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ASYMPTOTICALLY ALMOST PERIODIC AND ALMOST PERIODIC SOLUTIONS FOR A CLASS OF EVOLUTION EQUATIONS

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Abstract. In this paper we study the existence of asymptotically almost periodic and almost periodic solutions for the partial evolution equation

$$\frac{d}{dt}(x(t)+g(t,x(t))) = Ax(t)+f(t,Bx(t)),$$

where A is the infinitesimal generator of an analytic semigroup on a Banach space X, B is a closed linear operator, and f, g are given functions.

1. Introduction

The existence of almost periodic solutions for abstract evolution equation defined on abstract Banach spaces has been studied in various works, see for instance [2, 9, 10, 11, 12]. By using the semigroup theory and the contraction mapping principle, Zaidman studied in [10] the existence of almost periodic solutions for the integral equation associated to the abstract partial differential equation

$$x'(t) = Ax(t) + f(t,x(t)), \tag{1.1}$$

where A is the infinitesimal generator of a C_0-semigroup of bounded linear operators on a Banach space. Recently, Bahaj and Sidki studied in [2] the existence of almost periodic solution for (1.1).

The purpose of this paper is to discuss the existence of asymptotically almost periodic and almost periodic solutions for partial evolution equations of the form

$$\frac{d}{dt}(x(t)+g(t,x(t))) = Ax(t)+f(t,Bx(t)), \tag{1.2}$$

$$x(t_0) = y_0, \tag{1.3}$$

where A is the infinitesimal generator of an analytic semigroup of linear operators defined on a Banach space X, $B : D(B) \subset X \to X$ is a special type of closed operator and $f, g : I \times X \to X$ are give functions.

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We remark that the technical framework used in this work allow us, for instance, to study the partial differential equation

$$\frac{d}{dt}(u(t,\xi)+g(t,u(t,\xi))) = \frac{\partial^2 u(t,\xi)}{\partial^2 \xi} + f(t, \frac{\partial u(t,\xi)}{\partial \xi})).$$

In fact, it's well known, see [7], that for a class of operators A, there is a bounded linear operator $L : X \to X$ such that $\frac{\partial}{\partial \xi} = (-A)^{1/2} \circ L$. Additionally, we mention that by using the techniques used in this paper, it's possible to establish the existence of asymptotically almost periodic solutions for (1.2)-(1.3) without making additional regularity assumptions on the initial data. We refer to Bridges [1] and Rankin [7] for complementary remarks about this matter.

The results in this work are generalizations of the results in [2, 10] and our ideas and techniques can be used in the study of the existence of asymptotically almost periodic and almost periodic solutions of partial neutral functional differential equations and partial differential equations of Sobolev type, see Hernandez [5] for details. In general, our results are proved by using the semigroup theory of bounded linear operators, the theory of fractional power of closed operators and the contraction mapping principle.

This paper has four sections. In section 3 we study the existence of asymptotically almost periodic and almost periodic solutions for the integral equation associated to (2.2) and in section 4 we establish conditions under which these "mild" solutions are classical solutions. In section 5 an example is considered.

2. Preliminaries

In this section we mention a few results and establish notation needed for stating our results. In this paper, $(X, \|\cdot\|)$ is a Banach space and $A : D(A) \subset X \to X$ is the infinitesimal generator of a uniformly exponentially stable analytic semigroup of linear operators $(T(t))_{t\geq 0}$ on X such that $0 \in \rho(A)$. Throughout this work, M, δ are positive constants such that $\|T(t)\| \leq Me^{-\delta t}$ for every $t \geq 0$. Under these conditions it is possible to define the fractional power $(-A)^\alpha$, $0 < \alpha \leq 1$, as a closed linear operator on its domain $D((-A)^\alpha)$. Furthermore, $D((-A)^\alpha)$ is dense in X and the expression $\|x\|_\alpha = \|(-A)^\alpha x\|$ defines a norm in $D((-A)^\alpha)$. If X_α is the space $D((-A)^\alpha)$ endowed with the norm $\|\cdot\|_\alpha$, then the following properties hold, see [6].

Lemma 2.1. *Let $0 < \gamma \leq \vartheta \leq 1$. Then X_ϑ is a Banach space and $X_\vartheta \hookrightarrow X_\gamma$. Moreover, the function $t \to (-A)^\vartheta T(t)$ is continuous in the uniform operator topology on $(0,\infty)$ and there exist constants $C_\vartheta, C'_\vartheta$ such that*

$$\|(-A)^\vartheta T(t)\| \leq \frac{C_\vartheta e^{-\delta t}}{t^\vartheta} \quad \text{and} \quad \|(T(t)-I)(-A)^{-\vartheta}\| \leq C'_\vartheta t^\vartheta$$

for every $t > 0$.

Let $(Z, \|\cdot\|_Z)$ and $(W, \|\cdot\|_W)$ be abstract Banach spaces. In this work, we indicate by $\mathcal{L}\ (Z : W)$ the Banach space of bounded linear operator of Z into W and we abbreviate to $\mathcal{L}(Z)$ whenever $Z = W$. The notation $C(I : Z)$ represents the space of continuous function from I into Z endowed with the uniform convergence topology. As usual, $C_b([0,\infty) : Z)$ is the space of bounded continuous function from $[0,\infty)$ into Z endowed with the uniform convergence topology and $C_0([0,\infty) : Z)$ is the subspace of $C_b([0,\infty) : Z)$ formed by the functions which vanish at infinity.

Along this work, $B_r(x : Z)$, $x \in Z$, will denote the closed ball with center at x and radius $r > 0$ in Z. For a bounded and continuous function $\xi : (a, b) \to Z$ and $t \in (a, b)$, we will employ the notation $\|\xi\|_{a,t,Z}$ for

$$\|\xi\|_{a,t,Z} = \sup\{\|\xi(s)\|_Z : s \in (a, t]\}, \tag{2.1}$$

and we will write simply $\|\xi\|_{t,Z}$ when non confusion arise.

We remark that a function $f : [a, b] \to Z$ is σ-Hölder continuous, $0 < \sigma \le 1$, if there is a constant $\kappa > 0$ such that

$$\|f(s) - f(t)\| \le \kappa|t - s|^\sigma, \quad s, t \in [a, b].$$

We represent by $C^\sigma([a, b]; Z)$ the space of σ-Hölder continuous function from $[a, b]$ into Z endowed with the uniform convergence topology. The notation $C^\sigma((a, b]; Z)$ stands for the space of continuous function $f : [a, b] \to Z$ such that $f \in C^\sigma([\delta, b]; Z)$ for every $\delta > a$.

Next we make some remarks concerning almost periodic and asymptotically almost periodic functions.

Definition 2.2. A continuous function $f : \mathbb{R} \to Z$ is called almost periodic if for every $\epsilon > 0$ there exists a relatively dense subset of $\mathbb{R}$, denoted by $\mathcal{H}(\epsilon, f, Z)$, such that

$$\|f(t + \xi) - f(t)\|_Z < \epsilon,$$

for every $t \in \mathbb{R}$ and every $\xi \in \mathcal{H}(\epsilon, f, Z)$.

Definition 2.3. A continuous function $f : [0, \infty) \to Z$ is the called asymptotically almost periodic if there exists an almost periodic function $g(\cdot) : \mathbb{R} \to Z$ and a function $w(\cdot) \in C_0([0, \infty) : Z)$ such that $f(t) = g(t) + w(t)$ for every $t \ge 0$.

