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EXISTENCE AND UNIQUENESS FOR ONE-PHASE STEFAN PROBLEMS OF NON-CLASSICAL HEAT EQUATIONS WITH TEMPERATURE BOUNDARY CONDITION AT A FIXED FACE

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Abstract. We prove the existence and uniqueness, local in time, of a solution for a one-phase Stefan problem of a non-classical heat equation for a semi-infinite material with temperature boundary condition at the fixed face. We use the Friedman-Rubinstein integral representation method and the Banach contraction theorem in order to solve an equivalent system of two Volterra integral equations.

1. Introduction

The one-phase Stefan problem for a semi-infinite material for the classical heat equation requires the determination of the temperature distribution u of the liquid phase (melting problem) or of the solid phase (solidification problem), and the evolution of the free boundary $x = s(t)$. Phase-change problems appear frequently in industrial processes and other problems of technological interest [2, 3, 6, 8, 9, 10, 11, 12, 18, 29]. A large bibliography on the subject was given in [25].

Non-classical heat conduction problem for a semi-infinite material was studied in [4, 7, 17, 27, 28], e.g. problems of the type

$$\begin{gathered} u_t - u_{xx} = -F(u_x(0,t)), \quad x > 0,\ t > 0, \\ u(0,t) = 0, \quad t > 0 \\ u(x,0) = h(x), \quad x > 0 \end{gathered} \tag{1.1}$$

where $h(x), x > 0$, and $F(V), V \in \mathbb{R}$, are continuous functions. The function F, henceforth referred as control function, is assumed to satisfy the condition

(H1) $F(0) = 0$.

As observed in [27, 28], the heat flux $w(x,t) = u_x(x,t)$ for problem (1.1) satisfies a classical heat conduction problem with a nonlinear convective condition at $x = 0$,

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which can be written in the form

$$\begin{gathered} w_t - w_{xx} = 0, \quad x > 0,\ t > 0, \\ w_x(0,t) = F(w(0,t)), \quad t > 0, \\ w(x,0) = h'(x) \geq 0, \quad x > 0. \end{gathered} \tag{1.2}$$

The literature concerning problem (1.2) has increased rapidly since the publication of the papers [19, 21, 22]. Related problems have been also studied; see for example [1, 14, 16]. In [26], a one-phase Stefan problem for a non-classical heat equation for a semi-infinite material was presented. There the free boundary problem consists in determining the temperature $u = u(x,t)$ and the free boundary $x = s(t)$ with a control function F which depends on the evolution of the heat flux at the extremum $x = 0$ is given by the conditions

$$\begin{gathered} u_t - u_{xx} = -F(u_x(0,t)), \quad 0 < x < s(t),\ 0 < t < T, \\ u(0,t) = f(t) \geqslant 0, \quad 0 < t < T, \\ u(s(t),t) = 0, \quad u_x(s(t),t) = -\dot{s}(t), \quad 0 < t < T, \\ u(x,0) = h(x) \geq 0, \quad 0 \leqslant x \leqslant b = s(0) \quad (b > 0). \end{gathered} \tag{1.3}$$

The goal in this paper is to prove the existence and uniqueness, local in time, of a solution to the one-phase Stefan problem (1.3) for a non-classical heat equation with temperature boundary condition at the fixed face $x = 0$. First, we prove that problem (1.3) is equivalent to a system of two Volterra integral equations (2.4)-(2.5) following the Friedman-Rubinstein's method given in [13, 23]. Then, we prove that the problem (2.4)-(2.5) has a unique local solution by using the Banach contraction theorem.

2. Existence and Uniqueness of Solutions

We have the following equivalence for the existence of solutions to the non-classical free boundary problem (1.3).

Theorem 2.1. *The solution of the free-boundary problem* (1.3) *is*

$$\begin{aligned} u(x,t) = &\int_0^b G(x,t;\xi,0)h(\xi)d\xi + \int_0^t G_\xi(x,t;0,\tau)f(\tau)\,d\tau \\ &+ \int_0^t G(x,t;s(\tau),\tau)v(\tau)\,d\tau - \iint_{D(t)} G(x,t;\xi,\tau)F(V(\tau))d\xi\,d\tau, \end{aligned} \tag{2.1}$$

$$s(t) = b - \int_0^t v(\tau)\,d\tau\,, \tag{2.2}$$

where $D(t) = \{(x,\tau) : 0 < x < s(\tau), 0 < \tau < t\}$, *with* $f \in C^1[0,T)$, $h \in C^1[0,b]$, $h(b) = 0$, $h(0) = f(0)$, F *is a Lipschitz function over* $C^0[0,T]$, *and the functions* $v \in C^0[0,T]$, $V \in C^0[0,T]$ *defined by*

$$v(t) = u_x(s(t),t)\,, \quad V(t) = u_x(0,t) \tag{2.3}$$

must satisfy the following system of Volterra integral equations

$$\begin{aligned} v(t) = {} & 2\int_0^b N(s(t),t;\xi,0)h'(\xi)d\xi - 2\int_0^t N(s(t),t;0,\tau)\dot{f}(\tau)\,d\tau \\ & + 2\int_0^t G_x(s(t),t;s(\tau),\tau)v(\tau)\,d\tau \\ & + 2\int_0^t [N(s(t),t;s(\tau),\tau) - N(s(t),t;0,\tau)]F(V(\tau))\,d\tau. \end{aligned} \tag{2.4}$$

$$\begin{aligned} V(t) = {} & \int_0^b N(0,t;\xi,0)h'(\xi)d\xi \\ & - \int_0^t N(0,t;0,\tau)\dot{f}(\tau)\,d\tau + \int_0^t G_x(0,t;s(\tau),\tau)v(\tau)\,d\tau \\ & + \int_0^t [N(0,t;s(\tau),\tau) - N(0,t;0,\tau)]F(V(\tau))\,d\tau, \end{aligned} \tag{2.5}$$

where G, N are the Green and Neumann functions and K is the fundamental solution of the heat equation, defined respectively by

