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EXISTENCE OF MULTIPLE SOLUTIONS AND ESTIMATES OF EXTREMAL VALUES FOR A KIRCHHOFF TYPE PROBLEM WITH FAST INCREASING WEIGHT AND CRITICAL NONLINEARITY

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Abstract. In this article, we study the Kirchhoff type problem

$$-\Big(a+\epsilon\int_{\mathbb{R}^3} K(x)|\nabla u|^2dx\Big)\operatorname{div}(K(x)\nabla u)=\lambda K(x)f(x)|u|^{q-2}u+K(x)|u|^4u,$$

where $x\in\mathbb{R}^3$, $1<q<2$, $K(x)=\exp(|x|^\alpha/4)$ with $\alpha\geq 2$, $\epsilon>0$ is small enough, and the parameters $a,\lambda>0$. Under some assumptions on $f(x)$, we establish the existence of two nonnegative nontrivial solutions and obtain uniform lower estimates for extremal values of the problem via variational methods.

1. Introduction and statement of main results

In this article, we consider the existence of multiple solutions and estimates of extremal values for the Kirchhoff type problem

$$-\Big(a+\epsilon\int_{\mathbb{R}^3} K(x)|\nabla u|^2dx\Big)\operatorname{div}(K(x)\nabla u)=\lambda K(x)f(x)|u|^{q-2}u+K(x)|u|^4u, \tag{1.1}$$

where $x\in\mathbb{R}^3$, $1<q<2$, $K(x)=\exp(|x|^\alpha/4)$ with $\alpha\geq 2$, $\epsilon>0$ is small enough, the potential f has indefinite sign, and the parameters a,λ are positive.

It is commonly known that Kirchhoff type problems are presented by Kirchhoff in [11] as an extension of the classical d'Alembert wave equation for free vibrations of elastic strings. Kirchhoff type problems are often viewed as nonlocal because of the appearance of the term $\int K|\nabla u|^2dx$. This provokes some mathematical difficulties which make the study of such problems particularly interesting. When $K\equiv 1$, the general Kirchhoff type problem with critical exponent

$$-\Big(a+b\int_{\Omega}|\nabla u|^2dx\Big)\Delta u=\lambda f(x,u)+u^5,\quad x\in\Omega, \tag{1.2}$$

has been studied extensively. For the case $\Omega\subset\mathbb{R}^3$ is a bounded domain, some interesting works can be founded in [5, 12, 17, 22]. In particular, Sun and Liu [22] studied that $f(x,u)=u^q$, where $0<q<1$, and proved the existence of at least one

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positive solution when $0 < \lambda < T$ for some $T = T(a) > 0$. There are also several existence results for (1.2) on unbounded domain, that is $\Omega = \mathbb{R}^3$. For this case, we refer the interested readers to [1, 15, 16, 14, 24].

As pointed in [4, 9], one of the motivations for investigating problem (1.1) is that for $\alpha = q = 2, \epsilon = 0$ and $f(x) = 1/5$, (1.1) arises naturally when one tries to seek self-similar solutions of the form

$$w(t,x) = t^{-1/5}u(xt^{-1/2})$$

to the evolution equation

$$w_t - \Delta w = |w|^4 w \quad \text{on } (0,\infty)\times\mathbb{R}^3.$$

For a more detailed description, see [4, 9].

Recently, Furtado et al. [7] studied the equation

$$-\operatorname{div}(K\nabla u) = a(x)K|u|^{q-2}u + b(x)K|u|^{2^*-2}u, \quad x\in\mathbb{R}^N, \tag{1.3}$$

where $2^* = 2N/(N-2)$, $N \geq 3$, and $1 < q < 2$. Under certain assumptions on the potentials a and b, the authors obtained two nonnegative nontrivial solutions for (1.3). Subsequently, based on the result of [7], Qian and Chen [20] obtained another two sign-changing solutions of (1.3) with some slightly stronger conditions on a and b. More results of related problem, please see [2, 6, 8, 7, 18, 19] and the references therein.

We also mention that, by using minimization argument and Mountain Pass Theorem, Lei et al. [13] obtained two positive solutions for (1.1) when $K \equiv 1$ and $\mathbb{R}^3$ is replaced by a bounded domain $\Omega \subset \mathbb{R}^3$ and λ is sufficiently small. However, in [13], the authors did not show any information on estimates of extremal values for the problem, which is just our purpose here. More precisely, our aim in this paper is to prove the existence and multiplicity of nonnegative solutions of (1.1), and establish uniform lower estimates for extremal values.

Let H denote the Hilbert space obtained as the completion of $C_c^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\| = \Big(\int_{\mathbb{R}^3} K|\nabla u|^2 dx\Big)^{1/2}.$$

Define the weighted Lebesgue spaces

$$L_K^s(\mathbb{R}^3) = \big\{u \text{ measurable in } \mathbb{R}^3 : \int_{\mathbb{R}^3} K|u|^s dx < \infty\big\}$$

with the norm

$$\|u\|_s = \Big(\int_{\mathbb{R}^3} K|u|^s dx\Big)^{1/s}.$$

From [6], we know that the embedding $H \hookrightarrow L_K^r(\mathbb{R}^3)$ is continuous for $2 \leq r \leq 6$, and compact for $2 \leq r < 6$. This enables us to define for any $r \in [2,6]$

$$S_r = \inf\big\{\int_{\mathbb{R}^3} K|\nabla u|^2 dx : u \in H, \int_{\mathbb{R}^3} K|u|^r dx = 1\big\}. \tag{1.4}$$

In particular, if $r = 6$, we put $S = S_6$ for simplicity. It is worth mentioning that this constant is equal to the best constant of the embedding $\mathcal{D}^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, see [2]. For each $r > 1$, we shall denote by r' its Hölder conjugated exponent, namely $1/r + 1/r' = 1$. In this paper, we will always assume f satisfies:

(A1) $f \in L_K^{\sigma_q}(\mathbb{R}^3)$ for some $(2/q) \leq \sigma_q' < (6/q)$;

(A2) the set $\Omega_f^+ := \{x \in \mathbb{R}^3 : f > 0\}$ has an interior point.

By (A1) and the above embedding, it is easy to see the functional

$$I(u)=\frac{a}{2}\|u\|^2+\frac{\epsilon}{4}\|u\|^4-\frac{\lambda}{q}\int_{\mathbb{R}^3}Kf|u|^q dx-\frac{1}{6}\int_{\mathbb{R}^3}K|u|^6dx$$

is well defined on H and $I\in C^1(H,\mathbb{R})$. As we all know, there exists a one to one correspondence between the critical points and the weak solutions of (1.1). Here, we say $u\in H$ is a weak solution of (1.1), if for all $\phi\in H$, it holds

$$(a+\epsilon\|u\|^2)\int_{\mathbb{R}^3}K\nabla u\nabla\phi dx-\lambda\int_{\mathbb{R}^3}Kf|u|^{q-2}u\phi dx-\int_{\mathbb{R}^3}K|u|^4u\phi dx=0.$$

Define the Nehari-type set of (1.1),

$$\Lambda=\{u\in H:\langle I'(u),u\rangle=0\},$$

and then split Λ into three subsets of Λ,

$$\begin{gathered}\Lambda^0=\{u\in\Lambda:a(2-q)\|u\|^2+\epsilon(4-q)\|u\|^4-(6-q)\|u\|_6^6=0\},\\ \Lambda^+=\{u\in\Lambda:a(2-q)\|u\|^2+\epsilon(4-q)\|u\|^4-(6-q)\|u\|_6^6>0\},\\ \Lambda^-=\{u\in\Lambda:a(2-q)\|u\|^2+\epsilon(4-q)\|u\|^4-(6-q)\|u\|_6^6<0\}.\end{gathered}$$

Set

$$T_{q,f,a}=\frac{4}{6-q}\frac{1}{\|f\|_{\sigma_q}}\Big(\frac{2-q}{6-q}\Big)^{\frac{2-q}{4}}a^{\frac{6-q}{4}}S_{q\sigma_q'}^{q/2}S^{\frac{3(2-q)}{4}}.$$

Our main results are stated belows.

Theorem 1.1. *Assume that $a>0$, $1<q<2$ and $\epsilon>0$ is sufficiently small. Under assumptions* (A1) *and* (A2)*, if $\lambda\in(0,T_{q,f,a})$, then problem* (1.1) *has at least two nonnegative nontrivial solutions $u_*\in\Lambda^+$, $\tilde{u}_*\in\Lambda^-$ with $\|u_*\|<\|\tilde{u}_*\|$.*

Let

$$\lambda^*=\sup\big\{\lambda>0:(1.1)\text{ has at least two nonnegative nontrivial solutions}\big\}.$$

Then, as a consequence of Theorem 1.1, we have the following lower bound for λ^*.

