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EXISTENCE OF QUASI-PERIODIC INVARIANT TORI FOR COUPLED VAN DER POL EQUATIONS

LIN LU, SHUJUAN LIU, JUN WU

Abstract. This work focuses on the existence of quasi-periodic invariant tori for coupled van der Pol equations. Using averaging method, a series of reversible transformations and KAM techniques, we prove that there exist quasi-periodic invariant tori for most parameters. The results in this article can be regarded as a generalization of those in [22].

1. Introduction

The van der Pol oscillators have been investigated by the authors in the areas of mechanics [15, 24] and biology [29, 33, 35], and were extensively discussed as a host of a rich class of dynamical behavior [8, 31, 36, 38, 39, 42]. Feng and Gao studied the first integrals of the Duffing-van der Pol equations in [10, 11, 14]. By homotopy perturbation method, Chen and Jiang [5] investigated the periodic solution of the Duffing-van der Pol oscillator. Hirano and Rybicki [16], by S^1-degree theory, discussed the existence of limit cycles of coupled van der Pol system. Pastor et al. [28] analyzed the ordered and chaotic behavior of two coupled van der Pol equations. Rand and Holmes [34], by perturbation methods, considered the bifurcations of phase-locked periodic motions in two weakly coupled van der Pol oscillators. Dieci et al. [9] presented the numerical results for two weakly linearly coupled van der Pol systems. Gilsinn [13] constructed the invariant tori for two weakly nonlinearly coupled van der Pol equations. Beregov and Melkikh [2] considered two inductively coupled van der Pol generators and established the presence of metastable chaos, a strange non-chaotic attractor, and several stable limiting cycles. Zhang and Gu [42] investigated the dynamics of two weakly coupled van der Pol equations with time delay, and derived the explicit expression for determining the direction of the Hopf bifurcations and the stability of the bifurcating periodic solutions by the theory of normal form and the center manifold theorem.

In this article, we consider the coupled van der Pol equations

$$
\begin{aligned}
\ddot{x}_1 + \varepsilon(x_1^2 - 1)\dot{x}_1 + bx_1 &= a(x_1 - x_2) + \mu(x_1^3 - x_2^3), \\
\ddot{x}_2 + \varepsilon(x_2^2 - 1)\dot{x}_2 + bx_2 &= a(x_2 - x_1) + \mu(x_2^3 - x_1^3),
\end{aligned}
\tag{1.1}
$$

where the dot denotes the derivative with respect to the time t, ε and μ are small parameters, a and b are linear couple parameters, $b > 2a$ and $b > 0$.

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Many authors have considered some special forms of (1.1). For the case of $a = \mu = 0$ and $b = 1$, Equation (1.1) has been studied in [37]. When $\mu = 0$ and $b = 1$, using a perturbation method, Storti and Rand [35] studied the steady state behavior of the strongly coupled van der Pol equations

$$\begin{gathered}\ddot{x}_1 + \varepsilon(x_1^2 - 1)\dot{x}_1 + x_1 = a(x_1 - x_2),\\ \ddot{x}_2 + \varepsilon(x_2^2 - 1)\dot{x}_2 + x_2 = a(x_2 - x_1) + \Delta x_2,\end{gathered}$$

where $\varepsilon \ll 1$, a and Δ are small parameters. When $\mu = 0$ and $b = 1$, Nohara and Arimoto [27] considered the existence of the out-of-phase and in-phase solutions of the coupled system (1.1). Recently, by the homotopy analysis method (HAM), Li et al. [22] further discussed series solutions of (1.1) with $b = 1$, and obtained that there exist either in-phase or out-of-phase periodic solutions.

Since a periodic motion is a special case of a quasi-periodic one, possessing just one basic frequency, inspired by the above works, we can speculate that (1.1) should have quasi-periodic solutions with frequency depending on some parameters. Our aim is, by Kolmogorov-Arnold-Moser (KAM) theory, theoretically to prove that (1.1) has quasi-periodic solutions with the frequency depending on the parameters a and b. The above result can be regarded as a generalization of [22]. Since μ is a small real parameter, we may set $\mu = \varepsilon^2\tilde{\mu}$. With $\tilde{\mu}$ again denoted by μ, letting $\dot{x_1} = x_3$, $\dot{x_2} = x_4$, we can easily write (1.1) as

$$\begin{gathered}\dot{x}_1 = x_3,\\ \dot{x}_3 = (a-b)x_1 + \varepsilon x_3 - a x_2 + \varepsilon^2\mu(x_1^3 - x_2^3) - \varepsilon x_3 x_1^2,\\ \dot{x}_2 = x_4,\\ \dot{x}_4 = -a x_1 + (a-b)x_2 + \varepsilon x_4 + \varepsilon^2\mu(x_2^3 - x_1^3) - \varepsilon x_4 x_2^2,\end{gathered} \tag{1.2}$$

and prove that for sufficiently small ε, the autonomous system (1.2) has quasi-periodic solutions (i.e., invariant tori) for most values of the parameters a and b.

It is well known that KAM theory can be used to study quasi-periodic motions in nearly integrable Hamiltonian Systems [23, 32, 40], dissipative systems [4, 7, 17, 41] and mapping systems [3, 12]. By now KAM theory has blossomed into an enormous and somewhat complicated collection of ideas and methods where small divisors, degeneracy, reducibility, quasi-periodicity and invariant tori are the critical concepts. For instance, Li, Llave and Yuan considered the existence of quasi-periodic solutions of delay differential equations in [19, 20]. Several general surveys on the degenerate KAM theory were presented in [1, 6, 25]. Li [21] discussed the persistence of quasi-periodic invariant 2-tori and 3-tori for the double Hopf bifurcation and obtained that under appropriate conditions, the full system has quasi-periodic invariant 2-tori and 3-tori for most of the parameters in a sufficiently small neighborhood of the bifurcation point. The reducibility of nonlinear systems under quasi-periodic perturbations was studied by Jorba and Simo [15] for the case of suitable hypothesis of analyticity, non-resonance and non-degeneracy with respect to a small real parameter ε.

In this context, we shall write equation (1.2) as a quasi-periodic system under small perturbations, where frequencies are non-degenerate and depend on parameters a and b. Then we obtain quasi-periodic solutions by KAM techniques. Some ideas seeking for the quasi-periodic solutions in this paper could be found in [7, 18, 20, 41].

This paper is arranged as follows. In Section 2, by a series of transformations, the system (1.2) is changed into the normal form, some ideas of KAM steps and main result are outlined. Section 3 contains an iterative lemma which is very important in the proof of KAM theory. We present a KAM theorem in Section 4, which is devoted to obtaining the quasi-periodic solutions of (1.2). Some technical lemmas are provided in the appendix.

2. Normal form and Main result

2.1. **Normal form.** Let

$$A=\begin{pmatrix}0 & 1 & 0 & 0\\ a-b & \varepsilon & -a & 0\\ 0 & 0 & 0 & 1\\ -a & 0 & a-b & \varepsilon\end{pmatrix}.$$

Since the eigenvalues of A are

$$\lambda_1=\frac{\varepsilon}{2}+\frac{i\sqrt{4b-8a-\varepsilon^2}}{2},\quad \lambda_2=\frac{\varepsilon}{2}-\frac{i\sqrt{4b-8a-\varepsilon^2}}{2},$$
$$\lambda_3=\frac{\varepsilon}{2}+\frac{i\sqrt{4b-\varepsilon^2}}{2},\quad \lambda_4=\frac{\varepsilon}{2}-\frac{i\sqrt{4b-\varepsilon^2}}{2},$$

we can make a complex linear transformation such that the coefficient matrix A is diagonal. Let

$$\begin{pmatrix}x_1\\ x_3\\ x_2\\ x_4\end{pmatrix}=\begin{pmatrix}-1 & -1 & 1 & 1\\ -\lambda_1 & -\lambda_2 & \lambda_3 & \lambda_4\\ 1 & 1 & 1 & 1\\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4\end{pmatrix}\begin{pmatrix}y_1\\ y_2\\ y_3\\ y_4\end{pmatrix}.$$

Then system (1.2) is transformed into

$$\begin{aligned}\dot y_1&=\lambda_1y_1+g_1(y_1,y_2,y_3,y_4),\\ \dot y_2&=\lambda_2y_2+g_2(y_1,y_2,y_3,y_4),\\ \dot y_3&=\lambda_3y_3+g_3(y_1,y_2,y_3,y_4),\\ \dot y_4&=\lambda_4y_4+g_4(y_1,y_2,y_3,y_4),\end{aligned}\tag{2.1}$$

where

$$\begin{aligned}g_1(y_1,y_2,y_3,y_4)=\Big[&-2\varepsilon^2\mu(y_1+y_2)^3-6\varepsilon^2\mu(y_1+y_2)(y_3+y_4)^2\\ &+\varepsilon(\lambda_1y_1+\lambda_2y_2)(y_1+y_2)^2\\ &+2\varepsilon(\lambda_3y_3+\lambda_4y_4)(y_1+y_2)(y_3+y_4)\\ &+\varepsilon(\lambda_1y_1+\lambda_2y_2)(y_3+y_4)^2\Big](\lambda_2-\lambda_1)^{-1},\\ g_2(y_1,y_2,y_3,y_4)=\Big[&2\varepsilon^2\mu(y_1+y_2)^3+6\varepsilon^2\mu(y_1+y_2)(y_3+y_4)^2\\ &-\varepsilon(\lambda_1y_1+\lambda_2y_2)(y_1+y_2)^2\\ &-2\varepsilon(\lambda_3y_3+\lambda_4y_4)(y_1+y_2)(y_3+y_4)\\ &-\varepsilon(\lambda_1y_1+\lambda_2y_2)(y_3+y_4)^2\Big](\lambda_2-\lambda_1)^{-1},\end{aligned}$$

$$
\begin{aligned}
g_3(y_1,y_2,y_3,y_4) &= \Big[2\varepsilon(y_1+y_2)(\lambda_1 y_1+\lambda_2 y_2)(y_3+y_4)+\varepsilon(\lambda_3 y_3+\lambda_4 y_4)(y_1+y_2)^2 \\
&\quad +\varepsilon(\lambda_3 y_3+\lambda_4 y_4)(y_3+y_4)^2\Big](\lambda_4-\lambda_3)^{-1}, \\
&g_4(y_1,y_2,y_3,y_4) \\
&= \Big[-2\varepsilon(y_1+y_2)(\lambda_1 y_1+\lambda_2 y_2)(y_3+y_4)-\varepsilon(\lambda_3 y_3+\lambda_4 y_4)(y_1+y_2)^2 \\
&\quad -\varepsilon(\lambda_3 y_3+\lambda_4 y_4)(y_3+y_4)^2\Big](\lambda_4-\lambda_3)^{-1}.
\end{aligned}
$$

We set

$$
y_1 = r_1 e^{i\theta_1}, \quad y_2=\overline{y_1}, \quad y_3=r_2e^{i\theta_2}, \quad y_4=\overline{y_3}. \tag{2.2}
$$

Then (2.1) can be expressed as

$$
\begin{aligned}
\dot r_1 &= \varepsilon A_1(r,\theta,\varepsilon,a,b)=\varepsilon\frac{r_1(1-r_1^2-2r_2^2)}{2}+\varepsilon f_1(r,\theta,\varepsilon,a,b), \\
\dot r_2 &= \varepsilon A_2(r,\theta,\varepsilon,a,b)=\varepsilon\frac{r_2(1-r_2^2-2r_1^2)}{2}+\varepsilon f_2(r,\theta,\varepsilon,a,b), \\
\dot\theta_1 &= \Omega_1+\varepsilon B_1(r,\theta,\varepsilon,a,b) \\
&= \Omega_1+\frac{3\varepsilon^2-12\varepsilon^2\mu}{4\Omega_1}r_1^2+\frac{3\varepsilon^2-12\varepsilon^2\mu}{2\Omega_1}r_2^2+\varepsilon f_3(r,\theta,\varepsilon,a,b), \\
\dot\theta_2 &= \Omega_2+\varepsilon B_2(r,\theta,\varepsilon,a,b)=\Omega_2+\frac{3\varepsilon^2}{4\Omega_2}r_2^2+\frac{3\varepsilon^2}{2\Omega_2}r_1^2+\varepsilon f_4(r,\theta,\varepsilon,a,b),
\end{aligned} \tag{2.3}
$$

where

$$
\Omega_1=\frac{\sqrt{4b-8a-\varepsilon^2}}{2}, \quad \Omega_2=\frac{\sqrt{4b-\varepsilon^2}}{2},
$$

$r=(r_1,r_2)$, $\theta=(\theta_1,\theta_2)$, $f_i(r,\theta,\varepsilon,a,b)$ satisfies $f_i(r,\theta,\varepsilon,a,b)=O(r^3)$ and

$$
\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} f_i(r,\theta,\varepsilon,a,b)d\theta_1 d\theta_2=0, \quad i=1,2,3,4.
$$

Here $O(r^k)$ denotes a function which is analytic in r and θ, sufficiently smooth in parameters a and b in some bounded closed set and vanishes with r-derivatives up to order $k-1$ for $r=0$.