$$\begin{gathered} G(x,t,\xi,\tau) = K(x,t,\xi,\tau) - K(-x,t,\xi,\tau), \\ N(x,t,\xi,\tau) = K(x,t,\xi,\tau) + K(-x,t,\xi,\tau), \\ K(x,t,\xi,\tau) = \begin{cases} \frac{1}{2\sqrt{\pi(t-\tau)}}\exp\big(-\frac{(x-\xi)^2}{4(t-\tau)}\big) & t > \tau \\ 0 & t \le \tau\,, \end{cases} \end{gathered}$$

where $s(t)$ is given by (2.2),

Proof. Let $u(x,t)$ be the solution to (1.3). We integrate, on the domain $D_{t,\varepsilon} = \{(\xi,\tau) : 0 < \xi < s(\tau), \varepsilon < \tau < t-\varepsilon\}$, the Green identity

$$(Gu_\xi - uG_\xi)_\xi - (Gu)_\tau = GF(u_\xi(0,\tau))\,. \tag{2.6}$$

Now we let $\varepsilon \to 0$, to obtain the following integral representation for $u(x,t)$,

$$\begin{aligned} u(x,t) = {} & \int_0^b G(x,t;\xi,0)h(\xi)d\xi + \int_0^t G_\xi(x,t;0,\tau)f(\tau)\,d\tau \\ & + \int_0^t G(x,t;s(\tau),\tau)\;u_\xi(s(\tau),\tau)\,d\tau - \iint_{D(t)} G(x,t;\xi,\tau)F(u_\xi(0,\tau))d\xi\,d\tau\,. \end{aligned}$$

From the definition of $v(t)$ and $V(t)$ by (2.3), we obtain (2.1) and (2.2). If we differentiate $u(x,t)$ in variable x and we let $x \to 0^+$ and $x \to s(t)$, by using the jump relations, we obtain the integral equations for v and V given by (2.4) and (2.5).

Conversely, the function $u(x,t)$ defined by (2.1) where v and V are the solutions of (2.4)and (2.5), satisfy the conditions (1.3) (i),(ii),(iv) and (v). In order to prove condition (1.3) (iii) we define $\psi(t) = u(s(t),t)$. Taking into account that u satisfy the conditions (1.3) (i),(ii),(iv) and (v), if we integrate the Green identity (2.6) over the domain $D_{t,\varepsilon}$, ($\varepsilon > 0$) and we let $\varepsilon \to 0$ we obtain that

$$\begin{aligned} u(x,t) = &\int_0^b G(x,t;\xi,0)h(\xi)d\xi + \int_0^t G(x,t;s(\tau),\tau)v(\tau)\,d\tau \\ &+ \int_0^t \psi(\tau)[G_x(x,t;s(\tau),\tau) - G(x,t;s(\tau),\tau)v(\tau)]\,d\tau \\ &+ \int_0^t G_\xi(x,t;0,\tau)f(\tau)\,d\tau - \iint_{D(t)} G(x,t;\xi,\tau)F(V(\tau))d\xi\,d\tau. \end{aligned}$$

Then, if we compare this last expression with (2.1), we deduce that

$$M(x,t) = \int_0^t \psi(\tau)[G_x(x,t;s(\tau),\tau) - G(x,t;s(\tau),\tau)v(\tau)]\,d\tau \equiv 0 \tag{2.7}$$

for $0 < x < s(t)$, $0 < t < \sigma$. We let $x \to s(t)$ in (2.7) and by using the jump relations we have that ψ satisfy the integral equation

$$\frac{1}{2}\psi(t) + \int_0^t \psi(\tau)[G_x(s(t),t;s(\tau),\tau) - G(s(t),t;s(\tau),\tau)v(\tau)]\,d\tau = 0\,.$$

Then we deduce that

$$\begin{aligned} |\psi(t)| &\le C\int_0^t \frac{|\psi(\tau)|}{\sqrt{t-\tau}}\,d\tau \\ &\le C^2\int_0^t \frac{d\tau}{\sqrt{t-\tau}}\int_0^\tau \frac{|\psi(\eta)|}{\sqrt{\tau-\eta}}d\eta \\ &= C^2\int_0^t |\psi(\eta)|d\eta\int_\eta^t \frac{d\tau}{[(t-\tau)(\tau-\eta)]^{1/2}} \\ &= \pi C^2\int_0^t |\psi(\eta)|d\eta \end{aligned}$$

where $C = C(t)$; therefore by using the Gronwall inequality we have that $\psi(t) = 0$ over $[0,\sigma]$. $\square$

Next, we use the Banach fixed point theorem in order to prove the local existence and uniqueness of solution $v, V \in C^0[0,\sigma]$ to the system of two Volterra integral equations (2.4)-(2.5) where σ is a positive small number. Consider the Banach space

$$C_{M,\sigma} = \Big\{\vec{w} = \begin{pmatrix} v \\ V \end{pmatrix} : v, V : [0,\sigma] \to \mathbb{R},\ \text{continuous, with } \|\vec{w}\|_\sigma \le M\Big\}$$

with

$$\|\vec{w}\|_\sigma := \|v\|_\sigma + \|V\|_\sigma := \max_{t\in[0,\sigma]}|v(t)| + \max_{t\in[0,\sigma]}|V(t)|$$

