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GLOBAL WELL-POSEDNESS OF THE CAUCHY PROBLEM OF A HIGHER-ORDER SCHRÖDINGER EQUATION

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Abstract. We apply the I-method to prove that the Cauchy problem of a higher-order Schrödinger equation is globally well-posed in the Sobolev space $H^s(\mathbb{R})$ with $s > 6/7$.

1. Introduction

This paper concerns the Cauchy problem of the higher order Schrödinger equation

$$\begin{gathered}\partial_t u + ia\,\partial_x^2 u + b\partial_x^3 u + ic|u|^2 u + d|u|^2\partial_x u + eu^2\partial_x \bar{u} = 0 \quad \text{in } \mathbb{R}^2,\\ u(x,0) = \varphi(x) \quad for \quad x \in \mathbb{R},\end{gathered} \tag{1.1}$$

where a, b, c, d and e are real constants with $be \neq 0$, and the unknown function u is a complex-valued function.

Hasegawa and Kodama [9, 12] proposed (1.1) as the model for propagation of pulse in optical fiber. It is easy to see that cubic nonlinear Schrödinger equation, nonlinear Schrödinger equation with derivative and complex modified KdV equation are particular cases of (1.1). Therefore, in the literature, this model is also called the Airy-Schrödinger equation.

Well-posedness of the Cauchy problem of (1.1) in Sobolev spaces has been investigated by a few authors; see for instance [2, 10, 13, 15, 16]. Laurey [13] proved that the Cauchy problem of (1.1) is locally well-posed in $H^s(\mathbb{R})$ for $s > 3/4$. Laurey's result was improved by Staffilani [15], who obtained the local well-posedness in $H^s(\mathbb{R})$ with $s \geq \frac{1}{4}$. This local well-posedness combined with mass and energy conservation laws naturally yields that (1.1) is globally well-posed in $H^1(\mathbb{R})$. Recently, using I-method introduced by Colliander, Kell, Staffilani, Takaoka and Tao [3, 4, 5], Carvajal [2] established global well-posedness in $H^s(\mathbb{R})$ with $s > \frac{1}{4}$ under the relation $c = \frac{(d-e)a}{3b}$. Our aim of this paper is to get global well-posedness in $H^s(\mathbb{R})$ with $s > \frac{6}{7}$ without the above restriction condition.

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Without loss of generality, we may assume that $a = 0$ in the sequel. In fact, when $a \neq 0$ we may utilize the gauge transform [2]

$$v(x,t) = \exp(i\frac{a}{3b}x + i\frac{a^3}{27b^2}t)u(x + \frac{a^2}{3b}t, t),$$

then u satisfies (1.1) if and only if v is such that

$$\begin{gathered} \partial_t v + b\partial_x^3 v + i(c - \frac{(d-e)a}{3b})|v|^2 v + d|v|^2\partial_x v + ev^2\partial_x \bar{v} = 0 \quad \text{in } \mathbb{R}^2, \\ v(x,0) = e^{i\frac{ax}{3b}}\varphi(x) \quad \text{for } x \in \mathbb{R}. \end{gathered} \tag{1.2}$$

Note that when $c = \frac{(d-e)a}{3b}$, (1.2) is the complex mKdV equation satisfying a scaling invariant property. It is well known that real mKdV possesses global well-posedness in $H^s(\mathbb{R})$ with $s > 1/4$ [6]. Using the same argument as the one in [6] the same result was obtained for the complex case [2]. Since in our case a scaling invariance disappears, thus we must modify I-method suitably. Similar results as the one of this paper were also obtained for some other nonlinear dispersive systems and equations (e.g., [14, 17] and therein).

To precisely state our main result, we first introduce some notation. We use the notation $a+$ and $a-$ to respectively denote expressions of the forms $a+\varepsilon$ and $a-\varepsilon$, where $0 < \varepsilon << 1$. We denote by D_x^s the Riesz potential of order $-s$, or the Fourier multiplier with symbol $|\xi|^s$ ($s > 0$). Recall that the Sobolev space $H^s(\mathbb{R})$ is defined by

$$f \in H^s(\mathbb{R}) \Leftrightarrow \|f\|_{H^s(\mathbb{R})} := \|\langle\xi\rangle^s \hat{f}(\xi)\|_{L^2_\xi(\mathbb{R})} < \infty,$$

where $\langle\xi\rangle^s := (1 + |\xi|^2)^{s/2}$, and $\hat{f}$ represents the Fourier transformation in one variable of f. We define the space $X_{s,\alpha}(\mathbb{R}^2)$ (as in [1, 11]) by

$$u \in X_{s,\alpha}(\mathbb{R}^2) \Leftrightarrow \|u\|_{X_{s,\alpha}(\mathbb{R}^2)} := \|\langle\xi\rangle^s \langle\tau - \xi^3\rangle^\alpha \tilde{u}(\xi,\tau)\|_{L^2_\tau L^2_\xi} < \infty,$$

where $\tilde{u}$ represents the Fourier transformation in two variables of u. For any given interval L, we define the space $X_{s,\alpha}(L \times \mathbb{R})$ to be the restriction of $X_{s,\alpha}(\mathbb{R}^2)$ on $L \times \mathbb{R}$, with norm

$$\|u\|_{X_{s,\alpha}(L\times\mathbb{R})} = \inf\{\|U\|_{X_{s,\alpha}(\mathbb{R}^2)} : U|_{L\times\mathbb{R}} = u\}.$$

If $L = [0,T]$ (resp. $[0,\delta]$), we use $X^T_{s,\alpha}$ (resp. $X^\delta_{s,\alpha}$) to abbreviate $X_{s,\alpha}(L \times \mathbf{R})$.