Theorem 1.2. *Assume that $a>0$, $1<q<2$ and $\epsilon>0$ is sufficiently small. Under assumptions* (A1) *and* (A2)*, we have*

$$\lambda^*>T_{q,f,a}=\frac{4}{6-q}\frac{1}{\|f\|_{\sigma_q}}\Big(\frac{2-q}{6-q}\Big)^{\frac{2-q}{4}}a^{\frac{6-q}{4}}S_{q\sigma_q'}^{q/2}S^{\frac{3(2-q)}{4}}.$$

Inspired by [23, 21], we consider the following two minimization problems

$$c_1=\inf_{\Lambda^0\cup\Lambda^+}I,\quad c_2=\inf_{\Lambda^-}I$$

and expect to find two solutions, one in Λ^+ and one in Λ^-. As we now face the critical problem (1.1) in a unbounded domain, our main difficulty is to prove that the energy level belongs to the range where (PS) condition hold. Due to the presence of the term $\int K|\nabla u|^2dx$, the methods employed in [23, 21] cannot be directly used here. In fact, for the first solution, we may easily show that $c_1<0$ and then obtain compactness condition by standard argument with some modification. For the second one, we cannot proceed as in the preceding proof, since we can only provide that

$$c_2<I(u_*)+\frac{1}{3}\sqrt{a^3S^3},$$

where u_* is the first solution and $u_* \in \Lambda^+$. We also remark that the method used in [13] by letting λ sufficiently small do not apply here, since our aim is to establish uniform estimates of extremal values for λ. We overcome this new difficulty by developing some techniques applied in [10, 3].

This article is organized as follows. In the next section, we give some notation and preliminaries. Then we prove our main results in Section 3.

2. Notation and Preliminaries

Throughout this paper, we write $\int u$ instead of $\int_{\mathbb{R}^3} u(x)dx$. The dual space of a Hilbert space H will be denoted by H^{-1}. $B_r(x)$ denotes the ball centered at x with radius $r > 0$. Let $\to$ and $\rightharpoonup$ denote strong convergence and weak convergence, respectively. All limitations hold as $n \to \infty$ unless otherwise stated. C and C_i denote various positive constants whose values may vary from line to line.

For any $u \in H$ and $u \neq 0$, set

$$t_{\max} = \Big[\frac{(2-q)\|u\|^2}{(6-q)\int K|u|^6}\Big]^{1/4}.$$

Then, the following lemma holds.

Lemma 2.1. *Let $\lambda \in (0, T_{q,f,a})$. For any $u \in H$ and $u \neq 0$, there is a unique $t^+ = t^+(u) > t_{\max} > 0$ such that $t^+u \in \Lambda^-$ and $I(t^+u) = \max_{t \geq t_{\max}} I(tu)$. Moreover, if $\int Kf|u|^q > 0$, there is a unique $0 < t^- = t^-(u) < t_{\max}$ such that $t^-u \in \Lambda^+$ and $I(t^-u) = \min_{0 \leq t \leq t^+} I(tu)$.*

Proof. A simple calculation shows that

$$\frac{\partial I}{\partial t}(tu) = t^{q-1}\Big(t^{2-q}a\|u\|^2 + t^{4-q}\epsilon\|u\|^4 - \lambda\int Kf|u|^q - t^{6-q}\int K|u|^6\Big).$$

For any $u \in H$, $u \neq 0$, define

$$\psi(t) = t^{2-q}a\|u\|^2 + t^{4-q}\epsilon\|u\|^4 - t^{6-q}\int K|u|^6, \quad \text{for all } t > 0,$$

$$\psi_1(t) = t^{2-q}a\|u\|^2 - t^{6-q}\int K|u|^6, \quad \text{for all } t > 0.$$

Since $1 < q < 2$, it is easy to see that $\lim_{t\to 0^+} \psi_1(t) = 0$ and $\lim_{t\to+\infty} \psi_1(t) = -\infty$. Moreover, $\psi_1(t)$ is concave and achieves its maximum at the point $t_{\max}$. Now we notice that

$$\psi_1(t_{\max}) = \Big(\frac{4}{6-q}\Big)\Big(\frac{2-q}{6-q}\Big)^{\frac{2-q}{4}}\Big[\frac{(a\|u\|^2)^{6-q}}{(\int K|u|^6)^{2-q}}\Big]^{1/4}$$

and consequently from (1.4) we obtain

$$\psi(t_{\max}) \geq \psi_1(t_{\max}) \geq \Big(\frac{4}{6-q}\Big)\Big(\frac{2-q}{6-q}\Big)^{\frac{2-q}{4}} a^{\frac{6-q}{4}} S^{\frac{3(2-q)}{4}}\|u\|^q.$$

Therefore if $\int Kf|u|^q \leq 0$, then there is a unique $t^+ > t_{\max}$ such that $\psi(t^+) = \int Kf|u|^q$ and $\psi'(t^+) < 0$. Equivalently $t^+u \in \Lambda^-$ and $I(t^+u) \geq I(tu)$, for all $t \geq t_{max}$.

In the case $\int Kf|u|^q > 0$, using (A1) and Hölder's inequality, we deduce that for $\lambda \in (0, T_{q,f,a})$

$$\lambda\int Kf|u|^q \leq \lambda\|f\|_{\sigma_q}\|u\|^q_{q\sigma_q'} \leq \lambda\|f\|_{\sigma_q} S^{-q/2}_{q\sigma_q'}\|u\|^q < \psi(t_{\max}).$$

This implies that there are $t^+ > t_{\max} > t^- > 0$ such that

$$\psi(t^+) = \int Kf|u|^q = \psi(t^-),$$
$$\psi'(t^+) < 0 < \psi'(t^-).$$

That is, $t^+u \in \Lambda^-$ and $t^-u \in \Lambda^+$. Additionally, $I(t^+u) \geq I(tu)$, for all $t \geq t_{max}$ and $I(t^-u) \leq I(tu)$, for all $t \in [0, t^+]$. $\square$

Lemma 2.2. *If $\lambda \in (0, T_{q,f,a})$, then $\Lambda^0 = \{0\}$.*

Proof. Suppose to the contrary that there is $0 \neq w \in \Lambda^0$ such that $a(2-q)\|w\|^2 + \epsilon(4-q)\|w\|^4 - (6-q)\|w\|_6^6 = 0$. Since $w \in \Lambda^0 \subset \Lambda$, we have $-4a\|w\|^2 - 2\epsilon\|w\|^4 + \lambda(6-q)\int Kf|w|^q = 0$ and even further

$$\lambda \int Kf|w|^q = \frac{4a}{6-q}\|w\|^2 + \frac{2\epsilon}{6-q}\|w\|^4. \tag{2.1}$$

On the other hand, noticing that

$$a(2-q)\|w\|^2 < a(2-q)\|w\|^2 + \epsilon(4-q)\|w\|^4 = (6-q)\|w\|_6^6,$$

we obtain

$$\int K|w|^6 > \frac{2-q}{6-q}a\|w\|^2. \tag{2.2}$$

From this, (2.1) and (2.2), one has

$$\begin{aligned}
&\Big(\frac{4}{6-q}\Big)\Big(\frac{2-q}{6-q}\Big)^{\frac{2-q}{4}}\Big[\frac{(a\|w\|^2)^{6-q}}{(\int K|w|^6)^{2-q}}\Big]^{1/4} - \lambda\int Kf|w|^q \\
&\leq \Big(\frac{4}{6-q}\Big)\Big(\frac{2-q}{6-q}\Big)^{\frac{2-q}{4}}\Big[\frac{(a\|w\|^2)^4}{(\frac{2-q}{6-q})^{2-q}}\Big]^{1/4} - \frac{4a}{6-q}\|w\|^2 - \frac{2\epsilon}{6-q}\|w\|^4 \\
&= -\frac{2\epsilon}{6-q}\|w\|^4 < 0
\end{aligned}$$

which is impossible since $\lambda \in (0, T_{q,f,a})$. $\square$

Lemma 2.3. *Let $\lambda \in (0, T_{q,f,a})$, then there is a gap structure in Λ:*

$$\|\tilde{u}\| > A(0) > A(\lambda) > \|u\|, \quad \textit{for all } u \in \Lambda^+,\ \tilde{u} \in \Lambda^-,$$

where

$$A(0) = \Big(\frac{2-q}{6-q}\Big)^{1/4} a^{1/4} S^{3/4}, \quad A(\lambda) = \Big(\frac{6-q}{4a}\lambda\|f\|_{\sigma_q}\Big)^{\frac{1}{2-q}} S_{q\sigma_q'}^{\frac{-q}{2(2-q)}}.$$

Proof. For $\tilde{u} \in \Lambda^-$, one has

$$a(2-q)\|\tilde{u}\|^2 \leq a(2-q)\|\tilde{u}\|^2 + \epsilon(4-q)\|\tilde{u}\|^4 < (6-q)\int K|\tilde{u}|^6 \leq (6-q)S^{-3}\|\tilde{u}\|^6$$

which implies $\|\tilde{u}\| > A(0)$.

Similarly, for $u \in \Lambda^+$, from Hölder's inequality it follows that

$$4a\|u\|^2 \leq 4a\|u\|^2 + 2\epsilon\|u\|^4 < \lambda(6-q)\int Kf|u|^q \leq \lambda(6-q)\|f\|_{\sigma_q} S_{q\sigma_q'}^{-q/2}\|u\|^q$$

and consequently $\|u\| < A(\lambda)$.