We define $A : C_{M,\sigma} \longrightarrow C_{M,\sigma}$, such that

$$\vec{\tilde{w}}(t) = A(\vec{w}(t)) = \begin{pmatrix} A_1(v(t),V(t)) \\ A_2(v(t),V(t)) \end{pmatrix}$$

where

$$A_1(v(t),V(t)) = F_0(v(t)) + 2\int_0^t [N(s(t),t,s(\tau),\tau) - N(s(t),t,0,\tau)]F(V(\tau))\,d\tau \tag{2.8}$$

with

$$\begin{aligned} F_0(v(t)) &= 2\int_0^b N(s(t),t,\xi,0)h'(\xi)d\xi - 2\int_0^t N(s(t),t,0,\tau)\dot f(\tau)\,d\tau \\ &\quad + 2\int_0^t G_x(s(t),t,s(\tau),\tau)v(\tau)\,d\tau \end{aligned}$$

and

$$\begin{aligned} A_2(v(t),V(t)) &= \int_0^b N(0,t,\xi,0)h'(\xi)d\xi - \int_0^t N(0,t,0,\tau)\dot f(\tau)\,d\tau \\ &\quad + \int_0^t G_x(0,t,s(\tau),\tau)v(\tau)\,d\tau \\ &\quad + \int_0^t [N(0,t,s(\tau),\tau) - N(0,t,0,\tau)]F(V(\tau))\,d\tau. \end{aligned} \tag{2.9}$$

Lemma 2.2. *If $v \in C^0[0,\sigma]$, $\max_{t\in[0,\sigma]}|v(t)| \le M$ and $2M\sigma \le b$ then $s(t)$ defined by (2.2) satisfies*

$$|s(t)-s(\tau)| \le M|t-\tau| \quad |s(t)-b| \le \frac{b}{2}, \quad \forall t,\tau \in [0,\sigma].$$

To prove the following Lemmas we need the inequality

$$\exp\Big(\frac{-x^2}{\alpha(t-\tau)}\Big)/(t-\tau)^{n/2} \le \Big(\frac{n\alpha}{2ex^2}\Big)^{n/2}, \quad \alpha, x>0,\ t>\tau,\ n\in\mathbb{N}. \tag{2.10}$$

Lemma 2.3. *Let $\sigma \le 1$, $M \ge 1$, $f \in C^1[0,T)$, $h \in C^1[0,b]$, F a Lipschitz function over $C^0[0,T]$. Under the hypothesis of Lemma 2.2, we have the following properties:*

$$\int_0^t |N(s(t),t,0,\tau)||\dot f(\tau)|\,d\tau \le \|\dot f\|_t C_1(b)t \tag{2.11}$$

$$\int_0^t |G_x(s(t),t,s(\tau),\tau)||v(\tau)|\,d\tau \le M^2C_2(b)\sqrt{t} \tag{2.12}$$

$$\int_0^b |N(s(t),t,\xi,0)||h'(\xi)|d\xi \le \|h'\| \tag{2.13}$$

$$\int_0^t |N(s(t),t,s(\tau),\tau) - N(s(t),t,0,\tau)||F(V(\tau))|\,d\tau \le C_4(L)M\sqrt{t} \tag{2.14}$$

$$\int_0^b |N(0,t,\xi,0)||h'(\xi)|d\xi \le \|h'\| \tag{2.15}$$

$$\int_0^t |N(0,t,0,\tau)||\dot f(\tau)|\,d\tau \le \frac{2\|\dot f\|_\sigma}{\sqrt{\pi}}\sqrt{t} \tag{2.16}$$

$$\int_0^t |G_x(0,t,s(\tau),\tau)||v(\tau)|\,d\tau \le C_3(b)Mt \tag{2.17}$$

$$\int_0^t |N(0,t,s(\tau),\tau) - N(0,t,0,\tau)||F(V(\tau))|\,d\tau \le C_4(L)M\sqrt{t} \tag{2.18}$$

where L is the Lipschitz constant of F and

$$\begin{gathered}
C_1(b)=(\frac{8}{eb^2})^{1/2}\frac{1}{\sqrt{\pi}},\quad C_2(b)=\frac{1}{2\sqrt{\pi}}+\frac{3b}{4\sqrt{\pi}}(\frac{2}{3eb^2})^{3/2}\\
C_3(b)=\frac{3b}{8\sqrt{\pi}}(\frac{24}{eb^2})^{3/2},\quad C_4(L)=\frac{4L}{\sqrt{\pi}}\ .
\end{gathered}\tag{2.19}$$

Proof. To prove (2.11), we have

$$\begin{aligned}
|N(s(t),t,0,\tau)|&=|K(s(t),t,0,\tau)+K(-s(t),t,0,\tau)|=2K(s(t),t,0,\tau)\\
&=\exp\big(\frac{-s^2(t)}{4(t-\tau)}\big)\frac{(t-\tau)^{-1/2}}{\sqrt{\pi}}\\
&\le\exp\big(\frac{-b^2}{16(t-\tau)}\big)\frac{(t-\tau)^{-1/2}}{\sqrt{\pi}}\\
&\le(\frac{8}{eb^2})^{1/2}\frac{1}{\sqrt{\pi}}=C_1(b)
\end{aligned}$$

then (2.11) holds. To prove (2.12), we have

$$\begin{aligned}
|G_x(s(t),t,s(\tau),\tau)|&=\big|K_x(s(t),t,s(\tau),\tau)+K_x(-s(t),t,s(\tau),\tau)\big|\\
&=\frac{(t-\tau)^{-3/2}}{4\sqrt{\pi}}\Big|(s(t)-s(\tau))\exp\big(\frac{-(s(t)-s(\tau))^2}{4(t-\tau)}\big)\\
&\quad-(s(t)+s(\tau))\exp\big(\frac{-(s(t)+s(\tau))^2}{4(t-\tau)}\big)\Big|\\
&\le\frac{(t-\tau)^{-3/2}}{4\sqrt{\pi}}\Big(M(t-\tau)+3b\exp\big(\frac{-9b^2}{4(t-\tau)}\big)\Big)\\
&\le\frac{1}{4\sqrt{\pi}}\Big(M(t-\tau)^{-1/2}+3b(\frac{2}{3eb^2})^{3/2}\Big)\,.
\end{aligned}$$