For given $N >> 1$ and $s < 1$, we define the multiplier operator $I^s_N : H^s(\mathbb{R}) \to H^1(\mathbb{R})$ by

$$(I^s_N u)\hat{}(\xi) := m_{s,N}(\xi)\hat{u}(\xi), \quad u \in H^s(\mathbb{R}),$$

where $m_{s,N}(\xi)$ is an even C^∞ function, non-increasing in $|\xi|$, and

$$m_{s,N}(\xi) = \begin{cases} 1, & |\xi| \le N, \\ \big(|\xi|/N\big)^{s-1}, & |\xi| > 2N. \end{cases}$$

In the sequel, for simplicity of notation we shall omit the superscripts and subscripts s, N of the operator I^s_N and the multiplier $m_{s,N}(\xi)$.

It is obvious that for some positive constant C,

$$C^{-1}\|u\|_{H^s(\mathbb{R})} \le \|Iu\|_{H^1(\mathbb{R})} \le CN^{1-s}\|u\|_{H^s(\mathbb{R})}.$$

We denote by $\|\cdot\|_{X_{s,\alpha,N}(\mathbb{R}^2)}$ the equivalent norm in $X_{s,\alpha}(\mathbb{R}^2)$ defined by

$$\|u\|_{X_{s,\alpha,N}(\mathbb{R}^2)} := \|Iu\|_{X_{1,\alpha}(\mathbb{R}^2)}.$$

The space $X_{s,\alpha}(\mathbb{R}^2)$ endowed with this norm will be re-denoted as $X_{s,\alpha,N}(\mathbb{R}^2)$. Clearly, there also hold the inequalities

$$C^{-1}\|u\|_{X_{s,\alpha}(\mathbb{R}^2)} \le \|Iu\|_{X_{1,\alpha}(\mathbb{R}^2)} \le CN^{1-s}\|u\|_{X_{s,\alpha}(\mathbb{R}^2)}.$$

The notation $X^{\delta}_{s,\alpha,N}$ denotes the restriction of $X_{s,\alpha,N}(\mathbb{R}^2)$ on $\mathbb{R}\times[0,\delta]$.

Next we give a local well-posedness result. This local result is a variant of that of [10, 16], with precise estimates on the lifespan and the norm of the solution and it can be established by the same argument as [10, 16] and the interpolation lemma in [7].

Theorem 1.1. *For $\frac{6}{7} < s < 1$, the initial value problem of (1.1) is locally well-posed in $H^s(\mathbb{R})$. More precisely, for given $\varphi \in H^s(\mathbb{R})$ and $N >> 1$, there exists a corresponding $\delta > 0$ such that (1.1) has a unique solution $u \in X^{\delta}_{s,\frac{1}{2}+,N} \subseteq C([0,\delta], H^s(\mathbb{R}))$ satisfying the condition $u(0,\cdot) = \varphi$. Moreover, the lifespan satisfies the estimate*

$$\delta \sim \|I\varphi\|^{-\theta}_{H^1(\mathbb{R})}, \quad \theta = 12+ \tag{1.3}$$

and the solution satisfies the estimate

$$\|Iu\|_{X^{\delta}_{1,\frac{1}{2}+}} \le C\|I\varphi\|_{H^1(\mathbb{R})}. \tag{1.4}$$

Finally, we state our main result of this paper as follows:

Theorem 1.2. *The Cauchy problem of the equation (1.1) is globally well-posed in $H^s(\mathbb{R})$ for $s > 6/7$. More precisely, let $\varphi \in H^s(\mathbb{R})$ with $s > 6/7$. Then for any $T > 0$ the equation (1.1) has a unique solution $u \in X^T_{s,\frac{1}{2}+} \subseteq C([0,T], H^s(\mathbb{R}))$ satisfying the initial condition $u(0,\cdot) = \varphi$, and the mapping $\varphi \to u(t,\cdot)$ belongs to $C(H^s(\mathbb{R}), X^T_{s,\frac{1}{2}+}) \subseteq C(H^s(\mathbb{R}), C([0,T], H^s(\mathbb{R})))$.*

We note that the improvement of θ in Theorem 1.1 will directly lead to a better Sobolev index s in Theorem 1.2. Here we do not pursue this although it is possible to get a smaller θ by more precise trilinear estimates of nonlinear terms in (1.1).