Obviously, the averaged system of (2.3) is

$$
\begin{aligned}
\dot r_1 &= \varepsilon\frac{r_1(1-r_1^2-2r_2^2)}{2}, \\
\dot r_2 &= \varepsilon\frac{r_2(1-r_2^2-2r_1^2)}{2}, \\
\dot\theta_1 &= \Omega_1+\frac{3\varepsilon^2-12\varepsilon^2\mu}{4\Omega_1}r_1^2+\frac{3\varepsilon^2-12\varepsilon^2\mu}{2\Omega_1}r_2^2, \\
\dot\theta_2 &= \Omega_2+\frac{3\varepsilon^2}{4\Omega_2}r_2^2+\frac{3\varepsilon^2}{2\Omega_2}r_1^2.
\end{aligned} \tag{2.4}
$$

It is easy to see that system (2.4) has an equilibrium solution $r_1=r_2=\sqrt{3}/3$, and that

$$
\begin{aligned}
r_1 &= \frac{\sqrt3}{3}, \\
r_2 &= \frac{\sqrt3}{3},
\end{aligned}
$$

$$\theta_1 = \theta_{10} + \big(\Omega_1 + \frac{\varepsilon^2 - 4\varepsilon^2\mu}{4\Omega_1} + \frac{\varepsilon^2 - 4\varepsilon^2\mu}{2\Omega_1}\big)t,$$
$$\theta_2 = \theta_{20} + \big(\Omega_2 + \frac{\varepsilon^2}{4\Omega_2} + \frac{\varepsilon^2}{2\Omega_2}\big)t$$

are quasi-periodic solutions of (2.4), where θ_{10} and θ_{20} represent initial values.

We want to look for quasi-periodic solutions of the perturbed system (2.3) using KAM theory. To do this, we have to introduce some notation. We shall denote by C a universal positive constant which is independent of the KAM iteration and may be different in different places.

(i) Let $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$ and $\widehat{\mathbb{T}}^2 = \mathbb{C}^2/(2\pi\mathbb{Z})^2$;

(ii) For $\sigma > 0$ and $s > 0$, let

$$U(\sigma) = \{\varphi \in \widehat{\mathbb{T}}^2 : |\operatorname{Im}\varphi| := \max_{1\le i\le 2} |\operatorname{Im}\varphi_i| \le \sigma\},$$
$$W(s,\sigma) = \{(h,\varphi) \in \mathbb{C}^2 \times \widehat{\mathbb{T}}^2 : |h| \le s,\ |\operatorname{Im}\varphi| \le \sigma\};$$

(iii) Let $\Pi \subseteq \mathbb{R}^2$ be a parameter set with positive Lebesgue measure. Define $\Pi^\eta = \{\xi \in \mathbb{R}^2 : \operatorname{dist}(\xi,\Pi) < \eta\}$;

(iv) For given $0 < \varepsilon \ll 1$, if a map $F : W(s,\sigma) \times \Pi^\eta \to \mathbb{C}^2$ is real analytic in $(h,\varphi) \in W(s,\sigma)$ and C^1-smooth in $\xi \in \Pi^\eta$, and satisfies

$$\|F\|_{s,\sigma,\eta} := \max_{i=0,1} \sup_{W(s,\sigma)\times\Pi^\eta} |\partial_\xi^i F(h,\varphi,\xi)| \le C\varepsilon,$$

then we write $F = O_{s,\sigma,\eta}(\varepsilon)$, where the notation ∂_ξ denotes the partial derivative with respect to ξ;

(v) If a map $H : W(s,\sigma) \times \Pi^\eta \to \mathbb{C}^2$ is real analytic in $(h,\varphi) \in W(s,\sigma)$ and C^1-smooth in $\xi \in \Pi^\eta$, vanishes with its h-derivative for $h = 0$, and satisfies

$$\sup_{W(s,\sigma)\times\Pi^\eta} |\partial_h^j \partial_\xi^i H(h,\varphi,\xi)| \le C, \quad i = 0,1,\ j = 0,1,2,$$

then we write $H = O_{s,\sigma,\eta}(h^2)$.

Let

$$a = \xi_1, \quad b = \xi_2, \quad \xi = (\xi_1,\xi_2)^T. \tag{2.5}$$

Since $b > 2a$ and $b > 0$, we can assume that $(\xi_1,\xi_2) \in [1,2] \times [5,6] := \Pi$ without loss of generality.

We expand $A_i(\rho,\varphi,\varepsilon,\xi)$ and $B_i(\rho,\varphi,\varepsilon,\xi)$ into Fourier series in φ and truncate them by the operator Γ_{K_0}:

$$\Gamma_{K_0} A_i(\rho,\varphi,\varepsilon,\xi) = \sum_{|k|\le K_0} \hat{A}_i(k)(\rho,\varepsilon,\xi) e^{\sqrt{-1}(k,\varphi)},$$
$$\Gamma_{K_0} B_i(\rho,\varphi,\varepsilon,\xi) = \sum_{|k|\le K_0} \hat{B}_i(k)(\rho,\varepsilon,\xi) e^{\sqrt{-1}(k,\varphi)}, \quad i = 1,2,$$

where $k \in \mathbb{Z}^2$, $|k| = |k_1| + |k_2|$ and K_0 is a suitable positive integer satisfying

$$\begin{aligned} &\|(\mathrm{Id} - \Gamma_{K_0}) A_i(\rho,\varphi,\varepsilon,\xi)\|_{s_0',\frac{\sigma_0'}{2},\Pi^{\eta_0}} \le C\varepsilon, \\ &\|(\mathrm{Id} - \Gamma_{K_0}) B_i(\rho,\varphi,\varepsilon,\xi)\|_{s_0',\frac{\sigma_0'}{2},\Pi^{\eta_0}} \le C\varepsilon, \end{aligned} \tag{2.6}$$

here s_0', σ_0' and η_0 are positive constants, Id denotes the identity operator.

Using the transformation

$$
\begin{aligned}
r_1 &= \rho_1 + \varepsilon u_1(\rho, \varphi, \varepsilon, \xi), \\
r_2 &= \rho_2 + \varepsilon u_2(\rho, \varphi, \varepsilon, \xi), \\
\theta_1 &= \varphi_1 + \varepsilon w_1(\rho, \varphi, \varepsilon, \xi), \\
\theta_2 &= \varphi_2 + \varepsilon w_2(\rho, \varphi, \varepsilon, \xi)
\end{aligned} \tag{2.7}
$$

in (2.3), we deduce the homological equations

$$
\begin{gathered}
\partial_\varphi u_i \cdot \omega = \Gamma_{K_0} A_i(\rho, \varphi, \varepsilon, \xi) - \hat{A}_i(0)(\rho, \varepsilon, \xi), \\
\partial_\varphi w_i \cdot \omega = \Gamma_{K_0} B_i(\rho, \varphi, \varepsilon, \xi) - \hat{B}_i(0)(\rho, \varepsilon, \xi), \quad i = 1, 2,
\end{gathered} \tag{2.8}
$$

where $\omega = (\Omega_1, \Omega_2)^T$, $\rho = (\rho_1, \rho_2)$, $\varphi = (\varphi_1, \varphi_2)$, and '$\cdot^T$' represents the transposition of a vector '$\cdot$'. Note that there is a small divisor in (2.8). Letting

$$
\Pi_0 = \big\{ \xi \in \Pi : |\big(k, \omega(\xi)\big)| \geq \frac{\gamma}{|k|^\tau},\ 0 < |k| \leq K_0 \big\},
$$

where $0 \neq k \in \mathbb{Z}^2$ and $\tau \geq 3$, by Lemma 4.4, we have

$$
\operatorname{meas} \Pi_0 = \operatorname{meas} \Pi - O(\gamma).
$$

Solving equation (2.8), we obtain the estimates

$$
\begin{aligned}
\|u_i\|_{s_0', \frac{3\sigma_0'}{4}, \Pi_0^{\eta_0}} &\leq C\gamma^{-2} \|A_i\|_{s_0', \sigma_0', \Pi^{\eta_0}}, \\
\|\partial_\varphi u_i\|_{s_0', \frac{\sigma_0'}{2}, \Pi_0^{\eta_0}} &\leq C\gamma^{-2} \|A_i\|_{s_0', \sigma_0', \Pi^{\eta_0}}, \\
|w_i\|_{s_0', \frac{3\sigma_0'}{4}, \Pi_0^{\eta_0}} &\leq C\gamma^{-2} \|B_i\|_{s_0', \sigma_0', \Pi^{\eta_0}}, \\
\|\partial_\varphi w_i\|_{s_0', \frac{\sigma_0'}{2}, \Pi_0^{\eta_0}} &\leq C\gamma^{-2} \|B_i\|_{s_0', \sigma_0', \Pi^{\eta_0}}, \\
\|\partial_\rho u_i\|_{s_0', \frac{3\sigma_0'}{4}, \Pi_0^{\eta_0}} &\leq C\gamma^{-2} \|\partial_\rho A_i\|_{s_0', \sigma_0', \Pi^{\eta_0}}, \\
\|\partial_\rho w_i\|_{s_0', \frac{3\sigma_0'}{4}, \Pi_0^{\eta_0}} &\leq C\gamma^{-2} \|\partial_\rho B_i\|_{s_0', \sigma_0', \Pi^{\eta_0}}.
\end{aligned} \tag{2.9}
$$

Similar homological equations will be solved in Lemma 3.1. From (2.3), we obtain

$$
\begin{gathered}
A_i(\rho, \varphi, \varepsilon, \xi) = O(\rho), \quad B_i(\rho, \varphi, \varepsilon, \xi) = O(\rho^2), \\
\hat{B}_i(0)(\rho, \varepsilon, \xi) = O(\varepsilon), \quad \hat{B}_i(0)(\rho, \varepsilon, \xi) = O(\rho^2), \quad i = 1, 2.
\end{gathered} \tag{2.10}
$$