It is easy to checked that $A(0) > A(\lambda)$, if $\lambda \in (0, T_{q,f,a})$. $\square$

Lemma 2.4. *Given $u \in \Lambda^{\pm}$, there exist $\rho_u > 0$ and a continuous function $g_{\rho_u} : B_{\rho_u}(0) \to \mathbb{R}^+$ defined for $w \in H$, $w \in B_{\rho_u}(0)$ satisfying*

$$g_{\rho_u}(0) = 1, \quad g_{\rho_u}(w)(u-w) \in \Lambda^{\pm},$$

$$\langle g'_{\rho_u}(0), \phi\rangle = \frac{(2a+4\epsilon\|u\|^2)\int K\nabla u\nabla\phi - 6\int K|u|^4 u\phi - q\lambda\int Kf|u|^{q-2}u\phi}{a(2-q)\|u\|^2 + \epsilon(4-q)\|u\|^4 - (6-q)\int K|u|^6}.$$

Proof. We only prove the case $u \in \Lambda^-$. The case $u \in \Lambda^+$ can be proved by a similar argument. Fix $u \in \Lambda^-$ and define $F : \mathbb{R}^+ \times H \to \mathbb{R}$ by

$$F(t,w) = t^{2-q}a\|u-w\|^2 + t^{4-q}\epsilon\|u-w\|^4 - t^{6-q}\int K|u-w|^6 - \lambda\int Kf|u-w|^q.$$

Since $u \in \Lambda^- \subset \Lambda$, we obtain $F(1,0) = 0$ and

$$F_t(1,0) = a(2-q)\|u\|^2 + \epsilon(4-q)\|u\|^4 - (6-q)\int K|u|^6 < 0.$$

Applying implicit function theorem for F at the point $(1,0)$, we can deduce that there exists $\bar\rho_u > 0$ such that for $w \in H$, $\|w\| < \bar\rho_u$, the equation $F(t,w) = 0$ has a unique continuous solution $t = g_{\rho_u}(w) > 0$ with $g_{\rho_u}(0) = 1$. Since $F(g_{\rho_u}(w), w) = 0$ for $w \in H$, $\|w\| < \bar\rho_u$, one gets

$$\begin{aligned}
&g_{\rho_u}^{2-q}(w)a\|u-w\|^2 + g_{\rho_u}^{4-q}(w)\epsilon\|u-w\|^4 - g_{\rho_u}^{6-q}(w)\|u-w\|_6^6 - \lambda\int Kf|u-w|^q \\
&= \big[a\|g_{\rho_u}(w)(u-w)\|^2 + \epsilon\|g_{\rho_u}(w)(u-w)\|^4 - \int K|g_{\rho_u}(w)(u-w)|^6 \\
&\quad - \lambda\int Kf|g_{\rho_u}(w)(u-w)|^q\big]/\big[g_{\rho_u}^q(w)\big] = 0;
\end{aligned}$$

that is, $g_{\rho_u}(w)(u-w) \in \Lambda$ for all $w \in H$ and $\|w\| < \bar\rho_u$. Since $F_t(1,0) < 0$ and

$$\begin{aligned}
F_t(g_{\rho_u}(w), w) &= a(2-q)g_{\rho_u}^{1-q}(w)\|u-w\|^2 + \epsilon(4-q)g_{\rho_u}^{3-q}(w)\|u-w\|^2 \\
&\quad - (6-q)g_{\rho_u}^{5-q}(w)\int K|u-w|^6 \\
&= \Big(a(2-q)\|g_{\rho_u}(w)(u-w)\|^2 + \epsilon(4-q)\|g_{\rho_u}(w)(u-w)\|^4 \\
&\quad - (6-q)\|g_{\rho_u}(w)(u-w)\|_6^6\Big)/g_{\rho_u}^{1+q}(w),
\end{aligned}$$

we can take $\rho_u > 0$ small enough ($\rho_u < \bar\rho_u$) such that for $w \in H$, $\|w\| < \rho_u$,

$$a(2-q)\|g_{\rho_u}(w)(u-w)\|^2 + \epsilon(4-q)\|g_{\rho_u}(w)(u-w)\|^4 - (6-q)\|g_{\rho_u}(w)(u-w)\|^6 < 0,$$

which yields $g_{\rho_u}(w)(u-w) \in \Lambda^-$, for all $w \in H$, $\|w\| < \rho_u$. Furthermore, for any $\phi \in H$, $r > 0$, one has

$$\begin{aligned}
&F(1, 0 + r\phi) - F(1,0) \\
&= a\|u - r\phi\|^2 + \epsilon\|u - r\phi\|^4 - \int K|u-r\phi|^6 - \int Kf|u-r\phi|^q \\
&\quad - a\|u\|^2 - \epsilon\|u\|^4 + \int K|u|^6 + \int Kf|u|^q \\
&= -a\int K(2r\nabla u\nabla\phi - r^2|\nabla\phi|^2) - \epsilon\Big[2\int K|\nabla u|^2\int K(2r\nabla u\nabla\phi - r^2|\nabla\phi|^2) \\
&\quad - \Big(\int K(2r\nabla u\nabla\phi - r^2|\nabla\phi|^2)\Big)^2\Big] - \int K\big(|u-r\phi|^6 - |u|^6\big)
\end{aligned}$$

$$- \int Kf\big(|u-r\phi|^q - |u|^q\big)$$

and hence

$$\begin{aligned}
&\langle F_w, \phi\rangle|_{t=1,w=0}\\
&= \lim_{r\to 0} \frac{F(1,0+r\phi)-F(1,0)}{r}\\
&= -(2a+4\epsilon\|u\|^2)\int K\nabla u\nabla\phi + 6\int K|u|^4u\phi + q\lambda\int Kf|u|^{q-2}u\phi.
\end{aligned}$$

Thus,

$$\begin{aligned}
\langle g'_{\rho_u}(0),\phi\rangle &= -\frac{\langle F_w,\phi\rangle}{F_t}\Big|_{t=1,w=0}\\
&= \frac{(2a+4\epsilon\|u\|^2)\int K\nabla u\nabla\phi - 6\int K|u|^4u\phi - q\lambda\int Kf|u|^{q-2}u\phi}{a(2-q)\|u\|^2+\epsilon(4-q)\|u\|^4-(6-q)\int K|u|^6}.
\end{aligned}$$

This completes the proof. □

Lemma 2.5. *Let $\lambda\in(0,T_{q,f,a})$. Then*

(i) *the functional I is coercive and bounded from below on Λ;*
(ii) *$c_1=\inf_{\Lambda^+\cup\Lambda^0} I=\inf_{\Lambda^+} I\in(-\infty,0)$.*

Proof. (i) For $u\in\Lambda$, it follows from Hölder's inequality and (1.4) that

$$\begin{aligned}
I(u) &= \frac{a}{2}\|u\|^2+\frac{\epsilon}{4}\|u\|^4-\frac{\lambda}{q}\int Kf|u|^q-\frac16\int K|u|^6\\
&= \frac{a}{3}\|u\|^2+\frac{\epsilon}{12}\|u\|^4-\lambda\frac{6-q}{6q}\int Kf|u|^q\\
&\ge \frac{a}{3}\|u\|^2+\frac{\epsilon}{12}\|u\|^4-\lambda\frac{6-q}{6q}\|f\|_{\sigma_q}S_{q\sigma'_q}^{-q/2}\|u\|^q.
\end{aligned}$$

Hence the coercivity and lower boundedness of I hold.

(ii) For $u\in\Lambda^+$, one has

$$\begin{aligned}
I(u) &= \frac{a}{2}\|u\|^2+\frac{\epsilon}{4}\|u\|^4-\frac{\lambda}{q}\int Kf|u|^q-\frac16\int K|u|^6\\
&= a\Big(\frac12-\frac1q\Big)\|u\|^2+\epsilon\Big(\frac14-\frac1q\Big)\|u\|^4+\Big(\frac1q-\frac16\Big)\int K|u|^6\\
&= -\frac{2-q}{2q}a\|u\|^2-\frac{4-q}{4q}\epsilon\|u\|^4+\frac{6-q}{6q}\int K|u|^6\\
&< \frac{-a(2-q)\|u\|^2-\epsilon(4-q)\|u\|^4+(6-q)\int K|u|^6}{6q}<0.
\end{aligned}$$

Combining this with Lemma 2.2, we obtain $\inf_{\Lambda^+\cup\Lambda^0} I=\inf_{\Lambda^+} I<0$. By (i), we can further obtain $\inf_{\Lambda^+\cup\Lambda^0} I\neq-\infty$. In conclusion, $\inf_{\Lambda^+\cup\Lambda^0} I\in(-\infty,0)$. □

Lemma 2.6. *Let $\lambda\in(0,T_{q,f,a})$, then $\Lambda^0\cup\Lambda^+$ and Λ^- are closed.*

