\mathcal{G}}$ is a $(0,\widetilde{G}_1,\widetilde{G}_2,\widetilde{G}_3)$ operation, where $\widetilde{G}_1$, $\widetilde{G}_2$, and $\widetilde{G}_3$ can be chosen to depend only on G_1, G_2, G_3, C_1, and C_2. Set $B_0(t) := R(t)A_0(t)$, and consider the differential equation*

$$\begin{gathered}\frac{\partial}{\partial t}B(t,w) = -\operatorname{diag}(\lambda_1(t),\dots,\lambda_m(t))\frac{\partial}{\partial w}B(t,w) + \widetilde{\mathcal{G}}(B)(t,w),\\ B(t,0) = B_0(t).\end{gathered} \tag{6.8}$$

Then, solutions to (6.5) *and* (6.8) *are in bijective correspondence in the sense that $A(t,w)$ satisfies* (6.5) *if and only if $B(t,w) = R(t)A(t,w)$ satisfies* (6.8).

The above lemma is immediate from the definitions.

Proof of Proposition 6.2. In light of Lemmas 6.3 and 6.4 it suffices to prove the result when $L = 0$, $\mathcal{M}(t) = \operatorname{diag}(\lambda_1(t),\dots,\lambda_m(t))$.
Write $\mathcal{G}(A)(t,w) = (\mathcal{G}_1(A)(t,w),\dots,\mathcal{G}_m(A)(t,w))$, then (6.5) can be written as the system of differential equations

$$\frac{\partial}{\partial t}A_j(t,w) = -\lambda_j(t)\frac{\partial}{\partial w}A_j(t,w) + \mathcal{G}_j(A)(t,w), \quad A(t,0) = A_0(t). \tag{6.9}$$

Here, $A_0(t) \in C([0,\epsilon_1];\mathbb{R}^m)$, and the goal is to find a solution $A(t,w) \in C([0,\epsilon_1] \times [0,\delta \wedge \epsilon_2];\mathbb{R}^m)$ to (6.9) for some $\delta > 0$. The condition $A(t,0) = A_0(t)$ does not uniquely specify the solution to (6.9). We will prove the existence of a solution to (6.9) that, in addition, satisfies $A(0,w) = A_0(0)$.

Let $\delta > 0$, to be chosen later, and set $\delta_0 = \delta \wedge \epsilon_2$. We consider (t,w) in $[0,\epsilon_1] \times [0,\delta_0]$. For each $j \in \{1,\dots,m\}$, let $V_j := \frac{\partial}{\partial t} + \lambda_j(t)\frac{\partial}{\partial w}$. For $u \geq 0$, let $\phi_{j,u}(v) := \int_u^{u+v} \lambda_j(r)\,dr$ and define

$$\psi_{j,u}(r) := \begin{cases} \phi_{j,u}^{-1}(r), & \text{if } \int_u^{\epsilon_1} \lambda_j(s)\,ds \geq r,\\ \epsilon_1, & \text{if } \int_u^{\epsilon_1} \lambda_j(s)\,ds \leq r.\end{cases}$$

V_j foliates $[0,\epsilon_1] \times [0,\delta_0]$ into integral curves. We parameterize these integral curves by $u \in [-\delta_0,\epsilon_1]$: when $u \leq 0$ we use the integral curve starting at $(0,-u)$ and when $u \geq 0$ we use the integral curve starting at $(u,0)$.

More precisely, set

$$\begin{aligned}U^j_{\epsilon_1,\delta_0} := \big\{(u,v) : u \in [-\delta_0,\epsilon_1],\ &\text{and if } u \leq 0 \text{ then } v \in [0,\psi_{j,0}(\delta_0+u)],\\ &\text{and if } u \geq 0 \text{ then } v \in [0,\psi_{j,u}(\delta_0)]\big\}.\end{aligned}$$

Note that for $(u,v)\in U^j_{\epsilon_1,\delta_0}$, $v\le \delta_0/c_0\le \delta/c_0$. Define $H_j: U^j_{\epsilon_1,\delta_0}\to[0,\epsilon_1]\times[0,\delta_0]$ by

$$H_j(u,v):=\begin{cases}(v,-u+\int_0^v\lambda_j(r)\,dr), & \text{if } u\le 0,\\ (u+v,\int_u^{u+v}\lambda_j(r)\,dr), & \text{if } u\ge 0.\end{cases}$$

Then, for each $u\in[-\delta_0,\epsilon_1]$, $H_j(u,\cdot)$ parameterizes and integral curve of V_j: when $u\le 0$, it parameterizes the curve starting at $(0,-u)$ and when $u\ge 0$, it parameterizes the curve starting at $(u,0)$. As such, $H_j: U^j_{\epsilon_1,\delta_0}\to[0,\epsilon_1]\times[0,\delta_0]$ is a homeomorphism.

Define $L_0\in C([-\delta_0,\epsilon_1];\mathbb{R}^m)$ by $L_0(u)=A_0(u)$ for $u\ge 0$ and $L_0(u)=A_0(0)$ for $u\le 0$. We consider $L=(L_1,\dots,L_m)$ with $L_j(u,v)\in C(U^j_{\epsilon_1,\delta_0})$. We related L and A by the correspondence $L_j(u,v)=A_j\circ H_j(u,v)$. We consider the system of differential equations

$$\frac{\partial}{\partial v}L_j(u,v)=\mathcal{G}_j(L_1\circ H_1^{-1},\dots,L_m\circ H_m^{-1})(H_j(u,v)),\quad L_j(u,0)=L_{0,j}(u), \tag{6.10}$$

where $L_0=(L_{0,1},\dots,L_{0,m})$. Note that if L satisfies (6.10), then A satisfies (6.9) and has $A(0,w)=A_0(0)$. Thus, we complete the proof by finding $\delta>0$ such that there is a solution to (6.10). To do this, we utilize the contraction mapping principle.

For $M>0$, let

$$\mathcal{F}_{M,\epsilon_1,\delta_0}:=\{L=(L_1,\dots,L_j):L_j\in C(U^j_{\epsilon_1,\delta_0}),\|L_j\|_{C^0}\le M\},$$

and we give $\mathcal{F}_{M,\epsilon_1,\delta_0}$ the metric $\rho(L,\widetilde{L})=\max_{1\le j\le m}\|L_j-\widetilde{L}_j\|_{C^0}$, making $\mathcal{F}_{M,\epsilon_1,\delta_0}$ into a complete metric space.