Then

$$\begin{aligned}
\int_0^t|G_x(s(t),t,s(\tau),\tau)||v(\tau)|\,d\tau&\le\frac{M}{4\sqrt{\pi}}\Big(2M\sqrt{t}+3b(\frac{2}{3eb^2})^{3/2}t\Big)\\
&\le M^2\sqrt{t}\Big(\frac{1}{2\sqrt{\pi}}+\frac{3b}{M4\sqrt{\pi}}(\frac{2}{3eb^2})^{3/2}\Big)\\
&\le M^2C_2(b)\sqrt{t},
\end{aligned}$$

which implies (2.12). To prove (2.13), we have

$$\int_0^b|N(s(t),t,\xi,0)||h'(\xi)|d\xi\le\|h'\|\int_0^\infty|N(s(t),t,\xi,0)|d\xi\le\|h'\|$$

because

$$\int_0^\infty|N(s(t),t,\xi,0)|d\xi\le 1.$$

To prove (2.14), by taking into account that

$$|N(s(t),t,s(\tau),\tau)-N(s(t),t,0,\tau)|\le\frac{2}{\sqrt{\pi(t-\tau)}}$$

we obtain

$$\begin{aligned}\int_0^t |N(s(t),t,s(\tau),\tau)-N(s(t),t,0,\tau)||F(V(\tau))|\,d\tau &\le LM\int_0^t \frac{2}{\sqrt{\pi(t-\tau)}}\,d\tau\\ &= C_4(L)M\sqrt{t}.\end{aligned}$$

The inequality (2.15) is prove in the same way as (2.13). To prove (2.16), we have

$$\begin{aligned}\int_0^t |N(0,t,0,\tau)||\dot f(\tau)|\,d\tau &\le \|\dot f\|_\sigma \int_0^t |N(0,t,0,\tau)|\,d\tau\\ &= \|\dot f\|_\sigma \int_0^t \frac{1}{\sqrt{\pi(t-\tau)}}\,d\tau\\ &= \frac{\|\dot f\|_\sigma}{\sqrt{\pi}}2\sqrt{t}.\end{aligned}$$

To prove (2.17), we have

$$\begin{aligned}|G_x(0,t,s(\tau),\tau)| &= \frac{(t-\tau)^{-3/2}}{4\sqrt{\pi}}\,s(\tau)\exp\Big(\frac{-(s(\tau))^2}{4(t-\tau)}\Big)\\ &\le \frac{3b}{8\sqrt{\pi}}(t-\tau)^{-3/2}\exp\Big(\frac{-b^2}{16(t-\tau)}\Big)\\ &\le \frac{3b}{8\sqrt{\pi}}(\frac{24}{eb^2})^{3/2}\,.\end{aligned}$$

To prove (2.18), as in (2.14), we prove that

$$|N(0,t,s(\tau),\tau)-N(0,t,0,\tau)| \le \frac{2}{\sqrt{\pi(t-\tau)}}$$

and therefore (2.18) holds. ◻

Lemma 2.4. *Let s_1, s_2 be the functions corresponding to v_1, v_2 in $C^0[0,\sigma]$, respectively, with $\max_{t\in[0,\sigma]}|v_i(t)|\le M$, $i=1,2$, Then we have*

$$\begin{gathered}|s_2(t)-s_1(t)|\le t\|v_2-v_1\|_t,\\ |s_i(t)-s_i(\tau)|\le M|t-\tau|,\quad i=1,2,\\ \frac{b}{2}\le s_i(t)\le\frac{3b}{2},\quad \forall t\in[0,\sigma],\ i=1,2.\end{gathered}\tag{2.20}$$

Lemma 2.5. *Let $f \in C^1[0,T)$, $h \in C^1[0,b]$, F a Lipschitz function in $C^0[0,T]$. We have*

$$|F_0(v_2(t)) - F_0(v_1(t))| \le E(b,h,f)\sqrt{t}\|v_2 - v_1\|_t; \tag{2.21}$$

$$\begin{aligned} &\int_0^t |N(s_2(t),t,s_2(\tau),\tau) - N(s_2(t),t,0,\tau)||F(V_2(\tau)) - F(V_1(\tau))|\,d\tau \\ &\le C_4(L)\sqrt{t}\|V_2 - V_1\|_t; \end{aligned} \tag{2.22}$$

$$\begin{aligned} &\int_0^t |N(s_2(t),t,0,\tau) - N(s_1(t),t,0,\tau)||F(V_1(\tau))|\,d\tau \\ &\le C_5(b,L,M)t\|v_2 - v_1\|_t; \end{aligned} \tag{2.23}$$

$$\begin{aligned} &\int_0^t |N(s_2(t),t,s_2(\tau),\tau) - N(s_1(t),t,s_1(\tau),\tau)||F(V_1(\tau))|\,d\tau \\ &\le [C_6(L,M)\sqrt{t} + C_7(b,L,M)t]\|v_2 - v_1\|_t; \end{aligned} \tag{2.24}$$

$$\int_0^t |G_x(0,t,s_2(\tau),\tau)||v_2(\tau) - v_1(\tau)|\,d\tau \le C_8(b)t\|v_2 - v_1\|_t; \tag{2.25}$$

$$\begin{aligned} &\int_0^t |G_x(0,t,s_2(\tau),\tau)v_2(\tau) - G_x(0,t,s_1(\tau),\tau)v_1(\tau)|\,d\tau \\ &\le (C_8(b)t + C_9(b,M)t^2)\|v_2 - v_1\|_t; \end{aligned} \tag{2.26}$$

$$\begin{aligned} &\int_0^t \Big|[N(0,t,s_2(\tau),\tau) - N(0,t,0,\tau)]F(V_2(\tau)) \\ &- [N(0,t,s_1(\tau),\tau) - N(0,t,0,\tau)]F(V_1(\tau))\Big|\,d\tau \\ &\le C_4(L)\sqrt{t}\|V_2 - V_1\|_t + C_5(b,L,M)t^2\|v_2 - v_1\|_t, \end{aligned} \tag{2.27}$$

where the constants are defined by

$$\begin{gathered} C_4(L) = \frac{4L}{\sqrt{\pi}}, \quad C_5(b,L,M) = LM\frac{3b}{8\sqrt{\pi}}(\frac{24}{eb^2})^{3/2}, \\ C_6(L,M) = \frac{LM^3}{\sqrt{\pi}}, \quad C_7(b,L,M) = (\frac{6}{eb^2})^{3/2}\frac{3bLM^2}{2\sqrt{\pi}}, \\ C_8(b) = \frac{3}{4\sqrt{\pi}}(\frac{24}{eb^2})^{3/2}, \quad C_9(b,M) = [(\frac{40}{eb^2})^{\frac{5}{2}}\frac{9b^2}{16\sqrt{\pi}} + \frac{1}{2\sqrt{\pi}}(\frac{24}{eb^2})^{3/2}]\frac{M}{2}. \end{gathered} \tag{2.28}$$