Proof. Let $\{\tilde u_n\}$ be a sequence in Λ^- with $\tilde u_n\to\tilde u_0$ in H. Then it follows from $\{\tilde u_n\}\subset\Lambda^-\subset\Lambda$ that

$$a\|\tilde u_0\|^2+\epsilon\|\tilde u_0\|^4=\lim_{n\to\infty}[a\|\tilde u_n\|^2+\epsilon\|\tilde u_n\|^4]$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \Big[\lambda \int Kf|\tilde{u}_n|^q + \int K|\tilde{u}_n|^6\Big] \\
&= \lambda \int Kf|\tilde{u}_0|^q + \int K|\tilde{u}_0|^6
\end{aligned}
$$

and

$$
\begin{aligned}
& a(2-q)\|\tilde{u}_0\|^2 + \epsilon(4-q)\|\tilde{u}_0\|^4 - (6-q)\int K|\tilde{u}_0|^6 \\
&= \lim_{n\to\infty} \Big[a(2-q)\|\tilde{u}_n\|^2 + \epsilon(4-q)\|\tilde{u}_n\|^4 - (6-q)\int K|\tilde{u}_n|^6\Big] \le 0,
\end{aligned}
$$

that is, $\tilde{u}_0 \in \Lambda^- \cup \Lambda^0$. Moreover, for $\lambda \in (0, T_{q,f,a})$, we deduce from Lemmas 2.2 and 2.3 that $\tilde{u}_0 \notin \Lambda^0$. In turn, $\tilde{u}_0 \in \Lambda^-$ and hence, Λ^- is closed for $\lambda \in (0, T_{q,f,a})$. The same argument can show that $\Lambda^0 \cup \Lambda^+$ is closed and thus we complete the proof. □

3. Proof of main results

3.1. **Existence of the first solution.** Thanks to Lemmas 2.5 and 2.6, we can apply Ekeland variational principle to construct a minimizing sequence $\{u_n\} \subset \Lambda^+ \cup \Lambda^0$ with the following properties:

(1) $I(u_n) \to c_1$,
(2) $I(z) \ge I(u_n) - \frac{1}{n}\|u_n - z\|$ for all $z \in \Lambda^+ \cup \Lambda^0$.

Since $I(|u|) = I(u)$, we can assume that $u_n \ge 0$ on $\mathbb{R}^3$. Again using Lemma 2.5, it follows that $\{u_n\}$ is bounded in H, and thus we may assume

$$
\begin{gathered}
u_n \rightharpoonup u_* \quad \text{in } H, \\
u_n \to u_* \quad \text{in } L^r_K(\mathbb{R}^3),\ 2 \le r < 6, \\
u_n \to u_* \quad \text{a.e. on } \mathbb{R}^3.
\end{gathered}
$$

In what follows we prove that u_* is a nonnegative nontrivial solution of (1.1). The proof will be complete in five steps.

Step 1. $u_* \not\equiv 0$. If, to the contrary, that $u_* \equiv 0$. By the properties of $\{u_n\}$ and $\lim_{n\to\infty}\int Kf|u_n|^q = \int Kf|u_*|^q$ (see [7]), one has

$$
c_1 = I(u_n) + o(1) = \frac{a}{2}\|u_n\|^2 + \frac{\epsilon}{4}\|u_n\|^4 - \frac{1}{6}\int K|u_n|^6 + o(1). \tag{3.1}
$$

Noting that $u_n \in \Lambda^+$ for n large enough, we obtain

$$
a(2-q)\|u_n\|^2 + \epsilon(4-q)\|u_n\|^4 - (6-q)\int K|u_n|^6 > 0. \tag{3.2}
$$

In view of (3.1), (3.2) that $c_1 < 0$, we have

$$
0 < \frac{6(6-q) - 2(2-q)}{2} a\|u_n\|^2 + \frac{6(6-q) - 4(2-q)}{4}\epsilon\|u_n\|^4 < 6(6-q)c_1 + o(1) < 0
$$

which gives a contradiction. This completes the proof of step 1.

Step 2. There is a positive number $C_1 > 0$ such that

$$
4a\|u_n\|^2 + 2\epsilon\|u_n\|^4 - \lambda(6-q)\int Kf|u_n|^q < -C_1. \tag{3.3}
$$

Obviously, to prove (3.3), it suffices to show that

$$4a\liminf_{n\to\infty}\|u_n\|^2+2\epsilon\liminf_{n\to\infty}\|u_n\|^4-\lambda(6-q)\int Kf|u_*|^q<0. \tag{3.4}$$

Since $u_n\in\Lambda^+\cup\Lambda^0$, we have

$$4a\|u_n\|^2+2\epsilon\|u_n\|^4-\lambda(6-q)\int Kf|u_n|^q\le 0$$

and so

$$4a\liminf_{n\to\infty}\|u_n\|^2+2\epsilon\liminf_{n\to\infty}\|u_n\|^4\le\lambda(6-q)\int Kf|u_*|^q\,, \tag{3.5}$$

$$4a\limsup_{n\to\infty}\|u_n\|^2+2\epsilon\limsup_{n\to\infty}\|u_n\|^4\le\lambda(6-q)\int Kf|u_*|^q. \tag{3.6}$$

Combining (3.4) and (3.5), we can argue indirectly and suppose that

$$4a\liminf_{n\to\infty}\|u_n\|^2+2\epsilon\liminf_{n\to\infty}\|u_n\|^4=\lambda(6-q)\int Kf|u_*|^q.$$

This and (3.6) imply that there exists a positive constant $A_1>0$ such that $\|u_n\|^2\to A_1$, where A_1 satisfies

$$4aA_1+2\epsilon A_1^2=\lambda(6-q)\int Kf|u_*|^q. \tag{3.7}$$

Furthermore, it follows from $u_n\in\Lambda$ that

$$\int K|u_n|^6=a\|u_n\|^2+\epsilon\|u_n\|^4-\lambda\int Kf|u_n|^q\to\frac{2-q}{6-q}aA_1+\frac{4-q}{6-q}\epsilon A_1^2. \tag{3.8}$$

By (3.7) and (3.8), we obtain that if $\lambda\in(0,T_{q,f,a})$, then

$$\begin{aligned}
0&<\Big(\frac{4}{6-q}\Big)\Big(\frac{2-q}{6-q}\Big)^{\frac{2-q}{4}}\Big[\frac{(a\|u_n\|^2)^{6-q}}{(\int K|u_n|^6)^{2-q}}\Big]^{1/4}-\lambda\int Kf|u_n|^q\\
&\to\Big(\frac{4}{6-q}\Big)\Big(\frac{2-q}{6-q}\Big)^{\frac{2-q}{4}}\Big[\frac{(aA_1^2)^{6-q}}{\Big(\frac{2-q}{6-q}aA_1+\frac{4-q}{6-q}\epsilon A_1^2\Big)^{2-q}}\Big]^{1/4}-\frac{4a}{6-q}A_1+\frac{2\epsilon}{6-q}A_1^2\\
&<\Big(\frac{4}{6-q}\Big)\Big(\frac{2-q}{6-q}\Big)^{\frac{2-q}{4}}\Big[\frac{(aA_1^2)^{6-q}}{(\frac{2-q}{6-q}aA_1)^{2-q}}\Big]^{1/4}-\frac{4a}{6-q}A_1+\frac{2\epsilon}{6-q}A_1^2\\
&=-\frac{2\epsilon}{6-q}A_1^2<0
\end{aligned}$$

which leads to a contradiction. Thus, (3.3) holds and the proof of step 2 is complete.

Step 3. $I'(u_n)\to 0$ in H^{-1}. Let $0<\rho<\rho_n\equiv\rho_{u_n}$, $g_n\equiv g_{u_n}$, where ρ_{u_n} and g_{u_n} are defined according to Lemma 2.4. Let $v_\rho=\rho u$ with $\|u\|=1$. Fix n and let $z_\rho=g_n(v_\rho)(u_n-v_\rho)$. Since $z_\rho\in\Lambda^+$, by the properties of $\{u_n\}$,

$$I(z_\rho)-I(u_n)\ge-\frac{1}{n}\|z_\rho-u_n\|.$$

It then from Mean value Theorem it follows that

$$\langle I'(u_n),z_\rho-u_n\rangle+o(\|z_\rho-u_n\|)\ge-\frac{1}{n}\|z_\rho-u_n\|.$$

Thus,

$$\langle I'_\lambda(u_n), -v_\rho + \big(g_n(v_\rho)-1\big)(u_n - v_\rho)\rangle \geq -\frac{1}{n}\|z_\rho - u_n\| + o(\|z_\rho - u_n\|)$$

which yields

$$-\rho\langle I'(u_n), u\rangle + \big(g_n(v_\rho)-1\big)\langle I'_\lambda(u_n), u_n - v_\rho\rangle \geq -\frac{1}{n}\|z_\rho - u_n\| + o(\|z_\rho - u_n\|).$$

Therefore,

$$\langle I'(u_n), u\rangle \leq \frac{1}{n}\frac{\|z_\rho - u_n\|}{\rho} + \frac{o(\|z_\rho - u_n\|)}{\rho} + \frac{g_n(v_\rho)-1}{\rho}\langle I'_\lambda(u_n), u_n - v_\rho\rangle. \tag{3.9}$$

By step 2, Lemma 2.4 and the boundedness of $\{u_n\}$, it is easily verified that

$$\lim_{\rho\to 0}\frac{|g_n(v_\rho)-1|}{\rho} \leq \|g'_n(0)\| \leq C$$

for some positive constant $C > 0$, independent of n. For fixed n, since $\|z_\rho - u_n\| \leq \rho + |g_n(v_\rho) - 1|C_2$, $\langle I'(u_n), u_n\rangle = 0$ and $(u_n - v_\rho) \to u_n$ as $\rho \to 0$, by letting $\rho \to 0$ in (3.9) we can deduce that

$$\langle I'(u_n), u\rangle \leq \frac{C}{n},$$

which shows that $I'(u_n) \to 0$. This completes the proof of step 3.