For $L\in\mathcal{F}_{M,\epsilon_1,\delta_0}$, define $\mathcal{T}(L)=(\mathcal{T}_1(L),\dots,\mathcal{T}_m(L))$, where $\mathcal{T}_j(L)\in C(U^j_{\epsilon_1,\delta_0})$ is defined by

$$\mathcal{T}_j(L)(u,v):=L_{0,j}(u)+\int_0^v\mathcal{G}_j(L_1\circ H_1^{-1},\dots,L_m\circ H_m^{-1})(H_j(u,v'))\,dv'.$$

We wish to pick M and δ so that $\mathcal{T}:\mathcal{F}_{M,\epsilon_1,\delta_0}\to\mathcal{F}_{M,\epsilon_1,\delta_0}$ is a strict contraction. First, we pick M and δ so that $\mathcal{T}:\mathcal{F}_{M,\epsilon_1,\delta_0}\to\mathcal{F}_{M,\epsilon_1,\delta_0}$. Indeed, we have

$$\begin{aligned}|\mathcal{T}_j(L)(u,v)|&\le\|A_0\|_{C^0}+\int_0^v G_1(\sqrt{m}M)\,dr\\ &=\|A_0\|_{C^0}+vG_1(\sqrt{m}M)\le C_4+\frac{\delta}{c_0}G_1(\sqrt{m}M),\end{aligned}$$

where in the last step we have used $v\le\frac{\delta}{c_0}$, as noted earlier. Set $M=2C_4$, then if $\delta\le c_0C_4G_1(\sqrt{m}M)^{-1}$, we have $\mathcal{T}:\mathcal{F}_{M,\epsilon_1,\delta_0}\to\mathcal{F}_{M,\epsilon_1,\delta_0}$.

We now wish to show that if we make δ sufficiently small, $\mathcal{T}$ is a strict contraction. Consider, for $L,\widetilde{L}\in\mathcal{F}_{M,\epsilon_1,\delta_0}$ we have

$$|\mathcal{T}_j(L)(u,v)-\mathcal{T}_j(\widetilde{L})(u,v)|\le\int_0^v G_2(\sqrt{m}M)\rho(L,\widetilde{L})\,dr\le\frac{\delta}{c_0}G_2(\sqrt{m}M)\rho(L,\widetilde{L}),$$

where we have again used $v\le\frac{\delta}{c_0}$. Thus, $\rho(\mathcal{T}(L),\mathcal{T}(\widetilde{L}))\le\frac{\delta}{c_0}G_2(\sqrt{m}M)\rho(L,\widetilde{L})$. Thus, if $\delta=\left(\frac{1}{2}c_0G_2(\sqrt{m}M)^{-1}\right)\wedge\left(c_0C_4G_1(\sqrt{m}M)^{-1}\right)$, $\mathcal{T}:\mathcal{F}_{M,\epsilon_1,\delta_0}\to\mathcal{F}_{M,\epsilon_1,\delta_0}$ is a strict contraction.

The contraction mapping principle applies to show that there is a fixed point $L \in \mathcal{F}_{M,\epsilon_1,\delta}$ with $\mathcal{T}(L) = L$. This L is the desired solution to (6.10), which completes the proof. □

Proof of Proposition 6.1. We begin with the existence of $\delta > 0$ and $A(t,w) \in C^{0,2}([0,\epsilon_1] \times [0,\delta \wedge \epsilon_2];\mathbb{R}^m)$ satisfying (6.1). Proposition 4.18 shows that (6.1) is of the form covered by the case $L = 2$ of Proposition 6.2. Thus, the existence of δ and A follow from Proposition 6.2.

Let f be given by (6.2), so that for $x > 0$,

$$\frac{\partial}{\partial t} f(t,x) = \frac{1}{x} \int_0^{\delta\wedge\epsilon_2} e^{-w/x} \widehat{P}(t, A(t,\cdot), A(t,0))\, dw.$$

From here, (6.3) follows from Lemma 4.17.

Finally, suppose $\tilde{f}$ is as in the statement of the proposition, and set $g(t,x) = f(t,x) - \tilde{f}(t,x)$. Since $f(t,0) = \tilde{f}(t,0) = A_0(t)$, combining (6.3) with (6.4) shows that there exists a bounded function $\widehat{G}(t,x) : [0,\epsilon_1] \times (0,\epsilon_0] \to \mathbb{R}^m$ such that for $x \in (0,\epsilon_0]$,

$$\begin{aligned} \frac{\partial}{\partial t} g(t,x) &= \frac{P(t,x,f(t,x),A_0(t)) - P(t,x,\tilde{f}(t,x),A_0(t))}{x} + e^{-\delta_1/x}\widehat{G}(t,x) \\ &= -\frac{1}{x}\mathcal{M}(t,x) + e^{-\delta_1/x}\widehat{G}(t,x), \end{aligned} \tag{6.11}$$

where $\mathcal{M}(t,x) = -\int_0^1 d_y P(t,x,sf(t,x) + (1-s)\tilde{f}(t,x), A_0(t))\, ds$. In particular, note that $\mathcal{M}(t,0) = \mathcal{M}(t)$, since $f(t,0) = \tilde{f}(t,0) = A_0(t)$. Solving (6.11) we have

$$\begin{aligned} g(t,x) &= \overleftarrow{\exp}\Big(-\frac{1}{x} \int_0^t \mathcal{M}(s,x)\, ds\Big) g(0,x) \\ &\quad + e^{-\delta_1/x} \int_0^t \overleftarrow{\exp}\Big(-\frac{1}{x} \int_s^t \mathcal{M}(r,x)\, dr\Big) \widehat{G}(s,x)\, ds \end{aligned}$$

Applying Proposition 5.2, we have for all $\gamma \in [0,1)$,

$$\begin{aligned} |g(t,x)| &\lesssim e^{-\frac{\gamma}{x}\int_0^t \lambda_0(s)\, ds} |g(0,x)| + e^{-\delta_1/x} \int_0^t e^{-\frac{\gamma}{x}\int_s^t \lambda_0(r)\, dr}\, ds \\ &= O\Big(e^{-\delta_1/x} + e^{-\frac{\gamma}{x}\int_0^t \lambda_0(s)\, ds}\Big), \end{aligned}$$

completing the proof. □

Proof of Theorem 2.2. Let $\tilde{f}(t,x) \in C([0,\epsilon_1] \times [0,\epsilon_0];\mathbb{R}^m)$ be the function $f(t,x)$ from Proposition 6.1. Thus, $\tilde{f}$ satisfies (6.3) for some function $\widetilde{G}(t,x) \in C([0,\epsilon_1] \times [0,\epsilon_0];\mathbb{R}^m)$.

For some $\delta_0 > 0$, we will construct $f(t,x) \in C([0,\epsilon_1] \times [0,\delta_0];\mathbb{R}^m)$ as in the statement of the theorem. We do this by considering $f(t,x)$ of the form $f(t,x) = \tilde{f}(t,x) + g(t,x)$, where $g(t,x) \in C([0,\epsilon_1] \times [0,\delta_0];\mathbb{R}^m)$. Notice that $f(t,x)$ satisfies the conclusions of the theorem if $g(t,x)$ satisfies the following:

- $g(t,0) = 0$, for all $t \in [0,\epsilon_1]$ (so that $f(t,0) = \hat{f}(t,0) = A_0(t)$).
- $g(0,x) = g_0(x)$, where $g_0(x) = f_0(x) - \tilde{f}(0,x) \in C([0,\epsilon_0];\mathbb{R}^m)$. Since $f_0(0) = A_0(0) = \tilde{f}(0,0)$, we have $g_0(0) = 0$.

- $$\frac{\partial}{\partial t}g(t,x) = \frac{P(t,x,\tilde{f}(t,x)+g(t,x),A_0(t)) - P(t,x,\tilde{f}(t,x),A_0(t))}{x} \\ + G_1(t,x,g(t,x)), \tag{6.12}$$

  where $G_1(t,x,g(t,x)) = G(t,x,\tilde{f}(t,x)+g(t,x),A_0(t)) - \frac{1}{x^2}e^{-(\delta\wedge\epsilon_2)/x}\widetilde{G}(t,x)$ and δ is as in Proposition 6.1.