Using (2.6)–(2.8), (2.10) and Taylor's formula, we can rewrite equation (2.3) as

$$
\begin{aligned}
&(1 + \varepsilon \frac{\partial u_1}{\partial \rho_1}) \dot{\rho}_1 + \varepsilon \frac{\partial u_1}{\partial \rho_2} \dot{\rho}_2 + \varepsilon \frac{\partial u_1}{\partial \varphi_1} (\dot{\varphi}_1 - \Omega_1) + \varepsilon \frac{\partial u_1}{\partial \varphi_2} (\dot{\varphi}_2 - \Omega_2) \\
&= \varepsilon \hat{A}_1(0)(\rho, \varepsilon, \xi) + \varepsilon \big[A_1(\rho + \varepsilon u, \varphi + \varepsilon v, \varepsilon, \xi) - A_1(\rho, \varphi, \varepsilon, \xi) \\
&\quad + (\mathrm{Id} - \Gamma_{K_0}) A_1(\rho, \varphi, \varepsilon, \xi) \big] \\
&=: \varepsilon \hat{A}_1(0)(\rho, \varepsilon, \xi) + \varepsilon^2 A_1^{(1)}(\rho, \varphi, \varepsilon, \xi), \\
&\varepsilon \frac{\partial u_2}{\partial \rho_1} \dot{\rho}_1 + (1 + \varepsilon \frac{\partial u_2}{\partial \rho_2}) \dot{\rho}_2 + \varepsilon \frac{\partial u_2}{\partial \varphi_1} (\dot{\varphi}_1 - \Omega_1) + \varepsilon \frac{\partial u_2}{\partial \varphi_2} (\dot{\varphi}_2 - \Omega_2) \\
&= \varepsilon \hat{A}_2(0)(\rho, \varepsilon, \xi) + \varepsilon \big[A_2(\rho + \varepsilon u, \varphi + \varepsilon v, \varepsilon, \xi) - A_2(\rho, \varphi, \varepsilon, \xi) \\
&\quad + (\mathrm{Id} - \Gamma_{K_0}) A_2(\rho, \varphi, \varepsilon, \xi) \big] \\
&=: \varepsilon \hat{A}_2(0)(\rho, \varepsilon, \xi) + \varepsilon^2 A_2^{(1)}(\rho, \varphi, \varepsilon, \xi),
\end{aligned}
$$

$$
\begin{aligned}
&\varepsilon\frac{\partial w_1}{\partial\rho_1}\dot\rho_1+\varepsilon\frac{\partial w_1}{\partial\rho_2}\dot\rho_2+(1+\varepsilon\frac{\partial w_1}{\partial\varphi_1})(\dot\varphi_1-\Omega_1)+\varepsilon\frac{\partial w_1}{\partial\varphi_2}(\dot\varphi_2-\Omega_2)\\
&=\varepsilon\hat B_1(0)(\rho,\varepsilon,\xi)+\varepsilon\big[B_1(\rho+\varepsilon u,\varphi+\varepsilon v,\varepsilon,\xi)-B_1(\rho,\varphi,\varepsilon,\xi)\\
&\quad+(\mathrm{Id}-\Gamma_{K_0})B_1(\rho,\varphi,\varepsilon,\xi)\big]\\
&=:\varepsilon\hat B_1(0)(\rho,\varepsilon,\xi)+\varepsilon^2B_1^{(1)}(\rho,\varphi,\varepsilon,\xi),\\
&\varepsilon\frac{\partial w_2}{\partial\rho_1}\dot\rho_1+\varepsilon\frac{\partial w_2}{\partial\rho_2}\dot\rho_2+\varepsilon\frac{\partial w_2}{\partial\varphi_1}(\dot\varphi_1-\Omega_1)+(1+\varepsilon\frac{\partial w_2}{\partial\varphi_2})(\dot\varphi_2-\Omega_2)\\
&=\varepsilon\hat B_2(0)(\rho,\varepsilon,\xi)+\varepsilon[B_2(\rho+\varepsilon u,\varphi+\varepsilon v,\varepsilon,\xi)-B_2(\rho,\varphi,\varepsilon,\xi)\\
&\quad+(\mathrm{Id}-\Gamma_{K_0})B_2(\rho,\varphi,\varepsilon,\xi)]\\
&=:\varepsilon\hat B_2(0)(\rho,\varepsilon,\xi)+\varepsilon^2B_2^{(1)}(\rho,\varphi,\varepsilon,\xi),
\end{aligned}
$$

where

$$A_i^{(1)}(\rho,\varphi,\varepsilon,\xi)=O(\rho),\quad B_i^{(1)}(\rho,\varphi,\varepsilon,\xi)=O(\rho^2),\quad i=1,2.$$

Solving the above equations, we obtain

$$
\begin{gathered}
\dot\rho_1=\varepsilon\hat A_1(0)(\rho,\varepsilon,\xi)+\varepsilon^2X_1(\rho,\varphi,\varepsilon,\xi),\\
\dot\rho_2=\varepsilon\hat A_2(0)(\rho,\varepsilon,\xi)+\varepsilon^2X_2(\rho,\varphi,\varepsilon,\xi),\\
\dot\varphi_1=\Omega_1+\varepsilon^2Y_1(\rho,\varphi,\varepsilon,\xi),\\
\dot\varphi_2=\Omega_2+\varepsilon^2Y_2(\rho,\varphi,\varepsilon,\xi),
\end{gathered}\tag{2.11}
$$

where $X_i(\rho,\varphi,\varepsilon,\xi)=O(\rho)$, $Y_i(\rho,\varphi,\varepsilon,\xi)=O(\rho^2)$, $i=1,2$. Let

$$
\begin{gathered}
\rho_1=\frac{\sqrt3}{3}-\varepsilon^{1/3}L_1+\varepsilon^{1/3}L_2,\quad \rho_2=\frac{\sqrt3}{3}+\varepsilon^{1/3}L_1+\varepsilon^{1/3}L_2,\\
h=(L_1,L_2)^T,\quad \varphi=(\varphi_1,\varphi_2)^T.
\end{gathered}\tag{2.12}
$$

Using (2.9) and Taylor's formula and dropping the parameter ε from functions for simplicity, we can rewrite (2.11) as

$$
\begin{gathered}
\dot h=\varepsilon[\Lambda h+M(\varphi,\xi)+Q(\varphi,\xi)h+F(h,\varphi,\xi)],\\
\dot\varphi=\omega(\xi)+\varepsilon^{5/3}[N(\varphi,\xi)+G(h,\varphi,\xi)],
\end{gathered}\tag{2.13}
$$

where

$$
\begin{gathered}
\omega(\xi)=(\Omega_1,\Omega_2)^T,\quad \Lambda:=\operatorname{diag}\big(\lambda_1,\lambda_2\big)=\operatorname{diag}\Big(\frac13,-1\Big),\\
M=O_{\sigma_0,\eta_0}(\varepsilon^{1/6}),\quad Q=O_{\sigma_0,\eta_0}(\varepsilon^{1/6}),\quad N=O_{\sigma_0,\eta_0}(\varepsilon^{1/6}),\\
F=\varepsilon^{1/6}O_{s_0,\sigma_0,\eta_0}(h^2),\quad G=\varepsilon^{1/6}O_{s_0,\sigma_0,\eta_0}(h),
\end{gathered}
$$

where s_0 and σ_0 are constants satisfying $s_0>0$ and $\sigma_0=\sigma_0'/2$.

2.2. **Outline of KAM steps.** To obtain quasi-periodic solutions of (2.13), we perform some changes to simplify (2.13) by Newton iteration and KAM techniques. Firstly, the terms $M(\varphi,\xi)$, $Q(\varphi,\xi)$ and $N(\varphi,\xi)$ will be eliminated by means of a family of quasi-periodic changes of variables. More precisely, substituting the change of variables

$$h=h_1+v_1(\phi_1,\xi)+v_2(\phi_1,\xi)h_1,\quad \varphi=\phi_1+v_3(\phi_1,\xi)\tag{2.14}$$

into (2.13) and dropping the parameter ξ from functions for simplicity, we obtain

$$
\begin{aligned}
&(\mathrm{Id}+v_2)\dot{h}_1\\
&=\varepsilon\Big[\Lambda h_1+\Lambda v_1+\Lambda v_2 h_1+M(\phi_1+v_3)+Q(\phi_1+v_3)h_1+Q(\phi_1+v_3)v_1\\
&\quad+Q(\phi_1+v_3)v_2h_1+F(h_1+v_1+v_2h_1,\phi_1+v_3)\Big]\\
&\quad-\partial_{\phi_1}v_1\cdot\dot{\phi}_1-\partial_{\phi_1}v_2\cdot\dot{\phi}_1h_1,\\
&(\mathrm{Id}+\partial_{\phi_1}v_3)\dot{\phi}_1=\omega+\varepsilon^{5/3}\big[N(\phi_1+v_3)+G(h_1+v_1+v_2h_1,\phi_1+v_3)\big].
\end{aligned}
\tag{2.15}
$$

Denote by $Q^{ij}(\phi_1)$ $(i,j=1,2)$ the matrix elements of operator $Q(\phi_1)$. Denote by $\widehat{Q}^{ii}(k)$ $(i=1,2)$ and $\widehat{N}(k)$ the kth-Fourier coefficient of $Q^{ii}(\phi_1)$ and $N(\phi_1)$, respectively. Then the transformation (2.14) will be obtained by solving the homological equations

$$\partial_{\phi_1}v_1\cdot\omega=\varepsilon(\Gamma_K M+\Lambda v_1), \tag{2.16}$$
$$\partial_{\phi_1}v_2\cdot\omega=\varepsilon\big[\Lambda v_2-v_2\Lambda+\Gamma_K M-\mathrm{diag}\,\big(\widehat{Q}^{11}(0),\widehat{Q}^{22}(0)\big)\big], \tag{2.17}$$
$$\partial_{\phi_1}v_3\cdot\omega=\varepsilon^{5/3}\big[\Gamma_K N-\widehat{N}(0)\big]. \tag{2.18}$$

Once equations (2.16)–(2.18) are solved, using Taylor's formula for $F(h_1+v_1+v_2h_1,\phi_1+v_3)$ and $G(h_1+v_1+v_2h_1,\phi_1+v_3)$, we can rewrite the system (2.15) as

$$
\begin{gathered}
\dot{h}_1=\varepsilon[\Lambda_1(\xi)h_1+M_1(\phi_1,\xi)+Q_1(\phi_1,\xi)h_1+F_1(h_1,\phi_1,\xi)],\\
\dot{\phi}_1=\omega_1(\xi)+\varepsilon^{5/3}[N_1(\phi_1,\xi)+G_1(h_1,\phi_1,\xi)],
\end{gathered}
$$

where

$$
\begin{gathered}
\Lambda_1(\xi)=\Lambda+\mathrm{diag}\,\Big(\widehat{Q}^{11}(0),\widehat{Q}^{22}(0)\Big),\\
\omega_1(\xi)=\omega(\xi)+\varepsilon^{5/3}\widehat{N}(0),\\
N_1(\phi_1,\xi)=(\mathrm{Id}+\partial_{\phi_1}v_3)^{-1}\Big[-\partial_{\phi_1}v_3\widehat{N}(0)+(\mathrm{Id}-\Gamma_K)N\\
+G(v_1,\phi_1+v_3)+N(\phi_1+v_3)-N\Big],\\
G_1(h_1,\phi_1,\xi)=(\mathrm{Id}+\partial_{\phi_1}v_3)^{-1}[G(h_1+v_1+v_2h_1,\phi_1+v_3)-G(v_1,\phi_1+v_3)],\\
M_1(\phi_1,\xi)=(\mathrm{Id}+v_2)^{-1}\Big[(\mathrm{Id}-\Gamma_K)M+M(\phi_1+v_3)-M+Q(\phi_1+v_3)v_1\\
-\varepsilon^{2/3}\partial_{\phi_1}v_1(\widehat{N}(0)+N_1)+F(v_1,\phi_1+v_3)\Big],\\
Q_1(\phi_1,\xi)=(\mathrm{Id}+v_2)^{-1}\Big[(\mathrm{Id}-\Gamma_K)Q-v_2(\Lambda_1-\Lambda)+Q(\phi_1+v_3)-Q\\
+Q(\phi_1+v_3)v_2-\varepsilon^{2/3}\partial_{\phi_1}v_2(\widehat{N}(0)+N_1)+\partial_h F(v_1,\phi_1+v_3)(\mathrm{Id}+v_2)\\
-\varepsilon^{2/3}\partial_{\phi_1}v_1(\mathrm{Id}+\partial_{\phi_1}v_3)^{-1}\partial_h G(v_1,\phi_1+v_3)(\mathrm{Id}+v_2)\Big],
\end{gathered}
$$

$$
\begin{aligned}
&F_1(h_1,\phi_1,\xi)\\
&= (\mathrm{Id}+v_2)^{-1}\Big\{-\varepsilon^{2/3}\partial_{\phi_1}v_1(\mathrm{Id}+\partial_{\phi_1}v_3)^{-1}\big[G(h_1+v_1+v_2h_1,\phi_1+v_3)\\
&\quad - G(v_1,\phi_1+v_3)-\partial_h G(v_1,\phi_1+v_3)(\mathrm{Id}+v_2)h_1\big]\\
&\quad -\varepsilon^{2/3}\partial_{\phi_1}v_2(\mathrm{Id}+\partial_{\phi_1}v_3)^{-1}\big[G(h_1+v_1+v_2h_1,\phi_1+v_3)-G(v_1,\phi_1+v_3)\big]h_1\\
&\quad + F(h_1+v_1+v_2h_1,\phi_1+v_3)-F(v_1,\phi_1+v_3)\\
&\quad -\partial_h F(v_1,\phi_1+v_3)(\mathrm{Id}+v_2)h_1\Big\}.
\end{aligned}
$$

In this way, after n steps, (2.13) becomes

$$
\begin{gathered}
\dot h_n=\varepsilon[\Lambda_n(\xi)h_n+M_n(\phi_n,\xi)+Q_n(\phi_n,\xi)h_n+F_n(h_n,\phi_n,\xi)],\\
\dot\phi_n=\omega_n(\xi)+\varepsilon^{5/3}[N_n(\phi_n,\xi)+G_n(h_n,\phi_n,\xi)].
\end{gathered}
$$

If the norms of M_n, Q_n and N_n tend to zero with a super-exponential velocity, then the composition of transformations is convergent, and the above equation converges to the form

$$
\begin{gathered}
\dot{\tilde h}=\varepsilon[\Lambda_\infty(\xi)\tilde h+F_\infty(\tilde h,\tilde\phi,\xi)],\\
\dot{\tilde\phi}=\omega_\infty(\xi)+\varepsilon^{5/3}G_\infty(\tilde h,\tilde\phi,\xi),
\end{gathered}
$$

where $F_\infty(\tilde h,\tilde\phi,\xi)=\varepsilon^{1/6}O(\tilde h^2)$ and $G_\infty(\tilde h,\tilde\phi,\xi)=\varepsilon^{1/6}O(\tilde h)$. Obviously, $\tilde h=0,\ \tilde\phi=\phi_*+\omega_\infty(\xi)t$ is a trivial solution of the above equation, where ϕ_* represents an initial value. It means that for the original system (2.13), there exists a quasi-periodic solution with the frequency $\omega_\infty(\xi)$.