In this paper, $AP(Z)$ and $AAP(Z)$ are the spaces

$$AP(Z) = \{u \in C_b(\mathbb{R} : Z) : u \text{ is almost periodic } \},$$

$$AAP(Z) = \{u \in C_b([0, \infty) : Z) : u \text{ is asymptotically almost periodic } \},$$

provided with the norm of the uniform convergence. It' s well known that $AP(Z)$ and $AAP(Z)$ are Banach spaces, see [13].

Lemma 2.4 (Characterization of asymptotically almost periodic function [13, Theorem 5])**.** *Let $F([0, \infty) : Z)$ be the subspace of $C_b([0, \infty) : Z)$ formed by the functions $f(\cdot)$ which satisfy the following property: for every $\epsilon > 0$ there exists $L(\epsilon, f, Z) > 0$ and a relatively dense subset of $[0, \infty)$, denoted by $\mathcal{T}(\epsilon, f, Z)$, such that*

$$\|f(t + \xi) - f(t)\|_Z < \epsilon,$$

for every $t \ge L(\epsilon, f, Z)$ and every $\xi \in \mathcal{T}(\epsilon, f, Z)$. Then, $F([0, \infty) : Z) = AAP(Z)$.

The next definitions and properties are essential for establishing our results.

Definition 2.5. Let $\Omega \subset W$ be a open set and $F : \mathbb{R} \times \Omega \to Z$ be a continuous function.

(1) F is called pointwise almost periodic (pointwise a.p.), if $F(\cdot, x) \in AP(Z)$ for every $x \in \Omega$.

(2) F is called uniformly almost periodic (u.a.p.), if for every $\epsilon > 0$ and every compact $K \subset \Omega$ there exists a relatively dense subset of $\mathbb{R}$, denoted by $\mathcal{H}(\epsilon, F, K, Z)$, such that

$$\|F(t + \xi, y) - F(t, y)\|_Z \le \epsilon,$$

for every $(t, \xi, y) \in \mathbb{R} \times \mathcal{H}(\epsilon, F, K, Z) \times K$.

Definition 2.6. Let $\Omega \subset W$ be a open set and $F : [0,\infty) \times \Omega \to Z$ be a continuous function.

(1) F is called pointwise asymptotically almost periodic (pointwise a.a.p.), if $F(\cdot, x) \in AAP(Z)$ for every $x \in \Omega$.
(2) F is called uniformly asymptotically almost periodic (u.a.a.p.), if for every $\epsilon > 0$ and every compact $K \subset \Omega$ there exists a relatively dense subset of $[0,\infty)$, denoted by $\mathcal{T}(\epsilon, F, K, Z)$, and $L(\epsilon, F, K, Z) > 0$ such that

$$\|F(t+\xi, y) - F(t, y)\|_Z \leq \epsilon,$$

for every $t \geq L(\epsilon, F, K, Z)$ and every $(\xi, y) \in \mathcal{T}(F, \epsilon, K, Z) \times K$.

For details concerning the next two lemmas, see [8, Theorem 1.2.7] and [10].

Lemma 2.7. *Let $\Omega \subset W$ be a open set and $F : \mathbb{R} \times \Omega \to Z$ be a continuous function. Then the following properties hold.*

(1) *If F is pointwise a.p. and satisfies a local Lipschitz condition at $x \in \Omega$, uniformly at t, then F is u.a.p.*
(2) *If F is u.a.p. and $y \in AP(W)$ is such that $\overline{\{y(t) : t \in \mathbb{R}\}}^W \subset \Omega$, then $F(t, y(t)) \in AP(Z)$.*

Lemma 2.8. *Let $\Omega \subset W$ be a open set and $F : [0,\infty) \times \Omega \to Z$ be a continuous function. Then the following properties hold.*

(1) *If F is pointwise a.a.p. and satisfies a local Lipschitz condition at $x \in \Omega$, uniformly at t, then F is u.a.a.p.*
(2) *If F is u.a.a.p. and $y \in AAP(W)$ is such that $\overline{\{y(t) : t \in [0,\infty)\}}^W \subset \Omega$, then $F(t, y(t)) \in AAP(Z)$.*

Throughout this paper, $0 < \alpha, \beta \leq 1$ are fixed numbers and $(Y, \|\cdot\|_Y)$ is a Banach space such that $X_\eta \hookrightarrow Y \hookrightarrow X$ for every $\eta \in (0,1)$. To obtain our results we will use the following technical conditions.

(H1) The function $s \to T(s)y \in C([0,\infty); Y)$ for every $y \in Y$ and there are $\widetilde{M} > 0, \tilde{\delta} > 0$ such that $\|T(s)\|_{\mathcal{L}(Y)} \leq \widetilde{M} e^{-\tilde{\delta} s}$ for every $s \geq 0$. Moreover, the functions $s \to (-A)^{1-\beta} T(s)$, $s \to (-A)^\alpha T(s)$ defined from $(0,\infty)$ into $\mathcal{L}(X, Y)$ are strongly measurable and there are non-decreasing functions H_β, H_α and numbers $\omega_i < 0$, $i = 1, 2$, such that $e^{\omega_1 s} H_\beta(s) \in \mathrm{L}^1([0,\infty))$, $e^{\omega_2 s} H_\alpha(s) \in \mathrm{L}^1([0,\infty))$ and

$$\|(-A)^{1-\beta} T(s)\|_{\mathcal{L}(X:Y)} \leq e^{\omega_1 s} H_\beta(s), \quad s > 0,$$
$$\|(-A)^{\alpha} T(s)\|_{\mathcal{L}(X:Y)} \leq e^{\omega_2 s} H_\alpha(s), \quad s > 0.$$

(H2) The function $g(\cdot)$ is X_β-valued, $(-A)^\beta g : \mathbb{R} \times Y \to X$ is continuous, $(-A)^\beta g(s, 0) = 0$ for every $s \geq 0$ and there is a continuous function $L_g : [0,\infty) \to (0,\infty)$ such that $L_g(0) = 0$ and

$$\|(-A)^\beta g(t_1, y_1) - (-A)^\beta g(t_2, y_2)\| \leq L_g(r)(|t_1 - t_2| + \|y_1 - y_2\|_Y),$$

for every $(t_i, y_i) \in \mathbb{R} \times B_r(0, Y)$.

(H3) The map $B : D(B) \subset X \to X$ is a closed linear operator such that $D((-A)^\alpha) \subset D(B)$ and there are continuous functions $\tilde{f} : \mathbb{R} \times Y \to X$,

$L_{\tilde{f}} : [0,\infty) \to [0,\infty)$ such that $L_{\tilde{f}}(0) = 0$, $\tilde{f}(s,0) = 0$ for every $s \geq 0$, $\tilde{f}(\mathbb{R} \times X_\alpha) \subset X_\alpha$, $(-A)^\alpha \tilde{f}(t,x) = f(t,Bx)$ for every $(t,x) \in \mathbb{R} \times X_\alpha$ and

$$\|\tilde{f}(t_1,y_1) - \tilde{f}(t_2,y_2)\| \leq L_{\tilde{f}}(r)(|t_1 - t_2| + \|y_1 - y_2\|_Y),$$

when $(t_i,y_i) \in \mathbb{R} \times B_r(0,Y)$.

Remark 2.9. For examples of semigroups of linear operators and functions verifying the previous assumption, see Bridges [1], Hagen & Turi [4] and Rankin [7].

Following Hernandez [5] and Rankin [7] we introduce the next concepts.