Set
$$\widetilde{\mathcal{M}}(t,x,z) := -\int_0^1 (d_yP)(t,x,\tilde{f}(t,x)+sz,A_0(t))\,ds,$$
so that
$$\widetilde{\mathcal{M}}(t,x,z)z = P(t,x,\tilde{f}(t,x),A_0(t)) - P(t,x,\tilde{f}(t,x)+z,A_0(t)).$$
Using this, (6.12) can be re-written as
$$\frac{\partial}{\partial t}g(t,x) = -\frac{1}{x}\widetilde{\mathcal{M}}(t,x,g(t,x))g(t,x) + G_1(t,x,g(t,x)).$$
Also note note that $\widetilde{\mathcal{M}}(t,0,0) = \mathcal{M}(t)$, where $\mathcal{M}(t)$ is as in the statement of the theorem. From here, the existence of $g(t,x)$ follows from Proposition 5.4, completing the proof. □

Proof of Theorem 2.4. The representation (2.2) follows by applying Proposition 6.1 with f playing the role $\tilde{f}$, and δ_0 playing the role of ϵ_0. The uniqueness of the representation follows from Corollary 8.4. □

7. Uniqueness

The purpose of this section is to prove Theorems 2.5, 2.6, Proposition 2.8, and Theorem 2.10. The main remaining ingredient needed is an abstract uniqueness result, which we present first.

7.1. An abstract uniqueness result.

Proposition 7.1. *Let $m \geq 1$, $\epsilon_1, \epsilon_2 > 0$. Let $\mathcal{M}(t) \in C([0,\epsilon_1];\mathbb{M}^{m\times m})$ be such that there exists $R(t) \in C^1([0,\epsilon_1];\mathrm{GL}_m)$ with $R(t)\mathcal{M}(t)R(t)^{-1} = \mathrm{diag}(\lambda_1(t),\dots,\lambda_m(t))$ where each $\lambda_j(t) > 0$, for all $t \in [0,\epsilon_1]$. Suppose $g(t,w) \in C([0,\epsilon_1]\times[0,\epsilon_2];\mathbb{R}^m)$ satisfies the differential equation*
$$\frac{\partial}{\partial t}g(t,w) = \mathcal{M}(t)\frac{\partial}{\partial w}g(t,w) + F(t,w), \quad g(0,w) = 0, \forall w,$$
where $F(t,w)$ satisfies $|F(t,w)| \leq C\sup_{0\leq r\leq w}|g(t,r)|$.
Set $\gamma_0(t) := \max_{1\leq j\leq m}\int_0^t \lambda_j(s)\,ds$, and
$$\delta_0 := \begin{cases} \gamma_0^{-1}(\epsilon_2), & \text{if } \gamma_0(\epsilon_1) \geq \epsilon_2, \\ \epsilon_1, & \text{otherwise.}\end{cases}$$
Then $g(t,0) = 0$ for $0 \leq t \leq \delta_0$.

Proof. We begin by showing that it suffices to prove the result in the case when $\mathcal{M}(t) = \mathrm{diag}(\lambda_1(t),\dots,\lambda_m(t))$. Indeed, if $g(t,w)$ is as above and $h(t,w) = R(t)g(t,w)$, then $h(t,w)$ satisfies
$$\frac{\partial}{\partial t}h(t,w) = \mathrm{diag}(\lambda_1(t),\dots,\lambda_m(t))\frac{\partial}{\partial w}h(t,w) + R(t)F(t,w) + \dot{R}(t)R(t)^{-1}h(t,w),$$

$$h(0,w)=0,\forall w.$$

Thus, if we have the result for h, the result for g follows.

For the rest of the proof, we assume $\mathcal{M}(t)=\operatorname{diag}(\lambda_1(t),\dots,\lambda_m(t))$. Write $g(t,w)=(g_1(t,w),\dots,g_m(t,w))$ and $F(t,w)=(F_1(t,w),\dots,F_m(t,w))$. Thus we are interested in the system of equations

$$\frac{\partial}{\partial t}g_j(t,w)=\lambda_j(t)\frac{\partial}{\partial w}g_j(t,w)+F_j(t,w),\quad g_j(0,w)=0, \tag{7.1}$$

under the hypothesis $|F_j(t,w)|\le C\sup_{0\le r\le w}|g(t,r)|$. For each $j\in\{1,\dots,m\}$, set $\gamma_j(t)=\int_0^t\lambda_j(s)\,ds$, and let $Y_j=\frac{\partial}{\partial t}-\lambda_j(t)\frac{\partial}{\partial w}$. Let $H_j(u,v)=\big(v,u-\int_0^v\lambda_j(s)\,ds\big)$ (we will be more precise about the domain of H_j in a moment). Note that H_j is invertible with $H_j^{-1}(v,r)=\big(r+\int_0^v\lambda_j(s)\,ds,v\big)$. Finally set

$$\delta_j:=\begin{cases}\gamma_j^{-1}(\epsilon_2), & \text{if } \gamma_j(\epsilon_1)\ge\epsilon_2,\\ \epsilon_1, & \text{otherwise.}\end{cases}$$

For $0\le j\le m$, set $W_j:=\{(t,w):0\le t\le\delta_j,0\le w\le\epsilon_2-\gamma_j(t)\}$, and note that for $j\in\{1,\dots,m\}$, $W_0\subseteq W_j$. Furthermore, for $j\in\{1,\dots,m\}$, Y_j foliates W_j into the integral curves of Y_j. Indeed, for $u\in[0,\epsilon_2]$, define

$$r_j(u):=\begin{cases}\gamma_j^{-1}(u), & \text{if } \gamma_j(\epsilon_1)\ge u,\\ \epsilon_1, & \text{otherwise.}\end{cases}$$

Note that $r_j(\epsilon_2)=\delta_j$. As v ranges from 0 to $r_j(u)$, $H_j(u,v)$ parameterizes the integral curve of Y_j in W_j which starts at $(0,u)$. Let $U_j:=\{(u,v):u\in[0,\epsilon_2],v\in[0,r_j(u)]\}$. By the above discussion, $H_j:U_j\to W_j$ is a homeomorphism. Set $V_j:=H_j^{-1}(W_0)\subseteq U_j$.

For $v\in[0,\delta_0]$, we define

$$E(v):=\sup\{|g(v,w)|:(v,w)\in W_0\}=\sup\{|g(v,w)|:w\in[0,\epsilon_2-\gamma_0(v)]\}.$$

Clearly $E(0)=0$, since $g(0,w)=0$. We will show $E(v)=0$ for $v\in[0,\delta_0]$, which will complete the proof.

