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The Optimal Order of Convergence for Stable Evaluation of Differential Operators *

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Abstract

An optimal order of convergence result, with respect to the error level in the data, is given for a Tikhonov-like method for approximating values of an unbounded operator. It is also shown that if the choice of parameter in the method is made by the discrepancy principle, then the order of convergence of the resulting method is suboptimal. Finally, a modified discrepancy principle leading to an optimal order of convergence is developed.

1 Introduction

Suppose that $L : \mathcal{D}(L) \subseteq H_1 \to H_2$ is a closed densely defined unbounded linear operator from a Hilbert space H_1 into a Hilbert space H_2. The problem of computing values $y = Lx$, for $x \in \mathcal{D}(L)$, is then ill–posed in the sense that small perturbations in x may lead to data x^δ satisfying $\|x - x^\delta\| \leq \delta$, but $x^\delta \notin \mathcal{D}(L)$, or, even if $x^\delta \in \mathcal{D}(L)$, it may happen that $Lx^\delta \not\to Lx$ as $\delta \to 0$, since the operator L is unbounded.

Morozov has studied a stable method for approximating the value Lx when only approximate data x^δ is available (see [7] for information on Morozov's work). This method takes as an approximation to $y = Lx$ the vector $y_\alpha^\delta = Lz_\alpha^\delta$, where z_α^δ minimizes the functional

$$\|z - x^\delta\|^2 + \alpha\|Lz\|^2 \qquad (\alpha > 0) \tag{1.1}$$

over $\mathcal{D}(L)$. This is equivalent to

$$y_\alpha^\delta = L(I + \alpha L^* L)^{-1} x^\delta. \tag{1.2}$$

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Morozov shows that if $\alpha = \alpha(\delta) \to 0$ as $\delta \to 0$, in such a way that $\frac{\delta}{\sqrt{\alpha}} \to 0$, then $y_\alpha^\delta \to Lx$ as $\delta \to 0$. He also develops an a posteriori method, the *discrepancy principle*, for choosing the parameter α, depending on the data x^δ, that leads to a stable convergent approximation scheme for Lx.

As a simple concrete example of this type of approximation, consider differentiation in $L^2(\mathbf{R})$. That is, the operator L is defined on $H^1(\mathbf{R})$, the Sobolev space of functions possessing a weak derivative in $L^2(\mathbf{R})$, by $Lx = x'$. For a given data function $x^\delta \in L^2(\mathbf{R})$ satisfying $\|x - x^\delta\| \le \delta$, the stabilized approximate derivative (1.2) is easily seen (using Fourier transform analysis) to be given by

$$y_\alpha^\delta(s) = \int_{-\infty}^{\infty} \sigma_\alpha(s-t) x^\delta(t)\, dt$$

where the kernel σ_α is given by

$$\sigma_\alpha(t) = -\frac{\operatorname{sign}(t)}{2\alpha} \exp(-|t|/\sqrt{\alpha}).$$

Another concrete example of this stable evaluation method is provided by the Dirichlet to Neumann map. Consider for simplicity the unit disk D and unit circle ∂D. For a given function g on ∂D we denote by u the function which is harmonic in D and takes boundary values g. The operator L is then defined by $Lg = \frac{\partial u}{\partial n}$. To be more specific, L is the closed operator defined on the dense subspace

$$\mathcal{D}(L) = \left\{ g \in L^2(\partial D) : \sum_{n \in \mathbf{Z}} |n|^2 |\hat{g}(n)|^2 < \infty \right\}$$

of $L^2(\partial D)$ by

$$(Lg)(e^{i\theta}) = \sum_{n \in \mathbf{Z}} |n| \hat{g}(n) \exp(in\theta)$$

where

$$\hat{g}(n) = \frac{1}{2\pi} \int_0^{2\pi} g(t) e^{-int}\, dt\,.$$

The stable approximation (1.2) for Lx, given approximate data x^δ, is then

$$y_\alpha^\delta(e^{i\theta}) = \sum_{n \in \mathbf{Z}} \left(\frac{n}{1 + \alpha n^2} \right) \hat{x^\delta}(n) \exp(in\theta).$$