2.3. **Main result.** The small divisor conditions

$$
\begin{gathered}
\big|\sqrt{-1}\big(k,\omega(\xi)\big)-\varepsilon\lambda_i(\xi)\big|\ge\frac{\gamma}{|k|^\tau},\ i=1,2,\\
\big|\sqrt{-1}\big(k,\omega(\xi)\big)+\varepsilon\lambda_1(\xi)-\varepsilon\lambda_2(\xi)\big|\ge\frac{\gamma}{|k|^\tau},\\
\big|\big(k,\omega(\xi)\big)\big|\ge\frac{\gamma}{|k|^\tau}
\end{gathered}
$$

are needed in the process of solving equations (2.16)–(2.18), respectively, where $0\ne k\in\mathbb{Z}^2$ and $\tau\ge 3$. Hence, we need to take out some small (in the sense of Lebesgue measure) parameter sets and control the measure of the resonant sets on each step of the iteration. Moreover, to estimate the measure of the resonant sets, we require that the frequency $\omega(\xi)$ satisfies the following non-degeneracy condition.

Definition 2.1. Assume that $\omega(\xi)$ is continuously differentiable in $\xi\in\Pi\subseteq\mathbb{R}^2$. We say that $\omega(\xi)$ satisfies the non-degeneracy condition on the set Π, if there exists a constant $\chi>0$ such that

$$
\inf_{\xi\in\Pi}\big|\det\frac{\partial\omega}{\partial\xi}\big|\ge\chi.
$$

With these preliminaries, we state the main result in this article.

Theorem 2.2. *Suppose that $\xi=(a,b)\in[1,2]\times[5,6]=\Pi$. Then for given $0<\gamma\ll 1$, there is a sufficiently small positive number ε_0^* ($\varepsilon_0^*=O(\gamma^{s_*}), s_*\ge 29$), such that if $0<\varepsilon<\varepsilon_0^*$, then there is a Cantorian subset $\Pi_\infty\subset\Pi$ with Lebesgue* *measure $1-O(\gamma)$, and for any $\xi\in\Pi_\infty$, the equation (2.13) possesses a real analytic quasi-periodic solution with the frequency $\omega_\infty(\xi)$ satisfying*

$$\sup_{\xi\in\Pi_\infty}\|\omega_\infty(\xi)-\omega(\xi)\|=O(\varepsilon^{43/24}).$$

Moreover, system (1.2) also possesses a real analytic quasi-periodic solution with the frequency $\omega_\infty(\xi)$ for $\xi\in\Pi_\infty$.

Remark 2.3. Equation (2.13) could not be dealt with Moser theorem [26] directly, although some ideas are similar to it. Based on modifying terms method and keeping frequencies fixed, Moser [26] discussed the existence of quasi-periodic solutions. However, the present proofs rest on measure estimates method originated by Pöschel [30]. We need introduce parameter ξ to the system and the frequency $\omega_\infty(\xi)$ of the quasi-periodic solution we obtained satisfies $\sup_{\xi\in\Pi_\infty}\|\omega_\infty(\xi)-\omega(\xi)\|=O(\varepsilon^{\frac{43}{24}})$ compared with [26].

3. Iterative lemma

Theorem 2.2 is proved using an iterative procedure. To state and prove the iterative lemma, we first introduce some iterative constants and notation. Let $\sigma_0,s_0,\chi_0,\zeta_0,c_0,d_0$ and τ be positive constants and $\tau\ge 3$. For all $m\ge 1$,

(1) $\varepsilon_0=\varepsilon,\ \varepsilon_m=\varepsilon_{m-1}^{5/4}$ (ε_m bounds the size of the perturbation after the m-th iteration);
(2) $\nu_0=0$, $\nu_m=(1^{-2}+\dots+m^{-2})/(2\sum_{j=1}^\infty j^{-2})$;
(3) $\sigma_m=(1-\nu_m)\sigma_0$ (σ_m measures the size of the analytic domain in the angle variables after the mth iteration);
(4) $s_m=(1-\nu_m)s_0$ (s_m measures the size of the analytic domain in the action variable after the mth iteration);
(5) $\chi_m=\chi_0-\varepsilon^{5/3}\sum_{i=0}^{m-1}\varepsilon_i^{1/8}$ ($\chi_m\ge\chi_0/2$ if ε is sufficiently small);
(6) $\zeta_m=\zeta_0+\sum_{i=0}^{m-1}\varepsilon_i^{1/8}$ ($\zeta_m\le 2\zeta_0$ if ε is sufficiently small);
(7) $c_m=c_0-\sum_{i=0}^{m-1}\varepsilon_i^{1/8}$ ($c_m\ge c_0/2$ if ε is sufficiently small);
(8) $d_m=d_0+\varepsilon^{5/3}\sum_{i=0}^{m-1}\varepsilon_i^{1/8}$ ($d_m\le 2d_0$ if ε is sufficiently small);
(9) $\kappa_m=\frac13(\sigma_m-\sigma_{m+1})=\sigma_0/[6(m+1)^2\sum_{j=1}^\infty j^{-2}]$;
(10) $K_m=-\frac{1}{\sigma_0}(m+1)^2 2^{m+2}\ln\varepsilon$ (K_m determines the number of Fourier coefficients we must consider at the mth step of the iteration);
(11) $\gamma_m=\gamma/(m+1)^2$, $\gamma_0=\gamma$, $\varepsilon_0^*=O(\gamma^{s_*}), s_*\ge 29$;
(12) $\eta_m=\frac{\gamma_{m-1}}{4d_0K_m^{\tau+1}}$; ($\eta_m$ is used to extend the closed parameter set Π_m to a small neighborhood at the mth step of the iteration);

Lemma 3.1. *Suppose that there is a sequence of closed parameter sets $\mathbb{R}^2\supset\Pi_0\supset\Pi_1\supset\dots\supset\Pi_l$ and a family of equations defined on $W(s_m,\sigma_m)\times\Pi_m^{\eta_m}$, for $m=0,1,\dots,l$, by $(Eq)_m$:*

$$\begin{gathered}\dot h_m=\varepsilon_0[\Lambda_m(\xi)h_m+M_m(\phi_m,\xi)+Q_m(\phi_m,\xi)h_m+F_m(h_m,\phi_m,\xi)],\\ \dot\phi_m=\omega_m(\xi)+\varepsilon_0^{5/3}[N_m(\phi_m,\xi)+G_m(h_m,\phi_m,\xi)],\end{gathered}\tag{3.1}$$

where $\Lambda_m(\xi)=\operatorname{diag}\big(\lambda_1^m(\xi),\,\lambda_2^m(\xi)\big)$ and $\omega_m(\xi)=\big(\omega_1^m(\xi),\,\omega_2^m(\xi)\big)^T$. Assume that for $m=0,1,\dots,l$ the following conditions are satisfied:

(A1) *for $\xi \in \Pi_m^{\eta_m}$, the frequency $\omega_m(\xi)$ satisfies the non-degeneracy conditions*

$$\inf_{\xi\in\Pi_0^{\eta_0}} \big|\det \frac{\partial \omega_0}{\partial \xi}\big| \geq \chi_0, \quad \inf_{\xi\in\Pi_m^{\eta_m}} \big|\det \frac{\partial \omega_m}{\partial \xi}\big| \geq \chi_m \geq \frac{1}{2}\chi_0, \tag{3.2}$$

and $\omega_m(\xi)$ and $\lambda_j^m(\xi)$ satisfy

$$\begin{gathered} \|\omega_0(\xi)\|_{\eta_0} \leq d_0, \quad \|\omega_m(\xi)\|_{\eta_m} \leq d_m \leq 2d_0, \\ \inf_{\xi\in\Pi_m^{\eta_m}} |\lambda_1^m(\xi) - \lambda_2^m(\xi)| \geq c_m \geq \frac{1}{2}c_0, \quad \|\lambda_1^m(\xi) - \lambda_2^m(\xi)\|_{\eta_m} \leq \zeta_m \leq 2\zeta_0, \\ \inf_{\xi\in\Pi_m^{\eta_m}} |\lambda_j^m(\xi)| \geq c_m \geq \frac{1}{2}c_0, \quad \|\lambda_j^m(\xi)\|_{\eta_m} \leq \zeta_m \leq 2\zeta_0, \quad j=1,2; \end{gathered} \tag{3.3}$$

(A2) *the terms $M_m(\phi_m,\xi)$, $Q_m(\phi_m,\xi)$ and $N_m(\phi_m,\xi)$ are real analytic in $\phi_m \in U(\sigma_m)$ and C^1-smooth in $\xi \in \Pi_m^{\eta_m}$, and satisfy the following estimates*

$$M_m = O_{\sigma_m,\eta_m}(\varepsilon_m^{1/6}), \quad Q_m = O_{\sigma_m,\eta_m}(\varepsilon_m^{1/8}), \quad N_m = O_{\sigma_m,\eta_m}(\varepsilon_m^{1/8}); \tag{3.4}$$

(A3) *the terms $F_m(h_m,\phi_m,\xi)$ and $G_m(h_m,\phi_m,\xi)$ are real analytic in $(h_m,\phi_m) \in W(s_m,\sigma_m)$ and C^1-smooth in $\xi \in \Pi_m^{\eta_m}$, and satisfy the following estimates*

$$F_m = \varepsilon_0^{1/6} O_{s_m,\sigma_m,\eta_m}(h^2), \quad G_m = \varepsilon_0^{1/6} O_{s_m,\sigma_m,\eta_m}(h); \tag{3.5}$$

(A4) *there is a constant $C_0 > 0$ such that the Lebesgue measure of Π_m satisfies*

$$\operatorname{meas}\Pi_{m+1} \geq \operatorname{meas}\Pi_m(1 - C_0\gamma_m). \tag{3.6}$$

Then there exists a closed subset $\Pi_{l+1} \subset \Pi_l$ and a change of variables $W(s_{l+1},\sigma_{l+1}) \times \Pi_{l+1}^{\eta_{l+1}} \to W(s_l,\sigma_l) \times \Pi_l^{\eta_l}$ of the form $\mathcal{T}_l$:

$$h_l = h_{l+1} + v_1^l(\phi_{l+1},\xi) + v_2^l(\phi_{l+1},\xi)h_{l+1}, \quad \phi_l = \phi_{l+1} + v_3^l(\phi_{l+1},\xi), \quad \xi = \xi, \tag{3.7}$$

where h_{l+1} and ϕ_{l+1} are new variables, v_j^l $(j = 1,2,3)$ are real analytic in $\phi_{l+1} \in U(\sigma_{l+1})$ and C^1-smooth in $\xi \in \Pi_{l+1}^{\eta_{l+1}}$, and satisfy the following estimates

$$\begin{gathered} v_1^l = O_{\sigma_{l+1},\eta_{l+1}}\Big(\frac{\varepsilon_l^{1/6}}{\gamma_l^2\kappa_l^{2\tau+3}}\Big), \quad v_2^l = O_{\sigma_{l+1},\eta_{l+1}}\Big(\frac{\varepsilon_l^{1/8}}{\gamma_l^2\kappa_l^{2\tau+3}}\Big), \\ v_3^l = O_{\sigma_{l+1},\eta_{l+1}}\Big(\frac{\varepsilon_0^{5/3}\varepsilon_l^{1/8}}{\gamma_l^2\kappa_l^{2\tau+3}}\Big), \end{gathered} \tag{3.8}$$

such that by the change of variable $\mathcal{T}_l$, equation $(Eq)_l$ is transformed into equation $(Eq)_{l+1}$:

$$\begin{gathered} \dot h_{l+1} = \varepsilon_0[\Lambda_{l+1}(\xi)h_{l+1} + M_{l+1}(\phi_{l+1},\xi) + Q_{l+1}(\phi_{l+1},\xi)h_{l+1} + F_{l+1}(h_{l+1},\phi_{l+1},\xi)], \\ \dot\phi_{l+1} = \omega_{l+1}(\xi) + \varepsilon_0^{5/3}[N_{l+1}(\phi_{l+1},\xi) + G_{l+1}(h_{l+1},\phi_{l+1},\xi)], \end{gathered} \tag{3.9}$$

and conditions (A1)–(A4) are satisfied when replacing m by $l+1$.