Proof. The proof of (2.21) can be found in [13]. To prove (2.22), we have

$$|N(s_2(t),t,s_2(\tau),\tau) - N(s_2(t),t,0,\tau)| \le \frac{2}{\sqrt{\pi(t-\tau)}}.$$

Then

$$\begin{aligned} &\int_0^t |N(s_2(t),t,s_2(\tau),\tau) - N(s_2(t),t,0,\tau)||F(V_2(\tau)) - F(V_1(\tau))|\,d\tau \\ &\le \frac{4L}{\sqrt{\pi}}\sqrt{t}\|V_2 - V_1\|_t \end{aligned}$$

To prove (2.23), we use the mean value theorem: There exists $c = c(t, \tau)$ between $s_1(t)$ and $s_2(t)$ such that

$$
\begin{aligned}
&|N(s_2(t), t, 0, \tau) - N(s_1(t), t, 0, \tau)||F(V_1(\tau))| \\
&= |N_x(c, t, 0, \tau)||s_2(\tau) - s_1(\tau)||F(V_1(\tau))| \\
&\le |c| \exp\big(-\frac{c^2}{4(t-\tau)}\big) \frac{(t-\tau)^{-3/2}}{2\sqrt{\pi}} LM\tau |v_2(\tau) - v_1(\tau)| \\
&\le \frac{3b}{4\sqrt{\pi}} \exp\big(-\frac{b^2}{16(t-\tau)}\big)(t-\tau)^{-3/2} LM\tau |v_2(\tau) - v_1(\tau)| \\
&\le \frac{3b}{4\sqrt{\pi}} (\frac{24}{eb^2})^{3/2} LM\tau |v_2(\tau) - v_1(\tau)|\,.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\int_0^t |N(s_2(t), t, 0, \tau) - N(s_1(t), t, 0, \tau)||F(V_1(\tau))|\, d\tau \\
&\le \frac{3b}{8\sqrt{\pi}} (\frac{24}{eb^2})^{\frac{3}{2}} LMt \|v_2 - v_1\|_t = C_5(b, L, M) t \|v_2 - v_1\|_t.
\end{aligned}
$$

To prove (2.24), we have

$$
\begin{aligned}
&N(s_2(t), t, s_2(\tau), \tau) - N(s_1(t), t, s_1(\tau), \tau) \\
&= K(s_2(t), t, s_2(\tau), \tau) - K(s_1(t), t, s_1(\tau), \tau) \\
&\quad + K(-s_2(t), t, s_2(\tau), \tau) - K(-s_1(t), t, s_1(\tau), \tau)\,.
\end{aligned}
$$

As in [24], for each (t, τ), $0 < \tau < t$, we define

$$
f_{t,\tau}(x) = \exp\big(\frac{-x^2}{4(t-\tau)}\big).
$$

Then we have

$$
\begin{aligned}
&K(s_2(t), t, s_2(\tau), \tau) - K(s_1(t), t, s_1(\tau), \tau) \\
&= \frac{(t-\tau)^{-1/2}}{2\sqrt{\pi}} \big[\exp\big(-\frac{(s_2(t) - s_2(\tau))^2}{4(t-\tau)}\big) - \exp\big(-\frac{(s_1(t) - s_1(\tau))^2}{4(t-\tau)}\big)\big] \\
&= \frac{(t-\tau)^{-1/2}}{2\sqrt{\pi}} \big[f_{t,\tau}(s_2(t) - s_2(\tau)) - f_{t,\tau}(s_1(t) - s_1(\tau))\big]
\end{aligned}
$$

and

$$
\begin{aligned}
&K(-s_2(t), t, s_2(\tau), \tau) - K(-s_1(t), t, s_1(\tau), \tau) \\
&= \frac{(t-\tau)^{-1/2}}{2\sqrt{\pi}} \big[\exp\big(-\frac{(s_2(t) + s_2(\tau))^2}{4(t-\tau)}\big) - \exp\big(-\frac{(s_1(t) + s_1(\tau))^2}{4(t-\tau)}\big)\big] \\
&= \frac{(t-\tau)^{-1/2}}{2\sqrt{\pi}} \big[f_{t,\tau}(s_2(t) + s_2(\tau)) - f_{t,\tau}(s_1(t) + s_1(\tau))\big]
\end{aligned}
$$

By the mean value theorem there exists $c = c(t,\tau)$ between $s_2(t) - s_2(\tau)$ and $s_1(t) - s_1(\tau)$ such that

$$\begin{aligned}
&f_{t,\tau}(s_2(t) - s_2(\tau)) - f_{t,\tau}(s_1(t) - s_1(\tau)) \\
&= f'_{t,\tau}(c)(s_2(t) - s_2(\tau) - s_1(t) + s_1(\tau)) \\
&= \frac{-c}{2(t-\tau)} \exp(-\frac{c^2}{4(t-\tau)})(s_2(t) - s_2(\tau) - s_1(t) + s_1(\tau))
\end{aligned}$$

Taking into account that

$$|c| \le \max\{|s_i(t) - s_i(\tau)|, i = 1,2\} \le M(t-\tau)$$

it results

$$\begin{aligned}
|f_{t,\tau}(s_2(t) - s_2(\tau)) - f_{t,\tau}(s_1(t) - s_1(\tau))| &\le \frac{M}{2}[|s_2(t)) - s_1(t)| + |s_2(\tau) - s_1(\tau)|] \\
&\le M^2 \|v_2 - v_1\|_t \,.
\end{aligned}$$