Our aim is to provide an order of convergence result for $\{y_\alpha^\delta\}$ and to show that this order of convergence is essentially best possible. Our approach, which is inspired by a work of Lardy [6] on generalized inverses of unbounded operators, is based on spectral analysis of certain *bounded* operators associated with L (see also [5] where other consequences of this approach are investigated). We also determine the best possible order of convergence when the discrepancy principle is used to determine α. This order of convergence turns out to be suboptimal. For results of a similar nature pertaining to Tikhonov regularization for solving first kind equations involving bounded operators, see [4, Chapter 3]. Finally, we propose a modification of the discrepancy principle for approximating values of an unbounded operator that leads to an optimal convergence rate.

2 Order of Convergence

To establish the order of convergence of (1.2) it will be convenient to reformulate (1.2) as

$$y_\alpha^\delta = L\check{L}[\alpha I + (1-\alpha)\check{L}]^{-1}x^\delta \tag{2.1}$$

where $\check{L} = (I + L^*L)^{-1}$ and $L\check{L}$ are known to be bounded everywhere defined linear operators and $\check{L}$ is self–adjoint with spectrum $\sigma(\check{L}) \subseteq [0,1]$ (see, e.g. [8, p.307]).

Because x^δ in (2.1) is operated upon by a product of bounded operators, we see that for fixed $\alpha > 0$, y_α^δ depends continuously on x^δ, that is, the approximations $\{y_\alpha^\delta\}$ are stable. The representation (2.1) has certain advantages in that the dependence of y_α^δ on bounded operators ($\check{L}$ and $L\check{L}$), which are independent of α, is explicit. To further simplify the presentation, we introduce the functions

$$T_\alpha(t) = (\alpha + (1-\alpha)t)^{-1}, \quad \alpha > 0,\ t \in [0,1].$$

We then have

$$y_\alpha^\delta = L\check{L}T_\alpha(\check{L})x^\delta.$$

The approximation with no data error will be denoted by y_α:

$$y_\alpha = L\check{L}T_\alpha(\check{L})x.$$

Theorem 2.1 If $x \in \mathcal{D}(LL^*L)$ and $\alpha = \alpha(\delta)$ satisfies $\frac{\alpha^3}{\delta^2} \to C > 0$ as $\delta \to 0$, then $\|y_\alpha^\delta - Lx\| = O\big(\delta^{\frac{2}{3}}\big)$.

Proof: Let $w = (I + LL^*)Lx$. Then $Lx = \hat{L}w$, where $\hat{L} = (I + LL^*)^{-1}$. Since

$$L\check{L} = L(I + L^*L)^{-1} = (I + LL^*)^{-1}L = \hat{L}L$$

and $Lx = (I + LL^*)^{-1}w = \hat{L}w$, we obtain from (2.1)

$$\begin{aligned} y_\alpha - Lx &= L\left(\check{L} - [\alpha I + (1-\alpha)\check{L}]\right)[\alpha I + (1-\alpha)\check{L}]^{-1}x \\ &= \alpha L(\check{L} - I)[\alpha I + (1-\alpha)\check{L}]^{-1}x \\ &= \alpha(\hat{L} - I)[\alpha I + (1-\alpha)\hat{L}]^{-1}Lx \\ &= \alpha(\hat{L} - I)T_\alpha(\hat{L})\hat{L}w. \end{aligned}$$

