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UNIQUENESS THEOREMS FOR STURM-LIOUVILLE OPERATORS WITH INTERIOR TWIN-DENSE NODAL SET

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ABSTRACT. We study Inverse problems for the Sturm-Liouville operator with Robin boundary conditions. We establish two uniqueness theorems from the twin-dense nodal subset $W_S([\frac{1-\varepsilon}{2}, \frac{1}{2}])$, $0 < \varepsilon \leq 1$, together with parts of either one spectrum, or the minimal nodal subset $\{x_n^1\}_{n=1}^{\infty}$ on the interval $[0, \frac{1}{2}]$. In particular, if one spectrum is given a priori, then the potential q on the whole interval $[0, 1]$ can be uniquely determined by $W_S([\frac{1-\varepsilon}{2}, \frac{1}{2}])$ for any S and arbitrarily small ε.

1. Introduction

Consider the Sturm-Liouville operator $L := L(q, h, H)$ defined by

$$-u'' + q(x)u = \lambda u, \quad x \in (0, 1) \tag{1.1}$$

with boundary conditions

$$U_0(u) := u'(0, \lambda) - hu(0, \lambda) = 0, \tag{1.2}$$

$$U_1(u) := u'(1, \lambda) + Hu(1, \lambda) = 0, \tag{1.3}$$

where $h, H \in \mathbb{R}$, $q(x)$ is a real-valued function and $q \in L^1[0, 1]$.

The inverse nodal problem is to reconstruct this operator from the given nodal points(zeros) of its eigenfunctions. Inverse nodal problems for differential operators have many applications in many areas, such as mathematics, physics, engineering, etc (see [1, 2, 3, 4, 5, 8, 11, 15, 16, 18, 21, 22, 25, 27, 28, 29, 30] and the references therein). Inverse spectral problems for (1.1)-(1.3) consist in recovering this operator from the given data (refer to [6, 7, 10, 12, 13, 14, 17, 19, 20, 23, 24, 26, 31] and other works). In particular, McLaughlin [18] discussed the inverse nodal problem for (1.1)-(1.3) and showed that a dense subset of nodal points of its eigenfunctions is sufficient to determine the potential q up to its mean value and coefficients h, H of boundary conditions. From the physical point of view this corresponds to finding, e.g., the density of a string or a beam from the zero-amplitude positions of their eigenvibrations. Later, X.F. Yang [29] presented an interesting theorem for (1.1)-(1.3) and showed that the s-dense nodal subset on the interval $[0, b]$, $\frac{1}{2} < b \leq 1$, is sufficient to determine the potential q up to its mean value and coefficients h, H of boundary conditions by the Gesztesy-Simon theorem [7]. Then Cheng et al [4]

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improved the Yang's theorem by the twin-dense nodal subset (Similar to definition 2.1) instead of the s-dense nodal subset. Yang [27] presented a counterexample, which illustrates that two operators have the same spectrum and in the subinterval $[0, \frac{1-\alpha}{2}] \cup [\frac{1+\alpha}{2}, 1]$ for any $\alpha, 0 < \alpha < 1$, their nodal points are the same, but $q(x) \neq \tilde{q}(x)$ on the interval $(\frac{1-\alpha}{2}, \frac{1+\alpha}{2})$. In [8, 9], Guo and Wei showed that only the twin-dense nodal data on a small interval $[a, b]$ containing the midpoint $\frac{1}{2}$ suffices to determine the differential operator (potential functions plus boundary constants h and H) uniquely. Their method is inspired by analysis of Weyl m-functions in the work of Gesztesy-Simon[7]. The result of Guo-Wei is a big step forward from those in [29, 4], where nodal data on more that half of the interval are needed.

In this note, we plan to follow the method of Guo-Wei to show two uniqueness results. We shall concentrate on the situation when only information of the twin-dense nodal subset $W_S([a, \frac{1}{2}])$ on the left portion $[a, \frac{1}{2}]$, still an interior subinterval. As discussed in [8], this is not enough. We add some more information (part of the eigenvalues λ_n, or the sequence of first nodal point $x^1_{n_k}$). They suffice to guarantee the uniqueness of the potential function. There are four types of boundary conditions, we shall only concentrate on Case IV: $h, H \in \mathbb{R}$ in [8]. Moreover we shall simplify part of their proof (cf. proof of Lemma 3.1 below).

This article is organized as follows. In Section 2, we present preliminaries. We introduce our main results in Section 3, which will be proved in Section 4.

2. Preliminaries

Let $S(x, \lambda)$, $C(x, \lambda)$, $u_-(x, \lambda)$ and $u_+(x, \lambda)$ be solutions of (1.1) with the initial conditions:

$$
\begin{gathered}
S(0, \lambda) = 0, \quad S'(0, \lambda) = 1, C(0, \lambda) = 1, C'(0, \lambda) = 0, \\
u_-(0, \lambda) = 1, \quad u_-'(0, \lambda) = h, \quad u_+(1, \lambda) = 1, \; u_+'(1, \lambda) = -H.
\end{gathered}
$$

Clearly, $U_0(u_-) = U_1(u_+) = 0$ and

$$
\begin{gathered}
u_-(x, \lambda) = C(x, \lambda) + hS(x, \lambda), \\
u_+(x, \lambda) = U_1(S)C(x, \lambda) - U_1(C)S(x, \lambda).
\end{gathered}
$$