Definition 2.10. A function $u \in C([t_0,r) : Y)$ is a Y-mild solution of the abstract Cauchy problem (1.2)-(1.3) if $u(t_0) = y_0$; the functions $s \to AT(t-s)g(s,u(s))$, $s \to (-A)^\alpha T(t-s)\tilde{f}(s,u(s))$ belong to $\mathrm{L}^1([t_0,t] : Y)$ for every $t_0 \leq t < r$ and

$$\begin{aligned} u(t) = T(t-t_0)(y_0 + g(t_0,y_0)) - g(t,u(t)) - \int_{t_0}^t AT(t-s)g(s,u(s))ds \\ + \int_{t_0}^t (-A)^\alpha T(t-s)\tilde{f}(s,u(s))ds, \quad t \in [t_0,r]. \end{aligned} \tag{2.2}$$

Definition 2.11. A function $u \in C([t_0,r) : X)$ is a mild solution of (1.2)-(1.3) if $u(t_0) = y_0$; $u \in C((t_0,r) : X_\alpha)$; the function $s \to AT(t-s)g(s,u(s))$ belongs to $\mathrm{L}^1([t_0,t] : X)$ for every $t \in [t_0,r)$ and

$$\begin{aligned} u(t) = T(t-t_0)(y_0 + g(t_0,y_0)) - g(t,u(t)) - \int_{t_0}^t AT(t-s)g(s,u(s))ds \\ + \int_{t_0}^t T(t-s)f(s,Bu(s))ds, \quad t \in [t_0,r). \end{aligned}$$

The next definition has been introduced in Hernandez [5].

Definition 2.12. A function $u \in C([t_0,r] : X)$ is an S-classical (Semi-classical) solution of (1.2)-(1.3) if $u(t_0) = y_0$, $\frac{d}{dt}(u(t) + g(t,u(t)))$ is continuous on (t_0,r), $u(t) \in D(A)$ for all $t \in (t_0,r]$ and $u(\cdot)$ satisfies (1.2)-(1.3) on (t_0,r).

In relation to asymptotically almost periodic and almost periodic solutions we introduce the following definitions.

Definition 2.13. A function $u \in AP(Y)$ is an almost periodic Y-mild solution of (1.2)-(1.3) if the functions $s \to AT(t-s)g(s,u(s))$, $s \to (-A)^\alpha T(t-s)\tilde{f}(s,u(s))$ belong to $\mathrm{L}^1((-\infty,t] : Y)$ for every $t \in \mathbb{R}$ and

$$u(t) = -g(t,u(t)) - \int_{-\infty}^t AT(t-s)g(s,u(s))ds + \int_{-\infty}^t (-A)^\alpha T(t-s)\tilde{f}(s,u(s))ds,$$

for every $t \in \mathbb{R}$.

Definition 2.14. A function $u \in AP(X)$ is an almost periodic mild solution of (1.2)-(1.3) if $u \in C(\mathbb{R} : X_\alpha)$, the function $s \to AT(t-s)g(s,u(s))$ belongs to $\mathrm{L}^1((-\infty,t] : X)$ for every $t \in \mathbb{R}$ and

$$u(t) = -g(t,u(t)) - \int_{-\infty}^t AT(t-s)g(s,u(s))ds + \int_{-\infty}^t T(t-s)f(s,Bu(s))ds, \quad t \in \mathbb{R}.$$

Definition 2.15. A function $u \in AP(X)$ is a S-classical solution of (1.2)-(1.3) on $\mathbb{R}$, if u is a S-classical solution of (1.2)-(1.3) on every interval $[t_0, t_0+\sigma] \subset \mathbb{R}$, with $t_0 \in \mathbb{R}$ and $\sigma > 0$.

Definition 2.16. A function $u \in AAP(Y)$ is an asymptotically almost periodic Y-mild solution of (1.2)-(1.3) if $u(0) = y_0$, the functions $s \to AT(t-s)g(s,u(s))$, $s \to (-A)^\alpha T(t-s)\tilde{f}(s,u(s))$ belong to $\mathrm{L}^1((0,t] : Y)$ for every $t \in [0,\infty)$ and

$$\begin{aligned} u(t) &= T(t)(y_0 + g(t_0,y_0)) - g(t,u(t)) - \int_0^t AT(t-s)g(s,u(s))ds \\ &\quad + \int_0^t (-A)^\alpha T(t-s)\tilde{f}(s,u(s))ds, \quad t \in [0,\infty). \end{aligned}$$

Definition 2.17. A function $u \in AAP(X)$ is a mild solution of (1.2)-(1.3) if $u(0) = y_0$, $u \in C((0,\infty) : X_\alpha)$, the function $s \to AT(t-s)g(s,u(s))$ belongs to $\mathrm{L}^1([0,t] : X)$ for every $t \in [0,\infty)$ and

$$\begin{aligned} u(t) &= T(t)(y_0 + g(t_0,y_0)) - g(t,u(t)) - \int_0^t AT(t-s)g(s,u(s))ds \\ &\quad + \int_0^t T(t-s)f(s,Bu(s))ds, \quad t \in [0,\infty). \end{aligned}$$

Definition 2.18. A function $u \in AAP(X)$ is a S-classical solution of (1.2)-(1.3) if u is a S-classical solution of (1.2)-(1.3) on $[0,r]$ for every $r > 0$.

3. Existence results of Y-mild solutions

In this section we establish the existence of asymptotically almost periodic and almost periodic Y-mild solutions for (1.2)-(1.3). First, we need the next result.

Proposition 3.1. *Let $\mu \in (0,1)$, $v(\cdot) \in AAP(X_\mu)$ and assume that there is $\omega < 0$ and a non-increasing function $H_\mu(\cdot)$ so that $e^{\omega s}H_\mu(s) \in \mathrm{L}^1([0,\infty))$ and $\|(-A)^{1-\mu}T(t)\|_{\mathcal{L}(X:Y)} \le e^{\omega t}H_\mu(t)$ for every $t > 0$. If $u(\cdot)$ is the function defined by*

$$u(t) = \int_0^t AT(t-s)v(s)ds, \quad t \ge 0, \tag{3.1}$$

then $u(\cdot) \in AAP(Y)$.

Proof. From Lemma 2.4, it's sufficient to prove that $u \in F(\mathbb{R}^+ : Y)$. Let $\epsilon > 0$ given and $\mathcal{T}(\epsilon, v, X_\mu)$, $L = L(\epsilon, v, X_\mu)$ be as in Lemma 2.4. If $t \ge L(\epsilon, v, X_\mu) + 1$ and $\xi \in \mathcal{T}(\epsilon, v, X_\mu)$, then

$$\begin{aligned} &\|u(t+\xi) - u(t)\|_Y \\ &\le \int_0^\xi \|(-A)^{1-\mu}T(t+\xi-s)(-A)^\mu v(s)\|_Y ds \\ &\quad + \int_0^t \|(-A)^{1-\mu}T(t-s)\left((-A)^\mu v(s+\xi) - (-A)^\mu v(s)\right)\|_Y ds \\ &= I_1(t,\xi) + I_2(t,\xi). \end{aligned}$$

Now, we estimate each term $I_i(t,\xi)$ separately. For the first term we get

$$I_1(t,\xi) \le \|(-A)^\mu v\|_{AAP(X)} \int_0^\xi e^{\omega(t+\xi-s)}H_\mu(t+\xi-s)ds$$

$$\leq e^{\omega t}\|(-A)^{\mu}v\|_{AAP(X)}\int_0^{\xi}e^{\omega(\xi-s)}H_{\mu}(\xi-s)ds,$$

and hence, there exits $d_1>0$ independent of ξ such that

$$I_1(t,\xi)\leq c_1e^{\omega t}, \tag{3.2}$$

for every $t\geq L(\epsilon,v,X_{\mu})+1$. On the other hand, for the second term we see that

$$\begin{aligned}
I_2(t,\xi)\leq&\int_0^{L+1}\|(-A)^{1-\mu}T(t-s)\left((-A)^{\mu}v(s+\xi)-(-A)^{\mu}v(s)\right)\|_Y ds\\
&+\int_{L+1}^{t}\|(-A)^{1-\mu}T(t-s)(-A)^{\mu}(v(s+\xi)-v(s))\|_Y ds\\
\leq&\, 2\|(-A)^{\mu}v\|_{AAP(X)}e^{\omega(t-L-1)}\int_0^{L+1}e^{\omega(L+1-s)}H_{\mu}(L+1-s)ds\\
&+\epsilon\int_{L+1}^{t}\|(-A)^{1-\mu}T(t-s)\|_{\mathcal{L}(X;Y)}ds\\
\leq&\, 2\|(-A)^{\mu}v\|_{AAP(X)}e^{\omega(t-L-1)}\int_0^{\infty}e^{\omega s}H_{\mu}(s)ds+\epsilon\int_0^{\infty}e^{\omega s}H_{\mu}(s)ds.
\end{aligned}$$