Since $\|T_\alpha(\hat{L})\hat{L}\| \le 1$, we find that

$$\|y_\alpha - Lx\| = O(\alpha). \tag{2.2}$$

Also,

$$\begin{aligned} \|y_\alpha^\delta - y_\alpha\|^2 &= (L^*L\check{L}T_\alpha(\check{L})(x^\delta - x), \check{L}T_\alpha(\check{L})(x^\delta - x)) \\ &= ((I-\check{L})T_\alpha(\check{L})(x^\delta - x), \check{L}T_\alpha(\check{L})(x^\delta - x)) \\ &\le \|I-\check{L}\|\tfrac{\delta^2}{\alpha} \end{aligned} \tag{2.3}$$

since $\|T_\alpha(\check{L})\| \le \frac{1}{\alpha}$. Therefore,

$$\|y_\alpha^\delta - y_\alpha\| = O\left(\frac{\delta}{\sqrt{\alpha}}\right).$$

We then have

$$\|y_\alpha^\delta - Lx\| = O(\alpha) + O\left(\frac{\delta}{\sqrt{\alpha}}\right) = O\big(\delta^{\frac{2}{3}}\big),$$

since $\frac{\alpha^3}{\delta^2} \to C > 0$. □

This theorem shows that under the regularity condition $x \in \mathcal{D}(LL^*L)$ on the exact data the order of convergence $O\left(\delta^{\frac{2}{3}}\right)$ is attainable by the approximation (2.1) using approximate data with error level δ. In the next section we show that this order is best possible, except for the trivial case when $x \in N(L)$, i.e., when $Lx = 0$.

3 Optimality

We begin by showing that any improvement in the order $O\big(\delta^{\frac{2}{3}}\big)$ entails a certain convergence rate for the parameter α.

Lemma 3.1 If $x \notin N(L)$ and $\|y_\alpha^\delta - Lx\| = o\big(\delta^{\frac{2}{3}}\big)$ for all x^δ satisfying $\|x - x^\delta\| \le \delta$, then $\alpha = o\big(\delta^{\frac{2}{3}}\big)$.

Proof: Let $x^\delta = x - \delta u$, where u is a unit vector and let $e_\alpha^\delta = y_\alpha^\delta - Lx$. Then

$$\begin{aligned} [\alpha I + (1-\alpha)\hat{L}]e_\alpha^\delta &= [\alpha I + (1-\alpha)\hat{L}]\left(L\check{L}(\alpha I + (1-\alpha)\check{L})^{-1}x - Lx\right) - \\ &\qquad \delta[\alpha I + (1-\alpha)\hat{L}]L\check{L}(\alpha I + (1-\alpha)\check{L})^{-1}u \\ &= \alpha(\hat{L} - I)Lx - \delta L\check{L}u. \end{aligned}$$

Since $\|e_\alpha^\delta\| = o\big(\delta^{\frac{2}{3}}\big)$, by assumption, and since

$$\|\delta L\check{L}u\| \le \delta\|L\check{L}\| = o\big(\delta^{\frac{2}{3}}\big),$$

we find that
$$\frac{\alpha}{\delta^{\frac{2}{3}}}\|(\hat{L}-I)Lx\| \to 0, \text{ as } \delta \to 0.$$

But $x \notin N(L) = N((\hat{L}-I)L)$ and hence $\alpha = o\left(\delta^{\frac{2}{3}}\right)$. □

We now show that for a wide class of operators the order of convergence $O\left(\delta^{\frac{2}{3}}\right)$ can not be improved. We will consider the important class of operators L^*L which have a divergent sequence of eigenvalues. Such is the case if L is the derivative operator, when $-L^*L$ is the Laplacian operator, or more generally whenever L is a differential operator for which $\check{L}$ is compact.

Theorem 3.1 If L^*L has eigenvalues $\mu_n \to \infty$ and $\|y_\alpha^\delta - Lx\| = o\left(\delta^{\frac{2}{3}}\right)$ for all x^δ with $\|x - x^\delta\| \le \delta$, then $x \in N(L)$.