Denote $\lambda = \rho^2$ and $\tau = |\mathrm{Im}\rho|$. We have the asymptotic formulae (see [31]).

$$
u_-(x, \lambda) = \cos \rho x + \Big(h + \frac{1}{2}\int_0^x q(t)\mathrm{d}t\Big)\frac{\sin \rho x}{\rho} + o\big(\frac{\mathrm{e}^{\tau x}}{\rho}\big), \quad 0 \le x \le 1 \tag{2.1}
$$

$$
u_-'(x, \lambda) = -\rho \sin \rho x + O(\mathrm{e}^{\tau x}), \quad 0 \le x \le 1, \tag{2.2}
$$

$$
\begin{gathered}
u_+(x, \lambda) = \cos \rho(1-x) + \Big(H + \frac{1}{2}\int_x^1 q(t)\mathrm{d}t\Big)\frac{\sin \rho(1-x)}{\rho} + o\big(\frac{\mathrm{e}^{\tau(1-x)}}{\rho}\big), \\
\text{for } 0 \le x \le 1
\end{gathered}
$$

$$
u_+'(x, \lambda) = \rho \sin \rho(1-x) + O(\mathrm{e}^{\tau(1-x)}), \quad 0 \le x \le 1.
$$

The following formula is called the Green's formula

$$
\int_0^1 (yL(z) - zL(y)) = [y, z](1) - [y, z](0), \tag{2.3}
$$

where $[y, z](x) := y(x)z'(x) - y'(x)z(x)$ is the Wronskian of y and z.

Denote

$$
\Delta(\lambda) := [u_+, u_-](x, \lambda).
$$

Then $\Delta(\lambda)$ does not depend on x and

$$\Delta(\lambda) = U_1(u_-) = -U_0(u_+),$$

which is called the characteristic function of L. Hence

$$\Delta(\lambda) = -\rho \sin \rho + O(\mathrm{e}^{\tau}). \tag{2.4}$$

Let $\sigma(L) := \{\lambda_n\}_{n=0}^{\infty}$ be the set of all eigenvalues of (1.1)-(1.3). It is well known that all zeros λ_n of $\Delta(\lambda)$ are real and simple. For sufficiently large n, we have asymptotic formula for eigenvalues λ_n of (1.1)-(1.3)

$$\sqrt{\lambda_n} = n\pi + \frac{\omega}{n\pi} + o\big(\frac{1}{n}\big), \tag{2.5}$$

where $\omega = h + H + \frac{1}{2}\int_0^1 q(t)\mathrm{d}t$. Denote $G_\delta := \{\rho : |\rho - k\pi| > \delta, k \in \mathbb{Z}\}$. For sufficiently small δ, then there exists a constant C_δ such that for sufficiently large $|\lambda|$,

$$|\Delta(\lambda)| \geq C_\delta |\rho| \mathrm{e}^{\tau}, \quad \forall \rho \in G_\delta. \tag{2.6}$$

We define the Weyl m-function $m_\pm(x, \lambda)$ by

$$m_\pm(x, \lambda) = \pm \frac{u'_\pm(x, \lambda)}{u_\pm(x, \lambda)}.$$

From [17, 7], we get the following asymptotic formulae:

$$m_\pm(x, \lambda) = i\rho + o(1), \quad \frac{1}{m_\pm(x, \lambda)} = -\frac{i}{\rho} + o\big(\frac{1}{\rho^2}\big) \tag{2.7}$$

uniformly in $x \in [0, 1-\delta]$ for $m_+(x, \lambda)$ (resp., $x \in [\delta, 1]$ for $m_-(x, \lambda)$), $\delta > 0$ as $|\lambda| \to \infty$ in any sector $\varepsilon < \arg(\lambda) < \pi - \varepsilon$ for $\varepsilon > 0$.

Let $u_-(x, \lambda_n)$ be the eigenfunction corresponding to the n-th eigenvalue λ_n of eqrefE1.1-(1.3) and x_n^j be the nodal points of the eigenfunction $u_-(x, \lambda_n)$, i.e., $u_-(x_n^j, \lambda_n) = 0$, where $0 < x_n^1 < x_n^2 < \dots < x_n^j < \dots < x_n^n < 1, n \geq 1$. Denote $x_n^0 = 0$ and $x_n^{n+1} = 1$. Additionally, for $j = \overline{0, n}$, let I_n^j be the nodal interval by $I_n^j = (x_n^j, x_n^{j+1})$ and l_n^j be the nodal length of the interval I_n^j by $l_n^j = x_n^{j+1} - x_n^j$. Denote $X := \{x_n^j\}$ be the set of nodal points of (1.1)-(1.3), where $j = j(n), j = \overline{0, n}$.

For sufficiently large n, we have asymptotic formulae for zeros x_n^j of the eigenfunction $u_-(x, \lambda_n)$ of (1.1)-(1.3) (see [22])

$$\begin{aligned} x_n^j = {} & \frac{j - \frac{1}{2}}{n} + \frac{1}{2(n\pi)^2}\Big(2h + \int_0^{x_n^j} q(t)\mathrm{d}t\Big) \\ & - \frac{j - \frac{1}{2}}{2n^3\pi^2}\Big(2\omega - \int_0^1 q(t)\cos(2n\pi t)\mathrm{d}t\Big) + o\big(\frac{1}{n^2}\big). \end{aligned} \tag{2.8}$$

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N}_2 = \mathbb{N} \backslash \{1\}$, and $S := \{n_k \in \mathbb{N}_2 : n_k < n_{k+1}, k = 1, 2, \dots, \infty\}$.

