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EXISTENCE OF PERIODIC SOLUTIONS OF A DELAYED PREDATOR-PREY SYSTEM ON TIME SCALES

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Abstract. In this paper, we prove the existence of periodic solutions of a delayed periodic predator-prey system based on continuation theorem of coincidence degree.

1. Introduction

In recent years, the predator-prey models together with many kinds of functional responses have been of great interest to both applied mathematicians and ecologists [7, 9, 11, 15, 16, 18]. In 2006, Yu Yang et al. [17] considered the delayed system with general functional response in Gilpin model

$$\begin{aligned} x_1'(t) &= x_1(t)\big[r(t) - b(t)x_1^{\theta}(t-\tau_1(t)) - \frac{\alpha(t)x_1^{p-1}(t)}{1+mx_1^{p}(t)}x_2(t-\sigma(t))\big],\\ x_2'(t) &= x_2(t)\big[-d(t) - a(t)x_2(t-\tau_2(t)) + \frac{\beta(t)x_1^{p}(t-\tau_3(t))}{1+mx_1^{p}(t-\tau_3(t))}\big], \end{aligned} \tag{1.1}$$

where $x_1(t), x_2(t)$ represent the densities of the prey population and predator population at time t, respectively. They obtained a sufficient condition on the existence of positive periodic solutions of (1.1) by using the continuation theorem of coincidence degree theory.

In order to unify differential and difference equations, people have done a lot of research about dynamic equations on time scales [2, 3, 4, 8, 14], since the theory of time scales is introduced by hilger in [12]. To the best of our knowledge, only a few results can be found in the literature for predator-prey system by using coincidence degree theorem on time scales.

Motivated by [12, 17], the aim of this paper is to explore the existence of periodic solutions of the delayed predator-prey system with general functional response,

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which the prey population growth satisfies Gilpin model on time scales

$$
\begin{aligned}
z_1^{\Delta}(t) &= r(t) - b(t)\exp\{\theta z_1(t-\tau_1(t))\} - \frac{\alpha(t)\exp\{(p-1)z_1(t) + z_2(t-\sigma(t))\}}{1+m\exp\{pz_1(t)\}},\\
z_2^{\Delta}(t) &= -d(t) - a(t)\exp\{z_2(t-\tau_2(t))\} + \frac{\beta(t)\exp\{pz_1(t-\tau_3(t))\}}{1+m\exp\{pz_1(t-\tau_3(t))\}},
\end{aligned}
\tag{1.2}
$$

for $t \in \mathbb{T}$. As we see, if $x_1(t) = \exp z_1(t)$, $x_2(t) = \exp z_2(t)$, and $\mathbb{T} = \mathbb{R}$, then (1.2) reduces to (1.1).

The rest of this paper is organized as follows. In section 2, we present some preliminaries, including basic definitions time scales and coincidence degree theorems. We give our main result in section 3 based on the continuation theorem of coincidence degree theorem [10]. In the last section, we present an example to illustrate our main result. Also the numerical simulations are given to support the theoretical findings.

2. Preliminaries

The study of dynamic equation on time scales goes back to its founder Stefan Hilger [12] and it is a new area of still fairly theoretical exploration in mathematics.

For convenience, we first introduce some definitions and the theory of calculus on timescales, which are needed later. For more details on timescales, please see [1, 5, 6, 12, 13].

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers $\mathbb{R}$. The operators σ and ρ from $\mathbb{T}$ to $\mathbb{T}$, defined by [12],

$$
\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\} \in \mathbb{T}, \quad \text{and} \quad \rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\} \in \mathbb{T}
$$

are called the forward jump operator and the backward jump operator, respectively. In this definition

$$
\inf \emptyset := \sup \mathbb{T}, \quad \sup \emptyset := \inf \mathbb{T}.
$$

The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively.

Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$ (assume t is not left-scattered if $t = \sup \mathbb{T}$), then the delta derivative of f at the point t is defined to be the number $f^{\Delta}(t)$ (provided it exists) with the property that for each $\epsilon > 0$ there is a neighborhood U of t such that

$$
|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \le |\sigma(t) - s|, \quad \text{for all} \quad s \in U.
$$

A function f is said to be delta differentiable on $\mathbb{T}$ if f^{Δ} exists for all $t \in \mathbb{T}$. A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^{\Delta} = f(t)$ for all $t \in \mathbb{T}$. Then we define

$$
\int_a^b f(t)\Delta t = F(b) - F(a), \quad \text{for } a, b \in \mathbb{T}.
$$

Notation. Throughout this paper, $\mathbb{T}$ denotes a time scale. Let $\omega > 0$, the time scale $\mathbb{T}$ is assumed to be $\omega-$periodic, i.e., $t \in \mathbb{T}$ implies $t + \omega \in \mathbb{T}$. Let $\kappa = \min\{\mathbb{R}^+ \cap \mathbb{T}\}$, and $I_\omega = [\kappa, \kappa + \omega] \cap \mathbb{T}$. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at right-dense points in $\mathbb{T}$ and it left-sided limits exist (finite)at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{rd}(\mathbb{T})$.

Lemma 2.1. *If $a, b \in \mathbb{T}$, $\alpha, \beta \in \mathbb{R}$ and $f, g \in C_{rd}(\mathbb{T})$, then*

(a)
$$\int_a^b [\alpha f(t) + \beta g(t)]\Delta t = \alpha \int_a^b f(t)\Delta t + \beta \int_a^b g(t)\Delta t;$$

(b) *if $f(t) \geq 0$ for all $a \leq t \leq b$, then $\int_a^b f(t)\Delta t \geq 0$;*

(c) *if $|f(t)| \leq g(t)$ on $[a, b) := \{t \in \mathbb{T} : a \leq t < b\}$, then*
$$\Big|\int_a^b f(t)\Delta t\Big| \leq \int_a^b g(t)\Delta t.$$

Throughout of this paper, for (1.2) we assume that

(H) For $i = 1, 2$: $a(t), b(t), \alpha(t), \beta(t), \sigma(t), \tau_i(t) : \mathbb{R} \to [0, +\infty)$ are rd-continuous positive periodic functions with period ω and $\alpha(t) \neq 0$, $\beta(t) \neq 0$; $r(t), d(t) : \mathbb{R} \to \mathbb{R}$ are rd-continuous functions of period ω and $\int_\kappa^{\kappa+\omega} d(t)\Delta t > 0$, $\int_\kappa^{\kappa+\omega} r(t)\Delta t > 0$; p is a positive constant and $p \geq 1$; m and θ are positive constants.

In view of the actual applications of system (1.2), we consider the initial value problem
$$\begin{gathered} z_i(s) = \Phi_i(s), s \in [\kappa - \tau, \kappa] \cap \mathbb{T}, \Phi_i(\kappa) > 0, \\ \Phi_i(s) \in C_{rd}([\kappa - \tau, \kappa] \cap \mathbb{T}, \mathbb{R}^+), \quad i = 1, 2, \end{gathered}$$
where $\tau = \max_{t \in [\kappa, \kappa+\omega]}\{\tau_1(t), \tau_2(t), \tau_3(t), \sigma(t)\}$.

Next we give some fundamental definitions about coincidence degree theorem. These concepts will be used for proving the existence of solutions of (1.2).

Let X and Z be two Banach spaces, $L : \operatorname{Dom} L \subset X \to Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index Zero if $\dim \ker L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Z. If L is a Fredholm mapping of index zero and there follows that $L|\operatorname{Dom} L \cap \ker P : (I - P)X \to ImL$ is invertible. We denote the inverse of that map by K_p. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I - Q)N : \overline{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\ker L$, there exists an isomorphism $J : \operatorname{Im} Q \to \ker L$.

The following Lemma is important for the proof of our main results.

Lemma 2.2. *(Continuation Theorem [1]) Let L be a Fredholm mapping of index zero and let N be L-compact on Ω. Suppose*

(a) *for each $\lambda \in (0, 1)$, every solution x of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$;*

(b) *$QNx \neq 0$ for each $x \in \partial\Omega \cap \ker L$ and*
$$\deg\{JQN, \Omega \cap \ker L, 0\} \neq 0.$$

Then the equation $Lx = Nx$ has at least one solution lying in $\operatorname{Dom} L \cap \overline{\Omega}$.

The following lemma will be used in the proof of our results. The proof is similar to that of Lemma 3.2 established in [16]. So we omit it here.

Lemma 2.3. *Let $t_1, t_2 \in \mathbb{T}$ and $t \in \mathbb{T}$. If $g : \mathbb{T} \to \mathbb{R} \in C_{rd}(\mathbb{T})$ is $\omega-$periodic, then*
$$g(t) \leq g(t_1) + \int_\kappa^{\kappa+\omega} |g^\Delta(s)|\Delta s, \quad \textit{and} \quad g(t) \geq g(t_2) - \int_\kappa^{\kappa+\omega} |g^\Delta(s)|\Delta s.$$

By simple calculation, we get the following two lemmas.