Proof: If $x \notin N(L)$, then $\alpha = o\left(\delta^{\frac{2}{3}}\right)$, by Lemma 3.1. Let $e_\alpha^\delta = y_\alpha^\delta - Lx$, then
$$\|e_\alpha^\delta\|^2 = \|y_\alpha - Lx\|^2 + 2(y_\alpha - Lx, y_\alpha^\delta - y_\alpha) + \|y_\alpha^\delta - y_\alpha\|^2$$
and by hypothesis $\frac{\|y_\alpha - Lx\|^2}{\delta^{\frac{4}{3}}} \to 0$ as $\delta \to 0$ (since $x^\delta = x$ satisfies $\|x - x^\delta\| \le \delta$). Therefore we must have
$$\frac{2(y_\alpha - Lx, y_\alpha^\delta - y_\alpha) + \|y_\alpha^\delta - y_\alpha\|^2}{\delta^{\frac{4}{3}}} \to 0 \text{ as } \delta \to 0. \tag{3.1}$$

Suppose that $\{u_n\}$ are orthonormal eigenvectors of L^*L associated with $\{\mu_n\}$. Then $\{u_n\}$ are eigenvectors of $\check{L}$ associated with the eigenvalues $\lambda_n = \frac{1}{1+\mu_n}$ and $\lambda_n \to 0$ as $n \to \infty$. Now let $x^\delta = x + \delta u_n$. Then
$$\begin{aligned}
\|y_\alpha^\delta - y_\alpha\|^2 &= \delta^2\left(\check{L}(\alpha I + (1-\alpha)\check{L})^{-1}u_n, L^*L\check{L}(\alpha I + (1-\alpha)\check{L})^{-1}u_n\right)\\
&= \delta^2\lambda_n^2\mu_n(\alpha + (1-\alpha)\lambda_n)^{-2}\\
&= \delta^2\lambda_n(1-\lambda_n)(\alpha + (1-\alpha)\lambda_n)^{-2}.
\end{aligned}$$

Therefore, if $\delta = \delta_n = \lambda_n^{\frac{3}{2}}$, then $\delta_n \to 0$ as $n \to \infty$ and
$$\frac{\|y_\alpha^{\delta_n} - y_\alpha\|^2}{\delta_n^{\frac{4}{3}}} = (1-\lambda_n)\left(\frac{\alpha}{\delta_n^{\frac{2}{3}}} + 1 - \alpha\right)^{-2} \to 1 \text{ as } n \to \infty. \tag{3.2}$$

Finally, we have
$$\frac{|(y_\alpha - Lx, y_\alpha^{\delta_n} - y_\alpha)|}{\delta_n^{\frac{4}{3}}} \le \frac{\|y_\alpha - Lx\|}{\delta_n^{\frac{2}{3}}}\frac{\|y_\alpha^{\delta_n} - y_\alpha\|}{\delta_n^{\frac{2}{3}}} \to 0.$$

This, along with (3.2), contradicts (3.1) and hence $x \in N(L)$. □

4 The Discrepancy Principle

We may write the approximation y_α^δ to Lx as

$$y_\alpha^\delta = Lz_\alpha^\delta \quad \text{where} \quad z_\alpha^\delta = \check{L}T_\alpha(\check{L})x^\delta. \tag{4.1}$$

Morozov [7, p.125] has shown that if $\|x^\delta\| > \delta$ (i.e., the signal–to–noise ratio is greater than one), then there is a unique $\alpha = \alpha(\delta) > 0$ such that

$$\|z_{\alpha(\delta)}^\delta - x^\delta\| = \delta. \tag{4.2}$$

Moreover, he showed that $y_{\alpha(\delta)}^\delta \to Lx$ as $\delta \to 0$. We now provide an order of convergence for this method and show that, in general, it can not be improved.

Theorem 4.1 If $x \in \mathcal{D}(L^*L)$ and $x \notin N(L)$, then $\|y_{\alpha(\delta)}^\delta - Lx\| = O(\sqrt{\delta})$.