Definition 2.1. Take $a \in [0, \frac{1}{2})$. We call $W_S([a, \frac{1}{2}])$ a left twin-dense nodal subset on the interval $[a, \frac{1}{2}]$ if

(1) $W_S([a, \frac{1}{2}]) \subseteq X \cap [a, \frac{1}{2}]$.
(2) For all $n_k \in S$, there exists j_k such that both $x_{n_k}^{j_k}, x_{n_k}^{j_k+1} \in W_S([a, \frac{1}{2}])$.
(3) The set $W_S([a, \frac{1}{2}])$ is dense on $[a, \frac{1}{2}]$, i.e. $\overline{W_S([a, \frac{1}{2}])} = [a, \frac{1}{2}]$.

In the same way, we define a right twin-dense nodal subset $W_S([\frac{1}{2}, b])$ on the interval $[\frac{1}{2}, b]$ for some $b, \frac{1}{2} < b \leq 1$.

The following two lemmas are important for proofs of our main results.

Lemma 2.2 ([17]). *Let $m_+(\alpha,\lambda)$ (resp., $m_-(1-\alpha,\lambda)$), $\alpha\in[0,1)$, be the Weyl m-function of the problem (1.1)-(1.3). Then $m_+(\alpha,\lambda)$ (resp. $m_-(1-\alpha,\lambda)$) uniquely determines coefficient H (resp. h) of the boundary condition as well as q on the interval $[\alpha,1]$ (resp. $[0,1-\alpha]$).*

Lemma 2.3 ([19, Proposition B.6]). *Let $f(z)$ be an entire function such that*

1. $\sup_{|z|=R_k}|f(z)|\le C_1exp(C_2R_k^{\alpha})$ *for some $0<\alpha<1$, some sequence $R_k\to\infty$ as $k\to\infty$ and $C_1,C_2>0$.*
2. $\lim_{|x|\to\infty}|f(ix)|=0,\ x\in\mathbb{R}$.

Then $f\equiv 0$.

3. Main results

With L we consider here and in the sequel a boundary value problem $\widetilde{L}=L(\widetilde{q},\widetilde{h},\widetilde{H})$ of the same form but with different coefficients. If a certain symbol γ denotes an object related to L, then the corresponding symbol $\widetilde{\gamma}$ with tilde denotes the analogous object related to $\widetilde{L}$, and $\hat{\gamma}=\gamma-\widetilde{\gamma}$. The so-called $W_S([a,b])=\widetilde{W}_{\widetilde{S}}([a,b])$ means that for any $x_{n_k}^{j_k}\in W_S([a,b])$, then at least one of (3.1) and (3.2) holds. i.e.

$$x_{n_k}^{j_k}=\widetilde{x}_{\widetilde{n}_k}^{\widetilde{j}_k}\quad\text{and}\quad x_{n_k}^{j_k+1}=\widetilde{x}_{\widetilde{n}_k}^{\widetilde{j}_k+1},\quad \textit{or} \tag{3.1}$$

$$x_{n_k}^{j_k}=\widetilde{x}_{\widetilde{n}_k}^{\widetilde{j}_k}\quad\text{and}\quad x_{n_k}^{j_k-1}=\widetilde{x}_{\widetilde{n}_k}^{\widetilde{j}_k-1}, \tag{3.2}$$

where $x_{n_k}^{j_k+j}\in W_S([a,b])$ and $\widetilde{x}_{\widetilde{n}_k}^{\widetilde{j}_k+j}\in\widetilde{W}_{\widetilde{S}}([a,b])$ in this paper. i.e., for each fixed (n_k,j_k), there exists $(\widetilde{n}_k,\widetilde{j}_k)$ such that (3.1), or (3.2). Next, we present the following Lemma 3.1 (see [29, 4, 8]), however we prove it by an improved method.

Lemma 3.1. *If $W_S([\frac{1-\varepsilon}{2},\frac12])=\widetilde{W}_{\widetilde{S}}([\frac{1-\varepsilon}{2},\frac12])$, then*

$$q(x)-\widetilde{q}(x)=2\widehat{\omega}\quad \textit{a.e. on } [\frac{1-\varepsilon}{2},\frac12], \tag{3.3}$$

$$\begin{gathered}\lambda_{n_k}-\widetilde{\lambda}_{\widetilde{n}_k}=2\widehat{\omega}\quad \textit{for all } n_k\in S,\\ n_k=\widetilde{n}_k \textit{ except for a finite number of natural numbers } k.\end{gathered} \tag{3.4}$$

Adding the condition (3.6), we establish the following uniqueness theorem.