2. The almost conserved energy

Laurey [13] showed that the Cauchy problem of (1.1) has the following two conserved quantities

$$M(u) = \int_{\mathbb{R}} |u(x,t)|^2 dx := M_0, \tag{2.1}$$

$$\begin{aligned} E(u) = k_1 \int_{\mathbb{R}} |\partial_x u(x,t)|^2 dx + k_2 \int_{\mathbb{R}} |u(x,t)|^4 dx \\ + k_3 \operatorname{Im} \int_{\mathbb{R}} u(x,t)\overline{\partial_x u(x,t)} dx := E_0, \end{aligned} \tag{2.2}$$

where $k_1 = 3be$, $k_2 = -\frac{e(e+d)}{2}$ and $k_3 = 3bc$.

Applying Gagliardo-Nirenberg inequality, Young inequality and Hölder inequality, we have

$$\int_{\mathbb{R}} |u(x,t)|^4 dx \le C\|\partial_x u\|_{L^2_x}\|u\|^3_{L^2_x} \le \varepsilon\|\partial_x u\|^2_{L^2_x} + C(\varepsilon)\|\partial_x u\|^6_{L^2_x}, \tag{2.3}$$

$$\int_{\mathbb{R}} u(x,t)\overline{\partial_x u(x,t)} dx \le C\|\partial_x u\|_{L^2_x}\|u\|_{L^2_x} \le \varepsilon\|\partial_x u\|^2_{L^2_x} + C(\varepsilon)\|\partial_x u\|^2_{L^2_x}. \tag{2.4}$$

By (2.1)–(2.4), we obtain an a-priori bound of the H^1-norm of the solution u and an upper bound of E

$$\|\partial_x u\|_{L^2_x}^2 \le C(E_0 + M_0^3 + M_0), \tag{2.5}$$
$$|E(u)| \le C(\|\partial_x u\|_{L^2_x}^2 + M_0^3 + M_0). \tag{2.6}$$

From the local well-posedness and the a-priori bound (2.5), it follows that the Cauchy problem of (1.1) is globally well-posed in $H^1(\mathbb{R})$. However, we are searching solutions in $C(\mathbb{R}, H^s(\mathbb{R}))$ with $s < 1$, so we shall alteratively consider the modified energy $E(Iu)$ as in Colliander et al [3, 4, 5, 6]. We shall show the modified energy $E(Iu)$ is almost conserved, that is, it has a very slow increment in time if N is sufficiently large. First we give the precise expression of the increment of $E(Iu)$ in the following lemma.

Lemma 2.1. *If u is a solution of* (1.1) *on* $[0,\delta]$ *in the sense of Theorem 1.1, then*

$$\begin{aligned}
E(Iu(\delta)) - E(I\varphi) &= 2k_1 d\,\mathrm{Re}\int_0^\delta\int_{\mathbb{R}} \partial_x^2 I\bar u\Big(I(|u|^2\partial_x u) - |Iu|^2\partial_x Iu\Big)\,dx\,dt\\
&\quad + 2k_1 e\,\mathrm{Re}\int_0^\delta\int_{\mathbb{R}} \partial_x^2 I\bar u\Big(I(u^2\partial_x \bar u) - (Iu)^2\partial_x I\bar u\Big)\,dx\,dt\\
&\quad - 2k_1 c\,\mathrm{Im}\int_0^\delta\int_{\mathbb{R}} \partial_x^2 I\bar u\Big(I(|u|^2 u) - |Iu|^2 Iu\Big)\,dx\,dt\\
&\quad - 2k_3 e\,\mathrm{Im}\int_0^\delta\int_{\mathbb{R}} \partial_x I\bar u\Big(I(u^2\partial_x \bar u) - (Iu)^2\partial_x I\bar u\Big)\,dx\,dt\\
&\quad - 2k_3 d\,\mathrm{Im}\int_0^\delta\int_{\mathbb{R}} \partial_x I\bar u\Big(I(|u|^2\partial_x u) - |Iu|^2\partial_x Iu\Big)\,dx\,dt\\
&\quad - 2k_3 c\,\mathrm{Re}\int_0^\delta\int_{\mathbb{R}} \partial_x I\bar u\Big(I(|u|^2 u) - |Iu|^2 Iu\Big)\,dx\,dt\\
&\quad - 4k_2 d\,\mathrm{Re}\int_0^\delta\int_{\mathbb{R}} |Iu|^2 I\bar u\Big(I(|u|^2\partial_x u) - |Iu|^2\partial_x Iu\Big)\,dx\,dt\\
&\quad - 4k_2 e\,\mathrm{Re}\int_0^\delta\int_{\mathbb{R}} |Iu|^2 I\bar u\Big(I(u^2\partial_x \bar u) - (Iu)^2\partial_x I\bar u\Big)\,dx\,dt\\
&\quad + 4k_2 c\,\mathrm{Im}\int_0^\delta\int_{\mathbb{R}} |Iu|^2 I\bar u\Big(I(|u|^2 u) - |Iu|^2 Iu\Big)\,dx\,dt.
\end{aligned} \tag{2.7}$$