Thus, there exist positive constants d_2,d_3 independents of $t\geq L(\epsilon,v,X_{\mu})+1$ and $\xi\in\mathcal{T}(\epsilon,v,X_{\mu})$ such that

$$I_2(t,\xi)\leq d_2e^{\omega t}+\epsilon d_3. \tag{3.3}$$

From (3.2)-(3.3) we have

$$\|u(t+\xi)-u(t)\|_Y\leq d_4e^{\omega t}+\epsilon d_5,$$

where d_4,d_5 are positive constants independent of $t\geq L(\epsilon,v,X_{\mu})+1$ and $\xi\in\mathcal{T}(\epsilon,v,X_{\mu})$. Thus, for an appropriate $L(\epsilon,u)>L(\frac{\epsilon}{2d_5},v,X_{\mu})+1$, it follows

$$\|u(t+\xi)-u(t)\|_Y\leq\epsilon$$

for every $t\geq L(\epsilon,u)$ and all $\xi\in\mathcal{T}(\frac{\epsilon}{2d_5},v,X_{\mu})$, which shows that $u\in F(\mathbb{R}^+:Y)$ and completes the proof of this result. □

Proceeding as in the previous proof we can prove the next result.

Corollary 3.2. *Let $\mu\in(0,1)$ and $v\in AAP(X_{\mu})$. If $u(\cdot)$ is the function defined by 3.1, then $u(\cdot)\in AAP(X)$.*

In the next result we establish the existence of asymptotically almost periodic Y-mild solution of (1.2)-(1.3).

Theorem 3.3. *Let $\mathbf{H_1},\mathbf{H_2},\mathbf{H_3}$ be verified. Then, there exists $\epsilon>0$ such that for every $y_0\in B_{\epsilon}(0,Y)$ there exits an Y-mild solution $u(\cdot,y_0)\in C([0,\infty):Y)$ of (1.2)-(1.3). Moreover, if the functions $\tilde{f},(-A)^{\beta}g:[0,\infty)\times Y\to X$ are pointwise asymptotically almost periodic, then $u(\cdot,y_0)\in AAP(Y)$.*

Proof. Let $J:[0,\infty)\to\mathbb{R}$ be the function defined by

$$J(r)=L_g(r)\Big(\|(-A)^{-\beta}\|_{\mathcal{L}(X:Y)}+\int_0^{\infty}e^{\omega_1 s}H_{\beta}(s)ds\Big)+L_{\tilde{f}}(r)\int_0^{\infty}e^{\omega_2 s}H_{\alpha}(s)ds$$

and let $r>0$, $\gamma\in(0,1)$ be such that

$$\widetilde{M}\left(1+r\|(-A)^{-\beta}\|_{\mathcal{L}(X:Y)}L_g(r)\right)\gamma r+J(r)r<r. \tag{3.4}$$

Note that the assertion holds for $\epsilon = \gamma r$. To prove this statement we fix $y_0 \in B_\epsilon(0, Y)$ and define the operator $\Gamma : B_r(0, C_b([0,\infty) : Y)) \to C([0,\infty) : Y)$ by

$$\begin{aligned}\Gamma x(t) &= T(t)(y_0 + g(0, y_0)) - g(t, x(t)) + \int_0^t (-A)^{1-\beta} T(t-s)(-A)^\beta g(s, x(s))ds \\ &\quad + \int_0^t (-A)^\alpha T(t-s)\tilde{f}(s, x(s))ds.\end{aligned}$$

From the assumptions on the functions $s \to (-A)^\alpha T(s)$ and $s \to (-A)^{1-\beta} T(s)$, the estimates

$$\begin{aligned}\|(-A)^{1-\beta} T(s)(-A)^\beta g(s, x(s))\|_Y &\le \|(-A)^{1-\beta} T(s)\|_{\mathcal{L}(X;Y)} L_g(r) r \\ &\le e^{\omega_1 s} H_\beta(s) L_g(r) r,\end{aligned}$$

$$\|(-A)^\alpha T(s)\tilde{f}(s, x(s))\|_Y \le \|(-A)^\alpha T(s)\|_{\mathcal{L}(X;Y)} L_{\tilde{f}}(r) r \le e^{\omega_2 s} H_\alpha(s) L_{\tilde{f}}(r) r,$$

and the Bochner Theorem, we infer that $\Gamma x(t)$ is well defined and that $\Gamma x \in C([0,\infty); Y)$. Moreover, for $t \ge 0$ we get

$$\begin{aligned}&\|\Gamma x(t)\|_Y \\ &\le \widetilde{M}(\|y_0\|_Y + L_g(r)\|(-A)^{-\beta}\|_{\mathcal{L}(X:Y)}\|y_0\|_Y) + L_g(r)\|(-A)^{-\beta}\|_{\mathcal{L}(X:Y)}\|x(t)\|_Y \\ &\quad + \int_0^t \|(-A)^{1-\beta} T(s)\|_{\mathcal{L}(X;Y)} L_g(r) r ds + \int_0^t \|(-A)^\alpha T(s)\|_{\mathcal{L}(X;Y)} L_{\tilde{f}}(r) r ds \\ &\le \widetilde{M}(\gamma r + L_g(r)\|(-A)^{-\beta}\|_{\mathcal{L}(X:Y)}\gamma r) + J(r) r,\end{aligned}$$

which from (3.4) implies that $\Gamma(B_r(0, C_b([0,\infty) : Y))) \subset B_r(0, C_b([0,\infty) : Y))$.

Next, we prove that Γ is a contraction on $B_r(0, C_b([0,\infty) : Y))$. For functions $u, v \in B_r(0, C_b([0,\infty) : Y))$ we get

$$\begin{aligned}\|\Gamma u(t) - \Gamma v(t)\|_Y &\le L_g(r)\|(-A)^{-\beta}\|_{\mathcal{L}(X:Y)}\|u(t) - v(t)\|_Y \\ &\quad + L_g(r)\int_0^t \|(-A)^{1-\beta} T(t-s)\|_{\mathcal{L}(X;Y)}\|u(s) - v(s)\|_Y ds \\ &\quad + L_{\tilde{f}}(r)\int_0^t \|(-A)^\alpha T(t-s)\|_{\mathcal{L}(X;Y)}\|u(s) - v(s)\|_Y ds \\ &\le L_g(r)\Big(\|(-A)^{-\beta}\|_{\mathcal{L}(X:Y)} + \int_0^\infty e^{\omega_1 s} H_\beta(s) ds\Big)\|u - v\|_{0,t,Y} \\ &\quad + \Big(L_{\tilde{f}}(r)\int_0^\infty e^{\omega_2 s} H_\alpha(s) ds\Big)\|u - v\|_{0,t,Y} \\ &\le J(r)\|u - v\|_{0,t,Y},\end{aligned}$$

which proves that Γ is a contraction on $B_r(0, C_b([0,\infty) : Y))$ and that Γ has a unique fixed point $u(\cdot, y_0) \in B_r(0, C_b([0,\infty) : Y))$. Clearly, $u(\cdot, y_0)$ is a Y-mild solution of (1.2)-(1.3).