Proof: First note that

$$[\alpha I + (1-\alpha)\check{L}](z_\alpha^\delta - x^\delta) = \alpha(\check{L} - I)x^\delta.$$

Moreover, note that (cf. (2.1)) $\|\alpha I + (1-\alpha)\check{L}\| \le 1$.

Therefore, if α is chosen by (4.2), then

$$\alpha\|(\check{L} - I)x^\delta\| \le \|z_\alpha^\delta - x^\delta\| = \delta$$

and hence

$$\|(\check{L} - I)x^\delta\| \le \frac{\delta}{\alpha(\delta)}.$$

Since $x \notin N(L)$, we have $x \notin N(\check{L} - I)$ and hence

$$0 < \|(\check{L} - I)x\| \le \liminf_{\delta \to 0} \frac{\delta}{\alpha(\delta)}.$$

We therefore have

$$\alpha = O(\delta). \tag{4.3}$$

Since z_α^δ minimizes (1.1) over $\mathcal{D}(L)$ and $\alpha(\delta)$ satisfies (4.2), it follows that

$$\begin{aligned}
\delta^2 + \alpha(\delta)\|Lz_{\alpha(\delta)}^\delta\|^2 &= \|z_{\alpha(\delta)}^\delta - x^\delta\|^2 + \alpha(\delta)\|Lz_{\alpha(\delta)}^\delta\|^2 \\
&\le \|x - x^\delta\|^2 + \alpha(\delta)\|Lx\|^2 \\
&\le \delta^2 + \alpha(\delta)\|Lx\|^2
\end{aligned}$$

and hence $\|Lz_{\alpha(\delta)}^\delta\| \le \|Lx\|$. We then have

$$\begin{aligned}
\|y_{\alpha(\delta)}^\delta - Lx\|^2 &= \|y_{\alpha(\delta)}^\delta\|^2 - 2(y_{\alpha(\delta)}^\delta, Lx) + \|Lx\|^2 \\
&\le 2(Lx - y_{\alpha(\delta)}^\delta, Lx) = 2(x - z_{\alpha(\delta)}^\delta, L^*Lx) \\
&\le 4\delta\|L^*Lx\|
\end{aligned}$$

and hence $\|y^\delta_{\alpha(\delta)} - Lx\| = O(\sqrt{\delta})$. □

It turns out that if the parameter is chosen by the discrepancy method (4.2), then the order of convergence derived in Theorem 4.1 can not be improved in general. To see this, suppose that $\check{L}$ has a sequence of eigenvalues $\lambda_n \to 0$ and that $\{u_n\}$ is a corresponding sequence of orthonormal eigenvectors. Furthermore, let $\lambda_n = \frac{1}{1+\mu_n}$, $x = u_1$, and
$x^{\delta_n} = u_1 + \delta_n u_n$. An easy calculation then gives

$$\|y^{\delta_n}_\alpha - Lx\|^2 \geq \frac{\lambda_n^2}{(\alpha + (1-\alpha)\lambda_n)^2}\delta_n^2 \mu_n. \tag{4.4}$$

Now set $\delta_n = \frac{\mu_n}{(1+\mu_n)^2}$, then $\delta_n \to 0$ as $n \to \infty$. We will show that if α satisfies (4.3), then $\|y^{\delta_n}_\alpha - Lx\| = o(\sqrt{\delta_n})$ is not possible. Indeed, if this were the case, then by (4.4) we have

$$\left(\frac{\alpha}{\delta_n} + (1-\alpha)\frac{\lambda_n}{\delta_n}\right)^{-2} = \mu_n \lambda_n^2 \delta_n (\alpha + (1-\alpha)\lambda_n)^{-2} \to 0$$

and hence $\frac{\alpha}{\delta_n} + (1-\alpha)\frac{\lambda_n}{\delta_n} \to \infty$. But if α is chosen by (4.2), then by (4.3), $\frac{\alpha}{\delta_n}$ is bounded and hence $\frac{\lambda_n}{\delta_n} \to \infty$. But $\frac{\lambda_n}{\delta_n} = \frac{1}{\mu_n} + 1 \to 1$, a contradiction.