We claim that if $(u,v)\in V_j$, then for all $v'\in[0,v]$, $(u,v')\in V_j$. Indeed, note that

$$(u,v)\in V_j\Leftrightarrow v\in[0,\delta_0]\text{ and }0\le u-\int_0^v\lambda_j(s)\,ds\le\epsilon_2-\max_k\int_0^v\lambda_k(s)\,ds.$$

So if $(u,v)\in V_j$ and $v'\in[0,v]$, then clearly $v'\in[0,\delta_0]$ and adding $\int_{v'}^v\lambda_j(s)\,ds$ to the above equation, we see

$$\begin{aligned}0&\le\int_{v'}^v\lambda_j(s)\,ds\le u-\int_0^{v'}\lambda_j(s)\,ds\\ &\le\epsilon_2-\max_k\int_0^v\lambda_k(s)\,ds+\int_{v'}^v\lambda_j(s)\,ds\le\epsilon_2-\max_k\int_0^{v'}\lambda_k(s)\,ds.\end{aligned}$$

Thus, $(u,v')\in V_j$, proving the claim.

Set $l_j(u,v)=g_j\circ H_j(u,v)$. Then (7.1) shows that

$$\frac{\partial}{\partial v}l_j(u,v)=F_j\circ H_j(u,v),\quad l_j(u,0)=g_j(0,u)=0.$$

Hence, $l_j(u,v)=\int_0^v F_j\circ H_j(u,v')\,dv'$.

For $(u,v)\in V_j$, $H_j(u,v)\in W_0$ and therefore $u-\int_0^v \lambda_j(s)\,ds\leq \epsilon_2-\gamma_0(v)$. Hence, for $(u,v)\in V_j$,

$$|F_j\circ H_j(u,v)|\lesssim \sup_{0\leq r\leq u-\int_0^v \lambda_j(s)\,ds} |g(v,r)|\leq \sup_{0\leq r\leq \epsilon_2-\gamma_0(v)} |g(v,r)|=E(v).$$

Thus, for $(u,v)\in V_j$, if $v'\in[0,v]$ we have $(u,v')\in V_j$ and therefore $|F_j\circ H_j(u,v')|\lesssim E(v')$. We conclude, for $(u,v)\in V_j$,

$$|l_j(u,v)|=\Big|\int_0^v F_j\circ H_j(u,v')\,dv'\Big|\lesssim \int_0^v E(v')\,dv'.$$

Therefore, for $v\in[0,\delta_0]$,

$$\sup\{|g_j(v,w)|:(v,w)\in W_0\}=\sup\{|l_j(u,v)|:(u,v)\in V_j\}\lesssim \int_0^v E(v')\,dv',$$

and so $E(v)\lesssim \int_0^v E(v')\,dv'$. Grönwall's inequality implies $E(v)=0$ for $v\in[0,\delta_0]$, completing the proof. □

7.2. **Completion of proofs.**

Proof of Theorem 2.6. Set $\tilde f(t,x)=f(\epsilon_1-t,x)$, $\tilde A_0(t)=f(\epsilon_1-t,0)=\tilde f(t,0)$, $\widetilde P(t,x,y,z)=-P(\epsilon_1-t,x,y,z)$. $\tilde f$ satisfies, for all $\gamma\in[0,\epsilon_2)$,

$$\frac{\partial}{\partial t}\tilde f(t,x)=\frac{\widetilde P(t,x,\tilde f(t,x),\tilde f(t,0))-\widetilde P(t,0,\tilde f(t,0),\tilde f(t,0))}{x}+O(e^{-\gamma/x}),$$
$$\tilde f(t,0)=\tilde A_0(t).$$

By the hypotheses of the theorem, $\widetilde P$ and $\tilde A_0$ satisfy all the hypotheses of P and A_0 in Proposition 6.1. Here, $\tilde\lambda_j(t)=\lambda_j(\epsilon_1-t)$ plays the role of λ_j in that proposition. Thus, let δ be as in Proposition 6.1 and $\tilde A\in C^{0,2}([0,\epsilon_1]\times[0,\delta];\mathbb{R}^m)$ be A from Proposition 6.1 when applied to $\widetilde P$ and $\tilde A_0$. Proposition 6.1 shows that for all $\gamma\in[0,1)$, if $\tilde\lambda_0(t)=\min_{1\leq j\leq m}\tilde\lambda_j(t)$,

$$\frac{1}{x}\int_0^{\delta\wedge\epsilon_2} e^{-w/x}\tilde A(t,w)\,dw=\tilde f(t,x)+O\Big(e^{-\gamma(\epsilon_2\wedge\delta)/x}+e^{-\frac{\gamma}{x}\int_0^t\tilde\lambda_0(s)\,ds}\Big). \tag{7.2}$$

Define $A(t,x):=\tilde A(\epsilon_1-t,x)$. Replacing t with ϵ_1-t in (7.2) and using that $\tilde A$ satisfies (6.1) (with P and A_0 replaced by $\widetilde P$ and $\tilde A_0$), (2.4) and (2.5) follow. Finally, the stated uniqueness of (2.5) follows from Corollary 8.4. □

Proof of Proposition 2.8. Let $g(t,w)=A(t,w)-B(t,w)$. (2.6) combined with Proposition 4.19 shows

$$\frac{\partial}{\partial t}g(t,w)=\mathcal{M}(t)\frac{\partial}{\partial w}g(t,w)+F(t,w),\quad g(0,w)=0,$$

where $|F(t,w)|\lesssim\sup_{0\leq r\leq w}|g(t,r)|$. $\mathcal{M}(t)$ and $g(t,w)$ satisfy all the hypotheses of Theorem 7.1 (with ϵ_2 replaced by δ'), and the result follows from Theorem 7.1. □

Proof of Theorem 2.10. Applying Theorem 2.6 to f_1 and f_2 we see that there exists $\delta=\delta(m,D,c_0,C_0,C_1,C_2,C_3,C_4)>0$ and $A_1,A_2\in C^{0,2}([0,\epsilon_1]\times[0,\delta\wedge\epsilon_2];\mathbb{R}^m)$ such that for $k=1,2$, $\frac{\partial}{\partial t}A_k(t,w)=\widehat P(t,A_k(t,\cdot),A_k(t,0))(w)$, $A_k(t,0)=f_k(t,0)$, and for all $\gamma\in[0,1)$,

$$f_k(0,x)=\frac{1}{x}\int_0^{\delta\wedge\epsilon_2} e^{-w/x}A_k(0,w)\,dw+O\left(e^{-\gamma(\delta\wedge\epsilon_2)/x}+e^{-\frac{\gamma}{x}\int_0^{\epsilon_1}\lambda_0^k(s)\,ds}\right).$$

The uniqueness of this representation as described in Theorem 2.6, combined with (2.7), shows that $A_1(0,w) = A_2(0,w)$ for $w \in [0, \delta' \wedge r]$.

From here, Proposition 2.8 shows that $A_1(t,0) = A_2(t,0)$ for $t \in [0, \delta_0]$. Since $A_k(t,0) = f_k(t,0)$, the result follows. $\square$

Proof of Theorem 2.5. This follows from the reconstruction procedure discussed in Remark 2.9. $\square$

8. Appendix: Laplace transform

The purpose of this section is to discuss the following Paley-Wiener type theorem for the Laplace transform, which is presented in [22].

Theorem 8.1 ([22, Theorem A.2.2])**.** *Fix $\epsilon > 0$ and suppose $f, g \in L^1([0,\epsilon])$ and for some $s \in [0,\epsilon]$,*

$$\int_0^\epsilon e^{-\lambda t} f(t)\, dy = \int_0^\epsilon e^{-\lambda t} g(t)\, dy + O(e^{-s\lambda}), \quad \text{as } \lambda \uparrow \infty.$$

Then $f \equiv g$ on $[0,s)$.