Then we have

$$|K(s_2(t), t, s_2(\tau), \tau) - K(s_1(t), t, s_1(\tau), \tau)| \le \frac{M^2}{2\sqrt{\pi(t-\tau)}} \|v_2 - v_1\|_t.$$

In the same way we have

$$\begin{aligned}
&f_{t,\tau}(s_2(t) + s_2(\tau)) - f_{t,\tau}(s_1(t) + s_1(\tau)) \\
&= f'_{t,\tau}(c^*)(s_2(t) + s_2(\tau) - s_1(t) - s_1(\tau)) \\
&= \frac{-c^*}{2(t-\tau)} \exp(-\frac{c^{*2}}{4(t-\tau)})(s_2(t) + s_2(\tau) - s_1(t) - s_1(\tau))
\end{aligned}$$

where $c^* = c^*(t,\tau)$ is between $s_2(t) + s_2(\tau)$ and $s_1(t) + s_1(\tau)$. Since $s_1(t) + s_1(\tau) \le c^* \le s_2(t) + s_2(\tau)$, (or viceversa), we deduce that $b \le c^* \le 3b$, that is $\exp(-c^{*2}/4(t-\tau)) \le \exp(-b^2/4(t-\tau))$. Then we obtain

$$\begin{aligned}
&|K(-s_2(t), t, s_2(\tau), \tau) - K(-s_1(t), t, s_1(\tau), \tau)| \\
&= \frac{(t-\tau)^{-1/2}}{2\sqrt{\pi}} |f_{t,\tau}(s_2(t) + s_2(\tau)) - f_{t,\tau}(s_1(t) + s_1(\tau))| \\
&\le \frac{3b}{4\sqrt{\pi}(t-\tau)^{3/2}} \exp\big(-\frac{b^2}{4(t-\tau)}\big) 2M \|v_2 - v_1\|_t \\
&\le (\frac{6}{eb^2})^{3/2} \frac{3bM}{2\sqrt{\pi}} \|v_2 - v_1\|_t
\end{aligned}$$

and

$$\begin{aligned}
&|N(s_2(t), t, s_2(\tau), \tau) - N(s_1(t), t, s_1(\tau), \tau)| \\
&\le (\frac{M^2}{2\sqrt{\pi(t-\tau)}} + (\frac{6}{eb^2})^{3/2} \frac{3bM}{2\sqrt{\pi}}) \|v_2 - v_1\|_t.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_0^t |N(s_2(t),t,s_2(\tau),\tau)-N(s_1(t),t,s_1(\tau),\tau)||F(V_1(\tau))|\,d\tau\\
&\le \int_0^t \Big(\frac{M^2}{2\sqrt{\pi(t-\tau)}}+(\frac{6}{eb^2})^{3/2}\frac{3bM}{2\sqrt{\pi}}\Big)\|v_2-v_1\|_t|F(V_1(\tau))|\,d\tau\\
&\le LM\Big(\frac{M^2\sqrt{t}}{\sqrt{\pi}}+(\frac{6}{eb^2})^{3/2}\frac{3bM}{2\sqrt{\pi}}t\Big)\|v_2-v_1\|_t\\
&=(C_6(L,M)\sqrt{t}+C_7(L,M,b)t)\|v_2-v_1\|_t.
\end{aligned}$$

To prove (2.25), we take into account (2.10):

$$\begin{aligned}
G_x(0,t,s_2(\tau),\tau)&=K(0,t,s_2(\tau),\tau)\frac{s_2(\tau)}{t-\tau}\\
&=\exp\big(-\frac{s_2^2(\tau)}{4(t-\tau)}\big)\frac{(t-\tau)^{-3/2}}{2\sqrt{\pi}}s_2(\tau)\\
&\le \frac{1}{2\sqrt{\pi}}\big(\frac{24}{eb^2}\big)^{3/2}s_2(\tau)\le \frac{3b}{4\sqrt{\pi}}(\frac{24}{eb^2})^{\frac32}=C_8(b).
\end{aligned}$$

To prove (2.26), we have

$$\begin{aligned}
&|G_x(0,t,s_2(\tau),\tau)v_2(\tau)-G_x(0,t,s_1(\tau),\tau)v_1(\tau)|\\
&\le |G_x(0,t,s_2(\tau),\tau)||v_2(\tau)-v_1(\tau)|\\
&\quad+|G_x(0,t,s_2(\tau),\tau)-G_x(0,t,s_1(\tau),\tau)||v_1(\tau)|.
\end{aligned}$$

Using the mean value theorem there exists $c=c(\tau)$ between $s_2(\tau)$ and $s_1(\tau)$ such that $G_x(0,t,s_2(\tau),\tau)-G_x(0,t,s_1(\tau),\tau)=G_{x\xi}(0,t,c,\tau)(s_2(\tau)-s_1(\tau))$. Taking into account the following properties

$$\begin{gathered}
G_{x\xi}(0,t,c,\tau)=\frac{K(0,t,c,\tau)}{t-\tau}\big(\frac{c^2}{2(t-\tau)}+1\big),\\
\frac{K(0,t,c,\tau)}{t-\tau}=\frac{1}{2\sqrt{\pi}}\exp\big(-\frac{c^2}{4(t-\tau)}\big)(t-\tau)^{-3/2}\le\frac{1}{2\sqrt{\pi}}(\frac{24}{eb^2})^{3/2},\\
K(0,t,c,\tau)\frac{c^2}{2(t-\tau)^2}=\frac{1}{4\sqrt{\pi}}\exp\big(-\frac{c^2}{4(t-\tau)}\big)(t-\tau)^{-\frac52}c^2\le\frac{9b^2}{16\sqrt{\pi}}(\frac{40}{eb^2})^{5/2}
\end{gathered}$$

we have

$$\begin{aligned}
&|G_x(0,t,s_2(\tau),\tau)-G_x(0,t,s_1(\tau),\tau)||v_1(\tau)|\\
&\le (\frac{1}{2\sqrt{\pi}}(\frac{24}{eb^2})^{\frac32}+\frac{9b^2}{16\sqrt{\pi}}(\frac{40}{eb^2})^{\frac52})|s_2(\tau)-s_1(\tau)||v_1(\tau)|\\
&\le M(\frac{1}{2\sqrt{\pi}}(\frac{24}{eb^2})^{\frac32}+\frac{9b^2}{16\sqrt{\pi}}(\frac{40}{eb^2})^{\frac52})\tau|v_2(\tau)-v_1(\tau)|\,.
\end{aligned}$$