Proof. To simplify notation, we denote quantities referring to $l+1$ such as M_{l+1} by M_+, σ_{l+1} by σ_+, and those referring to l without the l such as N_l by N, σ_l by σ. By a little abuse of notation, we also denote $h_{l+1} = h_1$ and $\phi_{l+1} = \phi_1$ and drop the parameter ξ from functions whenever there is no confusion.

Substituting (3.7) in (3.1) with $m=l$, we have

$$\begin{aligned}
&(\mathrm{Id}+v_2)\dot h_1\\
&=\varepsilon_0[\Lambda h_1+\Lambda v_1+\Lambda v_2h_1+M(\phi_1+v_3)+Q(\phi_1+v_3)h_1+Q(\phi_1+v_3)v_1\\
&\quad+Q(\phi_1+v_3)v_2h_1+F(h_1+v_1+v_2h_1,\phi_1+v_3)]-\partial_{\phi_1}v_1\cdot\dot\phi_1-\partial_{\phi_1}v_2\cdot\dot\phi_1h_1,\\
&\quad(\mathrm{Id}+\partial_{\phi_1}v_3)\dot\phi_1=\omega+\varepsilon_0^{5/3}[N(\phi_1+v_3)+G(h_1+v_1+v_2h_1,\phi_1+v_3)].
\end{aligned}\tag{3.10}$$

Suppose that the transformation (3.7) can be solved with the homological equations

$$\partial_{\phi_1}v_1\cdot\omega=\varepsilon_0(\Gamma_KM+\Lambda v_1),\tag{3.11}$$

$$\partial_{\phi_1}v_2\cdot\omega=\varepsilon_0\big[\Lambda v_2-v_2\Lambda+\Gamma_KQ-\mathrm{diag}(\widehat Q^{11}(0),\widehat Q^{22}(0))\big],\tag{3.12}$$

$$\partial_{\phi_1}v_3\cdot\omega=\varepsilon_0^{5/3}\big[\Gamma_KN-\widehat N(0)\big].\tag{3.13}$$

Plugging (3.11)–(3.13) in (3.10) and using Taylor's formula for $F(h_1+v_1+v_2h_1,\phi_1+v_3)$ and $G(h_1+v_1+v_2h_1,\phi_1+v_3)$, we rewrite (3.10) as

$$\begin{aligned}
\dot h_1&=\varepsilon_0[\Lambda_+(\xi)h_1+M_+(\phi_1,\xi)+Q_+(\phi_1,\xi)h_1+F_+(h_1,\phi_1,\xi)],\\
\dot\phi_1&=\omega_+(\xi)+\varepsilon_0^{5/3}[N_+(\phi_1,\xi)+G_+(h_1,\phi_1,\xi)],
\end{aligned}\tag{3.14}$$

where

$$\Lambda_+(\xi)=\Lambda(\xi)+\mathrm{diag}\left(\widehat Q^{11}(0),\widehat Q^{22}(0)\right),\quad \omega_+(\xi)=\omega(\xi)+\varepsilon_0^{5/3}\widehat N(0),\tag{3.15}$$

$$\begin{aligned}
N_+(\phi_1,\xi)&=(\mathrm{Id}+\partial_{\phi_1}v_3)^{-1}\Big[-\partial_{\phi_1}v_3\widehat N(0)+(\mathrm{Id}-\Gamma_K)N\\
&\quad+G(v_1,\phi_1+v_3)+N(\phi_1+v_3)-N\Big],
\end{aligned}\tag{3.16}$$

$$\begin{aligned}
&G_+(h_1,\phi_1,\xi)\\
&=(\mathrm{Id}+\partial_{\phi_1}v_3)^{-1}\big[G(h_1+v_1+v_2h_1,\phi_1+v_3)-G(v_1,\phi_1+v_3)\big],
\end{aligned}\tag{3.17}$$

$$\begin{aligned}
M_+(\phi_1,\xi)&=(\mathrm{Id}+v_2)^{-1}\Big[(\mathrm{Id}-\Gamma_K)M+M(\phi_1+v_3)-M+Q(\phi_1+v_3)v_1\\
&\quad-\varepsilon_0^{2/3}\partial_{\phi_1}v_1(\widehat N(0)+N_+)+F(v_1,\phi_1+v_3)\Big],
\end{aligned}\tag{3.18}$$

$$\begin{aligned}
&Q_+(\phi_1,\xi)\\
&=(\mathrm{Id}+v_2)^{-1}\Big[(\mathrm{Id}-\Gamma_K)Q-v_2(\Lambda_+-\Lambda)+Q(\phi_1+v_3)-Q\\
&\quad+Q(\phi_1+v_3)v_2-\varepsilon_0^{2/3}\partial_{\phi_1}v_2(\widehat N(0)+N_+)+\partial_hF(v_1,\phi_1+v_3)(\mathrm{Id}+v_2)\\
&\quad-\varepsilon_0^{2/3}\partial_{\phi_1}v_1(\mathrm{Id}+\partial_{\phi_1}v_3)^{-1}\partial_hG(v_1,\phi_1+v_3)(\mathrm{Id}+v_2)\Big],
\end{aligned}\tag{3.19}$$

$$\begin{aligned}
&F_+(h_1,\phi_1,\xi)\\
&=(\mathrm{Id}+v_2)^{-1}\Big\{-\varepsilon_0^{2/3}\partial_{\phi_1}v_1(\mathrm{Id}+\partial_{\phi_1}v_3)^{-1}\big[G(h_1+v_1+v_2h_1,\phi_1+v_3)\\
&\quad-G(v_1,\phi_1+v_3)-\partial_hG(v_1,\phi_1+v_3)(\mathrm{Id}+v_2)h_1\big]\\
&\quad-\varepsilon_0^{2/3}\partial_{\phi_1}v_2(\mathrm{Id}+\partial_{\phi_1}v_3)^{-1}\big[G(h_1+v_1+v_2h_1,\phi_1+v_3)\\
&\quad-G(v_1,\phi_1+v_3)\big]h_1+F(h_1+v_1+v_2h_1,\phi_1+v_3)-F(v_1,\phi_1+v_3)\\
&\quad-\partial_hF(v_1,\phi_1+v_3)(\mathrm{Id}+v_2)h_1\Big\}.
\end{aligned}\tag{3.20}$$

To make the proof coherent, we divide the proof of Lemma 3.1 into several parts, with each part containing several computations and estimates that are based on the same idea.

3.1. **Solutions of** (3.11)–(3.13) **and estimates.** We first solve the equation (3.13). Let N^i and v_{3i} $(i = 1, 2)$ be the ith element of N and v_3, respectively. Expand N^i and v_{3i} into Fourier series in ϕ_1 and truncate them by operator Γ_K:

$$\Gamma_K N^i = \sum_{|k|\le K} \widehat{N}^i(k) e^{\sqrt{-1}(k,\phi_1)}, \quad \Gamma_K v_{3i} = \sum_{0<|k|\le K} \widehat{v}_{3i}(k) e^{\sqrt{-1}(k,\phi_1)},$$

where $k \in \mathbb{Z}^2$, the Fourier coefficients $\widehat{N}^i(k)$ and $\widehat{v}_{3i}(k)$ rely on the parameter ξ. Then from (3.13), we obtain

$$\widehat{v}_{3i}(k) = \frac{\varepsilon_0^{5/3}\widehat{N}^i(k)}{\sqrt{-1}(k,\omega)}, \quad 0 < |k| \le K,\ i = 1, 2. \tag{3.21}$$

Since $N(\phi_1)$ is real analytic in $\phi_1 \in U(\sigma)$ and C^1-smooth in $\xi \in \Pi^\eta$, we have

$$|\partial_\xi^s \widehat{N}^i(k)| \le \max_{s=0,1} \sup_{U(\sigma)\times\Pi^\eta} |\partial_\xi^s N^i| e^{-|k|\sigma}, \quad s = 0, 1. \tag{3.22}$$

Let

$$\Pi_+ = \big\{\xi \in \Pi : |(k,\omega)| \ge \frac{\gamma}{|k|^\tau},\ 0 < |k| \le K\big\}.$$

For every $\xi \in \Pi_+^{\eta_+}$, there is a $\xi^0 \in \Pi_+$ such that $|\xi - \xi^0| < \eta_+$, and for $0 < |k| \le K$, we have

$$\begin{aligned} |\big(k,\omega(\xi)\big)| &= \big|\big(k,\omega(\xi^0)\big) + \big(k,\omega(\xi)\big) - \big(k,\omega(\xi^0)\big)\big| \\ &\ge \frac{\gamma}{|k|^\tau} - 2d_0\eta_+K \ge \frac{\gamma}{2|k|^\tau} \end{aligned}$$

by (3.3). Differentiating (3.21) in ξ and using (3.3) again, we obtain

$$\max_{s=0,1} |\partial_\xi^s \widehat{v}_{3i}(k)| \le C\varepsilon_0^{5/3} \max_{s=0,1} |\partial_\xi^s \widehat{N}^i(k)| |k|^{2\tau+1}/\gamma^2 \le C\varepsilon_0^{5/3} \|N^i\|_{\sigma,\eta} e^{-|k|\sigma} |k|^{2\tau+1}/\gamma^2,$$

for $\xi \in \Pi_+^{\eta_+}$. Hence, by (3.4) with $m = l$ and Lemma 4.2, we have

$$\begin{aligned} \|v_3\|_{\sigma-\kappa,\eta_+} &\le \sum_{|k|\le K} (\max_{s=0,1} \sup_{\xi\in\Pi_+^{\eta_+}} |\partial_\xi^s \widehat{v}_3(k)|) e^{|k|(\sigma-\kappa)} \\ &\le C\varepsilon_0^{5/3} \|N\|_{\sigma,\eta} \sum_{k\in\mathbb{Z}^2} e^{-|k|\kappa} |k|^{2\tau+1}/\gamma^2 \\ &\le C\varepsilon_0^{5/3} \varepsilon^{1/8} \gamma^{-2} \kappa^{-(2\tau+3)}. \end{aligned} \tag{3.23}$$

By using the Cauchy inequality, it follows that

$$\partial_{\phi_1} v_3 = O_{\sigma_+,\eta_+}\big(\varepsilon_0^{5/3}\varepsilon^{1/8}\gamma^{-2}\kappa^{-(2\tau+4)}\big). \tag{3.24}$$