Theorem 3.2. *Suppose that the following two conditions are satisfied:*

1. $W_S([\frac{1-\varepsilon}{2},\frac12])=\widetilde{W}_{\widetilde{S}}([\frac{1-\varepsilon}{2},\frac12])$, *and*
$$\sharp\{n_k\in S: n_k\le n\}\ge(1-\varepsilon)n+\frac{3\varepsilon-1}{2} \tag{3.5}$$
for sufficiently large integer $n>0$.
2. *For the infinite set $\mathbb{N}_0\backslash S$,*
$$\lambda_n=\widetilde{\lambda}_n,\quad n\in\mathbb{N}_0\backslash S. \tag{3.6}$$

Then

$$q(x)=\widetilde{q}(x)\quad \textit{a.e. on } [0,1],\quad h=\widetilde{h}\quad \textit{and}\quad H=\widetilde{H}.$$

Remark 3.3. (1) For either case $(h,H)=(\infty,H)$, or (h,∞), or (∞,∞), if we modify the condition (3.5) suitably, then one obtains a similar results.

(2) We obtain an analogous results with the right twin-dense nodal subset on the interval $[\frac12,\frac{1+\varepsilon}{2}]$ instead of the left twin-dense nodal subset in Theorem 3.2.

We have the following corollary from Theorem 3.2, i.e. if one spectrum is given a priori, then potential q on the whole interval $[0,1]$ can be uniquely determined by $W_S([\frac{1-\varepsilon}{2},\frac{1}{2}])$ for any S and any arbitrarily small ε.

Corollary 3.4. *If one spectrum $\sigma(L)$ is given a priori, then the potential q and coefficients h,H can be uniquely determined by the left twin-dense nodal subset $W_S([\frac{1-\varepsilon}{2},\frac{1}{2}])$ for any S and arbitrarily small ε.*

For any $n\in\mathbb{N}$, let x_n^1 and x_n^n be the minimal and maximal nodal point of the corresponding eigenvalue λ_n, respectively. From the Sturm's oscillation theorem (see [29, Lemma 1.1.4, pp. 18]), we see that if $0<x_1^1\le\frac12$, then $0<x_n^1\le\frac12$ for all $n>1$ and if $\frac12\le x_1^1<1$, then $\frac12\le x_n^n<1$ for all $n>1$. Adding the condition $0<x_1^1\le\frac12$ and (3.7), we obtain the following uniqueness theorem.

Theorem 3.5. *If the following three conditions are satisfied:*

(1) *$H=\widetilde{H}$ and $0<x_1^1\le\frac12$,*
(2) *$W_S([\frac{1-\varepsilon}{2},\frac12])=\widetilde{W}_{\widetilde{S}}([\frac{1-\varepsilon}{2},\frac12])$ and (3.5) holds.*
(3) *For all $n\in\mathbb{N}\backslash S$,*

$$x_n^1=\widetilde{x}_{\widetilde{n}}^1, \tag{3.7}$$

then

$$q(x)-\int_0^1 q(t)\mathrm{d}t=\widetilde{q}(x)-\int_0^1\widetilde{q}(t)\mathrm{d}t \quad \textit{a.e. on } [0,1], \textit{ and } h=\widetilde{h}.$$

4. Proofs of main results

In this section, we present proofs of our main results. Firstly, we prove Lemma 3.1 by the improved method.

Proof of Lemma 3.1. For each fixed $x\in[\frac{1-\varepsilon}{2},\frac12]$, we choose $x_{n_k}^{j_k}\in W_S([\frac{1-\varepsilon}{2},\frac12])$ such that $\lim_{k\to\infty}x_{n_k}^{j_k}=x$. From (2.8), we have

$$\lim_{k\to\infty}\frac{j_k-\frac12}{n_k}=x.$$

By using the Riemann-Lebesgue lemma together with (2.8), we get

$$\begin{aligned}
f(x)&:=\lim_{k\to\infty}\big[2(n_k\pi)^2x_{n_k}^{j_k}-2n_k\pi^2\big(j_k-\frac12\big)\big]\\
&=\lim_{k\to\infty}\Big[2h+\int_0^{x_{n_k}^{j_k}}q(t)\mathrm{d}t-\frac{j_k-\frac12}{n_k}\Big(2\omega-\int_0^1 q(t)\cos(2n_k\pi t)\mathrm{d}t\Big)\\
&\quad+o(1)\Big]\\
&=\int_0^x q(t)\mathrm{d}t+2h-2\omega x,\quad x\in[\frac{1-\varepsilon}{2},\frac12].
\end{aligned}\tag{4.1}$$

Since $\int_0^x q(t)\mathrm{d}t+2h-x\int_0^1 q(t)\mathrm{d}t$ (a.e. on $x\in[\frac{1-\varepsilon}{2},\frac12]$) with respect to x is differentiable, $f(x)$ with respect to x is also differentiable. By taking derivatives for (4.1), we obtain

$$f'(x)=q(x)-2\omega\quad\text{a.e. on }[\frac{1-\varepsilon}{2},\frac12].$$

Since

$$W_S\big([\frac{1-\varepsilon}{2},\frac12]\big)=\widetilde{W}_{\widetilde{S}}\big([\frac{1-\varepsilon}{2},\frac12]\big),$$

it follows that $f(x)=\widetilde{f}(x)$ for $x\in[\frac{1-\varepsilon}{2},\frac{1}{2}]$. Therefore

$$f'(x)=\widetilde{f}'(x)\quad \text{a.e. on}\quad [\frac{1-\varepsilon}{2},\frac{1}{2}].$$