Proof. From (1.1), we have

$$\begin{gathered}
\partial_t Iu = -b\partial_x^3 Iu - icI(|u|^2 u) - dI(|u|^2\partial_x u) - eI(u^2\partial_x\bar u),\\
\partial_t\partial_x Iu = -b\partial_x^4 Iu - ic\partial_x I(|u|^2 u) - d\partial_x I(|u|^2\partial_x u) - e\partial_x I(u^2\partial_x\bar u),\\
\partial_t\partial_x I\bar u = -b\partial_x^4 I\bar u + ic\partial_x I(|u|^2 \bar u) - d\partial_x I(|u|^2\partial_x \bar u) - e\partial_x I(\bar u^2\partial_x u).
\end{gathered}$$

From the above equalities and using integration by part we obtain

$$\begin{aligned}
\frac{d}{dt}E(Iu) &= 2k_1\,\mathrm{Re}\int \partial_x I\bar u\partial_t\partial_x Iu\\
&\quad + 4k_2\,\mathrm{Re}\int |Iu|^2 I\bar u\partial_t Iu + k_3\,\mathrm{Im}\Big(\int \partial_x I\bar u\partial_t Iu + \int Iu\partial_t\partial_x I\bar u\Big)
\end{aligned}$$

$$
\begin{aligned}
&= 2k_1 d \operatorname{Re}\int \partial_x^2 I\bar{u} I(|u|^2\partial_x u) + 2k_1 e \operatorname{Re}\int \partial_x^2 IuI(u^2\partial_x\bar{u}) \\
&\quad - 4k_2 b \operatorname{Re}\int |Iu|^2 I\bar{u}\partial_x^3 Iu - 2k_1 c \operatorname{Im}\int \partial_x^2 I\bar{u} I(|u|^2 u) \\
&\quad - 2k_3 d \operatorname{Im}\int \partial_x I\bar{u} I(|u|^2\partial_x u) - 2k_3 e \operatorname{Im}\int \partial_x I\bar{u} I(u^2\partial_x\bar{u}) \\
&\quad - 2k_3 c \operatorname{Re}\int \partial_x I\bar{u} I(|u|^2 u) + 4k_2 c \operatorname{Im}\int |Iu|^2 I\bar{u} I(|u|^2 u) \\
&\quad - 4k_2 d \operatorname{Re}\int |Iu|^2 I\bar{u} I(|u|^2\partial_x u) - 4k_2 e \operatorname{Re}\int |Iu|^2 I\bar{u} I(|u|^2\partial_x\bar{u}).
\end{aligned}
$$

We note that

$$
\operatorname{Re}\int \partial_x^2 I\bar{u}(Iu)^2\partial_x I\bar{u} = \operatorname{Re}\int \partial_x^2 I\bar{u}|Iu|^2\partial_x Iu
$$

and

$$
\operatorname{Re}\int |Iu|^2 I\bar{u}\partial_x^3 Iu = -\operatorname{Re}\int \partial_x^2 I\bar{u}(Iu)^2\partial_x I\bar{u} - 2\operatorname{Re}\int \partial_x^2 I\bar{u}|Iu|^2\partial_x Iu.
$$

Hence

$$
\begin{aligned}
&2k_1 d \operatorname{Re}\int \partial_x^2 I\bar{u} I(|u|^2\partial_x u) + 2k_1 e \operatorname{Re}\int \partial_x^2 I\bar{u} I(u^2\partial_x\bar{u}) \\
&- 4k_2 b \operatorname{Re}\int |Iu|^2 I\bar{u}\partial_x^3 Iu \\
&= 2k_1 d \operatorname{Re}\int \partial_x^2 I\bar{u}\Big(I(|u|^2\partial_x u) - |Iu|^2\partial_x Iu\Big) \\
&\quad + 2k_1 e \operatorname{Re}\int \partial_x^2 I\bar{u}\Big(I(u^2\partial_x\bar{u}) - (Iu)^2\partial_x I\bar{u}\Big).
\end{aligned}
\tag{2.8}
$$

Integration by part yields

$$
\operatorname{Im}\int \partial_x^2 I\bar{u}|Iu|^2 Iu = -\operatorname{Im}\int (\partial_x I\bar{u})^2(Iu)^2.
$$

It follows from the above equality that

$$
\begin{aligned}
&- 2k_1 c \operatorname{Im}\int \partial_x^2 I\bar{u} I(|u|^2 u) - 2k_3 d \operatorname{Im}\int \partial_x I\bar{u} I(|u|^2\partial_x u) - 2k_3 e \operatorname{Im}\int \partial_x I\bar{u} I(u^2\partial_x\bar{u}) \\
&= -2k_1 c \operatorname{Im}\int \partial_x^2 I\bar{u}\Big(I(|u|^2 u) - |Iu|^2 Iu\Big) \\
&\quad - 2k_3 d \operatorname{Im}\int \partial_x I\bar{u}\Big(I(|u|^2\partial_x u) - |Iu|^2\partial_x Iu\Big) \\
&\quad - 2k_3 e \operatorname{Im}\int \partial_x I\bar{u}\Big(I(u^2\partial_x\bar{u}) - (Iu)^2\partial_x I\bar{u}\Big).
\end{aligned}
$$