In the next section we show how the discrepancy principle can be modified to recover the optimal order of convergence.

5 Optimal Discrepancy Methods

Engl and Gfrerer [1],[2],[3] have developed discrepancy principles of optimal order for approximating solutions of bounded linear operator equations of the first kind by Tikhonov regularization. In this section we investigate similar principles for approximating values of unbounded linear operators.

We begin by considering the function

$$\rho(\alpha) = \alpha^2 \|z^\delta_\alpha - x^\delta\|.$$

By using a spectral representation of the operator $\check{L}T_\alpha(\check{L})$ which defines z^δ_α via (4.1), it is easy to see that the function $\alpha \to \rho(\alpha)$ is continuous, strictly increasing and satisfies

$$\lim_{\alpha \to 0+} \rho(\alpha) = 0 \text{ and } \lim_{\alpha \to \infty} \rho(\alpha) = \infty$$

(we assume that $x^\delta \notin N(L)$, for otherwise the approximations are trivial). Therefore, there is a unique $\alpha = \alpha(\delta) > 0$ satisfying

$$\|z^\delta_\alpha - x^\delta\| = \frac{\delta^2}{\alpha^2}. \tag{5.1}$$

We will show that the modified discrepancy principle (5.1) leads, under suitable conditions, to an optimal order of convergence for the approximations y^δ_α to Lx.

Theorem 5.1 Suppose $x \in \mathcal{D}(L^*L)$ and $x \notin N(L)$. If $\alpha = \alpha(\delta)$ is chosen by condition (5.1), then $\frac{\delta^2}{\alpha^3(\delta)} \to \|L^*Lx\| > 0$ as $\delta \to 0$.

Proof: To simplify notation we set $\alpha = \alpha(\delta)$ in the proof. First we show that $\alpha \to 0$ as $\delta \to 0$. Since

$$[\alpha I + (1-\alpha)\check{L}](z_\alpha^\delta - x^\delta) = \alpha(\check{L} - I)x^\delta \text{ and } \|\check{L}\| \leq 1,$$

we have

$$\alpha\|(\check{L} - I)x^\delta\| \leq \|z_\alpha^\delta - x^\delta\| = \frac{\delta^2}{\alpha^2}. \tag{5.2}$$

Also, $(\check{L} - I)x^\delta \to (\check{L} - I)x \neq 0$ as $\delta \to 0$ since $\check{L}x = x$ implies $L^*Lx = 0$, i.e., $x \in N(L)$.

Therefore, from (5.2), we find that $\alpha \to 0$ as $\delta \to 0$.

Next we show that $\frac{\delta}{\alpha} \to 0$ as $\delta \to 0$. In fact,

$$\|z_\alpha - z_\alpha^\delta\| = \|\check{L}T_\alpha(\check{L})(x - x^\delta)\| \leq \delta$$

and

$$x - z_\alpha = x - \check{L}T_\alpha(\check{L})x \to 0 \text{ as } \alpha \to 0$$

since $N(\check{L}) = \{0\}$ and $tT_\alpha(t) \to 1$ as $\alpha \to 0$ for each $t \neq 0$. Therefore $\|x - z_\alpha^\delta\| \to 0$ as $\delta \to 0$. We then have

$$\frac{\delta^2}{\alpha^2} = \|z_\alpha^\delta - x^\delta\| \leq \|z_\alpha^\delta - x\| + \delta \to 0 \text{ as } \delta \to 0$$

and hence $\frac{\delta}{\alpha} \to 0$ as $\delta \to 0$.

We can now show that $L^*Lz_\alpha^\delta \to L^*Lx$ as $\delta \to 0$.