This implies

$$q(x)-\widetilde{q}(x)=2\widehat{\omega}\quad \text{a.e. on } [\frac{1-\varepsilon}{2},\frac{1}{2}]. \tag{4.2}$$

Consider two Dirichlet boundary value problems defined on the interval $[x_{n_k}^{j_k},x_{n_k}^{j_k+1}]\subseteq[\frac{1-\varepsilon}{2},\frac{1}{2}]$,

$$-u_-''(x,\lambda_{n_k})+q(x)u_-(x,\lambda_{n_k})=\lambda_{n_k}u_-(x,\lambda_{n_k}), \tag{4.3}$$

$$u_-(x_{n_k}^{j_k},\lambda_{n_k})=u_-(x_{n_k}^{j_k+1},\lambda_{n_k})=0, \tag{4.4}$$

and

$$-\widetilde{u}_-''(x,\widetilde{\lambda}_{\widetilde{n}_k})+\widetilde{q}(x)\widetilde{u}_-(x,\widetilde{\lambda}_{\widetilde{n}_k})=\widetilde{\lambda}_{\widetilde{n}_k}\widetilde{u}_-(x,\widetilde{\lambda}_{\widetilde{n}_k}), \tag{4.5}$$

$$\widetilde{u}_-(x_{n_k}^{j_k},\widetilde{\lambda}_{\widetilde{n}_k})=\widetilde{u}_-(x_{n_k}^{j_k+1},\widetilde{\lambda}_{\widetilde{n}_k})=0. \tag{4.6}$$

Multiplying (4.3) by $\widetilde{u}_-(x,\widetilde{\lambda}_{\widetilde{n}_k})$ and (4.5) by $u_-(x,\lambda_{n_k})$, subtracting and integrating it from $x_{n_k}^{j_k}$ to $x_{n_k}^{j_k+1}$ together (4.4) and (4.6), we have

$$\int_{x_{n_k}^{j_k}}^{x_{n_k}^{j_k+1}}[(q(x)-\widetilde{q}(x))-(\lambda_{n_k}-\widetilde{\lambda}_{\widetilde{n}_k})]u_-(x,\lambda_{n_k})\widetilde{u}_-(x,\widetilde{\lambda}_{\widetilde{n}_k})dx=0. \tag{4.7}$$

By (4.7) and $q(x)-\widetilde{q}(x)=2\widehat{\omega}$ a.e. on $[\frac{1-\varepsilon}{2},\frac{1}{2}]$, this yields

$$[2\widehat{\omega}-(\lambda_{n_k}-\widetilde{\lambda}_{\widetilde{n}_k})]\int_{x_{n_k}^{j_k}}^{x_{n_k}^{j_k+1}}u_-(x,\lambda_{n_k})\widetilde{u}_-(x,\widetilde{\lambda}_{\widetilde{n}_k})dx=0. \tag{4.8}$$

Since both $u_-(x,\lambda_{n_k})$ and $\widetilde{u}_-(x,\widetilde{\lambda}_{\widetilde{n}_k})$ have no zero in the interval $(x_{n_k}^{j_k},x_{n_k}^{j_k+1})$, we get

$$u_-(x,\lambda_{n_k})\widetilde{u}_-(x,\widetilde{\lambda}_{\widetilde{n}_k})>0\quad\text{or}\quad u_-(x,\lambda_{n_k})\widetilde{u}_-(x,\widetilde{\lambda}_{\widetilde{n}_k})<0\quad \text{for } x\in(x_{n_k}^{j_k},x_{n_k}^{j_k+1}).$$

This implies

$$\int_{x_{n_k}^{j_k}}^{x_{n_k}^{j_k+1}}u_-(x,\lambda_{n_k})\widetilde{u}_-(x,\widetilde{\lambda}_{\widetilde{n}_k})dx\neq 0. \tag{4.9}$$

Therefore,

$$\lambda_{n_k}=\widetilde{\lambda}_{\widetilde{n}_k}+2\widehat{\omega},\quad \forall n_k\in S. \tag{4.10}$$

By (4.10) and (2.5), for sufficiently large k, this yields $n_k=\widetilde{n}_k$. Thus, the proof of Lemma 3.1 is complete. □

Next we show that Theorem 3.2 holds.

Proof of Theorem 3.2. Denote $\Lambda=\{\lambda_n: n\in S,\lambda_n\in\sigma(L)\}$ and $N_\Lambda(t)=\sharp\{\lambda_n:\lambda_n\in\Lambda,\lambda_n\leq t,\lambda_n\in\sigma(L)\}$ for all sufficiently large $t\in\mathbb{R}$. By calculating $N_\Lambda(t)$, we have

$$N_\Lambda(t)\geq(1-\varepsilon)N_{\sigma(L)}(t)-\frac{1-\varepsilon}{2}, \tag{4.11}$$

By the assumption in Theorem 3.2, Lemma 3.1 yields

$$q(x)-\widetilde{q}(x)=2\widehat{\omega}\quad \text{a.e. on } [\frac{1-\varepsilon}{2},\frac{1}{2}], \tag{4.12}$$

$$\lambda_{n_k} - \widetilde{\lambda}_{\widetilde{n}_k} = 2\widehat{\omega}, \quad \forall n_k \in S. \tag{4.13}$$