Next we solve (3.12). Let Q^{ij} and $v_2^{ij}(i, j = 1, 2)$ be the elements of Q and v_2, respectively. Expand Q^{ij} and v_2^{ij} into Fourier series in ϕ_1 and truncate them by operator Γ_K:

$$\Gamma_K Q^{ij} = \sum_{|k|\le K} \widehat{Q}^{ij}(k) e^{\sqrt{-1}(k,\phi_1)}, \quad \Gamma_K v_2^{ij} = \sum_{|k|\le K} \widehat{v}_2^{ij}(k) e^{\sqrt{-1}(k,\phi_1)},$$

where $k \in \mathbb{Z}^2$, the Fourier coefficients $\widehat{Q}^{ij}(k)$ and $\widehat{v}_2^{ij}(k)$ rely on the parameter ξ. Then by (3.12), we obtain

$$\widehat{v}_2^{ij}(k) = \begin{cases} \frac{\varepsilon_0 \widehat{Q}^{ij}(k)}{\varepsilon_0\lambda_j - \varepsilon_0\lambda_i + \sqrt{-1}(k,\omega)}, & \text{if } i \neq j \\ \frac{\varepsilon_0 \widehat{Q}^{ii}(k)}{\sqrt{-1}(k,\omega)}, & \text{if } k \neq 0,\ i = j \end{cases} \tag{3.25}$$

for $|k| \le K$. Since ε_0, λ_1 and λ_2 are real, $Q(\phi_1)$ and $N(\phi_1)$ are real analytic in $\phi_1 \in U(\sigma)$, from (3.4) and (3.15) it follows that λ_1^+, λ_2^+ and ω_+ are real and satisfy condition (A1) with $m = l + 1$. Obviously, for $\xi \in \Pi_+$, $0 < |k| \le K$, we obtain

$$|\sqrt{-1}\big(k, \omega(\xi)\big) + \varepsilon_0\lambda_1(\xi) - \varepsilon_0\lambda_2(\xi)| \ge |\big(k, \omega(\xi)\big)| \ge \frac{\gamma}{|k|^\tau}.$$

Similarly, for $\xi \in \Pi_+^{\eta_+}$, $0 < |k| \le K$, by (3.3) we have

$$|\sqrt{-1}\big(k, \omega(\xi)\big) + \varepsilon_0\lambda_1(\xi) - \varepsilon_0\lambda_2(\xi)| \ge \frac{\gamma}{2|k|^\tau}.$$

Differentiating (3.25) in ξ and using (3.3), we obtain

$$\begin{aligned} \max_{s=0,1} |\partial_\xi^s \widehat{v}_2^{ij}(k)| &\le C\varepsilon_0 \max_{s=0,1} |\partial_\xi^s \widehat{Q}^{ij}(k)| |k|^{2\tau+1}/\gamma^2 \\ &\le C\varepsilon_0 \|Q^{ij}\|_{\sigma,\eta} e^{-|k|\sigma} |k|^{2\tau+1}/\gamma^2, \end{aligned} \tag{3.26}$$

for $\xi \in \Pi_+^{\eta_+}$. Hence, by (3.3) and (3.4) with $m = l$, (3.25) and Lemma 4.2 again, we have

$$\begin{aligned} \|v_2 - \widehat{v}_2(0)\|_{\sigma-\kappa,\eta_+} &\le \sum_{|k|\le K} (\max_{s=0,1} \sup_{\xi\in\Pi_+^{\eta_+}} |\partial_\xi^s \widehat{v}_2(k)|) e^{|k|(\sigma-\kappa)} \\ &\le C\varepsilon_0 \|Q\|_{\sigma,\eta} \sum_{k\in\mathbb{Z}^2} e^{-|k|\kappa} |k|^{2\tau+1}/\gamma^2 \\ &\le C\varepsilon_0 \varepsilon^{1/8} \gamma^{-2} \kappa^{-(2\tau+3)}, \end{aligned} \tag{3.27}$$

$$\|v_2\|_{\sigma-\kappa,\eta_+} \le \Big\|\frac{\widehat{Q}(0)}{\lambda_1 - \lambda_2}\Big\|_{\eta_+} + C\varepsilon_0\varepsilon^{1/8}\gamma^{-2}\kappa^{-(2\tau+3)} \le C\varepsilon^{1/8}\gamma^{-2}\kappa^{-(2\tau+3)}.$$

Then by the Cauchy inequality, it follows that

$$\partial_{\phi_1} v_2 = O_{\sigma_+,\eta_+}(\varepsilon_0\varepsilon^{1/8}\gamma^{-2}\kappa^{-(2\tau+4)}). \tag{3.28}$$

Finally, we solve (3.11). Let M^i and v_{1i} $(i = 1, 2)$ be the i-th elements of M and v_1, respectively. Expand M^i and v_{1i} into Fourier series in ϕ_1 and truncate them by operator Γ_K,

$$\Gamma_K M^i = \sum_{|k|\le K} \widehat{M}^i(k) e^{\sqrt{-1}(k,\phi_1)}, \quad \Gamma_K v_{1i} = \sum_{|k|\le K} \widehat{v}_{1i}(k) e^{\sqrt{-1}(k,\phi_1)},$$

where $k \in \mathbb{Z}^2$, the Fourier coefficients $\widehat{M}^i(k)$ and $\widehat{v}_{1i}(k)$ rely on the parameter ξ. Then by (3.11), we obtain

$$\widehat{v}_{1i}(k) = \frac{\varepsilon_0 \widehat{M}^i(k)}{\sqrt{-1}(k,\omega) - \varepsilon_0\lambda_i}, \quad |k| \le K,\ i = 1, 2. \tag{3.29}$$

By similar estimates for equation (3.12), we obtain

$$\begin{aligned}\|v_1-\widehat{v}_1(0)\|_{\sigma-\kappa,\eta_+} &= \max_{s=0,1}\sup_{U(\sigma-\kappa)\times\Pi_+^{\eta_+}}|\partial_\xi^s v_1| \\ &\le \sum_{|k|\le K}(\max_{s=0,1}\sup_{\xi\in\Pi_+^{\eta_+}}|\partial_\xi^s\widehat{v}_1(k)|)e^{|k|(\sigma-\kappa)} \\ &\le C\varepsilon_0\|M\|_{\sigma,\eta}\sum_{k\in\mathbb{Z}^2}e^{-|k|\kappa}|k|^{2\tau+1}/\gamma^2 \\ &\le C\varepsilon_0\varepsilon^{1/6}\gamma^{-2}\kappa^{-(2\tau+3)}, \\ \|v_1\|_{\sigma-\kappa,\eta_+} &\le \max_{i=1,2}\Big\|\frac{\widehat{M}(0)}{\lambda_i}\Big\|_{\eta_+}+C\varepsilon_0\varepsilon^{1/6}\gamma^{-2}\kappa^{-(2\tau+3)} \\ &\le C\varepsilon^{1/6}\gamma^{-2}\kappa^{-(2\tau+3)}.\end{aligned}\tag{3.30}$$

Using the Cauchy inequality, we obtain

$$\partial_{\phi_1}v_1=O_{\sigma_+,\eta_+}(\varepsilon_0\varepsilon^{1/6}\gamma^{-2}\kappa^{-(2\tau+4)}).\tag{3.31}$$

Obviously, the transformation $\mathcal{T}_l$ maps $W(s_+,\sigma_+)$ into $W(s-\kappa,\sigma)\subset W(s,\sigma)$ by estimates (3.8) when we choose ε_0 sufficiently small.

3.2. **Estimates for perturbations** (3.16)-(3.20)**.** Since $(\mathrm{Id}+v_2)^{-1}=\mathrm{Id}-v_2+v_2^2-v_2^3+\dots$, we obtain

$$\|(\mathrm{Id}+v_2)^{-1}\|_{\sigma_+,\eta_+}\le\sum_{i=0}^{\infty}\|v_2\|_{\sigma_+,\eta_+}^i\le C\tag{3.32}$$

by (3.8). Similarly, we have

$$\|(\mathrm{Id}+\partial_{\phi_1}v_3)^{-1}\|_{\sigma_+,\eta_+}\le\sum_{i=0}^{\infty}\|\partial_{\phi_1}v_3\|_{\sigma_+,\eta_+}^i\le C.\tag{3.33}$$

From (3.5) with $m=l$, (3.8), (3.17), (3.20), (3.28) and (3.31), it is easy to see that G_+ and F_+ satisfy the condition (A3) with $m=l+1$, and

$$\|G(v_1,\phi_1+v_3)\|_{\sigma_+,\eta_+}=\varepsilon_0^{1/6}O_{\sigma_+,\eta_+}(\|v_1\|)=O_{\sigma_+,\eta_+}(\varepsilon_0^{1/6}\varepsilon^{1/6}\gamma^{-2}\kappa^{-(2\tau+3)}).\tag{3.34}$$

From the Cauchy inequality, (3.4) with $m=l$ and (3.8), it follows that

$$\begin{aligned}\|N(\phi_1+v_3)-N\|_{\sigma_+,\eta_+} &\le C\kappa^{-1}\|N\|_{\sigma,\eta}\|v_3\|_{\sigma_+,\eta_+} \\ &\le C\varepsilon_0^{5/3}\varepsilon^{1/4}\gamma^{-2}\kappa^{-(2\tau+4)},\end{aligned}\tag{3.35}$$

$$\begin{aligned}\|M(\phi_1+v_3)-M\|_{\sigma_+,\eta_+} &\le C\kappa^{-1}\|M\|_{\sigma,\eta}\|v_3\|_{\sigma_+,\eta_+} \\ &\le C\varepsilon_0^{5/3}\varepsilon^{7/24}\gamma^{-2}\kappa^{-(2\tau+4)},\end{aligned}\tag{3.36}$$

$$\|Q(\phi_1+v_3)-Q\|_{\sigma_+,\eta_+}\le C\kappa^{-1}\|Q\|_{\sigma,\eta}\|v_3\|_{\sigma_+,\eta_+}\le C\varepsilon_0^{5/3}\varepsilon^{1/4}\gamma^{-2}\kappa^{-(2\tau+4)}.\tag{3.37}$$

Now we estimate $N_+(\phi_1,\xi)$, $M_+(\phi_1,\xi)$ and $Q_+(\phi_1,\xi)$. By (3.4) with $m=l$, (3.16), (3.24), (3.33)–(3.35) and Lemma 4.3, we obtain

$$\begin{aligned}\|N_+\|_{\sigma_+,\eta_+} &\le \|(\mathrm{Id}+\partial_{\phi_1}v_3)^{-1}\|_{\sigma_+,\eta_+}\big\|-\partial_{\phi_1}v_3\widehat{N}(0)+(\mathrm{Id}-\Gamma_K)N \\ &\quad +G(v_1,\phi_1+v_3)+N(\phi_1+v_3)-N\big\|_{\sigma_+,\eta_+} \\ &\le C\big(\|\partial_{\phi_1}v_3\|_{\sigma_+,\eta_+}\|N\|_{\sigma_+,\eta_+}+\|(\mathrm{Id}-\Gamma_K)N\|_{\sigma_+,\eta_+} \\ &\quad +\|N(\phi_1+v_3)-N\|_{\sigma_+,\eta_+}+\|G(v_1,\phi_1+v_3)\|_{\sigma_+,\eta_+}\big)\end{aligned}$$

$$
\begin{aligned}
&\le C\Big(\frac{\varepsilon_0^{5/3}\varepsilon^{1/4}}{\gamma^2\kappa^{2\tau+4}}+\|N\|_{\sigma,\eta_+}K^2e^{-\kappa K}+\frac{\varepsilon_0^{1/6}\varepsilon^{1/6}}{\gamma^2\kappa^{2\tau+3}}\Big)\\
&\le \varepsilon_+^{1/8}.
\end{aligned}
$$

Using (3.5) with $m=l$, we obtain

$$
\begin{aligned}
\|F(v_1,\phi_1+v_3)\|_{\sigma_+,\eta_+}&=\varepsilon_0^{1/6}O_{\sigma_+,\eta_+}(\|v_1\|^2_{\sigma_+,\eta_+})\\
&=O_{\sigma_+,\eta_+}(\varepsilon_0^{1/6}\varepsilon^{1/3}\gamma^{-4}\kappa^{-(4\tau+6)}),
\end{aligned}\tag{3.38}
$$

$$
\begin{aligned}
\|\partial_hF(v_1,\phi_1+v_3)\|_{\sigma_+,\eta_+}&=\varepsilon_0^{1/6}O_{\sigma_+,\eta_+}(\|v_1\|_{\sigma_+,\eta_+})\\
&=O_{\sigma_+,\eta_+}(\varepsilon_0^{1/6}\varepsilon^{1/6}\gamma^{-2}\kappa^{-(2\tau+3)}).
\end{aligned}\tag{3.39}
$$