Step 4. $u_n \to u_*$ in H. Write $v_n = u_n - u_*$ and we claim that $\|v_n\| \to 0$. Otherwise, up to a subsequence (still denoted by $\{v_n\}$), we may suppose $\|v_n\| \to l$ with $l > 0$. From step 3, we have that $\langle I'(u_n), u_*\rangle = o(1)$ and hence

$$0 = a\|u_*\|^2 + \epsilon(l^2 + \|u_*\|^2)\|u_*\|^2 - \lambda\int Kf|u_*|^q - \int K|u_*|^6. \tag{3.10}$$

Moreover, by $\langle I'(u_n), u_n\rangle = 0$, we can use Brezis-Lieb Lemma to obtain

$$\begin{aligned} 0 &= a(\|v_n\|^2 + \|u_*\|^2) + \epsilon(\|v_n\|^4 + 2\|v_n\|^2\|u_*\|^2 + \|u_*\|^4) \\ &\quad - \lambda\int Kf|u_*|^q - \int K|v_n|^6 - \int K|u_*|^6 + o(1). \end{aligned} \tag{3.11}$$

Combining (3.10) and (3.11), we obtain

$$o(1) = a\|v_n\|^2 + \epsilon\|v_n\|^4 + \epsilon\|v_n\|^2\|u_*\|^2 - \int K|v_n|^6 \tag{3.12}$$

and so, from (1.4) it follows that

$$a\|v_n\|^2 \leq a\|v_n\|^2 + \epsilon\|v_n\|^4 + \epsilon\|v_n\|^2\|u_*\|^2 = \int K|v_n|^6 + o(1) \leq S^{-3}\|v_n\|^6 + o(1).$$

Taking the limit as $n \to \infty$, we obtain that

$$l^2 \geq \sqrt{aS^3}. \tag{3.13}$$

By (3.10) and Hölder's inequality, we obtain

$$
\begin{aligned}
I(u_*) &= \frac{a}{2}\|u_*\|^2 + \frac{\epsilon}{4}\|u_*\|^4 - \frac{\lambda}{q}\int Kf|u_*|^q - \frac{1}{6}\int K|u_*|^6 \\
&= \frac{a}{3}\|u_*\|^2 + \frac{\epsilon}{12}\|u_*\|^4 - \lambda\frac{6-q}{6q}\int Kf|u_*|^q - \frac{\epsilon}{6}l^2\|u_*\|^2 \\
&\geq \frac{a}{3}\|u_*\|^2 - \lambda\frac{6-q}{6q}\|f\|_{\sigma_q}S_{q\sigma_q'}^{-q/2}\|u_*\|^q - \frac{\epsilon}{6}l^2\|u_*\|^2 \\
&\geq -\frac{a(2-q)}{3q}\Big[\frac{(6-q)\|f\|_{\sigma_q}S_{q\sigma_q'}^{-q/2}}{4a}\Big]^{\frac{2}{2-q}}\lambda^{\frac{2}{2-q}} - \frac{\epsilon}{6}l^2\|u_*\|^2.
\end{aligned}
\tag{3.14}
$$

Set

$$
\bar{T}_{q,f,a} = \frac{4}{6-q}\frac{1}{\|f\|_{\sigma_q}}\Big(\frac{q}{2-q}\Big)^{\frac{2-q}{2}} a^{\frac{6-q}{4}} S_{q\sigma_q'}^{q/2} S^{\frac{3(2-q)}{4}}.
$$

Obviously, $T_{q,f,a} < \bar{T}_{q,f,a}$. If $\lambda \in (0, \bar{T}_{q,f,a})$, using (3.12)-(3.14), we deduce

$$
\begin{aligned}
c_1 + o(1) &= I(u_n) \\
&= \frac{a}{2}\|u_n\|^2 + \frac{\epsilon}{4}\|u_n\|^4 - \frac{\lambda}{q}\int Kf|u_n|^q - \frac{1}{6}\int K|u_n|^6 \\
&= \frac{a}{2}\|u_*\|^2 + \frac{\epsilon}{4}\|u_*\|^4 - \frac{\lambda}{q}\int Kf|u_*|^q - \frac{1}{6}\int K|u_*|^6 \\
&\quad + \frac{a}{2}\|v_n\|^2 + \frac{\epsilon}{4}\|v_n\|^4 + \frac{\epsilon}{2}\|v_n\|^2\|u_*\|^2 - \frac{1}{6}\int K|v_n|^6 + o(1) \\
&= I(u_*) + \frac{a}{2}\|v_n\|^2 + \frac{\epsilon}{4}\|v_n\|^4 + \frac{\epsilon}{2}\|v_n\|^2\|u_*\|^2 - \frac{1}{6}\int K|v_n|^6 + o(1) \\
&= I(u_*) + \frac{a}{3}\|v_n\|^2 + \frac{\epsilon}{12}\|v_n\|^4 + \frac{\epsilon}{3}\|v_n\|^2\|u_*\|^2 + o(1) \\
&\geq I(u_*) + \frac{a}{3}l^2 + \frac{\epsilon}{6}l^2\|u_*\|^2 + o(1) \\
&\geq I(u_*) + \frac{\sqrt{a^3S^3}}{3} + \frac{\epsilon}{6}l^2\|u_*\|^2 + o(1) > 0
\end{aligned}
$$

contradicting Lemma 2.5. Thus, the claim follows, that is, $u_n \to u_*$ in H. This finishes the proof of step 4.

Step 5. u_* is a nonnegative nontrivial solution of (1.1) and $u_* \in \Lambda^+$. By steps 3 and 4, we have that $\langle I'(u_*), \phi\rangle = 0$ for all $\phi \in H$. That is, u_* is a solution of (1.1). Hence, $u_* \in \Lambda$. From Lemma 2.3, it follows that

$$
\|u_*\| \leq \liminf_{n\to\infty}\|u_n\| < A(\lambda)
$$

and so $u_* \notin \Lambda^-$. By Lemma 2.2 and step 1, we obtain $u_* \notin \Lambda^0$. In turn, we conclude $u_* \in \Lambda^+$. Moreover, we have $u_* \geq 0$ since $u_n \geq 0$ and $u_n \to u_*$ a.e. on $\mathbb{R}^3$.

3.2. **Existence of a second solution.** In this section, we will establish the existence of a second solution in Λ^-. By condition (A2), we may suppose that $0 \in int(\Omega_f^+)$. Moreover, there exists $\eta > 0$ satisfying $B_{2\eta}(0) \subset \Omega_f^+$. Let $\varphi(x) \in C_0^\infty(R^3)$ be a cut-off function satisfying $\varphi(x) \equiv 1$ in $B_\eta(0)$, $\varphi(x) \equiv 0$ outside $B_{2\eta}(0)$ and $0 \leq \varphi \leq 1$. Define

$$
u_\varepsilon(x) = K^{-1/2}\varphi(x)\Big(\frac{1}{\varepsilon + |x|^2}\Big)^{1/2},
$$

and set

$$v_\varepsilon(x) = \frac{u_\varepsilon(x)}{\|u_\varepsilon\|_6}.$$

Lemma 3.1. *Let $\lambda \in (0, T_{q,f,a})$ and w_1 be a nonnegative nontrivial solution of (1.1). Then for $\varepsilon > 0$ small we have:*

(i)

$$\int K f w_1^{q-1} v_\varepsilon = \begin{cases} O\big(\varepsilon^{1/4}\big), & \text{if } 1 < \sigma_q' < 3, \\ O\big(\varepsilon^{\frac{3}{2\sigma_q'} - \frac{1}{4}} |\ln \varepsilon|^{1/\sigma_q'}\big), & \text{if } \sigma_q' = 3, \\ O\big(\varepsilon^{\frac{3}{2\sigma_q'} - \frac{1}{4}}\big), & \text{if } 3 < \sigma_q' < 6, \end{cases}$$

(ii) $\int K |\nabla v_\varepsilon|^2 = S + O\big(\varepsilon^{1/2}\big)$,

(iii) $\int K w_1^5 v_\varepsilon = O\big(\varepsilon^{1/4}\big)$,

(iv) $\int K v_\varepsilon^5 \geq C\varepsilon^{1/4} + O(\varepsilon^{5/4})$.