Since the set $\mathbb{N}_0\backslash S$ is an infinite set , from (4.13), (3.6) and (2.5), we get

$$\widehat{\omega} = 0. \tag{4.14}$$

By (4.12)-(4.14), we have

$$q(x) - \widetilde{q}(x) = 0 \text{ a.e. on } [\frac{1-\varepsilon}{2}, \frac{1}{2}] \text{ and } \lambda_n - \lambda_{\widetilde{n}} = 0, \quad \forall\ n \in \mathbb{N}, \tag{4.15}$$

Denote

$$F(u_-, \widetilde{u}_-, x, \lambda) = [u_-, \widetilde{u}_-](x, \lambda).$$

Let us prove

$$F\big(u_-, \widetilde{u}_-, \frac{1-\varepsilon}{2}, \lambda_{n_k}\big) = 0, \quad \forall n_k \in S.$$

Indeed, since $W_S([\frac{1-\varepsilon}{2}, \frac{1}{2}]) = \widetilde{W}_{\widetilde{S}}([\frac{1-\varepsilon}{2}, \frac{1}{2}])$ is a left twin-dense nodal subset, we choose $x_{n_k}^{j_{n_k}} \in W_S([\frac{1-\varepsilon}{2}, \frac{1}{2}])$. By the Green's formula, we obtain

$$F\big(u_-, \widetilde{u}_-, \frac{1-\varepsilon}{2}, \lambda_{n_k}\big) = -\int_{\frac{1-\varepsilon}{2}}^{x_{n_k}^{j_{n_k}}} \widehat{q}(x) u_-(x, \lambda_{n_k}) \widetilde{u}_-(x, \lambda_{n_k}) \mathrm{d}x. \tag{4.16}$$

By (4.16) and $\widehat{q}(x) = 0$ a.e. on $[\frac{1-\varepsilon}{2}, \frac{1}{2}]$, we get

$$F\big(u_-, \widetilde{u}_-, \frac{1-\varepsilon}{2}, \lambda_{n_k}\big) = 0, \quad \forall n_k \in S. \tag{4.17}$$

Next we prove $q(x) - \widetilde{q}(x) = 0$ a.e. on $[0, 1]$, $h = \widetilde{h}$ and $H = \widetilde{H}$.

Without loss of generality, we assume that $\lambda_n \neq 0$ for all $n \in \sigma(L)$. Define the functions $G_S(\lambda)$ and $K_1(\lambda)$ by

$$G_S(\lambda) = \prod_{n_k \in S} \big(1 - \frac{\lambda}{\lambda_{n_k}}\big), \tag{4.18}$$

$$K_1(\lambda) = \frac{F\big(u_-, \widetilde{u}_-, \frac{1-\varepsilon}{2}, \lambda\big)}{G_S(\lambda)}. \tag{4.19}$$

Hence (4.17), (4.18) and (4.19) imply that $K_1(\lambda)$ is an entire function in λ. Note that

$$\begin{aligned} &F\big(u_-, \widetilde{u}_-, \frac{1-\varepsilon}{2}, \lambda\big) \\ &= u_-\big(\frac{1-\varepsilon}{2}, \lambda\big) \widetilde{u}'_-\big(\frac{1-\varepsilon}{2}, \lambda\big) - u'_-\big(\frac{1-\varepsilon}{2}, \lambda\big) \widetilde{u}_-\big(\frac{1-\varepsilon}{2}, \lambda\big) \\ &= u'_-\big(\frac{1-\varepsilon}{2}, \lambda\big) \widetilde{u}'_-\big(\frac{1-\varepsilon}{2}, \lambda\big) \Big(m_-^{-1}\big(\frac{1-\varepsilon}{2}, \lambda\big) - \widetilde{m}_-^{-1}\big(\frac{1-\varepsilon}{2}, \lambda\big)\Big). \end{aligned} \tag{4.20}$$

From (2.2), (2.7) and (4.20), we have

$$\big|F\big(u_-, \widetilde{u}_-, \frac{1-\varepsilon}{2}, \lambda\big)\big| = o\big(\mathrm{e}^{(1-\varepsilon)\tau}\big)$$

as $|\lambda| \to \infty$ in any sector $\varepsilon < \arg(\lambda) < \pi - \varepsilon$. This implies

$$\big|F\big(u_-, \widetilde{u}_-, \frac{1-\varepsilon}{2}, iy\big)\big| = o\big(\mathrm{e}^{(1-\varepsilon)\mathrm{Im}\sqrt{i}|y|^{1/2}}\big) \tag{4.21}$$

for sufficiently large $y \in \mathbb{R}$. We analogously calculate $G_S(iy)$ from (4.11) and get the following formula (see [7])

$$|G_S(iy)| \geq c|y|^{1/2} \mathrm{e}^{(1-\varepsilon)\mathrm{Im}\sqrt{i}|y|^{1/2}},$$

where c is a constant. Therefore

$$|K_1(iy)| = o\big(|y|^{-1/2}\big). \tag{4.22}$$

It is easy to prove the following formula (see [7]):

$$\sup_{|z|=R_k} |K_1(z)| \leq C_1 \exp(C_2 R_k^{\alpha}) \tag{4.23}$$

for some $0 < \alpha < 1$, some sequence $R_k \to \infty$ as $k \to \infty$ and $C_1, C_2 > 0$.