Indeed,

$$L^*Lz_\alpha - L^*Lx = (\check{L}T_\alpha(\check{L}) - I)L^*Lx \to 0 \text{ as } \delta \to 0 \tag{5.3}$$

and

$$L^*L(z_\alpha^\delta - z_\alpha) = (I - \check{L})T_\alpha(\check{L})(x^\delta - x),$$

therefore,

$$\|L^*L(z_\alpha^\delta - z_\alpha)\| \leq \|(I - \check{L})\|\frac{\delta}{\alpha}. \tag{5.4}$$

Since $\|T_\alpha(\check{L})\| \leq \frac{1}{\alpha}$. But, since $\frac{\delta}{\alpha} \to 0$, we find from (5.3) and (5.4) that

$$L^*Lz_\alpha^\delta \to L^*Lx \text{ as } \delta \to 0.$$

Finally, we have

$$x^\delta - z_\alpha^\delta = \alpha L^*Lz_\alpha^\delta$$

and hence, by (5.1)

$$\frac{\delta^2}{\alpha^3} = \|L^*Lz_\alpha^\delta\| \to \|L^*Lx\| \text{ as } \delta \to 0.$$

□

From Theorem 2.1 and 5.1 we immediately obtain

Corollary 5.1 If $x \in \mathcal{D}(LL^*L)$, $x \notin N(L)$ and $\alpha = \alpha(\delta)$ is chosen by (5.1), then $\|y^\delta_{\alpha(\delta)} - Lx\| = O\big(\delta^{\frac{2}{3}}\big)$.

The Corollary requires the "smoothness" condition $x \in \mathcal{D}(LL^*L)$ in order to guarantee the optimal convergence rate, but it is possible to obtain a "quasi–optimal" rate without any additional smoothness assumptions on the data x. It follows from the proof of Theorem 2.1 (specifically, from (2.3)), that

$$\frac{1}{2}\|y^\delta_\alpha - Lx\|^2 \le \|y_\alpha - Lx\|^2 + C\frac{\delta^2}{\alpha}. \tag{5.5}$$

Let $m(x,\delta)$ be the infimum, over $\alpha > 0$, of the right hand side of (5.5). It is possible, following ideas of Engl and Gfrerer [2], to choose a parameter $\alpha = \alpha(\delta)$ such that $\|y^\delta_{\alpha(\delta)} - Lx\|^2$ has the same order as $m(x,\delta)$ which we call the quasi-optimal rate. In fact, minimizing the right hand side of (5.5) leads to a condition of the form

$$f(\alpha, x) := \big([\alpha(I - \check{L})T_\alpha(\check{L})]^3 x, x\big) = C\delta^2.$$

If we denote the spectral resolution of the identity generated by the operator $\check{L}$ by $\{E_\lambda : \lambda \in [0,1]\}$, then

$$f(\alpha, z) = \int_0^1 \left[\frac{\alpha(1-\lambda)}{\alpha(1-\lambda)+\lambda}\right]^3 d\|E_\lambda z\|^2 .$$

¿From this it follows that for any $z \notin N(L)$, $f(\cdot, z)$ is a monotonically increasing continuous function satisfying

$$\lim_{\alpha\to 0} f(\alpha, z) = 0 \;\text{ and }\; \lim_{\alpha\to\infty} f(\alpha, z) = \|Pz\|^2,$$

where P is the orthogonal projector of H_1 onto $N(L)^\perp$. Therefore, for any $\delta > 0$ and $x^\delta \notin N(L)$ and any positive constant γ which is dominated by the signal-to-noise ratio of data x^δ, that is, satisfying

$$0 < \gamma < \|Px^\delta\|/\delta,$$

there is a unique choice of the parameter $\alpha = \alpha(\delta)$ satisfying

$$f(\alpha(\delta), x^\delta) = (\gamma\delta)^2.$$

It can be shown, but we will not provide the details, that, this a posteriori choice of the parameter always leads to the quasi-optimal rate $\|y^\delta_\alpha - Lx\|^2 = O(m(x,\delta))$, without any additional smoothness assumptions on the data x.

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