Observe that

$$
\operatorname{Re}\int \partial_x I\bar{u}|Iu|^2 Iu = 0, \tag{2.9}
$$

$$
\operatorname{Im}\int |Iu|^2 I\bar{u}|Iu|^2 Iu = 0, \tag{2.10}
$$

$$
\operatorname{Re}\int |Iu|^2 I\bar{u}|Iu|^2\partial_x Iu = 0, \tag{2.11}
$$

$$\operatorname{Re}\int |Iu|^2 I\bar{u}(Iu)^2\partial_x I\bar{u} = 0. \tag{2.12}$$

Hence by (2.8)-(2.12), we have

$$\begin{aligned}
\frac{d}{dt}E(Iu) &= 2k_1 d\operatorname{Re}\int_{\mathbb{R}} \partial_x^2 I\bar{u}\Big(I(|u|^2\partial_x u) - |Iu|^2\partial_x Iu\Big)dx \\
&\quad + 2k_1 e\operatorname{Re}\int_{\mathbb{R}} \partial_x^2 I\bar{u}\Big(I(u^2\partial_x\bar{u}) - (Iu)^2\partial_x I\bar{u}\Big)dx \\
&\quad - 2k_1 c\operatorname{Im}\int_{\mathbb{R}} \partial_x^2 I\bar{u}\Big(I(|u|^2 u) - |Iu|^2 Iu\Big)dx \\
&\quad - 2k_3 e\operatorname{Im}\int_{\mathbb{R}} \partial_x I\bar{u}\Big(I(u^2\partial_x\bar{u}) - (Iu)^2\partial_x I\bar{u}\Big)dx \\
&\quad - 2k_3 d\operatorname{Im}\int_{\mathbb{R}} \partial_x I\bar{u}\Big(I(|u|^2\partial_x u) - |Iu|^2\partial_x Iu\Big)dx \\
&\quad - 2k_3 c\operatorname{Re}\int_{\mathbb{R}} \partial_x I\bar{u}\Big(I(|u|^2 u) - |Iu|^2 Iu\Big)dx \\
&\quad - 4k_2 d\operatorname{Re}\int_{\mathbb{R}} |Iu|^2 I\bar{u}\Big(I(|u|^2\partial_x u) - |Iu|^2\partial_x Iu\Big)dx \\
&\quad - 4k_2 e\operatorname{Re}\int_{\mathbb{R}} |Iu|^2 I\bar{u}\Big(I(u^2\partial_x\bar{u}) - (Iu)^2\partial_x I\bar{u}\Big)dx \\
&\quad + 4k_2 c\operatorname{Im}\int_{\mathbb{R}} |Iu|^2 I\bar{u}\Big(I(|u|^2 u) - |Iu|^2 Iu\Big)dx.
\end{aligned}$$

Integrating both sides of the above expression, over the interval $[0,\delta]$, we obtain (2.7). $\square$

Next we apply Lemma 2.1 to deduce an exact estimate on the increment of the modified energy $E(Iu)$ in terms of the norm $\|Iu\|_{X^\delta_{1,\frac12+}}$. Before stating the result, we give a few simple preliminary estimates.

The following embedding inequality is established in [11]:

$$\|u\|_{L^8_{xt}} \le C\|u\|_{X_{0,\frac12+}}, \tag{2.13}$$

$$\|u\|_{L^\infty_t L^2_x} \le C\|u\|_{X_{0,\frac12+}}, \tag{2.14}$$

$$\|D_x^{\frac16} u\|_{L^6_{xt}} \le C\|u\|_{X_{0,\frac12+}}. \tag{2.15}$$

By Hölder inequality and (2.14), we have

$$\|u\|_{L^2_x L^2_t(\mathbb{R}\times[0,\delta])} \le \delta^{\frac12}\|u\|_{L^\infty_t([0,\delta],L^2_x)} \le C\delta^{\frac12}\|u\|_{X^\delta_{0,\frac12+}}. \tag{2.16}$$

Interpolating (2.16) with (2.13) we get

$$\|u\|_{L^4_x L^4_t(\mathbb{R}\times[0,\delta])} \le C\delta^{\frac16}\|u\|_{X^\delta_{0,\frac12+}}. \tag{2.17}$$

From [8] we have the following bilinear estimate:

$$\|D_x^{\frac12} I_-^{\frac12}(u_1,u_2)\|_{L^2_{xt}} \le C\|u_1\|_{X_{0,\frac12+}}\|u_2\|_{X_{0,\frac12+}}, \tag{2.18}$$

where

$$\big(I_-^\alpha(u_1u_2)\big)\tilde{}(\xi,\tau) = \int_{\xi=\xi_1+\xi_2,\,\tau=\tau_1+\tau_2} |\xi_1-\xi_2|^\alpha \tilde{u_1}(\xi_1,\tau_1)\tilde{u_2}(\xi_2,\tau_2)d\xi_1 d\tau_1.$$

Also, we need the following refined Strichartz estimate.