Proof. For the proofs of (ii) and (iii), please see [2, 20]. To prove (i), we refer to [20, Lemma 2.2] for the following estimates

$$\int K v_\varepsilon^r = \begin{cases} O\big(\varepsilon^{r/4}\big), & \text{if } 1 < r < 3, \\ O\big(\varepsilon^{3/2} - r/4 |\ln \varepsilon|\big), & \text{if } r = 3, \\ O\big(\varepsilon^{3/2 - r/4}\big), & \text{if } 3 < r < 6. \end{cases}$$

Hölder's inequality provides

$$\int K f w_1^{q-1} v_\varepsilon \leq C \|f\|_{\sigma_q} \Big(\int K |v_\varepsilon|^{\sigma_q'} \Big)^{1/\sigma_q'} \leq C_1 \Big(\int K |v_\varepsilon|^{\sigma_q'} \Big)^{1/\sigma_q'}.$$

This and the above estimates imply that (i) holds .

For the proof of (iv), we have

$$\begin{aligned} \int K |u_\varepsilon|^5 &= \int \frac{K K^{-5/2} \varphi^5(x)}{(\varepsilon + |x|^2)^{5/2}} \\ &\geq C \int \frac{\varphi^5(x)}{(\varepsilon + |x|^2)^{5/2}} \\ &= C\Big(\int \frac{1}{(\varepsilon + |x|^2)^{5/2}} + \int \frac{\varphi^5(x) - 1}{(\varepsilon + |x|^2)^{5/2}} \Big) \\ &= C\Big(\varepsilon^{-1} \int \frac{1}{(1 + |x|^2)^{5/2}} + \int \frac{\varphi^5(x) - 1}{(\varepsilon + |x|^2)^{5/2}} \Big) \\ &= C_1 \varepsilon^{-1} + O(1). \end{aligned}$$

According to [2], we have

$$\|u_\varepsilon\|_6^6 = \int K |u_\varepsilon|^6 = \varepsilon^{-3/2} B_0 + O(1)$$

where

$$B_0 = \int \frac{1}{(1 + |x|^2)^3},$$

and thus

$$\int K |v_\varepsilon|^5 = \frac{\int K |u_\varepsilon|^5}{\|u_\varepsilon\|_6^5} \geq \frac{C_1 \varepsilon^{-1} + O(1)}{C_2 \varepsilon^{-5/4} + O(\varepsilon^{1/4})} = C_3 \varepsilon^{1/4} + O\big(\varepsilon^{5/4}\big).$$

Hence, (iv) follows. This completes the proof of Lemma 3.1. $\square$

Set $E_1 = \{u : u = 0 \text{ or } \|u\| < t^+(u/\|u\|)\}$ and $E_2 = \{u : \|u\| > t^+(u/\|u\|)\}$. Obviously, $H - \Lambda = E_1 \cup E_1$ and $\Lambda^+ \subset E_1$.

Lemma 3.2. *Let $\lambda \in (0, T_{q,f,a})$, then we have*

$$c_2 \leq \sup_{t>0} I(u_* + tv_\varepsilon) < I(u_*) + \frac{1}{3}\sqrt{a^3S^3}.$$

where u_* is the first solution obtained in section 3.1.

Proof. First, we prove that $c_2 \leq \sup_{t>0} I(u_* + tv_\varepsilon)$. Let $R > 1$ and set $w_\varepsilon = u_* + Rv_\varepsilon$. Since u_* is a solution of (1.1), it follows from Lemma 3.1 that

$$\begin{aligned}
\|w_\varepsilon\|^2 &= \|u_*\|^2 + 2R\int K\nabla u_*\nabla v_\varepsilon + R^2\int K|\nabla v_\varepsilon|^2 \\
&= \|u_*\|^2 + 2R\Big(\lambda\int Kfu_*^{q-1}v_\varepsilon + \int Ku_*^5v_\varepsilon \\
&\quad - \varepsilon\int K|\nabla u_*|^2\int K\nabla u_*\nabla v_\varepsilon\Big) + R^2\int K|\nabla v_\varepsilon|^2 \\
&= \|u_*\|^2 + \xi(\varepsilon) + O(\varepsilon^{1/4}) + O(\varepsilon) + R^2\big(S + O(\varepsilon^{1/2})\big),
\end{aligned} \tag{3.15}$$

where

$$\xi(\varepsilon) = \begin{cases} O(\varepsilon^{1/4}), & \text{if } 1 < \sigma_q' < 3, \\ O\big(\varepsilon^{\frac{3}{2\sigma_q'}-\frac{1}{4}}|\ln\varepsilon|^{1/\sigma'_q}\big), & \text{if } \sigma_q' = 3, \\ O\big(\varepsilon^{\frac{3}{2\sigma_q'}-\frac{1}{4}}\big), & \text{if } 3 < \sigma_q' < 6. \end{cases}$$

Note that there exists $C > 0$ such that

$$(r+s)^6 - r^6 - s^6 - 6r^5s \geq Crs^5. \tag{3.16}$$

Thus, we obtain

$$\int K|w_\varepsilon|^6 \geq \int K|u_*|^6 + R^6\int K|v_\varepsilon|^6 + 6R\int Ku_*^5v_\varepsilon + CR^5\int Ku_*v_\varepsilon^5. \tag{3.17}$$

We distinguish two cases. In the case $R^2S < \|u_*\|^2$, using (3.15) and (3.17) we obtain for ε sufficiently small,

$$\|w_\varepsilon\|^2 = \int K|\nabla(u_* + Rv_\varepsilon)|^2 \leq 2\|u_*\|^2 + \xi(\varepsilon) \leq 2\|u_*\|^2 + 1,$$
$$\int K|w_\varepsilon|^6 \geq \int K|u_*|^6$$

respectively, which yield

$$\int K|\frac{w_\varepsilon}{\|w_\varepsilon\|}|^6 \geq \frac{\|u_*\|_6^6}{\big(2\|u_*\|^2 + 1\big)^3}.$$

In case the $R^2S \geq \|u_*\|^2$, by using $\|v_\varepsilon\|_6^6 = 1$ and (3.15) and (3.17), we obtain for ε small enough

$$\int K|\nabla(u_* + Rv_\varepsilon)|^2 \leq 2R^2S + \xi(\varepsilon) \leq R^2(2S+1),$$
$$\int K|w_\varepsilon|^6 \geq R^6\int K|v_\varepsilon|^6 = R^6,$$

which yield

$$\int K|\frac{w_\varepsilon}{\|w_\varepsilon\|}|^6 \geq \frac{R^6}{R^6\big(2S+1\big)^3} = \frac{1}{\big(2S+1\big)^3}.$$

In conclusion, there is ε_0 small enough such that

$$\int K|\frac{w_\varepsilon}{\|w_\varepsilon\|}|^6 \geq \min\Big\{\frac{\|u_*\|_6^6}{\big(2\|u_*\|^2+1\big)^3}, \frac{1}{\big(2S+1\big)^3}\Big\}, \quad \text{for all } \varepsilon \in (0,\varepsilon_0). \tag{3.18}$$

Set $t_\varepsilon = t^+\big(\frac{w_\varepsilon}{\|w_\varepsilon\|}\big)$ and let $\psi(t)$ be defined as in Lemma 2.1. To proceed, take $u = \frac{w_\varepsilon}{\|w_\varepsilon\|}$ in the definition of $\psi(t)$. By (3.18) and the structure of I, it is easy to see that t_ε is bounded from above. Namely, $t^+\big(\frac{w_\varepsilon}{\|w_\varepsilon\|}\big) \leq C_2$, for all $\varepsilon \in (0,\varepsilon_0)$. Hence there exists $\varepsilon_1 > 0$ sufficiently small ($\varepsilon_1 < \varepsilon_0$) such that

$$\|w_\varepsilon\|^2 \geq \|u_*\|^2 + \frac{SR^2}{2}, \quad \text{for all } \varepsilon \in (0,\varepsilon_1)$$

and thus there is $R_0 > 1$ satisfying

$$\|w_\varepsilon\|^2 > C_2 \geq t^+\big(\frac{w_\varepsilon}{\|w_\varepsilon\|}\big), \quad \text{for all } \varepsilon \in (0,\varepsilon_1), R \geq R_0,$$

which implies $u_* + R_0 v_\varepsilon \in E_2$. Since $u_* \in E_1$ and $H - \Lambda^- = E_1 \cup E_2$, by the continuity of $t^+(u)$ we conclude that $u_* + tR_0 v_\varepsilon$ for $t \in (0,1)$ must intersect Λ^- and so

$$c_2 \leq \sup_{t>0} I(u_* + tv_\varepsilon).$$

It remains to prove that

$$\sup_{t>0} I(u_* + tv_\varepsilon) < I(u_*) + \frac{1}{3}\sqrt{a^3S^3}.$$

By Hölder's inequality, (3.16), Lemma 2.3 and the fact u_* is a solution, one gets