By Lemma 2.3, (4.22) and (4.23), we have $K_1(\lambda) = 0$ for all $\lambda \in \mathbb{C}$. Therefore,

$$F\big(u_-, \widetilde{u}_-, \frac{1-\varepsilon}{2}, \lambda\big) = 0, \quad \forall \lambda \in \mathbb{C}. \tag{4.24}$$

This implies

$$m_-\big(\frac{1-\varepsilon}{2}, \lambda\big) = \widetilde{m}_-\big(\frac{1-\varepsilon}{2}, \lambda\big), \quad \forall \lambda \in \mathbb{C}. \tag{4.25}$$

From Lemma 2.2 and (4.25), we obtain

$$q(x) - \widetilde{q}(x) = 0 \quad \text{a.e. on } [0, \frac{1-\varepsilon}{2}] \quad \text{and } h = \widetilde{h}.$$

Therefore,

$$q(x) - \widetilde{q}(x) = 0 \quad \text{a.e. on } [0, \frac{1}{2}], \quad h = \widetilde{h}, \quad \lambda_n = \lambda_n, \quad n \in \mathbb{N}_0. \tag{4.26}$$

By the Hochstadt-Lieberman theorem [13] and (4.26), we get

$$q(x) - \widetilde{q}(x) = 0 \quad \text{a.e. on } [0, 1], \quad \text{and} \quad H = \widetilde{H}.$$

Thus the proof of Theorem 3.2 is complete. □

Proof of Theorem 3.5. From Lemma 3.1, we have

$$q(x) - \widetilde{q}(x) = 2\widehat{\omega} \quad \text{a.e. on } [\frac{1-\varepsilon}{2}, \frac{1}{2}] \tag{4.27}$$

$$\lambda_{n_k} - \widetilde{\lambda}_{\widetilde{n}_k} = 2\widehat{\omega}, \quad \forall n_k \in S. \tag{4.28}$$

Define the potential $\widetilde{q}_1(x)$ by $\widetilde{q}_1(x) = \widetilde{q}(x) + 2\widehat{\omega}$. This implies

$$q(x) - \widetilde{q}_1(x) = 0 \quad \text{a.e. on } [\frac{1-\varepsilon}{2}, \frac{1}{2}] \quad \text{and} \quad \lambda_{n_k} - \widetilde{\lambda}_{1,\widetilde{n}_k} = 0, \quad \forall n_k \in S, \tag{4.29}$$

where $\widetilde{\lambda}_{1,\widetilde{n}_k} = \widetilde{\lambda}_{\widetilde{n}_k} + 2\widehat{\omega}$, which is the eigenvalue of equation (1.1) corresponding to $\widetilde{q}_1$ with boundary conditions (1.2) and (1.3). Analogous to the proof in Theorem 3.2, we have

$$q(x) - \widetilde{q}_1(x) = 0 \quad \text{a.e. on } [0, \frac{1}{2}], \quad \text{and} \quad h = \widetilde{h}. \tag{4.30}$$

Next, we prove $\lambda_n = \widetilde{\lambda}_{1,\widetilde{n}_k}$, $n \geq 1$. From the assumption of Theorem 3.5, there exists the nodal point $x_{n_k}^1$ of the corresponding eigenvalue λ_{n_k} such that

$$x_{n_k}^1 = \widetilde{x}_{\widetilde{n}_k}^1, \quad \forall n_k \in \mathbb{N} \backslash S, \quad 0 < x_{n_k}^1 \leq \frac{1}{2}.$$

Let us consider two boundary value problems defined on the interval $[0, x_{n_k}^1]$,

$$-u_-''(x, \lambda_{n_k}) + q(x) u_-(x, \lambda_{n_k}) = \lambda_{n_k} u_-(x, \lambda_{n_k}), \quad x \in (0, x_{n_k}^1) \tag{4.31}$$

$$u'_-(0,\lambda_{n_k}) - hu_-(0,\lambda_{n_k}) = u_-(x^1_{n_k},\lambda_{n_k}) = 0, \tag{4.32}$$

and

$$-\widetilde{u}''_-(x,\widetilde{\lambda}_{1,\widetilde{n}_k}) + \widetilde{q}_1(x)\widetilde{u}_-(x,\widetilde{\lambda}_{1,\widetilde{n}_k}) = \overline{\lambda}_{\widetilde{n}_k}\widetilde{u}_-(x,\widetilde{\lambda}_{1,\widetilde{n}_k}), \quad x \in (0, x^1_{n_k}) \tag{4.33}$$

$$\widetilde{u}'_-(0,\widetilde{\lambda}_{1,\widetilde{n}_k}) - hu_-(0,\overline{\lambda}_{\widetilde{n}_k}) = \widetilde{u}_-(x^1_{n_k},\widetilde{\lambda}_{1,\widetilde{n}_k}) = 0. \tag{4.34}$$