From (3.4) with $m=l$, (3.8), (3.18), (3.31), (3.32), (3.36), (3.38), Lemma 4.3 and $\|N_+\|_{\sigma_+,\eta_+}$ it follows that

$$
\begin{aligned}
&\|M_+\|_{\sigma_+,\eta_+}\\
&\le \|(\mathrm{Id}+v_2)^{-1}\|_{\sigma_+,\eta_+}\Big(\|(\mathrm{Id}-\Gamma_K)M\|_{\sigma_+,\eta_+}+\|M(\phi_1+v_3)-M\|_{\sigma_+,\eta_+}\\
&\quad+\varepsilon_0^{2/3}\|\partial_{\phi_1}v_1(\widehat{N}(0)+N_+)\|_{\sigma_+,\eta_+}+\|Q(\phi_1+v_3)v_1\|_{\sigma_+,\eta_+}\\
&\quad+\|F(v_1,\phi_1+v_3)\|_{\sigma_+,\eta_+}\Big)\\
&\le C\Big(\|M\|_{\sigma,\eta_+}K^2e^{-\kappa K}+\frac{\varepsilon^{7/24}}{\gamma^2\kappa^{2\tau+3}}+\frac{\varepsilon_0^{1/6}\varepsilon^{1/3}}{\gamma^4\kappa^{4\tau+6}}\Big)\\
&\le \varepsilon_+^{1/6}.
\end{aligned}
$$

Similarly, by (3.4) and (3.5) with $m=l$, (3.8), (3.19), (3.28), (3.31)-(3.33), (3.37), (3.39) and Lemma 4.3, we obtain

$$
\begin{aligned}
&\|Q_+\|_{\sigma_+,\eta_+}\\
&=\|(\mathrm{Id}+v_2)^{-1}\|_{\sigma_+,\eta_+}\Big(\varepsilon_0^{2/3}\|\partial_{\phi_1}v_1(\mathrm{Id}+\partial_{\phi_1}v_3)^{-1}\partial_hG(v_1,\phi_1+v_3)(\mathrm{Id}+v_2)\|_{\sigma_+,\eta_+}\\
&\quad+\|(\mathrm{Id}-\Gamma_K)Q\|_{\sigma_+,\eta_+}+\|v_2(\Lambda_+-\Lambda)\|_{\sigma_+,\eta_+}+\|Q(\phi_1+v_3)-Q\|_{\sigma_+,\eta_+}\\
&\quad+\|Q(\phi_1+v_3)v_2\|_{\sigma_+,\eta_+}+\varepsilon_0^{2/3}\|\partial_{\phi_1}v_2(\widehat{N}(0)+N_+)\|_{\sigma_+,\eta_+}\\
&\quad+\|\partial_hF(v_1,\phi_1+v_3)(\mathrm{Id}+v_2)\|_{\sigma_+,\eta_+}\Big)\\
&\le C\Big(\frac{\varepsilon_0^{5/3}\varepsilon^{1/6}}{\gamma^2\kappa^{2\tau+4}}+\|Q\|_{\sigma,\eta_+}K^2e^{-\kappa K}+\frac{\varepsilon^{1/4}}{\gamma^2\kappa^{2\tau+3}}+\frac{\varepsilon_0^{1/6}\varepsilon^{1/6}}{\gamma^2\kappa^{2\tau+3}}\Big)\\
&\le \varepsilon_+^{1/8}.
\end{aligned}
$$

Thus, we verify condition (A2) with $m=l+1$.

3.3. **Non-degeneracy condition and measure of the non-resonant set.** From (3.4) and (3.15), it follows that

$$
\inf_{\xi\in\Pi_+^{\eta_+}}\big|\det\frac{\partial\omega_+}{\partial\xi}\big|=\inf_{\xi\in\Pi_+^{\eta_+}}\Big|\det(\frac{\partial\omega_0}{\partial\xi}+\varepsilon_0^{5/3}\sum_{m=0}^{l}\frac{\partial\widehat{N}_m(0)}{\partial\xi})\Big|\ge\chi_+\ge\frac12\chi_0,
$$

the non-degeneracy condition is held with $m=l+1$. Let

$$
R_k=\big\{\xi\in\Pi:|(k,\omega)|<\frac{\gamma}{|k|^\tau},\ 0<|k|\le K\big\},\quad R=\cup_{0\ne|k|\le K}R_k.
$$

Then

$$\Pi_+ = \Pi \backslash R.$$

By (A1) with $m = l$ and Lemma 4.4, we obtain

$$\operatorname{meas} R_k \le \frac{4\gamma}{\chi_0 |k|^{\tau+1}} \operatorname{meas} \Pi,$$

$$\operatorname{meas} R \le \frac{4}{\chi_0} \gamma \operatorname{meas} \Pi \sum_{0 \ne k \in \mathbb{Z}^2} |k|^{-(\tau+1)} \le C_0 \gamma \operatorname{meas} \Pi.$$

Therefore,

$$\operatorname{meas} \Pi_+ \ge \operatorname{meas} \Pi - \operatorname{meas} R \ge \operatorname{meas} \Pi (1 - C_0 \gamma),$$

and the condition (A4) is satisfied with $m = l + 1$. The proof of Lemma 3.1 is complete. □

4. A KAM theorem and Proof of Theorem 2.2

Theorem 4.1. *Suppose that*

$$\begin{aligned} \dot{h}_0 &= \varepsilon_0 [\Lambda_0(\xi) h_0 + M_0(\phi_0, \xi) + Q_0(\phi_0, \xi) h_0 + F_0(h_0, \phi_0, \xi)], \\ \dot{\phi}_0 &= \omega_0(\xi) + \varepsilon_0^{5/3} [N_0(\phi_0, \xi) + G_0(h_0, \phi_0, \xi)] \end{aligned} \tag{4.1}$$

satisfy conditions (A1)–(A4) in Lemma 3.1 with $m = 0$. Then for any given $0 < \gamma \ll 1$, there exists a sufficiently small positive number $\varepsilon_0^* = O(\gamma^{s_*})$ with $s_* \ge 29$, such that if $0 < \varepsilon < \varepsilon_0^*$, then there exists a Cantorian closed subset $\Pi_\infty \subset \Pi_0$ with $\operatorname{meas} \Pi_\infty = \operatorname{meas} \Pi_0 - O(\gamma)$ such that for any $\xi \in \Pi_\infty$, by the coordinate transformation

$$\mathcal{T}: \ h_0 = \widetilde{h} + V_1(\widetilde{\phi}, \xi) + V_2(\widetilde{\phi}, \xi) \widetilde{h}, \quad \phi_0 = \widetilde{\phi} + V_3(\widetilde{\phi}, \xi), \quad \xi = \xi$$

in which V_1, V_2, V_3 are real analytic in $\widetilde{\phi} \in U(\frac{\sigma_0}{2})$ and Lipschitz in $\xi \in \Pi_\infty$, equation (4.1) can be transformed into

$$\begin{aligned} \dot{\widetilde{h}} &= \varepsilon_0 [\Lambda_\infty(\xi) \widetilde{h} + F_\infty(\widetilde{h}, \widetilde{\phi}, \xi)], \\ \dot{\widetilde{\phi}} &= \omega_\infty(\xi) + \varepsilon_0^{5/3} G_\infty(\widetilde{h}, \widetilde{\phi}, \xi), \end{aligned} \tag{4.2}$$

where

$$F_\infty(\widetilde{h}, \widetilde{\phi}, \xi) = \varepsilon_0^{1/6} O_{\frac{s_0}{2}, \frac{\sigma_0}{2}, \Pi_\infty}(\widetilde{h}^2), \quad G_\infty(\widetilde{h}, \widetilde{\phi}, \xi) = \varepsilon_0^{1/6} O_{\frac{s_0}{2}, \frac{\sigma_0}{2}, \Pi_\infty}(\widetilde{h}),$$

$$\sup_{\xi \in \Pi_\infty} \|\Lambda_0(\xi) - \Lambda_\infty(\xi)\| \le C \varepsilon_0^{1/8}, \quad \sup_{\xi \in \Pi_\infty} \|\omega_0(\xi) - \omega_\infty(\xi)\| \le C \varepsilon_0^{43/24}.$$

Moreover, $\widetilde{h} = 0$, $\widetilde{\phi} = \phi_* + \omega_\infty(\xi) t$ is a trivial solution of (4.2), here ϕ_* represents an initial value of angle variables. It means that

$$h_0 = V_1(\phi_* + \omega_\infty(\xi) t, \xi), \quad \phi_0 = \phi_* + \omega_\infty(\xi) t + V_3(\phi_* + \omega_\infty(\xi) t, \xi)$$

is a real analytic quasi-periodic solution of (4.1). Meanwhile, the functions $V_1(\widetilde{\phi}, \xi)$, $V_2(\widetilde{\phi}, \xi)$ and $V_3(\widetilde{\phi}, \xi)$ satisfy the estimates

$$\sup_{\mathbb{T}^2 \times \Pi_\infty} \|V_1\| \le C \varepsilon_0^{2/25}, \quad \sup_{\mathbb{T}^2 \times \Pi_\infty} \|V_2\| \le C \varepsilon_0^{3/58}, \quad \sup_{\mathbb{T}^2 \times \Pi_\infty} \|V_3\| \le C \varepsilon_0^{41/24}.$$

Proof. Since conditions (A1)–(A4) with $m=0$ are satisfied for equation (4.1), we obtain a sequence of domains

$$W(s_0,\sigma_0)\supset W(s_1,\sigma_1)\supset\cdots\supset W(s_l,\sigma_l)\supset\cdots\supset W(\frac{s_0}{2},\frac{\sigma_0}{2})$$

by induction, a sequence of closed subsets

$$\Pi_0\supset\Pi_1\supset\cdots\supset\Pi_l\supset\dots$$

and a sequence of transformations $\mathcal{T}_l$ such that $(Eq)_l\circ\mathcal{T}_l=(Eq)_{l+1}$.

For given integer $l\geq 1$, using (3.15), we obtain

$$\begin{aligned}\Lambda_{l+1}(\xi)&=\Lambda_0(\xi)+\operatorname{diag}(\widehat{Q}_0^{11}(0),\widehat{Q}_0^{22}(0))+\cdots+\operatorname{diag}(\widehat{Q}_l^{11}(0),\widehat{Q}_l^{22}(0)),\\ \omega_{l+1}(\xi)&=\omega_0(\xi)+\varepsilon_0^{5/3}\widehat{N}_0(0)+\varepsilon_0^{5/3}\widehat{N}_1(0)+\cdots+\varepsilon_0^{5/3}\widehat{N}_l(0).\end{aligned}$$

From (3.4), it follows that $\{\Lambda_l(\xi)\}$ and $\{\omega_l(\xi)\}$ are uniformly convergent on Π_∞ and

$$\begin{gathered}\sup_{\xi\in\Pi_\infty}\|\Lambda_0(\xi)-\Lambda_{l+1}(\xi)\|\leq C\sum_{i=0}^{l}\varepsilon_i^{1/8}\leq C\sum_{i=0}^{\infty}\varepsilon_i^{1/8}\leq C\varepsilon_0^{1/8},\\ \sup_{\xi\in\Pi_\infty}\|\omega_0(\xi)-\omega_{l+1}(\xi)\|\leq C\varepsilon_0^{5/3}\sum_{i=0}^{l}\varepsilon_i^{1/8}\leq C\varepsilon_0^{5/3}\sum_{i=0}^{\infty}\varepsilon_i^{1/8}\leq C\varepsilon_0^{43/24}.\end{gathered}$$

Taking the limit as $l\to\infty$, we obtain

$$\sup_{\xi\in\Pi_\infty}\|\Lambda_0(\xi)-\Lambda_\infty(\xi)\|\leq C\varepsilon_0^{1/8},\quad \sup_{\xi\in\Pi_\infty}\|\omega_0(\xi)-\omega_\infty(\xi)\|\leq C\varepsilon_0^{43/24}.$$