Then

$$\begin{aligned}
&\int_0^t |G_x(0,t,s_2(\tau),\tau)-G_x(0,t,s_1(\tau),\tau)||v_1(\tau)|\,d\tau\\
&\le (\frac{1}{2\sqrt{\pi}}(\frac{24}{eb^2})^{\frac32}+\frac{9b^2}{16\sqrt{\pi}}(\frac{40}{eb^2})^{\frac52})\frac{Mt^2}{2}\|v_2-v_1\|_t.
\end{aligned}$$

Then (2.26) holds by using (2.25). To prove (2.27), we have

$$\begin{aligned} &[N(0,t,s_2(\tau),\tau) - N(0,t,0,\tau)]F(V_2(\tau)) \\ &- [N(0,t,s_1(\tau),\tau) - N(0,t,0,\tau)]F(V_1(\tau)) \\ &= [N(0,t,s_2(\tau),\tau) - N(0,t,0,\tau)][F(V_2(\tau)) - F(V_1(\tau))] \\ &\quad + [N(0,t,s_2(\tau),\tau) - N(0,t,s_1(\tau),\tau)]F(V_1(\tau)) \end{aligned} \tag{2.29}$$

Using $|N(0,t,s_2(\tau),\tau) - N(0,t,0,\tau)| \le \frac{2}{\sqrt{\pi(t-\tau)}}$ we get

$$|N(0,t,s_2(\tau),\tau) - N(0,t,0,\tau)||F(V_2(\tau)) - F(V_1(\tau))| \le \frac{2}{\sqrt{\pi(t-\tau)}} L|V_2(\tau) - V_1(\tau)|,$$

and

$$\begin{aligned} &\int_0^t |N(0,t,s_2(\tau),\tau) - N(0,t,0,\tau)||F(V_2(\tau)) - F(V_1(\tau))|\, d\tau \\ &\le \frac{4\sqrt{t}}{\sqrt{\pi}} L\|V_2 - V_1\|_t = C_4(L)\sqrt{t}\|V_2 - V_1\|_t. \end{aligned} \tag{2.30}$$

Furthermore,

$$|N(0,t,s_2(\tau),\tau) - N(0,t,s_1(\tau),\tau)| = |N_\xi(0,t,c,\tau)||s_2(\tau) - s_1(\tau)|$$

where $c = c(\tau)$ is between $s_2(\tau)$ and $s_1(\tau)$ and

$$\begin{aligned} |N_\xi(0,t,c,\tau)||s_2(\tau) - s_1(\tau)| &= |-G_x(0,t,c,\tau)||s_2(\tau) - s_1(\tau)| \\ &\le \frac{|c|}{2\sqrt{\pi}} (\frac{24}{eb^2})^{3/2} \tau |v_2(\tau) - v_1(\tau)| \\ &\le \frac{3b}{4\sqrt{\pi}} (\frac{24}{eb^2})^{3/2} \tau |v_2(\tau) - v_1(\tau)|. \end{aligned}$$

Then

$$\begin{aligned} &\int_0^t |N(0,t,s_2(\tau),\tau) - N(0,t,s_1(\tau),\tau)||F(V_1(\tau))|\, d\tau \\ &\le LM \frac{3b}{4\sqrt{\pi}} (\frac{24}{eb^2})^{3/2} \frac{t^2}{2} \|v_2 - v_1\|_t = C_5(L,M,b) t^2 \|v_2 - v_1\|_t \end{aligned} \tag{2.31}$$

Therefore, by (2.29), (2.30), and (2.31)), the inequality (2.27) holds. □

Theorem 2.6. *The map $A : C_{M,\sigma} \to C_{M,\sigma}$ is well defined and is a contraction map if σ satisfies the following inequalities:*

$$\sigma \le 1\,, 2M\sigma \le b \tag{2.32}$$

$$(2\|\dot f\|_\sigma C_1(b) + MC_3(b))\sigma + (2M^2 C_2(b) + \frac{2\|\dot f\|_\sigma}{\sqrt{\pi}} + 3MC_4(L))\sqrt{\sigma} \le 1 \tag{2.33}$$

$$D(b,f,h,L,M)\sqrt{\sigma} < 1, \tag{2.34}$$

where

$$M = 1 + 3\|h'\| \tag{2.35}$$

and

$$\begin{aligned} D_1(b,f,h,L,M) &= E(b,f,h)+2C_6(L,M)+3C_4(L) \\ D_2(b,L,M) &= 2[C_5(b,L,M)+2C_7(b,L,M)+C_8(b)] \\ D_3(b,L,M) &= C_9(b,M)+C_5(b,L,M) \\ D(b,f,h,L,M) &= D_1(b,f,h,L,M)+D_2(b,L,M)+D_3(b,L,M). \end{aligned}$$

Then there exists a unique solution on $C_{M,\sigma}$ to the system of integral equations (2.4)*, (2.5).*

Proof. Firstly we demonstrate that A maps $C_{\sigma,M}$ into itself, that is

$$\|A(\vec{w})\|_\sigma = \max_{t\in[0,\sigma]}|A_1(v(t),V(t))| + \max_{t\in[0,\sigma]}|A_2(v(t),V(t))| \le M \tag{2.36}$$