In this section, we offer a discussion of this result, along with two proofs. The first is closely related to the proof in [22], though may be somewhat simpler. This first proof uses complex analysis. The second proof uses only real analysis and is more constructive.

Lemma 8.2. *Fix $\epsilon > 0$ and suppose $a \in L^1([0,\epsilon])$. For each $\lambda \geq 1$, let $F(\lambda) := \int_0^\epsilon e^{-\lambda t} a(t)\, dt$. Suppose $|F(\lambda)| = O(e^{-\epsilon\lambda})$ as $\lambda \uparrow \infty$. Then, $a = 0$.*

Proof. For $\lambda \in \mathbb{C}$, set $G(\lambda) = \int_0^\epsilon e^{(\epsilon - t)\lambda} a(t)\, dt = e^{\epsilon\lambda} F(\lambda)$. We have:

(a) G is entire.
(b) $\sup_{\lambda \in \mathbb{R}} |G(i\lambda)| < \infty$.
(c) $\sup_{\lambda \in [0,\infty)} |G(\lambda)| < \infty$ (this is a restatement of the fact that $|F(\lambda)| = O(e^{-\epsilon\lambda})$).
(d) $\sup_{\lambda \in (-\infty,0]} |G(\lambda)| < \infty$.
(e) $|G(\lambda)| \leq C e^{\epsilon|\lambda|}$, for all $\lambda \in \mathbb{C}$.

Item (e) shows that we may apply the Phragmén-Lindelöf principle in sectors of angle less than π. (b), (c), and (d) show $|G(\lambda)|$ is bounded on each coordinate axis, and so the Phragmén-Lindelöf principle shows that G is bounded in each quadrant. We conclude that G is a bounded entire function and therefore Liouville's theorem implies that G is constant. Since $\lim_{\lambda \to -\infty} G(\lambda) = 0$, we see that $G(\lambda) = 0$ for all λ. Thus, $0 = F(\lambda) = \int_0^\epsilon e^{-t\lambda} a(t)\, dt$ for all λ. Standard theorems now show $a = 0$. $\square$

Proof of Theorem 8.1. This follows immediately from Lemma 8.2. $\square$

In this paper, we use Theorem 8.1 and Lemma 8.2 via the next corollary.

Corollary 8.3. *Suppose $a \in C([0,\epsilon])$ satisfies $|\lambda \int_0^\epsilon e^{-t\lambda} a(t)\, dt| = O(e^{-\epsilon\lambda})$, as $\lambda \uparrow \infty$. Then, $a = 0$.*

The above corollary follows immediately from Lemma 8.2.

Corollary 8.4. *Let $\epsilon, \epsilon' > 0$ and suppose $a, b \in C([0,\epsilon'])$ satisfy*
$$\frac{1}{x}\int_0^{\epsilon'} e^{-w/x} a(w)\,dw = \frac{1}{x}\int_0^{\epsilon'} e^{-w/x} b(w)\,dw + O(e^{-\epsilon/x}) \text{ as } x \downarrow 0.$$
Then, $a(w) = b(w)$ for $w \in [0, \epsilon \wedge \epsilon']$.

The above corollary follows from Corollary 8.3 by setting $\lambda = \frac{1}{x}$.

It is interesting to note that Lemma 8.2 (and therefore Theorem 8.1) can be easily proved without complex analysis, and we present this next. Thus, all of the results in this paper can be proved without complex analysis. First, we note that Corollary 8.3 actually implies Lemma 8.2.

Proof of Lemma 8.2 given Corollary 8.3. Suppose that $a \in L^1([0,\epsilon])$ and that $\int_0^\epsilon e^{-\lambda t} a(t)\,dt = O(e^{-\epsilon\lambda})$; we wish to show $a = 0$. Integration by parts shows
$$e^{-\lambda\epsilon}\int_0^\epsilon a(s)\,ds + \lambda \int_0^\epsilon e^{-\lambda t}\int_0^t a(s)\,ds\,dt = \int_0^\epsilon e^{-\lambda t} a(t)\,dt = O(e^{-\epsilon\lambda}).$$
Thus $\lambda \int_0^\epsilon e^{-\lambda t}\int_0^t a(s)\,ds\,dt = O(e^{-\epsilon\lambda})$, and Corollary 8.3 shows $\int_0^t a(s)\,ds = 0$, for all t. Thus, $a = 0$, as desired. □

Hence, to prove Lemma 8.2 using only real analysis, it suffices to prove Corollary 8.3 using only real analysis, to which we now turn.

Proposition 8.5. *Fix $\epsilon > 0$, and let $a \in C([0,\epsilon])$. Suppose*
$$\sup_{n\in\mathbb{N}} \big| n \int_0^\epsilon e^{nt} a(t)\,dt \big| < \infty.$$
Then, $a = 0$.

Remark 8.6. Two remarks are in order:

- If $\int_0^\epsilon e^{nt} a(t)\,dt = 0$, for all $n \in \mathbb{N}$, then the classical Weierstrass approximation easily yields that $a = 0$. It therefore makes sense to consider Theorem 8.5 a "quantitative Weierstrass approximation theorem."
- By replacing $a(t)$ with $a(\epsilon - t)$, Theorem 8.5 implies Corollary 8.3.

Lemma 8.7. *Fix $\epsilon > 0$, and let $a \in C([0,\epsilon])$. Suppose*
$$\sup_{n\in\mathbb{N}} \big| n \int_0^\epsilon e^{nt} a(t)\,dt \big| < \infty.$$
Then, $a(0) = 0$.

Proof of Theorem 8.5 given Lemma 8.7.
Let $\delta \in [0,\epsilon)$, and set $C = \sup_{n\in\mathbb{N}} |n \int_0^\epsilon e^{nt} a(t)\,dt|$. Then, we have
$$\begin{aligned} \big| n \int_\delta^\epsilon e^{nt} a(t)\,dt \big| &\le \big| n \int_0^\epsilon e^{nt} a(t)\,dt \big| + \big| n \int_0^\delta e^{nt} a(t)\,dt \big| \\ &\le C + \Big(\sup_{t\in[0,\delta]} |a(t)| \Big) n \int_0^\delta e^{nt}\,dt \le D e^{n\delta}, \end{aligned}$$
for some constant D which does not depend on n. Multiplying both sides of the above inequality by $e^{-n\delta}$ and applying the change of variables $s = t - \delta$, we have
$$\big| n \int_0^{\epsilon-\delta} e^{ns} a(s+\delta)\,ds \big| \le D, \quad \forall n \in \mathbb{N}.$$

Lemma 8.7 now implies $a(\delta) = 0$. As $\delta \in [0, \epsilon)$ was arbitrary, this completes the proof. $\square$

We close this appendix with a proof of Lemma 8.7. Fix $\epsilon > 0$. For $j, N \in \mathbb{N}$, define

$$A_j := \int_1^\infty y^{j-1} e^{-y}\, dy, \quad I_{j,N} := \int_1^{e^{\epsilon N}} y^{j-1} e^{-y}\, dy,$$

so that $A_j \le A_{j+1}$ and $\lim_{N\to\infty} I_{j,N} = A_j$. Set

$$f_{j,N}(t) := \frac{N}{I_{j,N}} e^{Njt} e^{-e^{Nt}}.$$

Lemma 8.8. *$f_{j,N}$ has the following properties.*

- $\int_0^\epsilon f_{j,N}(t)\, dt = 1$.
- *For j fixed,* $\lim_{N\to\infty} f_{j,N}(x) = 0$ *uniformly on compact subsets of* $(0, \epsilon]$.
- *For* $a \in C([0, \epsilon])$, $\lim_{N\to\infty} \int_0^\epsilon f_{j,N}(t) a(t)\, dt = a(0)$.