Since $(-A)^\beta g$ and $\tilde{f}$ are pointwise asymptotically almost periodic, it follows from Lemma 2.8 and Proposition 3.1 that each solution $u(\cdot, y_0)$, $y_0 \in B_\epsilon(0, Y)$, is an asymptotically almost periodic Y-mild solution of (1.2)-(1.3). The proof is now complete. $\square$

In the next result, we discuss the existence of almost periodic Y-mild solutions.

Theorem 3.4. *If the assumptions* $\mathbf{H_1}, \mathbf{H_2}, \mathbf{H_3}$ *are satisfied and the functions* $(-A)^{\beta} g, \tilde{f}$ *are pointwise almost periodic, then there exits an almost periodic* Y*-mild solution of* (1.2)-(1.3).

Proof. Let $\Gamma : AP(Y) \to AP(Y)$ be the map defined by

$$\Gamma u(t) = -g(t, u(t)) - \int_{-\infty}^{t} AT(t-s) g(s, u(s)) ds + \int_{-\infty}^{t} (-A)^{\alpha} T(t-s) \tilde{f}(s, u(s)) ds.$$

The same arguments used in the proof of Theorem 3.3 proves that $\Gamma u(t)$ is well defined and that $\Gamma u \in C_b(\mathbb{R}; Y)$. In order to prove that Γ is $AP(Y)$-valued, we fix $u \in AP(Y)$ and $\epsilon > 0$. We know from Zaidman [13, pp. 30] and Lemma 2.7, that $z(t) = (\tilde{f}(t, u(t)), (-A)^{\beta} g(t, u(t))) \in AP(X \times X)$. If $\xi \in \mathcal{H}(\epsilon, z(\cdot), X \times X)$ we get

$$\begin{aligned}
&\|\Gamma u(t+\xi) - \Gamma u(t)\|_Y \\
&\le \|(-A)^{-\beta}\|_{\mathcal{L}(X;Y)} \|(-A)^{\beta} g(t+\xi, u(t+\xi)) - (-A)^{\beta} g(t, u(t))\| \\
&\quad + \int_{-\infty}^{t} \|(-A)^{1-\beta} T(t-s) \left((-A)^{\beta} g(s+\xi, u(s+\xi)) - (-A)^{\beta} g(s, u(s))\right) \|_Y ds \\
&\quad + \int_{-\infty}^{t} \|(-A)^{\alpha} T(t-s) \left(\tilde{f}(s+\xi, u(s+\xi)) - \tilde{f}(s, u(s))\right) \|_Y ds \\
&\le \epsilon [\|(-A)^{-\beta}\|_{\mathcal{L}(X;Y)} + \int_0^{\infty} (\|(-A)^{1-\beta} T(s)\|_{\mathcal{L}(X;Y)} + \|(-A)^{\alpha} T(s)\|_{\mathcal{L}(X;Y)}) ds] \\
&\le \epsilon [\|(-A)^{-\beta}\|_{\mathcal{L}(X;Y)} + \int_0^{\infty} (e^{\omega_1 s} H_{\alpha}(s) + e^{\omega_2 s} H_{\beta}(s))\, ds],
\end{aligned}$$

which shows that $\Gamma u \in AP(Y)$. Thus, Γ is well defined and with values in $AP(Y)$.

Note that there exists $r_0 > 0$ small enough such that Γ is a contraction from $B_{r_0}(0, AP(Y))$ into $B_{r_0}(0, AP(Y))$. Let $r > 0$ and $u \in B_r(0, AP(Y))$. If $t \in \mathbb{R}$ we see that

$$\begin{aligned}
\|\Gamma u(t)\|_Y &\le \|(-A)^{-\beta}\|_{\mathcal{L}(X;Y)} \|g(t, u(t))\| \\
&\quad + L_g(r) \int_{-\infty}^{t} \|(-A)^{1-\beta} T(t-s)\|_{\mathcal{L}(X;Y)} \|u(s)\|_Y ds \\
&\quad + L_{\tilde{f}}(r) \int_{-\infty}^{t} \|(-A)^{\alpha} T(t-s)\|_{\mathcal{L}(X;Y)} \|u(s)\|_Y ds \\
&\le \|(-A)^{-\beta}\|_{\mathcal{L}(X;Y)} L_g(r) r + L_g(r) r \int_0^{\infty} e^{\omega_1 s} H_{\beta}(s) ds \\
&\quad + L_{\tilde{f}}(r) r \int_0^{\infty} e^{\omega_2 s} H_{\alpha}(s) ds,
\end{aligned}$$

and so that $\|\Gamma u\|_{AP(Y)} \le r J(r)$, where

$$J(r) = L_g(r) \Big(\|(-A)^{-\beta}\|_{\mathcal{L}(X;Y)} + \int_0^{\infty} e^{\omega_1 s} H_{\beta}(s) ds \Big) + L_{\tilde{f}}(r) \int_0^{\infty} e^{\omega_2 s} H_{\alpha}(s)\, ds.$$

Since $J(\cdot)$ is continuous and $J(0) = 0$, we can fix $r_0 > 0$ such that $J(r_0) < 1$. Obviously, $\Gamma(B_{r_0}(0, AP(Y))) \subseteq B_{r_0}(0, AP(Y))$. Moreover, for $u, v \in B_{r_0}(0, AP(Y))$ we get

$$\begin{aligned}
&\|\Gamma u(t) - \Gamma v(t)\|_Y \\
&\le \|(-A)^{-\beta}\|_{\mathcal{L}(X;Y)} \|(-A)^{\beta} g(t, u(t)) - (-A)^{\beta} g(t, v(t))\|
\end{aligned}$$

$$
\begin{aligned}
&+\int_{-\infty}^{t} \|(-A)^{1-\beta}T(t-s)\|_{\mathcal{L}(X;Y)}\|(-A)^{\beta}g(s,u(s))-(-A)^{\beta}g(s,v(s))\|ds\\
&+\int_{-\infty}^{t} \|(-A)^{\alpha}T(t-s)\|_{\mathcal{L}(X;Y)}\|\tilde{f}(s,u(s))-\tilde{f}(s,v(s))\|ds\\
\le\ & L_g(r_0)\|(-A)^{-\beta}\|_{\mathcal{L}(X;Y)}\|u-v\|_{AP(Y)}\\
&+\|u-v\|_{AP(Y)}\Big(L_g(r_0)\int_0^\infty e^{\omega_1 s}H_\beta(s)ds+L_{\tilde{f}}(r_0)\int_0^\infty e^{\omega_2 s}H_\alpha(s)ds\Big),\\
\le\ & J(r_0)\|u-v\|_{AP(Y)},
\end{aligned}
$$

which proves that Γ is a contraction on $B_{r_0}(0, AP(Y))$ and that there exists an almost periodic Y-mild solution of (1.2)-(1.3). The proof is finished. □

4. Existence and regularity of mild solutions

In this section we establish conditions under which an Y-mild solution of (1.2)-(1.3) is a mild solution. Then, we apply theses results to prove the existence of asymptotically almost periodic and almost periodic solutions for (1.2)-(1.3).

In the next results, $u(\cdot) \in C([0,b] : Y)$ is a Y-mild solution of (1.2)- (1.3) on $[0,b]$ and the next condition is always assumed.