Lemma 2.2. *Let* u_1, u_2 *be such that* $\operatorname{supp} u_1 \subset \{|\xi| \sim N\}$ *and* $\operatorname{supp} u_2 \subset \{|\xi| \ll N\}$*, then*

$$\|u_1u_2\|_{L^2_{xt}} \leq \frac{C}{N}\|u_1\|_{X_{0,\frac{1}{2}+}}\|u_2\|_{X_{0,\frac{1}{2}+}}. \tag{2.19}$$

It is not difficult to prove the above, using the same argument as the one of [3, Lemma 7.1], so we omit it.

Lemma 2.3. *If* u *is the solution of* (1.1) *on* $[0,\delta]$ *in the sense of Theorem 1.1, then*

$$\begin{aligned}
&|E(Iu(\delta)) - E(I\varphi)| \\
&\leq C(N^{-1+}\delta^{\frac{2}{3}} + N^{-2+})\|Iu\|^4_{X^\delta_{1,\frac{1}{2}+}} + C(N^{-\frac{5}{2}+}\delta^{\frac{1}{2}} + N^{-3+})\|Iu\|^6_{X^\delta_{1,\frac{1}{2}+}}.
\end{aligned}$$

Proof. We denote the nine terms on the right-hand side of (2.7) in their appearing order by J_1, J_2, $\dots$, J_9, respectively. In the sequel we only consider J_1 and J_7 because the other terms can be readily controlled by the bound of J_1 and J_7.

Estimate of J_1. It suffices to prove that for any $(u_1,u_2,u_3,u_4) \in C([0,\delta], S(\mathbb{R}))^4$ such that the frequency support of each u_j is located in a dyadic band $|\xi| \sim N_j$ (i.e., $C_1N_j \leq |\xi| \leq C_2N_j$) for some positive numbers N_j $(j = 1,2,3,4)$, there holds

$$\begin{aligned}
I_1 := \int_0^\delta \Big(\int_* \Big| &\frac{m(\xi_1+\xi_2+\xi_3) - m(\xi_1)m(\xi_2)m(\xi_3)}{m(\xi_1)m(\xi_2)m(\xi_3)} \Big| \\
&\times |\xi_3|\xi_4^2|\hat{u_1}(\xi_1,t)\hat{u_2}(\xi_2,t)\hat{u_3}(\xi_3,t)\hat{u_4}(\xi_4,t)| \Big) dt \\
\leq{}& C(N^{-1+}\delta^{\frac{2}{3}} + N^{-2+})N^{0-}_{\max}\prod_{j=1}^4 \|u_j\|_{X^\delta_{1,\frac{1}{2}+}}.
\end{aligned}$$

where $N_{\max} = \max\{N_1,N_2,N_3,N_4\}$ and $*$ denotes integration on the set $\sum_{j=1}^4 \xi_j = 0$. Indeed, once this estimate is proved, then the Littlewood-Paley decomposition immediately implies that

$$|J_1| \leq C(N^{-1+}\delta^{\frac{2}{3}} + N^{-2+})\|Iu\|^4_{X^\delta_{1,\frac{1}{2}+}}. \tag{2.20}$$

First. All the frequencies are equivalent, namely, $|\xi_1| \sim |\xi_2| \sim |\xi_3| \sim |\xi_4| \geq CN$. Using Hölder inequality and (2.17) we see that

$$\begin{aligned}
I_1 &\leq C(\frac{N_1}{N})^{3(1-s)}N_3N_4^2\prod_{j=1}^4 \|u_j\|_{L^4_xL^4_t(\mathbb{R}\times[0,\delta])} \\
&\leq C(\frac{N_1}{N})^{3(1-s)}N_3N_4^2(N_1N_2N_3N_4)^{-1}\delta^{\frac{2}{3}}\prod_{j=1}^4 \|u_j\|_{X^\delta_{1,\frac{1}{2}+}} \\
&\leq C\delta^{\frac{2}{3}}N^{-1+}N^{0-}_{\max}\prod_{j=1}^4 \|u_j\|_{X^\delta_{1,\frac{1}{2}+}}.
\end{aligned}$$

Second. Three of the frequencies are equivalent. We shall deal with the most difficult case $|\xi_1| \sim |\xi_3| \sim |\xi_4| \geq CN$ and $|\xi_2| \ll |\xi_1|, |\xi_3|, |\xi_4|$. The other two cases $|\xi_1| \sim |\xi_2| \sim |\xi_3| \geq CN$ and $|\xi_1| \sim |\xi_2| \sim |\xi_4| \geq CN$ can be solved easily by the same argument as the case 1° and the difficult case, respectively. Since

$\xi_1+\xi_2+\xi_3+\xi_4=0$, the largest two of the frequencies must have different sign. We may assume that they are ξ_1 and ξ_4 for the other cases can be considered similarly. Thus we have

$$N_4\sim|\xi_1-\xi_4|\sim N_3\sim|\xi_3+\xi_2|\sim|\xi_1+\xi_4|.$$

Utilizing (2.18) and (2.19), we obtain

$$\begin{aligned}
I_1&\le C(\frac{N_1}{N})^{2(1-s)}\langle(\frac{N_2}{N})^{1-s}\rangle N_3N_4\|D_x^{\frac12}I_-^{\frac12}(u_1,u_4)\|_{L^2_{xt}}\|u_2u_3\|_{L^2_{xt}}\\
&\le C(\frac{N_1}{N})^{2(1-s)}\langle(\frac{N_2}{N})^{1-s}\rangle N_3N_4N_1^{-1}\langle N_2\rangle^{-1}N_3^{-2}N_4^{-1}\prod_{j=1}^4\|u_j\|_{X^\delta_{1,\frac12+}}\\
&\le CN^{-2+}N_{\max}^{0-}\prod_{j=1}^4\|u_j\|_{X^\delta_{1,\frac12+}}.
\end{aligned}$$