Proof. The last property follows from the first two. The second property is immediate from the definitions. We prove the first property. Applying the change of variables $y = e^{Nt}$, we have

$$\int_0^\epsilon f_{j,N}(t)\, dt = \frac{1}{I_{j,N}} \int_0^{e^{\epsilon N}} y^{j-1} e^{-y}\, dy = 1.$$

$\square$

Proof of Lemma 8.7. Let a be as in the statement of the lemma, and set $C := \sup_{n\in\mathbb{N}} |n \int_0^\epsilon e^{nt} a(t)\, dt| < \infty$. Using Lemma 8.8, we have

$$\begin{aligned} |a(0)| &= \lim_{N\to\infty} \Big| \int_0^\epsilon f_{j,N}(t) a(t)\, dt \Big| \le \liminf_{N\to\infty} \frac{N}{I_{j,N}} \sum_{k=0}^\infty \Big| \int_0^\epsilon e^{Njt} \frac{(-e^{Nt})^k}{k!} a(t)\, dt \Big| \\ &\le \liminf_{N\to\infty} \frac{N}{I_{j,N}} \sum_{k=0}^\infty \frac{C}{N(k+j)(k!)} = \frac{1}{A_j} \sum_{k=0}^\infty \frac{C}{(k+j)(k!)}. \end{aligned}$$

Taking the limit of the above equation as $j \to \infty$ shows $a(0) = 0$, completing the proof. $\square$

9. Appendix: Pseudodifferential operators and the Calderón problem

The results in this paper can serve as a model case for a more difficult (and still open) problem involving pseudodifferential operators, which arises in the famous Calderón problem.

Let N be a smooth manifold of dimension $n \ge 2$, and let $\Psi\mathrm{DO}^s$ denote the space of standard pseudodifferential operators on N of order $s \in \mathbb{R}$. We use x to denote points in N. For $T \in \Psi\mathrm{DO}^s$, let $\sigma(T)$ denote the principal symbol of T. Let $t \mapsto \Gamma(t)$ be a smooth map $[0, \epsilon_1] \to \Psi\mathrm{DO}^1$ such that $\Gamma(t)$ is elliptic for all t, and such that

$$\sigma(\Gamma(t))(x, \xi) = \Big(|g(x,t)| \sum_{\alpha,\beta} g^{\alpha,\beta}(x,t) \xi_\alpha \xi_\beta \Big)^{1/2},$$

where $g_{\alpha,\beta}(\cdot, t)$ is a Riemannian metric on N for each $t \in [0, \epsilon_1]$, $|g(x,t)|$ denotes $\det g_{\alpha,\beta}(x,t)$, and ξ denotes the frequency variable. In what follows, we suppress the dependance on x. By taking principal symbols, the function $\Gamma(t) \mapsto |g(t)|g^{\alpha,\beta}(t)$ is well defined. Also, $\det(|g|g^{\alpha,\beta}) = |g|^{n-1}$, so (since $n \geq 2$), $\Gamma(t) \mapsto |g(t)|$ is well-defined. We conclude that $\Gamma(t) \mapsto g_{\alpha,\beta}(t)$ is well defined.

Let $\Delta_{g(t)}$ denote the Laplace-Beltrami operator associated to $g(t)$ (with the convention that $\Delta_{g(t)}$ is a negative operator). We consider the following, well-known, differential equation:

$$\frac{\partial}{\partial t}\Gamma(t) = |g(t)|^{1/2}\Big(|g(t)|^{-1/2}\Gamma(t)\Big)^2 - \Big(-|g(t)|^{1/2}\Delta_{g(t)}\Big). \tag{9.1}$$

Since $g(t)$ is a function of $\Gamma(t)$, (9.1) can be considered as a differential equation involving only $\Gamma(t)$.

Conjecture 9.1. *If N is compact and without boundary, the differential equation* (9.1) *has uniqueness. I.e., if $\Gamma_1(t)$ and $\Gamma_2(t)$ are as above and both satisfy* (9.1) *and $\Gamma_1(0) = \Gamma_2(0)$, then $\Gamma_1(t) = \Gamma_2(t)$, for all t.*

Note that the left hand side of (9.1) is in $\Psi\mathrm{DO}^1$, while the right hand side is a difference of two elements of $\Psi\mathrm{DO}^2$, but this is possible since the principal symbols of the two terms on the right hand side cancel. This makes this equation similar to the ones studied in this paper, as we discuss next.

Remark 9.2. Other than this cancelation, as far as the methods in this paper are concerned, there seems to be nothing particularly special about the form of (9.1) and one could state many other versions of Conjecture 9.1 using different polynomials. We will see in Section 9.2, and as is well-known, (9.1) arises naturally in the Calderón problem. Thus, if one replaces (9.1) with a more general polynomial differential equation, one creates a class of conjectures which "generalize" part of the Calderón problem. These generalizations move beyond the setting where any ingredient in the problem is linear.

9.1. **Translation invariant operators.** When $N = \mathbb{R}^n$, $n \geq 2$, if one replaces composition of pseudodifferential operators with multiplication of their symbols, then (9.1) is of the form covered by our main theorems. Another way of saying this is that if the operators were all assumed to be translation invariant on $\mathbb{R}^n$, then the equation (9.1) is of the form covered by our main theorems–and we describe this next. Thus, Conjecture 9.1 can be viewed as a noncommutative analog of Theorem 2.5.

Let $\Gamma(t)$ be as described in the previous section, satisfying (9.1) and assume that $\Gamma(t)$ is translation invariant. Thus, $g(t)$ does not depend on x and $\Gamma(t)$ is given by a multiplier:

$$\widehat{\Gamma(t)f}(\xi) = M(t,\xi)\hat{f}(\xi),$$

and M satisfies the differential equation

$$\frac{\partial}{\partial t}M(t,\xi) = |g(t)|^{-1/2}M(t,\xi)^2 - |g(t)|^{1/2}\sum_{\alpha,\beta} g^{\alpha,\beta}(t)\xi_\alpha\xi_\beta, \tag{9.2}$$

and satisfies

$$M(t,\xi) = \Big(|g(t)|\sum_{\alpha,\beta} g^{\alpha,\beta}(t)\xi_\alpha\xi_\beta\Big)^{1/2} + O(1), \quad \text{as } |\xi| \to \infty.$$

For $1 \leq \alpha \leq n$, let e_α denote the αth standard basis element. For a positive definite quadratic form

$$B(\xi) = |\tilde{g}| \sum_{\alpha,\beta} \tilde{g}^{\alpha,\beta} \xi_\alpha \xi_\beta,$$

where $\tilde{g}$ is a positive definite matrix, associate to B the vector v indexed by $1 \leq \alpha \leq \beta \leq n$ with $v_{\alpha,\beta} = \sqrt{B(e_\alpha + e_\beta)}$. Note that $v = (v_{\alpha,\beta})$ uniquely determines $\tilde{g}$, and therefore B, and the function $\mathcal{F}(v) := |\tilde{g}|^{-1/2}$ is well-defined and smooth (here we have used $n \geq 2$ and argued as in the previous section).