$$\begin{aligned}
&I(u_* + tv_\varepsilon)\\
&= \frac{a}{2}\|u_* + tv_\varepsilon\|^2 + \frac{\epsilon}{4}\|u_* + tv_\varepsilon\|^4 - \frac{1}{6}\|u_* + tv_\varepsilon\|_6^6 - \frac{\lambda}{q}\int Kf|u_* + tv_\varepsilon|^q\\
&= \frac{a}{2}\|u_*\|^2 + at\int K\nabla u_*\nabla v_\varepsilon + \frac{a}{2}t^2\|v_\varepsilon\|^2 + \frac{\epsilon}{4}\|u_*\|^4 + \epsilon t^2(\int K\nabla u_*\nabla v_\varepsilon)^2\\
&\quad + \frac{\epsilon}{4}t^4\|v_\varepsilon\|^4 + \varepsilon t\|u_*\|^2\int K\nabla u_*\nabla v_\varepsilon + \frac{\epsilon}{2}t^2\|u_*\|^2\|v_\varepsilon\|^2 + \epsilon t^3\|v_\varepsilon\|^2\int K\nabla u_*\nabla v_\varepsilon\\
&\quad - \frac{1}{6}\|u_* + tv_\varepsilon\|_6^6 - \frac{\lambda}{q}\int Kf|u_* + tv_\varepsilon|^q\\
&= I(u_*) + \frac{a}{2}t^2\|v_\varepsilon\|^2 + \epsilon t^2\Big(\int K\nabla u_*\nabla v_\varepsilon\Big)^2 + \frac{\epsilon}{4}t^4\|v_\varepsilon\|^4 + \frac{\epsilon}{2}t^2\|u_*\|^2\|v_\varepsilon\|^2\\
&\quad + \epsilon t^3\|v_\varepsilon\|^2\int K\nabla u_*\nabla v_\varepsilon - \frac{1}{6}\int K[|u_* + tv_\varepsilon|^6 - |u_*|^6 - 6tu_*^5 v_\varepsilon]\\
&\quad - \frac{\lambda}{q}\int Kf[|u_* + tv_\varepsilon|^q - |u_*|^q - qtu_*^{q-1}v_\varepsilon]\\
&\leq I(u_*) + \frac{a}{2}t^2\|v_\varepsilon\|^2 + \epsilon t^2\|u_*\|^2\|v_\varepsilon\|^2 + \frac{\epsilon}{4}t^4\|v_\varepsilon\|^4 + \frac{\epsilon}{2}t^2\|u_*\|^2\|v_\varepsilon\|^2\\
&\quad + \epsilon t^3\|v_\varepsilon\|^2\|u_*\|\|v_\varepsilon\| - \frac{1}{6}\int K[|u_* + tv_\varepsilon|^6 - |u_*|^6 - 6tu_*^5 v_\varepsilon]
\end{aligned}$$

$$
\begin{aligned}
&\le I(u_*) + \frac{a}{2}t^2\|v_\varepsilon\|^2 + \frac{3}{2}\epsilon t^2\|u_*\|^2\|v_\varepsilon\|^2 + \frac{\epsilon}{4}t^4\|v_\varepsilon\|^4 + \epsilon t^3\|v_\varepsilon\|^2\|u_*\|\|v_\varepsilon\| \\
&\quad - \frac{t^6}{6}\int K|v_\varepsilon|^6 - \frac{3}{6}t^5\int Ku_*|v_\varepsilon|^5 \\
&\le I(u_*) + \frac{a}{2}t^2\|v_\varepsilon\|^2 + \frac{3}{2}\epsilon t^2 A(\lambda)^2\|v_\varepsilon\|^2 + \frac{\epsilon}{4}t^4\|v_\varepsilon\|^4 + \epsilon t^3 A(\lambda)\|v_\varepsilon\|^3 \\
&\quad - \frac{t^6}{6} - C_4 t^5\int K|v_\varepsilon|^5.
\end{aligned}
$$

Let

$$
\begin{aligned}
h(t) = \frac{a}{2}t^2\|v_\varepsilon\|^2 + \frac{3}{2}\epsilon t^2 A(\lambda)^2\|v_\varepsilon\|^2 + \frac{\epsilon}{4}t^4\|v_\varepsilon\|^4 + \epsilon t^3 A(\lambda)\|v_\varepsilon\|^3 \\
- \frac{t^6}{6} - C_4 t^5\int K|v_\varepsilon|^5.
\end{aligned}
$$

Clearly, $\lim_{t\to 0^+} h(t) = 0$ and $\lim_{t\to+\infty} h(t) = -\infty$. Note that $I(u_* + tv_\varepsilon) \le I(u_*) + h(t)$. Thus, to complete the proof of Lemma 3.2, it suffices to prove that

$$
\sup_{t_1\le t\le t_2} I(u_* + tv_\varepsilon) < I(u_*) + \frac{1}{3}\sqrt{a^3S^3}
$$

for some $0 < t_1 < t_2 < \infty$. Let $\epsilon = \varepsilon$, by Lemma 3.1, for ε sufficiently small we have

$$
\begin{aligned}
\sup_{t_1\le t\le t_2} I(u_* + tv_\varepsilon) &\le I(u_*) + \sup_{t>0}\big\{\frac{a}{2}t^2\|v_\varepsilon\|^2 - \frac{t^6}{6}\big\} + \frac{3}{2}\varepsilon t_2^2 A(\lambda)^2\|v_\varepsilon\|^2 \\
&\quad + \frac{\varepsilon}{4}t_2^4\|v_\varepsilon\|^4 + \varepsilon t_2^3 A(\lambda)\|v_\varepsilon\|^3 - C_4 t_1^5\int K|v_\varepsilon|^5 \\
&= I(u_*) + \frac{1}{3}\sqrt{a^3}\|v_\varepsilon\|^3 + O(\varepsilon) - C_5\varepsilon^{1/4} + O(\varepsilon^{\frac{5}{4}}) \\
&= I(u_*) + \frac{1}{3}\sqrt{a^3S^3} + O(\varepsilon^{1/2}) + O(\varepsilon) - C_5\varepsilon^{1/4} + O(\varepsilon^{5/4}) \\
&< I(u_*) + \frac{1}{3}\sqrt{a^3S^3}.
\end{aligned}
$$

The proof of Lemma 3.2 is complete. □

As in section 3.1, we may use once more Ekeland variational principle to establish another minimizing sequence $\{\tilde{u}_n\} \subset \Lambda^-$ such that

(1) $I(\tilde{u}_n) \to c_2$,
(2) $I(z) \ge I(\tilde{u}_n) - \frac{1}{n}\|\tilde{u}_n - z\|$ for all $z \in \Lambda^-$.

Since $I(|u|) = I(u)$, we can assume that $\tilde{u}_n \ge 0$ on $\mathbb{R}^3$. By Lemma 2.5, it follows that $\{\tilde{u}_n\}$ is bounded in H, and thus we suppose that

$$
\begin{gathered}
\tilde{u}_n \rightharpoonup \tilde{u}_* \quad \text{in } H, \\
\tilde{u}_n \to \tilde{u}_* \quad \text{in } L_K^r(\mathbb{R}^3), \quad 2 \le r < 6, \\
\tilde{u}_n \to \tilde{u}_* \quad \text{a.e. on } \mathbb{R}^3.
\end{gathered}
$$

With the previous preparations, we can now prove that $\tilde{u}_*$ is a nonnegative nontrivial solution for (1.1) and $\tilde{u}_* \in \Lambda^-$. To this purpose, we divide the argument in five steps.

1. $\tilde{u}_* \not\equiv 0$. Suppose to the contrary that $\tilde{u}_* \equiv 0$. Since $\tilde{u}_n \in \Lambda^- \subset \Lambda$, one gets

$$a\|\tilde{u}_n\|^2 + \epsilon\|\tilde{u}_n\|^4 - \lambda \int Kf|\tilde{u}_n|^q - \int K|\tilde{u}_n|^6 = 0.$$

As $\tilde{u}_* \equiv 0$, we have $\int Kf|\tilde{u}_n|^q = o(1)$. It then follows from (1.4) that

$$a\|\tilde{u}_n\|^2 \le a\|\tilde{u}_n\|^2 + \epsilon\|\tilde{u}_n\|^4 = \int K|\tilde{u}_n|^6 + o(1) \le S^{-3}\|\tilde{u}_n\|^6 + o(1). \tag{3.19}$$

Assume that $\|\tilde{u}_n\|^2 \to \iota^2$. Since $\{\tilde{u}_n\} \subset \Lambda^-$, from Lemma 2.3 we have that $\iota^2 > 0$. Letting $n \to \infty$ in (3.19), we have $\iota^2 \ge \sqrt{aS^3}$ and so

$$\begin{aligned}
c_2 &= \lim_{n\to\infty} I(\tilde{u}_n) \\
&= \lim_{n\to\infty} \Big[\frac{a}{2}\|\tilde{u}_n\|^2 + \frac{\epsilon}{4}\|\tilde{u}_n\|^4 - \frac{\lambda}{q}\int Kf|\tilde{u}_n|^q - \frac{1}{6}\int K|\tilde{u}_n|^6\Big] \\
&= \lim_{n\to\infty} \Big[\frac{a}{3}\|\tilde{u}_n\|^2 + \frac{\epsilon}{12}\|\tilde{u}_n\|^4 - \lambda\frac{6-q}{6q}\int Kf|\tilde{u}_n|^q\Big] \\
&= \frac{a}{3}\iota^2 + \frac{\epsilon}{12}\iota^4 \\
&\ge \frac{a}{3}\iota^2 \ge \frac{1}{3}\sqrt{a^3S^3}
\end{aligned}$$

which is a contradiction to Lemma 3.2. This completes the proof of step 1.