Assumption (Afg). There are constants $0 < \sigma_1, \sigma_2 < 1$ such that

$$
\begin{aligned}
\|(-A)^{\beta}g(t,x)-(-A)^{\beta}g(s,y)\| &\le L_g(r)\left(|t-s|^{\sigma_1}+\|x-y\|_Y\right),\\
\|\tilde{f}(t,x)-\tilde{f}(s,y)\| &\le L_{\tilde{f}}(r)\left(|t-s|^{\sigma_2}+\|x-y\|_Y\right),
\end{aligned}
$$

for each $(t,s) \in \mathbb{R}^2$ and every $x, y \in B_r(0,Y)$. Moreover, $0 < \alpha < \beta \le 1$ and $\|(-A)^{-\beta}\|_{\mathcal{L}(X:Y)}L(\|u\|_{0,b,Y}) < 1$.

Remark 4.1. Observe that the solutions given by the Theorems 3.3 and 3.4 are such that $\|(-A)^{-\beta}\|_{\mathcal{L}(X:Y)}L(\|u\|_{\sigma,\sigma+\mu,Y}) < 1$ for all $\sigma \in \mathbb{R}$ and all $\mu > 0$.

Proposition 4.2. *Let condition (Afg) be satisfied and assume that there are positive constants d, d_1, d_2; $0 < \xi_1, \xi_2 < 1$ such that $\|(-A)^{1-\beta+\mu}T(s)\|_{\mathcal{L}(X;Y)} \le \frac{d_1}{s^{\xi_1}}$ and $\|(-A)^{\alpha+\mu}T(s)\|_{\mathcal{L}(X;Y)} \le \frac{d_2}{s^{\xi_2}}$ for every $s \in (0,b]$ and every $\mu \in [0,d]$. Then $u \in C^{\sigma}((0,b];Y)$ for $\sigma = \min\{d, 1-\alpha, \sigma_1, 1-\xi_1, 1-\xi_2\}$.*

Proof. We follow the ideas in Rankin [7]. Let $t \in (0,b)$ and $0 < h < 1$ such that $t+h \in (0,b]$. Then

$$
\begin{aligned}
&\|u(t+h)-u(t)\|_Y\\
&\le \|(-A)^{\alpha}T(\tfrac{t}{2})\|_{\mathcal{L}(X;Y)}\|(T(h)-I)T(\tfrac{t}{2})(-A)^{-\alpha}(y_0+g(0,y_0))\|\\
&\quad+\|(-A)^{-\beta}\|_{\mathcal{L}(X;Y)}\|(-A)^{\beta}g(t+h,u(t+h))-(-A)^{\beta}g(t,u(t))\|\\
&\quad+\int_0^t \|(-A)^{1-\beta+\mu}T(t-s)\left(T(h)-I\right)(-A)^{\beta-\mu}g(s,u(s))\|_Y ds\\
&\quad+\int_t^{t+h} \|(-A)^{1-\beta}T(t+h-s)\|_{\mathcal{L}(X;Y)}\|(-A)^{\beta}g(s,u(s))\|ds\\
&\quad+\int_0^t \|(-A)^{\alpha+\mu}T(t-s)\left(T(h)-I\right)(-A)^{-\mu}\tilde{f}(s,u(s))\|_Y ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_t^{t+h}\|(-A)^{\alpha}T(t+h-s)\|_{\mathcal{L}(X;Y)}\|\tilde f(s,u(s))\|ds\\
&\le \frac{2^{\xi_2}d_2}{t^{\xi_2}}MC'_{\alpha}h^{1-\alpha}\|y_0+g(0,y_0)\|\\
&\quad+\|(-A)^{-\beta}\|_{\mathcal{L}(X;Y)}L_g(\|u\|_{b,Y})\left(h^{\sigma_1}+\|u(t+h)-u(t)\|_Y\right)\\
&\quad+\int_0^t\frac{d_1}{(t-s)^{\xi_1}}\|(T(h)-I)(-A)^{\beta-\mu}g(s,u(s))\|ds+L_g(\|u\|_{b,Y})\|u\|_{b,Y}\frac{d_1h^{1-\xi_1}}{1-\xi_1}\\
&\quad+\int_0^t\frac{d_2}{(t-s)^{\xi_2}}\|(T(h)-I)(-A)^{-\mu}\tilde f(s,u(s))\|ds+L_{\tilde f}(\|u\|_{b,Y})\|u\|_{b,Y}\frac{d_2h^{1-\xi_2}}{1-\xi_2}\\
&\le \frac{2^{\xi_2}d_2}{t^{\xi_2}}MC'_{\alpha}h^{1-\alpha}\|y_0+g(0,y_0)\|\\
&\quad+\|(-A)^{-\beta}\|_{\mathcal{L}(X;Y)}L_g(\|u\|_{b,Y})\left(h^{\sigma_1}+\|u(t+h)-u(t)\|_Y\right)\\
&\quad+\tilde d_1h^{\mu}\int_0^t\frac{ds}{(t-s)^{\xi_1}}+\tilde d_2h^{1-\xi_1}+\tilde d_3h^{\mu}\int_0^t\frac{ds}{(t-s)^{\xi_2}}+\tilde d_4h^{1-\xi_2},
\end{aligned}
$$

and then

$$
\begin{aligned}
\|u(t+h)-u(t)\|_Y\le L_g(\|u\|_{b,Y})\|(-A)^{-\beta}\|_{\mathcal{L}(X;Y)}\|u(t+h)-u(t)\|_Y+\tilde d_5h^{1-\alpha}\\
+\tilde d_6h^{\sigma_1}+\tilde d_7h^{\mu}+\tilde d_2h^{1-\xi_1}+\tilde d_4h^{1-\xi_2},
\end{aligned}
$$

where the constants $\tilde d_i$, $i=1,2,\dots 7$, are independent of t,h and $\mu\in[0,d]$. Since $\|(-A)^{-\beta}\|_{\mathcal{L}(X;Y)}L(\|u\|_{b,Y})<1$ and t,h,μ are arbitrary, the last inequality proves that $u(\cdot)\in C^{\sigma}((0,b];Y)$ for $\sigma=\min\{d,1-\alpha,\sigma_1,1-\xi_1,1-\xi_2\}$. The proof is complete □

Proposition 4.3. *Under the assumptions of Proposition 4.2, $u(\cdot)\in C((0,b];X_{\gamma})$ for $\gamma=\min\{1-\alpha,\beta\}$.*

Proof. First we introduce the decomposition $u=\sum_{i=1}^{3}u_i$ where

$$
\begin{gathered}
u_1(t)=T(t)(u(0)+g(0,u(0)))-g(t,u(t)),\\
u_2(t)=\int_0^t(-A)^{1-\beta}T(t-s)(-A)^{\beta}g(s,u(s))ds,\\
u_3(t)=\int_0^t(-A)^{\alpha}T(t-s)\tilde f(s,u(s))ds.
\end{gathered}
$$