For $1 \leq \alpha \leq \beta \leq n$ and $x \geq 0$, we define

$$f_{\alpha,\beta}(t,x) := \begin{cases} xM\left(t, \frac{1}{x}(e_\alpha + e_\beta)\right) & \text{if } x > 0, \\ \sqrt{|g(t)|\left(g^{\alpha,\alpha}(t) + 2g^{\alpha,\beta}(t) + g^{\beta,\beta}(t)\right)} & \text{if } x = 0. \end{cases}$$

Rewriting (9.2) in terms of $f_{\alpha,\beta}$ we see $f_{\alpha,\beta}$ satisfies the system of differential equations

$$\begin{aligned} \frac{\partial}{\partial t} f_{\alpha,\beta}(t,x) &= \frac{|g(t)|^{-1/2} f_{\alpha,\beta}(t,x)^2 - |g(t)|^{-1/2} f_{\alpha,\beta}(t,0)^2}{x} \\ &= \frac{\mathcal{F}(f(t,0)) f_{\alpha,\beta}(t,x)^2 - \mathcal{F}(f(t,0)) f_{\alpha,\beta}(t,0)^2}{x}. \end{aligned} \tag{9.3}$$

Note that, by the assumption that $g(t)$ is positive definite, $f_{\alpha,\beta}(t,0) > 0$, for all t. It follows that (9.3) is of the form covered by Theorem 2.5, where we have used the polynomial $P = (P_{\alpha,\beta})$, where

$$P_{\alpha,\beta}(t,x,y,z) = \mathcal{F}(z) y_{\alpha,\beta}^2.$$

Thus, under the restriction that $\Gamma(t)$ is translation invariant, Conjecture 9.1 follows from Theorem 2.5.

Remark 9.3. It is not difficult to simplify the above equation using Liouville transformations to reduce the problem to considering, for instance, the case $P(t,x,y,z) = y^2$. However, the generality of our approach lets us avoid such reductions.

9.2. **Calderón problem.** In this section, we describe how (9.1) arises in the Calderón problem–which is well-known to experts. Let M be a smooth, compact Riemannian manifold with boundary of dimension $n+1 \geq 3$. Let G denote the metric on M. The Dirichet-to-Neumann map $\Lambda_G : C^\infty(\partial M) \to C^\infty(\partial M)$ is defined as follows. Given $f \in C^\infty(\partial M)$, let $u \in C^\infty(M)$ be the unique solution to $\Delta_G u = 0$ on M, $u\big|_{\partial M} = f$. Λ_G is then defined as $\Lambda_G f = \frac{\partial}{\partial \nu} f\big|_{\partial M}$, where ν denotes the outward unit normal to ∂M. The inverse problem is to construct G given Λ_G. There is one obvious obstruction: if $\Psi : M \to M$ is a diffeomorphism which fixes ∂M, then $\Lambda_G = \Lambda_{\Psi^* G}$ (where $\Psi^* G$ denotes the pull back of G via Ψ).[8] Calderón's problem then asks if this is the only obstruction.

Anisotropic Calderón Conjecture: Suppose $\Lambda_{G_1} = \Lambda_{G_2}$. Then there is a diffeomorphism $\Psi : M \to M$, which fixes the boundary, such that $G_1 = \Psi^* G_2$.

The above conjecture remains open, and has attracted a great deal of attention. It began with work of Calderón [6]. When $M \subset \mathbb{R}^{n+1}$ and in the so-called *isotropic* setting: $G_{i,j}(x) = c(x)\delta_{i,j}$, the problem is well understood [9, 12, 14, 13, 23, 24, 25, 21, 19, 10, 1].

[8] This obstruction was noted by Luc Tartar.

Moving to the general (anisotropic) setting, much less is known. In the real analytic category, the result is known in the affirmative [16, 17, 15]. In the smooth category, little progress has been made on the full anisotropic question. In a big step forward, recent work of Dos Santos Ferreira, Kenig, Salo, and Uhlmann [7, 11] have given some of the first results in this setting. However, they still require a special form of the metric G, and even then do not answer the full Calderón question.

Remark 9.4. When $n+1=2$, the problem takes a slightly different form, and is very well understood [20, 27, 26, 5, 2, 3]. Because of this, our main interest is the case $n+1 \geq 3$.

Following [16], we use boundary normal coordinates on a neighborhood of ∂M. This sees a neighborhood of ∂M in the form $\partial M \times [0, \epsilon)$. We use coordinates $(x,t) \in \partial M \times [0,\epsilon)$. M has dimension $n+1$ and ∂M has dimension n. In what follows, α, β range over the numbers $1, \dots, n$ while i, j index the numbers $1, \dots, n+1$. In boundary normal coordinates, $G_{i,j}$ satisfies $G_{n+1,n+1} = 1$, $G_{n+1,\beta} = 0$, $G_{\alpha,n+1} = 0$. Let $g_{\alpha,\beta}(x,t) = G_{\alpha,\beta}(x,t)$; in particular, $g_{\alpha,\beta}(x,t)$ is an $n \times n$ matrix and satisfies $\det g_{\alpha,\beta}(x,t) = \det G_{i,j}(x,t)$.

For each $t_0 \in [0,\epsilon)$, we shrink the manifold M but cutting off the part of the manifold $[0,t_0) \times \partial M$ (in boundary normal coordinates), yielding a new Riemannian manifold M_{t_0}. Let G_{t_0} denote the metric on M_{t_0} (given by restricting G to M_{t_0}). For each $t_0 \in [0,\epsilon)$, we think of $g(x,t_0)$ as a metric on $\partial M \cong \partial M_{t_0}$ (where we identify ∂M with ∂M_{t_0} in the obvious way). We sometimes suppress the variable x and write $g(t_0)$ to denote the metric, which depends smoothly on t_0.

For each t_0 we define the map $\Gamma(t_0) : C^\infty(\partial M) \to C^\infty(\partial M)$ as follows. Let u_{t_0} solve $\Delta_{G_{t_0}} u_{t_0} = 0$ in M_{t_0} with $u_{t_0}\big|_{\partial M_{t_0}} = f$ (here we are again identifying ∂M_{t_0} with ∂M in the obvious way). Then define

$$\Gamma(t_0) f(x) := -|g(x,t)|^{1/2} \frac{\partial}{\partial t}\Big|_{t=t_0} u_{t_0}(t,x). \tag{9.4}$$

Note that $\Gamma(0) = |g(0)|^{1/2} \Lambda_G$. Because it is well-known that Λ_G uniquely determines G on ∂M, the Calderón problem can be equivalently stated with Λ_G replaced by $\Gamma(0)$.