Third. Exact two of the frequencies are equivalent. We only consider the most difficult case $|\xi_1|\sim|\xi_4|\ge CN$ and $|\xi_2|,|\xi_3|\ll|\xi_1|,|\xi_4|$.

3.1° $|\xi_2|,|\xi_3|\le N$ Applying the mean value theorem and (2.19) yield

$$\begin{aligned}
I_1&\le C\frac{N_2+N_3}{N_1}N_3N_4^2\|u_1u_2\|_{L^2_{xt}}\|u_3u_4\|_{L^2_{xt}}\\
&\le C\frac{N_2+N_3}{N_1}N_3N_4^2N_1^{-2}N_4^{-2}\langle N_2\rangle^{-1}\langle N_3\rangle^{-1}\prod_{j=1}^4\|u_j\|_{X^\delta_{1,\frac12+}}\\
&\le CN^{-2+}N_{\max}^{0-}\prod_{j=1}^4\|u_j\|_{X^\delta_{1,\frac12+}}.
\end{aligned}$$

3.2° $|\xi_2|\ge N$ ($|\xi_3|\ge N$ can be considered with the same argument).

By (2.19) we obtain

$$\begin{aligned}
I_1&\le C(\frac{N_2}{N})^{1-s}\langle(\frac{N_3}{N})^{1-s}\rangle N_3N_4^2\|u_1u_2\|_{L^2_{xt}}\|u_3u_4\|_{L^2_{xt}}\\
&\le C(\frac{N_2}{N})^{1-s}\langle(\frac{N_3}{N})^{1-s}\rangle N_3N_4^2N_1^{-2}N_4^{-2}\langle N_2\rangle^{-1}\langle N_3\rangle^{-1}\prod_{j=1}^4\|u_j\|_{X^\delta_{1,\frac12+}}\\
&\le CN^{-2+}N_{\max}^{0-}\prod_{j=1}^4\|u_j\|_{X^\delta_{1,\frac12+}}.
\end{aligned}$$

Estimate of J_7. Similarly as before we only need to prove that for any triple $(u_1,u_2,\dots,u_6)$ similar as before there holds

$$\begin{aligned}
I_7&:=\int_0^\delta\Big(\int_*\Big|\frac{m(\xi_4+\xi_5+\xi_6)-m(\xi_4)m(\xi_5)m(\xi_6)}{m(\xi_4)m(\xi_5)m(\xi_6)}\Big||\xi_6|\prod_{j=1}^6|\hat{u_j}(\xi_j,t)|dt\\
&\le C(N^{-\frac52+}\delta^{\frac12}+N^{-3+})N_{\max}^{0-}\prod_{j=1}^6\|u_j\|_{X^\delta_{1,\frac12+}}.
\end{aligned}$$

where $N_{\max}=\max\{N_1,N_2,\dots,N_6\}$ and $*$ denotes integration on the set $\sum_{j=1}^6\xi_j=0$.

First. At least three of ξ_i's satisfy $|\xi_i| \geq CN$. Let the largest three of $|\xi_i|$ be N_1^*, N_2^* and N_3^*. Then by Hölder inequality $L^6_{x,t} - L^6_{x,t} - L^6_{x,t} - L^2_{x,t} - L^\infty_{x,t} - L^\infty_{x,t}$, (2.14), (2.15) and Sobolev embedding we have

$$\begin{aligned} I_7 &\leq C(\frac{N_1^*}{N})^{1-s}(\frac{N_2^*}{N})^{1-s}(\frac{N_3^*}{N})^{1-s} N_1^* N_1^{*-\frac{7}{6}} N_2^{*-\frac{7}{6}} N_3^{*-\frac{7}{6}} \delta^{\frac{1}{2}} \prod_{j=1}^{6} \|u_j\|_{X^\delta_{1,\frac{1}{2}+}} \\ &\leq C N_1^{*\frac{5}{6}-s} N_2^{*-\frac{1}{6}-s} N_3^{*-\frac{1}{6}-s} \delta^{\frac{1}{2}} \prod_{j=1}^{6} \|u_j\|_{X^\delta_{1,\frac{1}{2}+}} \\ &\leq C N^{-\frac{5}{2}+} \delta^{\frac{1}{2}} N_{\max}^{0-} \prod_{j=1}^{6} \|u_j\|_{X^\delta_{1,\frac{1}{2}+}}. \end{aligned}$$