It is obvious that $u_1\in C((0,b];X_{\beta})$. On the other hand, from Proposition 4.2 we know that $u(\cdot)\in C^{\sigma}((0,b];Y)$ for $\sigma=\min\{d,1-\alpha,\sigma_1,1-\xi_1,1-\xi_2\}$ which from the estimate

$$
\begin{aligned}
&\|(-A)^{\gamma+1}T(t-s)\left(g(s,u(s))-g(t,u(t))\right)\|\\
&\le\|(-A)^{\gamma+1-\beta}T(t-s)\|_{\mathcal{L}(X)}\|(-A)^{\beta}g(s,u(s))-(-A)^{\beta}g(t,u(t)\|\\
&\le\frac{C_{\gamma+1-\beta}}{(t-s)^{\gamma+1-\beta}}L_g(\|u\|_{b,Y})\left(|t-s|^{\sigma_1}+\|u(s)-u(t)\|_Y\right)\\
&\le\frac{\tilde d_1}{(t-s)^{\gamma+1-\beta-\sigma_1}}+\frac{\tilde d_2}{(t-s)^{\gamma+1-\beta-\sigma}},
\end{aligned}
$$

implies that the function

$$
v(s)=(-A)^{\gamma+1}T(t-s)\left(g(s,u(s))-g(t,u(t))\right),
$$

is integrable on $[0,t)$, $t\in[a,b]$, when $\gamma<\min\{\beta+\sigma_1,\beta+\sigma\}$. In particular, for $\gamma=\beta$ we find that

$$\begin{aligned}
&\int_0^t v(s)ds+(-A)^{\beta}g(t,u(t))\\
&=\int_0^t v(s)ds+(-A)\int_0^t(-A)^{\beta}T(t-s)g(t,u(t))ds+T(t)(-A)^{\beta}g(t,u(t))\\
&=\int_0^t(-A)^{\beta+1}T(t-s)\left(g(s,u(s))-g(t,u(t))\right)ds\\
&\quad+\int_0^t(-A)^{\beta+1}T(t-s)g(t,u(t))ds+T(t)(-A)^{\beta}g(t,u(t)),
\end{aligned}$$

which shows that $u_2(\cdot)\in C([0,b];X_\beta)$ since $(-A)^{\beta}$ is a closed operator.

Proceeding as in the previous case, we can prove that $u_3(\cdot)\in C([0,b];X_{1-\alpha})$. From theses remarks we conclude that $u(\cdot)\in C((0,b];X_\gamma)$ for $\gamma=\min\{1-\alpha,\beta\}$. The proof is complete □

Theorem 4.4. *Under the hypotheses of Proposition 4.3, if $\alpha\le 1-\alpha$, then $u(\cdot)$ is a mild solution of* (1.2)-(1.3).

The assertion of this theorem is a consequence of Assumption (H3) and Lemma 2.1. Next we establish conditions under which $u(\cdot)$ is a S-classical solution.

Proposition 4.5. *Let assumption in Theorem 4.4 be satisfied and assume that $L_g(\|u\|_{b,Y})\|(-A)^{\alpha-\beta}\|_{\mathcal{L}(X:Y)}\|(-A)^{-\alpha}\|_{\mathcal{L}(X)}<1$. Then $u\in C^{\sigma}((0,b]:X_\alpha)$ for $\sigma=\min\{\beta-\alpha,\sigma_1\}$.*

Proof. Using the fact that $u\in C((0,b]:X_\alpha)$ and Lemma 2.1, for $0<\delta<t<b$ and $h>0$ such that $t+h<b$ we find that

$$\begin{aligned}
&\|u(t+h)-u(t)\|_\alpha\\
&\le\|(-A)^{\alpha}(T(h)-I)T(t)(y_0+g(0,y_0))\|\\
&\quad+\|(-A)^{\alpha-\beta}\|_{\mathcal{L}(X)}\|(-A)^{\beta}g(t+h,u(t+h))-(-A)^{\beta}g(t,u(t))\|\\
&\quad+\int_0^t\|(-A)^{1-\beta+\alpha}\left(T(h)-I\right)T(t-s)(-A)^{\beta}g(s,u(s))\|ds\\
&\quad+\int_t^{t+h}\|(-A)^{1-\beta+\alpha}T(t+h-s)\|\|(-A)^{\beta}g(s,u(s))\|ds\\
&\quad+\int_0^t\|(-A)^{\alpha}\left(T(h)-I\right)T(t-s)f(s,u(s))\|ds\\
&\quad+\int_t^{t+h}\|(-A)^{\alpha}T(t+h-s)\|\|f(s,u(s))\|ds\\
&\le\frac{C'_{1-\alpha}h^{1-\alpha}}{\delta^{\alpha}}\|y_0+g(0,y_0)\|\\
&\quad+\|(-A)^{\alpha-\beta}\|_{\mathcal{L}(X)}L_g(\|u\|_{b,Y})\left[h^{\sigma_1}+\|u(t+h)-u(t)\|_Y\right]\\
&\quad+\int_0^t C'_{\beta-\alpha}h^{\beta-\alpha}\|(-A)^{1-\beta+\alpha}T(t-s)(-A)^{\beta}g(s,u(s))\|ds\\
&\quad+\int_t^{t+h}\frac{C_{1-\beta+\alpha}}{(t+h-s)^{1-\beta+\alpha}}\|(-A)^{\beta}g(s,u(s))\|ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t C'_{1-\alpha} h^{1-\alpha}\|(-A)^\alpha T(t-s) f(s,u(s))\| ds \\
&+\int_t^{t+h} \frac{C_\alpha}{(t+h-s)^\alpha}\|f(s,u(s))\|\, ds
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\|u(t+h)-u(t)\|_\alpha \\
&\le \|(-A)^{\alpha-\beta}\|_{\mathcal{L}(X)} L_g(\|u\|_{b,Y})\|(-A)^{-\alpha}\|_{\mathcal{L}(X,Y)}\|u(t+h)-u(t)\|_\alpha \\
&\quad + \tilde{d}_1 h^{\sigma_1} + \tilde{d}_2 h^{\beta-\alpha} + \tilde{d}_3 h^{1-\alpha},
\end{aligned}
$$

where the constants $\tilde{d}_i$ are independent of $t \ge \delta$ and h. This inequality completes the proof of this Proposition since $\beta > \alpha$. □

The next result is consequence of Proposition 4.5, [6, Theorem 4.3.2] and [5, Lemma 2].

Theorem 4.6. *Assume that the hypotheses of Proposition 4.5 are satisfied. If $g \in C(\mathbb{R}\times X : X_1)$ and $\beta + \min\{\beta-\alpha, \sigma_1\} > 1$, then $u(\cdot)$ is a S-classical solution of* (1.2)-(1.3).

Remark 4.7. It is clear that the previous results of regularity of Y-mild solutions are valid for every Y-mild solution $u \in C([\sigma,\sigma+\mu];Y), \sigma\in\mathbb{R}, \mu>0$.

As consequence of the Theorems 4.4, 4.6 and Remarks 4.1 and 4.7, we obtain the following existence result of asymptotically almost periodic and almost periodic solutions of (1.2)-(1.3). The proof of the next result will be omitted.

Theorem 4.8. *Let assumptions (H1)–(H3) and condition (Afg1) be satisfied; also assume that $\alpha \le 1-\alpha$ and $\beta+\min\{\beta-\alpha,\sigma_1\} > 1$. Then the following properties are satisfied.*

(1) *If the functions $f, (-A)^\beta g : [0,\infty)\times Y \to X$ are pointwise asymptotically almost periodic, then there exists $\epsilon > 0$ such that for every $y_0 \in B_\epsilon(0,Y)$ there exits an asymptotically almost periodic S-classical solution, $u(\cdot,y_0)$, of the system* (1.2)-(1.3) *such that $u(0,y_0) = y_0$.*
(2) *If the functions $f, (-A)^\beta g : [0,\infty)\times Y \to X$ are pointwise almost periodic, then there exits an almost periodic S-classical solution of the equation* (1.2)-(1.3).