Using the Lemmas 2.3, 2.4 and the definitions (2.8)-(2.9), we have

$$\begin{gathered} |A_1(v(t),V(t))| \le 2\|\dot{f}\|_\sigma C_1(b)t + 2M^2C_2(b)\sqrt{t} + 2\|h'\| + 2C_4(L)M\sqrt{t}, \\ |A_2(v(t),V(t))| \le \|h'\| + (\frac{2\|\dot{f}\|_\sigma}{\sqrt{\pi}} + C_4(L)M)\sqrt{t} + C_3(b)Mt. \end{gathered}$$

Then

$$\begin{aligned} \|A(\vec{w})\|_\sigma &= \max_{t\in[0,\sigma]}|A_1(v(t),V(t))| + \max_{t\in[0,\sigma]}|A_2(v(t),V(t))| \\ &\le 3\|h'\| + (2\|\dot{f}\|_\sigma C_1(b) + C_3(b)M)\sigma \\ &\quad + \Big(2M^2C_2(b) + \frac{2\|\dot{f}\|_\sigma}{\sqrt{\pi}} + 3MC_4(L)\Big)\sqrt{\sigma}. \end{aligned}$$

Selecting M by (2.35) and σ such that (2.32) and (2.33) hold, we obtain (2.36).

Now, we prove that

$$\|A(\vec{w_2}) - A(\overrightarrow{w_1})\|_\sigma \le D(b,h,f,L,M)\sqrt{\sigma}\|\vec{w_2}-\vec{w_1}\|_\sigma$$

where $\vec{w_1} = \binom{v_1}{V_1}$, $\vec{w_2} = \binom{v_2}{V_2}$. By selecting σ such that (2.34) holds, A becomes a contraction mapping on $C_{\sigma,M}$ and therefore it has a unique fixed point. To prove this assertion we consider

$$A(\vec{w_1})(t) - A(\vec{w_2})(t) = \begin{pmatrix} A_1(v_2(t),V_2(t)) - A_1(v_1(t),V_1(t)) \\ A_2(v_2(t),V_2(t)) - A_2(v_1(t),V_1(t)) \end{pmatrix}$$

where

$$\begin{aligned} &A_1(v_2(t),V_2(t)) - A_1(v_1(t),V_1(t)) \\ &= F_0(v_2(t)) - F_0(v_1(t)) + 2\int_0^t [N(s_2(t),t,s_2(\tau),\tau) - N(s_2(t),t,0,\tau)]F(V_2(\tau))\,d\tau \\ &\quad - 2\int_0^t [N(s_1(t),t,s_1(\tau),\tau) - N(s_1(t),t,0,\tau)]F(V_1(\tau))\,d\tau \end{aligned}$$

and

$$
\begin{aligned}
&A_2(v_2(t),V_2(t)) - A_2(v_1(t),V_1(t))\\
&= \int_0^t [G_x(0,t,v_2(\tau),\tau)v_2(\tau) - G_x(0,t,v_1(\tau),\tau)v_1(\tau)]\,d\tau\\
&\quad + \int_0^t \big\{[N(0,t,s_2(\tau),\tau) - N(0,t,0,\tau)]F(V_2(\tau))\\
&\quad - [N(0,t,s_1(\tau),\tau) - N(0,t,0,\tau)]F(V_1(\tau))\big\}\,d\tau\,.
\end{aligned}
$$

Taking into account the Lemmas 2.4 and 2.5 it results

$$
\begin{aligned}
&|A_1(v_2(t),V_2(t)) - A_1(v_1(t),V_1(t))|\\
&\le E(b,h,f)\sqrt{t}\|v_2 - v_1\|_t + 2C_4(L)\sqrt{t}\|V_2 - V_1\|_t\\
&\quad + 2C_5(b,L,M)t\|v_2 - v_1\|_t + 2[C_6(L,M)\sqrt{t} + C_7(b,L,M)t]\|v_2 - v_1\|_t,
\end{aligned}
$$

and

$$
\begin{aligned}
&|A_2(v_2(t),V_2(t)) - A_2(v_1(t),V_1(t))|\\
&\le (C_8(b)t + C_9(b,M)t^2)\|v_2 - v_1\|_t\\
&\quad + C_4(L)\sqrt{t}\|V_2 - V_1\|_t + C_5(b,L,M)t^2\|v_2 - v_1\|_t\,.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\|A(\vec{w_2}) - A(\vec{w_1})\|_\sigma\\
&\le \max_{t\in[0,\sigma]} |A_1(v_2(t),V_2(t)) - A_1(v_1(t),V_1(t))|\\
&\quad + \max_{t\in[0,\sigma]} |A_2(v_2(t),V_2(t)) - A_2(v_1(t),V_1(t))|\\
&\le \{D_1(b,f,h,L,M)\sqrt{\sigma} + D_2(b,L,M)\sigma + D_3(b,L,M)\sigma^2\}\|\vec{w_2} - \vec{w_1}\|_\sigma\\
&\le D(b,f,h,L,M)\sqrt{\sigma}\|\vec{w_2} - \vec{w_1}\|_\sigma\,.
\end{aligned}
$$

By hypothesis (2.34) we have that A is a contraction. □

Remark. If F satisfies the conditions

(H2) $F(V) > 0$, for all $V \neq 0$ and $F(0) = 0$,

then by the maximum principle [5], u is a sub-solution for the same problem with $F \equiv 0$, that is

$$u(x,t) \le u_0(x,t)\,, \quad s(t) \le s_0(t)$$

where $u_0(x,t)$ and $s_0(t)$ solve the classical Stefan problem

$$
\begin{gathered}
u_{0t} - u_{0xx} = 0, \quad 0 < x < s_0(t),\ 0 < t < T,\\
u_0(0,t) = f(t) \geqslant 0, \quad 0 < t < T,\\
u_0(s_0(t),t) = 0, \quad u_{0x}(s_0(t),t) = -\dot{s_0}(t). 0 < t < T,\\
u_0(x,0) = h(x) \quad 0 \leqslant x \leqslant b = s_0(0)\,.
\end{gathered}
$$

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