Second. Exactly two of $|\xi_i| \geq CN$ and the others $\ll N$. For example, $|\xi_4|, |\xi_6| \geq CN$. Then , using Sobolev embedding, (2.14) and (2.19), we get

$$\begin{aligned} I_7 &\leq C(\frac{N_4}{N})^{1-s}(\frac{N_6}{N})^{1-s} N_6 \|u_1 u_4\|_{L^2_{xt}} \|u_2 u_6\|_{L^2_{xt}} \|u_3\|_{L^\infty_{xt}} \|u_5\|_{L^\infty_{xt}} \\ &\leq C(\frac{N_4}{N})^{1-s}(\frac{N_6}{N})^{1-s} N_6 N_4^{-2} N_6^{-2} \prod_{j=1}^{6} \|u_j\|_{X^\delta_{1,\frac{1}{2}+}} \\ &\leq C N^{-3+} N_{\max}^{0-} \prod_{j=1}^{6} \|u_j\|_{X^\delta_{1,\frac{1}{2}+}}. \end{aligned}$$

□

3. Proof of Theorem 1.1

For completeness, we give the proof of Theorem 1.1 in this section (see also [14, 17]). For any fixed $T > 0$, we want to construct the solution of the time initial value (1.1) on the interval $[0, T]$.

Since $\|I\varphi\|^2_{H^1(\mathbb{R})} \leq CN^{2(1-s)}$, it follows from (2.6) that

$$|E(I\varphi)| \leq C' N^{2(1-s)} \leq 2C' N^{2(1-s)},$$

which, by (2.5), implies $\|I\varphi\|^2_{H^1(\mathbb{R})} \leq \hat{C} N^{2(1-s)}$ with $\hat{C} = \hat{C}(2C')$. Applying Theorem 1.1 we know that the solution u exists on $[0, \delta]$ with

$$\begin{gathered} \delta \geq C'' \|I\varphi\|^{-\theta}_{H^1(\mathbb{R})} \geq C''\big(\hat{C} N^{(1-s)}\big)^{-\theta} = C_0 N^{-(1-s)\theta}, \\ \|Iu(t)\|_{X^\delta_{1,\frac{1}{2}+}} \leq C \|I\varphi\|_{H^1(\mathbb{R})} \leq \hat{C} C N^{1-s} \quad \text{for } 0 \leq t \leq \delta. \end{gathered}$$

By Lemma 2.3, we have

$$|E(Iu(\delta)) - E(I\varphi)| \leq C''' [(N^{-1+}\delta^{\frac{2}{3}} + N^{-2+}) N^{4(1-s)} + (N^{-\frac{5}{2}+}\delta^{\frac{1}{2}} + N^{-3+}) N^{6(1-s)}],$$

where C''' depends only on $\hat{C}C$. As long as

$$C''' [(N^{-1+}\delta^{\frac{2}{3}} + N^{-2+}) N^{4(1-s)} + (N^{-\frac{5}{2}+}\delta^{\frac{1}{2}} + N^{-3+}) N^{6(1-s)}] \leq C' N^{2(1-s)},$$

we have

$$|E(Iu(\delta))| \leq 2C' N^{2(1-s)}.$$

It follows, by considering δ as the initial time, using $Iu(\delta)$ as the initial value, and applying Theorem 1.1, that the problem (1.1) has a solution on $\mathbb{R}\times[\delta,2\delta]$. In this way we succeed to extend the solution of (1.1) to the time interval $[0,2\delta]$.

The above argument can be repeated for K steps as long as the following condition on K is satisfied:

$$C'''[(N^{-1+}\delta^{\frac{2}{3}}+N^{-2+})N^{4(1-s)}+(N^{-\frac{5}{2}+}\delta^{\frac{1}{2}}+N^{-3+})N^{6(1-s)}]K\le C'N^{2(1-s)}.$$

In order to extend the solution to the time interval $[0,T]$, we must have $K\delta\ge T$, or $K\ge T\delta^{-1}$. Since the minimum of all such K satisfies $K\sim T\delta^{-1}$, to arrive at this goal we only need to have

$$CC'''[(N^{-1+}\delta^{\frac{2}{3}}+N^{-2+})N^{4(1-s)}+(N^{-\frac{5}{2}+}\delta^{\frac{1}{2}}+N^{-3+})N^{6(1-s)}]T\delta^{-1}\le C'N^{2(1-s)}.$$

Since $\delta\ge C_0N^{-(1-s)\theta}$, this can be fulfilled if we can choose a sufficiently large number N so that

$$\begin{aligned}
&CC'''C_0^{-1}[(N^{-1+}N^{\frac{(1-s)\theta}{3}}+N^{-2+}N^{(1-s)\theta})N^{4(1-s)}\\
&+(N^{-\frac{5}{2}+}N^{\frac{(1-s)\theta}{2}}+N^{-3+}N^{(1-s)\theta})N^{6(1-s)}]T\\
&\le C'N^{2(1-s)}.
\end{aligned}$$

Though direct computation, we know that the above condition is satisfied if $s>6/7$. Hence, the solution exists on $\mathbb{R}\times[0,T]$ for any $T>0$, and it belongs to and is unique in $X^T_{s,\frac{1}{2}+}$.

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