2. There is a positive constant $C_6 > 0$ such that

$$4a\|\tilde{u}_n\|^2 + 2\epsilon\|\tilde{u}_n\|^4 - \lambda(6-q)\int Kf|\tilde{u}_n|^q > C_6. \tag{3.20}$$

Clearly, to prove (3.20), it suffices to show that

$$4a\liminf_{n\to\infty}\|\tilde{u}_n\|^2 + 2\epsilon\liminf_{n\to\infty}\|\tilde{u}_n\|^4 - \lambda(6-q)\int Kf|\tilde{u}_*|^q > 0. \tag{3.21}$$

From $\tilde{u}_n \in \Lambda^-$, we obtain

$$4a\|\tilde{u}_n\|^2 + 2\epsilon\|\tilde{u}_n\|^4 - \lambda(6-q)\int Kf|\tilde{u}_n|^q > 0. \tag{3.22}$$

Arguing by contradiction we suppose that

$$4a\liminf_{n\to\infty}\|\tilde{u}_n\|^2 + 2\epsilon\liminf_{n\to\infty}\|\tilde{u}_n\|^4 = \lambda(6-q)\int Kf|\tilde{u}_*|^q.$$

This and Lemma 2.3 imply that $\int Kf|\tilde{u}_*|^q > 0$. Consequently,

$$\liminf_{n\to\infty}\Big[\frac{4a\|\tilde{u}_n\|^2 + 2\epsilon\|\tilde{u}_n\|^4}{\lambda(6-q)\int Kf|\tilde{u}_n|^q}\Big] = \frac{\liminf_{n\to\infty}[4a\|\tilde{u}_n\|^2 + 2\epsilon\|\tilde{u}_n\|^4]}{\lambda(6-q)\int Kf|\tilde{u}_*|^q} = 1. \tag{3.23}$$

Then from (3.22) it follows that

$$\frac{4a\|\tilde{u}_n\|^2 + 2\epsilon\|\tilde{u}_n\|^4}{\lambda(6-q)\int Kf|\tilde{u}_n|^q} > 1, \tag{3.24}$$

for n large enough. Combining with (3.23) and (3.24), there is a subsequence $\{\tilde{u}_{n_k}\}$ of $\{\tilde{u}_n\}$ satisfying

$$\frac{4a\|\tilde{u}_{n_k}\|^2 + 2\epsilon\|\tilde{u}_{n_k}\|^4}{\lambda(6-q)\int Kf|\tilde{u}_{n_k}|^q} \to 1, \quad \text{as } k \to \infty.$$

Namely,

$$4a\|\tilde{u}_{n_k}\|^2 + 2\epsilon\|\tilde{u}_{n_k}\|^4 \to \lambda(6-q)\int Kf|\tilde{u}_*|^q, \quad \text{as } k \to \infty.$$

Thus, we can assume $\|\tilde{u}_{n_k}\|^2 \to A_2$ as $k \to \infty$ such that

$$4aA_2 + 2\epsilon A_2^2 = \lambda(6-q)\int Kf|\tilde{u}_*|^q. \tag{3.25}$$

Moreover, from $\tilde{u}_{n_k} \in \Lambda$ we have

$$\int K|\tilde{u}_{n_k}|^6 = a\|\tilde{u}_{n_k}\|^2 + \epsilon\|\tilde{u}_{n_k}\|^4 - \lambda\int Kf|\tilde{u}_*|^q \to \frac{2-q}{6-q}aA_2 + \frac{4-q}{6-q}\epsilon A_2^2. \tag{3.26}$$

Applying (3.25) and (3.26), the same argument as in step 2 we arrive at a contradiction. This completes the proof of step 2.

Step 3. $I'(\tilde{u}_n) \to 0$ in H^{-1}. The proof is similar to step 3 in the previous section.

Step 4. $\tilde{u}_n \to \tilde{u}_*$ in H. Assume that

$$\|\tilde{u}_n - \tilde{u}_*\|^2 \to \beta^2, \quad \int K|\tilde{u}_n - \tilde{u}_*|^6 \to \gamma^6.$$

If $\beta = 0$, we are done. Hence, suppose $\beta > 0$. Define

$$\sigma(t) = I(t\tilde{u}_*) = \frac{a}{2}t^2\|\tilde{u}_*\|^2 + \frac{\epsilon}{4}t^4\|\tilde{u}_*\|^4 - \frac{\lambda}{q}t^q\int Kf|\tilde{u}_*|^q - \frac{1}{6}t^6\int K|\tilde{u}_*|^6,$$

$$\delta(t) = \frac{a}{2}t^2\beta^2 + \frac{\epsilon}{4}t^4\beta^4 + \frac{2\epsilon\beta^2\|\tilde{u}_*\|^2}{4}t^4 - \frac{\gamma^6}{6}t^6,$$

$$\delta_1(t) = \frac{a}{2}t^2\beta^2 - \frac{\gamma^6}{6}t^6,$$

and $\theta(t) = \sigma(t) + \delta(t)$, then $I(t\tilde{u}_n) \to \theta(t)$. Inspired by [10, 3], we consider the following three cases:

(i) $t^+(\tilde{u}_*) \le 1$,
(ii) $t^+(\tilde{u}_*) > 1$ and $\gamma > 0$,
(iii) $t^+(\tilde{u}_*) > 1$ and $\gamma = 0$.

Case (i). From step 2 and $t^+(\tilde{u}_*) \le 1$, we have $\sigma'(1) \le 0$. Since $\{\tilde{u}_n\} \subset \Lambda^-$, we obtain that $\theta'(1) = 0$. In turn, $\delta'(1) \ge 0$. Hence $\delta\big(t^+(\tilde{u}_*)\big) > 0$ and consequently

$$c_2 = \theta(1) \ge \theta\big(t^+(\tilde{u}_*)\big) = I\big(t^+(\tilde{u}_*)\tilde{u}_*\big) + \delta\big(t^+(\tilde{u}_*)\big) > I\big(t^+(\tilde{u}_*)\tilde{u}_*\big) \ge c_2,$$

which is impossible.

Case (ii). Let $t_* = \big(\frac{a\beta^2}{\gamma^6}\big)^{1/4}$. It is easily verified that $\delta_1(t)$ achieves its maximum at t_*, and $\delta_1'(t) > 0$ for $0 < t < t_*$ and $\delta_1'(t) < 0$ for $t > t_*$. Also, $\delta_1(t_*) = \frac{\sqrt{a^3\beta^6}}{3\gamma^3} \ge \frac{\sqrt{a^3S^3}}{3}$. We claim that $t_* < t^+(\tilde{u}_*)$. Otherwise, $1 < t^+(\tilde{u}_*) < t_*$. Since $0 > \theta'(t) = \sigma'(t) + \delta'(t)$ for all $t > 1$, we obtain $\sigma'(t) \le -\delta'(t) \le -\delta_1'(t) < 0$ for $t \in (1, t_*)$, which is a contradiction to $1 < t^+(\tilde{u}_*) < t_*$ and $\sigma'\big(t^+(\tilde{u}_*)\big) = 0$. Hence the claim is true and so

$$c_2 = \theta(1) \ge \theta(t_*) \ge I\big(t_*\tilde{u}_*\big) + \delta_1(t_*) \ge I\big(t^-(\tilde{u}_*)\tilde{u}_*\big) + \frac{\sqrt{a^3S^3}}{3} \ge I(\tilde{u}_*) + \frac{\sqrt{a^3S^3}}{3},$$

which contradicts Lemma 3.2.

Case (iii). Since $\gamma = 0$ and $\theta'(1) = 0$ and $\theta''(1) \leq 0$, we have $\sigma'(1) = -\delta'(1) = -a\beta^2 - \epsilon\beta^4 - 2\epsilon\beta^2\|\tilde{u}_*\|^2 < 0$ and $\sigma''(1) = \theta''(1) - \delta''(1) \leq -\delta''(1) = -a\beta^2 - 3\epsilon\beta^4 - 6\epsilon\beta^4\|\tilde{u}_*\|^2 < 0$, which is absurd in contrast to $t^+(\tilde{u}_*) > 1$.

Thus, $\tilde{u}_n \to \tilde{u}_*$ in H and step 4 follows. This completes the proof of step 4.

Step 5. $\tilde{u}_*$ is a nonnegative nontrivial solution of (1.1) and $\tilde{u}_* \in \Lambda^-$. We can proceed exactly as in the proof of Step 5 in the previous section and conclude that $\tilde{u}_*$ is a nonnegative nontrivial solution for (1.1). In addition, it is clear that $\tilde{u}_* \in \Lambda^-$ by Lemma 2.6 and step 4. The proof of step 5 is complete.

Finally, we can conclude from Lemma 2.3 that problem (1.1) has at least two nonnegative nontrivial solutions $u_* \in \Lambda^+$, $\tilde{u}_* \in \Lambda^-$ with $\|u_*\| < \|\tilde{u}_*\|$ when $\lambda \in (0, T_{q,f,a})$. At this point, Theorem 1.2 is a immediate consequence of Theorem 1.1 and the definition for λ^*.

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