5. Example

In this section we illustrate some of our results. Consider the first order evolution equation

$$
\frac{d}{dt}\Big[u(t,\xi) + \int_0^\pi a(t) b(\eta,\xi) u(t,\eta) d\eta\Big] = \frac{\partial^2 u(t,\xi)}{\partial \xi^2} + F(t,u(t,\xi)), \quad \xi \in I = [0,\pi] \tag{5.1}
$$

$$
u(t,0) = u(t,\pi) = 0, \quad t\in\mathbb{R}, \tag{5.2}
$$

where $a(\cdot)\in C_b(\mathbb{R},\mathbb{R})$, $a(0)=0$ and

(a) The functions $b(\eta,\xi)$, $\frac{\partial^i b(\eta,\xi)}{\partial \xi^i}$, $i=1,2$, are measurable, $b(\eta,\pi)=b(\eta,0)=0$ for every $\eta\in\mathbb{R}$ and

$$L_g = |a(\cdot)|_{C_b(\mathbb{R}:\mathbb{R})} \max\big\{ (\int_0^\pi \int_0^\pi (\frac{\partial^i b(\eta,\xi)}{\partial \xi^i})^2 d\eta d\xi)^{1/2} : i=0,1,2\big\} < 1; \tag{5.3}$$

(b) $F:\mathbb{R}^2\to\mathbb{R}$ is continuous and there is $\mu\in C_b(\mathbb{R},\mathbb{R}^+)$ such that $\mu(0)=0$ and

$$|F(t,x)-F(t,y)| \le \mu(t)|x-y|,$$

for every $t\in\mathbb{R}$ and every $(x,y)\in\mathbb{R}^2$.

Let $X=L^2([0,\pi])$ and $A:D(A)\subset X\to X$ be the operator $Ax=x''$ where

$$D(A) := \{x(\cdot)\in L^2([0,\pi]) : x''(\cdot)\in L^2([0,\pi]),\ x(0)=x(\pi)=0\}.$$

It is well known that A is the infinitesimal generator C_0-semigroup $(T(t))_{t\ge 0}$ on X. Moreover, the next Theorem is valid.

Theorem 5.1. *Under the above conditions, the following properties hold*

(1) *A has discrete spectrum, the eigenvalues are $-n^2$, $n\in\mathbb{N}$, with corresponding eigenvectors $z_n(\xi) := \big(\frac{2}{\pi}\big)^{1/2}\sin(n\xi)$ and the set $\{z_n : n\in\mathbb{N}\}$ is an orthonormal basis of X.*

(2) *For every $x\in X$, $T(t)x=\sum_{n=1}^\infty e^{-n^2 t}\langle x,z_n\rangle z_n$. Moreover, the semigroup $(T(t))_{t\ge0}$ is compact, analytic, self-adjoint and $\|T(t)\|\le e^{-t}$ for every $t\ge 0$.*

(3) *For $f\in X$, $(-A)^{-\theta}f=\sum_{n=1}^\infty n^{-2\theta}\langle f,z_n\rangle z_n$ and the operator $(-A)^\theta$ is given by $(-A)^\theta f=\sum_{n=1}^\infty n^{2\theta}\langle f,z_n\rangle z_n$ on*

$$D((-A)^\theta) = \{f\in X : \sum_{n=1}^\infty n^{2\theta} < f, z_n > z_n \in X\}.$$

Moreover, $\|(-A)^{-1/2}\|=1$ and $\|(-A)^{1/2}T(t)\| \le \frac{e^{\frac{-t}{2}} t^{-\theta}}{\sqrt{2}}$ for every $t>0$.

This theorem follow from [3, Theorem 2.3.5] and [7, Theorem 4].

By defining the functions $f(\cdot),g(\cdot):\mathbb{R}\times X\to X$

$$g(t,x)(\xi) = a(t)\int_0^\pi b(\eta,\xi)x(\eta)d\eta,$$
$$f(t,x)(\xi) = F(t,x(\xi)),$$

the system (5.1)-(5.2) can be written as the abstract differential equation (1.2)-(1.3). Moreover, f,g are continuous function, g is $D(A)$-valued, $Ag:\mathbb{R}\times X\to X$ is continuous and

$$\|(-A)^\theta g(t,\cdot)\|_{\mathcal{L}(X)} \le |a(t)|L_g, \quad \theta=0,\frac{1}{2},1,$$
$$\|f(t,x)-f(t,y)\| \le \mu(t)\|x-y\|,$$

for every $t\in\mathbb{R}$ and every $x,y\in X$. Obviously, our results can be applied in the case $Y=X$. In this particular case, the next results is consequence of Theorem 4.8.

Theorem 5.2. *Under the above conditions, the following properties are satisfied.*

(1) *Assume that* $f : [0, \infty) \times X \to X$ *is pointwise a.a.p. and that* $a(\cdot)$ *is asymptotically almost periodic. Then there exists* $\varepsilon > 0$ *such that for every* $y_0 \in B_\varepsilon(0, X)$ *there exists an asymptotically almost periodic S-classical solution,* $u(\cdot, y_0)$*, of* (1.2)-(1.3) *such that* $u(0, y_0)) = y_0$.

(2) *If* $f : [0, \infty) \times Y \to X$ *is pointwise almost periodic and* $a(\cdot)$ *is almost periodic, then there exists an almost periodic S-classical solution of* (1.2)-(1.3).

Remark 5.3. By using the results in this paper, in a forthcoming paper we will study the existence of almost periodic solutions for the Navier-Stokes equation

$$u'(t, x) = Au(t) + (u(t) \cdot \nabla)u(t) + g'(t) \tag{5.4}$$

where $g \in C(\mathbb{R} : V)$ and $V = \{u \in H_0^1 : \operatorname{div} u = 0\}$. See [1] for details about this matter.

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References

[1] Bridges, Thomas J; *The Hopf bifurcation with symmetry for the Navier-Stokes equations in* $(L_p(\Omega))^n$*, with application to plane Poiseuille flow.* Arch. Rational Mech. Anal. 106 (1989), no. 4, 335–376.

[2] Bahaj, Mohamed; Sidki, Omar; *Almost periodic solutions of semilinear equations with analytic semigroups in Banach spaces.* Electron. J. Differential Equations, Vol 2002 (2002), No. 98, 1–11.

[3] Curtain, Ruth F; Zwart, Hans; *An introduction to infinite-dimensional linear systems theory.* Texts in Applied Mathematics, 21. Springer-Verlag, New York, 1995.

[4] Hagen, Thomas; Turi, Janos; *A semigroup approach to a class of semilinear parabolic differential equations.* Nonlinear Anal. 34 (1998), no. 1, 17–35.

[5] Hernandez, Eduardo; *Existence results for a class of semi-linear evolution equations*, Electronic Journal of Differential Equations. Vol. 2001 (2001), No. 24, pp. 1-14.

[6] Pazy, A; *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New-York, (1983).

[7] Rankin, Samuel M., III; *Semilinear evolution equations in Banach spaces with application to parabolic partial differential equations.* Trans. Amer. Math. Soc. 336 (1993), no. 2, 523–535.

[8] Yoshizawa, T; *Stability theory and the existence of periodic solutions and almost periodic solutions.* Applied Mathematical Sciences, Vol. 14. Springer-Verlag, New York-Heidelberg, 1975.

[9] Zaidman, S; *Topics in abstract differential equations II.* Pitman Research Notes in Mathematics Series, 321. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1995.

[10] Zaidman, S; *A non linear abstract differential equation with almost-periodic solution*, Riv. Mat. Univ. Parma (4) Vol 10 (1984), 331-336.

[11] Zaidman, S; *Topics in abstract differential equations.* Pitman Research Notes in Mathematics Series, 304. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1994.

[12] Zaidman, S; *Topics in abstract differential equations.* Nonlinear Anal., 23 (1994), no. 7, 849–870.

[13] Zaidman, S; *Almost-periodic functions in abstract spaces.* Research Notes in Mathematics, 126. Pitman (Advanced Publishing Program), Boston, 1985.

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