We have

$$\Delta_G = \Delta_{g(t)} + |g(x,t)|^{-1/2} \frac{\partial}{\partial t} |g(x,t)|^{1/2} \frac{\partial}{\partial t}.$$

Differentiating (9.4) with respect to t, using the above formula for Δ_G, and using $\Delta_{G_{t_0}} u_{t_0} = 0$, we see that $\Gamma(t)$ satisfies the differential equation (9.1).

Hence, if Conjecture 9.1 were true, it would follow that $\Gamma(0)$ uniquely determines $g(t)$. I.e., that Λ_G uniquely determines G on a neighborhood of the boundary in boundary normal coordinates.

Remark 9.5. In the real analytic category, differential equations always have uniqueness, and the above argument shows that, for a real analytic manifold, Λ_G uniquely determines G on a neighborhood of the boundary, in boundary normal coordinates. This is equivalent to the first step of [16], where the same ideas are used to determine the Taylor series of g in the t-variable, centered at $t=0$.

9.3. **Acknowledgments.** This research was partially supported by NSF DMS-1401671.

References

[1] Giovanni Alessandrini, *Stable determination of conductivity by boundary measurements*, Appl. Anal. **27** (1988), no. 1-3, 153–172. MR 922775 (89f:35195)

[2] Kari Astala and Lassi Päivärinta, *Calderón's inverse conductivity problem in the plane*, Ann. of Math. (2) **163** (2006), no. 1, 265–299. MR 2195135 (2007b:30019)

[3] Kari Astala, Lassi Päivärinta, and Matti Lassas, *Calderón's inverse problem for anisotropic conductivity in the plane*, Comm. Partial Differential Equations **30** (2005), no. 1-3, 207–224. MR 2131051 (2005k:35421)

[4] Göran Borg, *Uniqueness theorems in the spectral theory of* $y'' + (\lambda - q(x))y = 0$, Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, Johan Grundt Tanums Forlag, Oslo, 1952, pp. 276–287. MR 0058063 (15,315a)

[5] Russell M. Brown and Gunther A. Uhlmann, *Uniqueness in the inverse conductivity problem for nonsmooth conductivities in two dimensions*, Comm. Partial Differential Equations **22** (1997), no. 5-6, 1009–1027. MR 1452176 (98f:35155)

[6] Alberto-P. Calderón, *On an inverse boundary value problem*, Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro, 1980), Soc. Brasil. Mat., Rio de Janeiro, 1980, pp. 65–73. MR 590275 (81k:35160)

[7] David Dos Santos Ferreira, Carlos E. Kenig, Mikko Salo, and Gunther Uhlmann, *Limiting Carleman weights and anisotropic inverse problems*, Invent. Math. **178** (2009), no. 1, 119–171. MR 2534094 (2010h:58033)

[8] Fritz Gesztesy and Barry Simon, *On local Borg-Marchenko uniqueness results*, Comm. Math. Phys. **211** (2000), no. 2, 273–287. MR 1754515 (2001b:34020)

[9] Victor Isakov, *On uniqueness of recovery of a discontinuous conductivity coefficient*, Comm. Pure Appl. Math. **41** (1988), no. 7, 865–877. MR 951742 (90f:35205)

[10] Victor Isakov, *Completeness of products of solutions and some inverse problems for PDE*, J. Differential Equations **92** (1991), no. 2, 305–316. MR 1120907 (92g:35044)

[11] Carlos E. Kenig, Mikko Salo, and Gunther Uhlmann, *Reconstructions from boundary measurements on admissible manifolds*, Inverse Probl. Imaging **5** (2011), no. 4, 859–877. MR 2852376 (2012k:58038)

[12] Robert Kohn and Michael Vogelius, *Determining conductivity by boundary measurements*, Comm. Pure Appl. Math. **37** (1984), no. 3, 289–298. MR 739921 (85f:80008)

[13] Robert V. Kohn and Michael Vogelius, *Identification of an unknown conductivity by means of measurements at the boundary*, Inverse problems (New York, 1983), SIAM-AMS Proc., vol. 14, Amer. Math. Soc., Providence, RI, 1984, pp. 113–123. MR 773707

[14] R. V. Kohn and M. Vogelius, *Determining conductivity by boundary measurements. II. Interior results*, Comm. Pure Appl. Math. **38** (1985), no. 5, 643–667. MR 803253 (86k:35155)

[15] Matti Lassas, Michael Taylor, and Gunther Uhlmann, *The Dirichlet-to-Neumann map for complete Riemannian manifolds with boundary*, Comm. Anal. Geom. **11** (2003), no. 2, 207–221. MR 2014876 (2004h:58033)

[16] John M. Lee and Gunther Uhlmann, *Determining anisotropic real-analytic conductivities by boundary measurements*, Comm. Pure Appl. Math. **42** (1989), no. 8, 1097–1112. MR 1029119 (91a:35166)

[17] Matti Lassas and Gunther Uhlmann, *On determining a Riemannian manifold from the Dirichlet-to-Neumann map*, Ann. Sci. École Norm. Sup. (4) **34** (2001), no. 5, 771–787. MR 1862026 (2003e:58037)

[18] V. A. Marčenko, *Some questions of the theory of one-dimensional linear differential operators of the second order. I*, Trudy Moskov. Mat. Obšč. **1** (1952), 327–420. MR 0058064 (15,315b)

[19] Adrian I. Nachman, *Reconstructions from boundary measurements*, Ann. of Math. (2) **128** (1988), no. 3, 531–576. MR 970610 (90i:35283)

[20] ———, *Global uniqueness for a two-dimensional inverse boundary value problem*, Ann. of Math. (2) **143** (1996), no. 1, 71–96. MR 1370758 (96k:35189)

[21] Adrian Nachman, John Sylvester, and Gunther Uhlmann, *An n-dimensional Borg-Levinson theorem*, Comm. Math. Phys. **115** (1988), no. 4, 595–605. MR 933457 (89g:35082)

[22] Barry Simon, *A new approach to inverse spectral theory. I. Fundamental formalism*, Ann. of Math. (2) **150** (1999), no. 3, 1029–1057. MR 1740987 (2001m:34185a)

[23] John Sylvester and Gunther Uhlmann, *A uniqueness theorem for an inverse boundary value problem in electrical prospection*, Comm. Pure Appl. Math. **39** (1986), no. 1, 91–112. MR 820341 (87j:35377)

[24] ______, *A global uniqueness theorem for an inverse boundary value problem*, Ann. of Math. (2) **125** (1987), no. 1, 153–169. MR 873380 (88b:35205)

[25] ______, *Inverse boundary value problems at the boundary—continuous dependence*, Comm. Pure Appl. Math. **41** (1988), no. 2, 197–219. MR 924684 (89f:35213)

[26] Ziqi Sun and Gunther Uhlmann, *Anisotropic inverse problems in two dimensions*, Inverse Problems **19** (2003), no. 5, 1001–1010. MR 2024685 (2004k:35415)

[27] John Sylvester, *An anisotropic inverse boundary value problem*, Comm. Pure Appl. Math. **43** (1990), no. 2, 201–232. MR 1038142 (90m:35202)

Brian Street

University of Wisconsin-Madison, Department of Mathematics, 480 Lincoln Dr., Madison, WI 53706, USA

E-mail address: `street@math.wisc.edu`