Multiplying equation (4.31) by $\widetilde{u}_-(x,\overline{\lambda}_{\widetilde{n}_k})$ and equation (4.33) by $u_-(x,\lambda_{n_k})$, subtracting and integrating it from 0 to $x^1_{n_k}$ together with (4.32) and (4.34), we have

$$\int_0^{x^1_{n_k}} [(q(x)-\widetilde{q}_1(x)) - (\lambda_{n_k} - \widetilde{\lambda}_{1,\widetilde{n}_k})]u_-(x,\lambda_{n_k})\widetilde{u}_-(x,\widetilde{\lambda}_{1,\widetilde{n}_k})dx = 0. \tag{4.35}$$

By (4.35) and $q(x)-\widetilde{q}_1(x) = 0$ a.e. on the interval $[0,\frac{1}{2}]$, this yields

$$(\lambda_{n_k} - \widetilde{\lambda}_{1,\widetilde{n}_k})\int_0^{x^1_{n_k}} u_-(x,\lambda_{n_k})\widetilde{u}_-(x,\widetilde{\lambda}_{1,\widetilde{n}_k})dx = 0. \tag{4.36}$$

Since both $u_-(x,\lambda_{n_k})$ and $\widetilde{u}_-(x,\widetilde{\lambda}_{1,\widetilde{n}_k})$ have no zero in the interval $(0,x^1_{n_k})$, we get

$$u_-(x,\lambda_{n_k})\widetilde{u}_-(x,\widetilde{\lambda}_{1,\widetilde{n}_k}) > 0 \quad \text{for } x \in (0,x^1_{n_k}).$$

This implies

$$\int_0^{x^1_{n_k}} u_-(x,\lambda_{n_k})\widetilde{u}_-(x,\widetilde{\lambda}_{1,\widetilde{n}_k})dx > 0. \tag{4.37}$$

By (4.36) and (4.37), this yields $\lambda_{n_k} = \widetilde{\lambda}_{1,\widetilde{n}_k}$ for all $n_k \in \mathbb{N}\backslash S$. Thus we obtain

$$\lambda_n = \widetilde{\lambda}_{1,\widetilde{n}_k}, \quad n = 1,2,\dots. \tag{4.38}$$

By [26, Theorem 2.1], or the related Theorem in [19, Section 4] together with (4.30), (4.38), and given coefficients $H = \widetilde{H}$, we have

$$q(x) - \widetilde{q}_1(x) = 0 \quad \text{a.e. on } [\frac{1}{2}, 1].$$

Therefore,

$$q(x) - \widetilde{q}_1(x) = 0 \quad \text{a.e. on } [0,1], \quad \text{and} \quad h = \widetilde{h}.$$

This completes the proof of Theorem 3.5. □

In the remainder of this section, we present an example for reconstructing the potential q from the twin-dense nodal subset. Let $\varepsilon = 1/4$ and

$$S_0 := \{2n : n \ge 10, n \in \mathbb{N}\} \cup \{2k_i - 1 : 2k_i - 1 > 10, k_i \in \mathbb{N}\}_{i=1}^{10}. \tag{4.39}$$

Example 4.1. Let $W_{S_0}([\frac{1}{4},\frac{1}{2}]) = \widetilde{W}_{\widetilde{S}_0}([\frac{1}{4},\frac{1}{2}]) \subseteq X = \{x^j_n\}$, $n \in \mathbb{N}$, $j = 1,2,\dots,n$, be the left twin-dense nodal subset of the operator $L(q,h,1)$, where

$$x^j_n = \frac{j-\frac{1}{2}}{n} + \frac{1}{2n^2\pi^2}\Big(2 + \big(\frac{j-\frac{1}{2}}{n}\big)^2\Big) - \frac{5(j-\frac{1}{2})}{2n^3\pi^2} + o\big(\frac{1}{n^2}\big), \quad \forall n \in S_0 \tag{4.40}$$

and

$$x^1_n = \frac{1}{2n} + \frac{1}{2n^2\pi^2}\big(2 + \frac{1}{4n^2}\big) - \frac{5}{4n^3\pi^2} + o\big(\frac{1}{n^2}\big) < \frac{1}{2} \tag{4.41}$$

for all $n \in \mathbb{N}\backslash S_0$. By (4.1) together with (4.40), we have

$$\begin{aligned} f_1(x) :=& \lim_{k\to\infty}\big[2(n_k\pi)^2 x^{j_k}_{n_k} - 2n_k\pi^2\big(j_k - \frac{1}{2}\big)\big] \\ =& x^2 + 2 - 5x, \quad x \in \big[\frac{1}{4},\frac{1}{2}\big]. \end{aligned} \tag{4.42}$$

By (4.1) and (4.42) again, this yields

$$h = 1 \quad \text{and} \quad \omega = \frac{5}{2}. \tag{4.43}$$

By the given condition $H = 1$ and (4.43), we get

$$\int_0^1 q(t)\mathrm{d}t = 1. \tag{4.44}$$

By taking derivatives for (4.42) together with (4.44), we obtain

$$q(x) = 2x \quad \text{a.e. on } [\frac{1}{4}, \frac{1}{2}]. \tag{4.45}$$

By (4.39)-(4.41), (4.45) and $W_{S_0}([\frac{1}{4}, \frac{1}{2}]) = \widetilde{W}_{\widetilde{S}_0}([\frac{1}{4}, \frac{1}{2}])$, we see that all assumptions in Theorem 3.5 hold. Thus we have

$$q(x) = 2x \quad \text{a.e. on } [0, 1], \quad \text{and} \quad h = 1.$$

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