Next we prove that the composition of the transformation (3.7) is convergent. For $\xi\in\Pi_{l+1}^{\eta_{l+1}}$, if ε is sufficiently small, then by (3.7) and (3.8) we can easily verify that $\mathcal{T}_l$ maps $U(\sigma_{l+1})$ into $U(\sigma_l)$, the composite transformation $\widetilde{\mathcal{T}_l}=\mathcal{T}_0\circ\mathcal{T}_1\circ\cdots\circ\mathcal{T}_l$ maps $U(\sigma_{l+1})$ into $U(\sigma_0)$ and can be explicitly written as

$$h_0=h_{l+1}+\widetilde{v_1^l}(\phi_{l+1},\xi)+\widetilde{v_2^l}(\phi_{l+1},\xi)h_{l+1},\quad \phi_0=\phi_{l+1}+\widetilde{v_3^l}(\phi_{l+1},\xi),\quad \xi=\xi,$$

where

$$\begin{gathered}\widetilde{v_1^l}(\phi_{l+1},\xi)=v_1^0(\phi_1,\xi)+\sum_{m=0}^{l-1}\Big\{\Big(\prod_{i=0}^{m}(\mathrm{Id}+v_2^i(\phi_{i+1},\xi))\Big)v_1^{m+1}(\phi_{m+2},\xi)\Big\},\\ \widetilde{v_2^l}(\phi_{l+1},\xi)=\prod_{i=0}^{l}(\mathrm{Id}+v_2^i(\phi_{i+1},\xi))-\mathrm{Id},\quad \widetilde{v_3^l}(\phi_{l+1},\xi)=\sum_{i=0}^{l}v_3^i(\phi_{i+1},\xi).\end{gathered}\tag{4.3}$$

Using (3.23), (3.27), (3.30), (3.32) and (4.3), we obtain

$$\begin{aligned}\sup_{U(\sigma_{l+1})\times\Pi_{l+1}^{\eta_{l+1}}}\|\widetilde{v_1^l}\|&\leq C\Big(\frac{\varepsilon_0^{1/6}}{\gamma_0^2\kappa_0^{2\tau+3}}+\frac{\varepsilon_1^{1/6}}{\gamma_1^2\kappa_1^{2\tau+3}}+\cdots+\frac{\varepsilon_l^{1/6}}{\gamma_l^2\kappa_l^{2\tau+3}}\Big)\\ &\leq C\varepsilon_0^{\frac16-\frac{2}{s_*}}\leq C\varepsilon_0^{\frac{2}{25}},\end{aligned}\tag{4.4}$$

$$\begin{aligned}\sup_{U(\sigma_{l+1})\times\Pi_{l+1}^{\eta_{l+1}}}\|\widetilde{v_2^l}\|&\leq C\Big(\frac{\varepsilon_0^{1/8}}{\gamma_0^2\kappa_0^{2\tau+3}}+\frac{\varepsilon_1^{1/8}}{\gamma_1^2\kappa_1^{2\tau+3}}+\cdots+\frac{\varepsilon_l^{1/8}}{\gamma_l^2\kappa_l^{2\tau+3}}\Big)\\ &\leq C\varepsilon_0^{\frac18-\frac{2}{s_*}}\leq C\varepsilon_0^{\frac{3}{58}},\end{aligned}\tag{4.5}$$

$$\begin{aligned}\sup_{U(\sigma_{l+1})\times\Pi_{l+1}^{\eta_{l+1}}}\|\widetilde{v_3^l}\| &\le C\Big(\frac{\varepsilon_0^{5/3}\varepsilon_0^{1/8}}{\gamma_0^2\kappa_0^{2\tau+3}}+\frac{\varepsilon_0^{5/3}\varepsilon_1^{1/8}}{\gamma_1^2\kappa_1^{2\tau+3}}+\dots+\frac{\varepsilon_0^{5/3}\varepsilon_l^{1/8}}{\gamma_l^2\kappa_l^{2\tau+3}}\Big)\\ &\le C\varepsilon_0^{\frac{43}{24}-\frac{2}{s_*}}\le C\varepsilon_0^{41/24},\end{aligned}\tag{4.6}$$

where $\widetilde{v_1^l}$, $\widetilde{v_2^l}$ and $\widetilde{v_3^l}$ are real analytic in $\phi_{l+1}\in U(\sigma_{l+1})$ and C^1-smooth in $\xi\in\Pi_{l+1}^{\eta_{l+1}}$. Let

$$\begin{gathered}V_1=\lim_{l\to\infty}\widetilde{v_1^l},\quad V_2=\lim_{l\to\infty}\widetilde{v_2^l},\quad V_3=\lim_{l\to\infty}\widetilde{v_3^l},\\ \mathcal{T}=\lim_{l\to\infty}\widetilde{\mathcal{T}_l},\quad \Pi_\infty=\cap_{m=0}^\infty\Pi_m.\end{gathered}\tag{4.7}$$

Then $\widetilde{v_1^l},\widetilde{v_2^l}$ and $\widetilde{v_3^l}$ converge uniformly to V_1,V_2 and V_3, respectively, and

$$\sup_{\mathbb{T}^2\times\Pi_\infty}\|V_1\|\le C\varepsilon_0^{\frac{2}{25}},\quad \sup_{\mathbb{T}^2\times\Pi_\infty}\|V_2\|\le C\varepsilon_0^{\frac{3}{58}},\ \sup_{\mathbb{T}^2\times\Pi_\infty}\|V_3\|\le C\varepsilon_0^{41/24}.$$

By (3.6), we obtain

$$\operatorname{meas}\Pi_\infty=\lim_{l\to\infty}\operatorname{meas}\Pi_l=\operatorname{meas}\Pi_0-O(\gamma).$$

Now we prove that V_1, V_2 and V_3 are Lipschitz in $\xi\in\Pi_\infty$. As $\lim_{l\to\infty}\varepsilon_l^{1/16}\eta_{l+1}^{-1}=0$, there exists a constant C such that $\varepsilon_l^{1/16}\eta_{l+1}^{-1}\le C$. For $\xi^1,\xi^2\in\Pi_{l+1}$, if $|\xi^1-\xi^2|\ge 2\eta_{l+1}$, from (3.8) we obtain

$$\begin{aligned}&\sup_{\phi_{l+1}\in U(\sigma_{l+1})}\|v_3^l(\phi_{l+1},\xi^1)-v_3^l(\phi_{l+1},\xi^2)\|\\ &\le C\varepsilon_0^{5/3}\varepsilon_l^{1/8}\gamma_l^{-2}\kappa_l^{-(2\tau+3)}\le C\varepsilon_0^{5/3}\varepsilon_l^{1/16}\gamma_l^{-2}\kappa_l^{-(2\tau+3)}|\xi^1-\xi^2|;\end{aligned}$$

if $|\xi^1-\xi^2|<2\eta_{l+1}$, by (3.8) and the mean value theorem, we obtain

$$\begin{aligned}&\sup_{\phi_{l+1}\in U(\sigma_{l+1})}\|v_3^l(\phi_{l+1},\xi^1)-v_3^l(\phi_{l+1},\xi^2)\|\\ &\le \|v_3^l\|_{\sigma_{l+1},\eta_{l+1}}|\xi^1-\xi^2|\le C\varepsilon_0^{5/3}\varepsilon_l^{1/16}\gamma_l^{-2}\kappa_l^{-(2\tau+3)}|\xi^1-\xi^2|.\end{aligned}$$

Thus, v_3^l is Lipschitz with respect to $\xi\in\Pi_{l+1}$. Similarly, by (3.8) again, we can obtain that v_1^l and v_2^l are Lipschitz with respect to $\xi\in\Pi_{l+1}$. Therefore, by (4.3) and (4.7), it implies that V_1, V_2 and V_3 are Lipschitz with respect to $\xi\in\Pi_\infty$. Then equation (4.1) can be changed into

$$\begin{gathered}\dot{\widetilde{h}}=\varepsilon_0[\Lambda_\infty(\xi)\widetilde{h}+F_\infty(\widetilde{h},\widetilde{\phi},\xi)],\\ \dot{\widetilde{\phi}}=\omega_\infty(\xi)+\varepsilon_0^{5/3}G_\infty(\widetilde{h},\widetilde{\phi},\xi)\end{gathered}$$

by the transformation $\mathcal{T}$, where $F_\infty(\widetilde{h},\widetilde{\phi},\xi)=\varepsilon_0^{1/6}O_{\frac{s_0}{2},\frac{\sigma_0}{2},\Pi_\infty}(\widetilde{h}^2)$, $G_\infty(\widetilde{h},\widetilde{\phi},\xi)=\varepsilon_0^{1/6}O_{\frac{s_0}{2},\frac{\sigma_0}{2},\Pi_\infty}(\widetilde{h})$. In particular, $\widetilde{h}=0$, $\widetilde{\phi}=\phi_*+\omega_\infty(\xi)t$ is a trivial solution of above equation, and

$$h_0=V_1(\phi_*+\omega_\infty(\xi)t,\xi),\quad \phi_0=\phi_*+\omega_\infty(\xi)t+V_3(\phi_*+\omega_\infty(\xi)t,\xi)$$

is a real analytic quasi-periodic solution of (4.1). □

Proof of Theorem 2.2. We only need to prove Theorem 2.2 for equation (2.13). Suppose that $\xi=(a,b)\in[1,2]\times[5,6]$, $\omega_0=(\Omega_1,\Omega_2)^T$,

$$\Lambda_0=\operatorname{diag}\big(\lambda_1^0,\lambda_2^0\big)=\operatorname{diag}\big(\frac{1}{3},-1\big),$$

$$\Omega_1 = \frac{\sqrt{4b - 8a - \varepsilon^2}}{2}, \quad \Omega_2 = \frac{\sqrt{4b - \varepsilon^2}}{2}.$$

Then there exist positive constants c_0, ζ_0, d_0 and χ_0 such that

$$\inf_{\xi \in \Pi_0^{\eta_0}} |\lambda_1^0(\xi) - \lambda_2^0(\xi)| \geq c_0, \quad \|\lambda_1^0 - \lambda_2^0\|_{\eta_0} \leq \zeta_0, \quad \inf_{\xi \in \Pi_0^{\eta_0}} \Big| \det \frac{\partial \omega_0}{\partial \xi} \Big| \geq \chi_0$$

$$\|\omega_0(\xi)\|_{\eta_0} \leq d_0, \quad \inf_{\xi \in \Pi_0^{\eta_0}} |\lambda_j^0(\xi)| \geq c_0, \quad \|\lambda_j^0\|_{\eta_0} \leq \zeta_0, \quad j = 1, 2$$

for $\xi \in \Pi_0^{\eta_0}$. Hence, the system (2.13) satisfies conditions (A1)–(A4) with $m = 0$. By Theorem 4.1, we obtain the result in Theorem 2.2. □

sectionAppendix

Lemma 4.2 ([3, 41]). *For $\delta > 0$ and $\mu > 0$, it holds*

$$\sum_{k \in \mathbb{Z}^n} e^{-2|k|\delta} |k|^\mu \leq \Big(\frac{\mu}{e}\Big)^\mu \frac{1}{\delta^{\mu+n}} (1+e)^n.$$

Lemma 4.3 ([30]). *Denote by $\mathfrak{A}^s$ the space of all functions on $\mathbb{T}^n$ bounded and analytic in the strip $\{\phi : |\operatorname{Im} \phi| \leq s\}$. If $v \in \mathfrak{A}^s$ and $K\sigma \geq 1$, then*

$$\|(Id - \Gamma_K)v\|_{s-\sigma} \leq CK^n e^{-K\sigma} \|v\|_s, \ 0 \leq \sigma \leq s,$$

where the constant C depends only on n.

Lemma 4.4. *Suppose that $\Pi \subset \mathbb{R}^n$ is a bounded closed set with positive Lebesgue measure and that Π^η is a neighborhood of Π, $f(x)$ is continuously differentiable in $x \in \Pi^\eta$. If there are two positive constants a and b such that*

$$\inf_{x \in \Pi^\eta} \big| \det \frac{\partial f}{\partial x} \big| \geq a, \quad \sup_{x \in \Pi^\eta} \|f(x)\| \leq b, \quad \sup_{x \in \Pi^\eta} \big\| \frac{\partial f}{\partial x} \big\| \leq b,$$

then

$$a \operatorname{meas} \Pi \leq \operatorname{meas} f\{\Pi\} \leq b \operatorname{meas} \Pi.$$

Proof of results similar to the one above can be found in [3, 19].

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Lin Lu
School of Mathematics and Computational Science, Hunan First Normal University, Changsha, Hunan 410205, China
Email address: 14726989141@163.com

Shujuan Liu (corresponding author)
Key Laboratory of High Performance Computing and Stochastic Information Processing, Department of Mathematics, Hunan Normal University, Changsha, Hunan 410083, China
Email address: 846706840@smail.hunnu.edu.cn

Jun Wu
Key Laboratory of High Performance Computing and Stochastic Information Processing, Department of Mathematics, Hunan Normal University, Changsha, Hunan 410083, China
Email address: wujun433123@163.com