Lemma 2.4. *The following algebraic equation*

$$\bar{b}\exp\{\theta z_1\} - \bar{r} = 0,$$
$$\bar{\beta}\frac{\exp\{pz_1\}}{1+m\exp\{pz_1\}} - \bar{a}\exp\{z_2\} - \bar{d} = 0,$$

has a unique solution.

Lemma 2.5. *If $y(t) > 0$ for $t \in \mathbb{T}$, then*

$$\frac{y^{p-1}(t)}{1+my^p(t)} \le \max\{\frac{1}{m}, 1\}.$$

3. Main result

For convenience, we denote

$$z_i(\xi_i) = \min_{t\in I_\omega} z_i(t), \quad z_i(\eta_i) = \max_{t\in I_\omega} z_i(t), \quad i = 1, 2. \tag{3.1}$$

Theorem 3.1. *Assume that condition (H) holds and*

$$\bar{a}\bar{r} - \max\{\frac{1}{m}, 1\}\overline{\alpha}\overline{\beta}\exp\{(\overline{D}+\overline{d})\omega\} > 0, \quad \frac{\overline{\beta}\exp\{pH_2\}}{1+m\exp\{pH_2\}} - \overline{d} > 0,$$

where

$$H_2 = \frac{1}{\theta}ln\Big(\frac{m\bar{a}\bar{r} - \max\{\frac{1}{m}, 1\}\bar{\alpha}\bar{\beta}exp\{(\bar{D}+\bar{d})\omega\}}{m\bar{a}\bar{b}}\Big) - (\bar{R}+\bar{r})\omega,$$
$$\bar{a} = \frac{1}{\omega}\int_\kappa^{\kappa+\omega} a(t)\Delta t, \quad \bar{r} = \frac{1}{\omega}\int_\kappa^{\kappa+\omega} r(t)\Delta t,$$
$$\bar{R} = \frac{1}{\omega}\int_\kappa^{\kappa+\omega} |r(t)|\Delta t, \quad \bar{\alpha} = \frac{1}{\omega}\int_\kappa^{\kappa+\omega} \alpha(t)\Delta t,$$
$$\bar{d} = \frac{1}{\omega}\int_\kappa^{\kappa+\omega} d(t)\Delta t, \quad \bar{D} = \frac{1}{\omega}\int_\kappa^{\kappa+\omega} |d(t)|\Delta t,$$
$$\bar{\beta}(t) = \frac{1}{\omega}\int_\kappa^{\kappa+\omega} \beta\Delta t,$$

then system (1.2)*has at least one $\omega-$periodic solution.*

Proof. Define

$$X = Z = \{(z_1, z_2)^T \in C(\mathbb{T}, \mathbb{R}^2) : z_i(t+\omega) = z_i(t),\ i = 1, 2,\ t \in \mathbb{T}\},$$
$$\|(z_1, z_2)^T\| = \sum_{i=1}^{2} \max|z_i(t)|, (z_1, z_2)^T \in X(Z).$$

then X, Z are both Banach spaces endowed with norm $\|\cdot\|$. Let

$$L : \operatorname{Dom} L \to Z, \quad L\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1^\Delta(t) \\ z_2^\Delta(t) \end{pmatrix},$$

where $\operatorname{Dom} L = X$, and $N : \operatorname{Dom} L \to Z$,

$$N\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} r(t) - b(t)\exp\{\theta z_1(t-\tau_1(t))\} - \frac{\alpha(t)\exp\{(p-1)z_1(t)\}}{1+m\exp\{pz_1(t)\}}\exp\{z_2(t-\sigma(t))\} \\ -d(t) - a(t)\exp\{z_2(t-\tau_2(t))\} + \frac{\beta(t)\exp\{pz_1(t-\tau_3(t))\}}{1+m\exp\{p(t-\tau_3(t))\}} \end{pmatrix},$$

$$P\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = Q\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\omega}\int_\kappa^{\kappa+\omega} z_1(t)\Delta t \\ \frac{1}{\omega}\int_\kappa^{\kappa+\omega} z_2(t)\Delta t \end{pmatrix},$$

where $(z_1, z_2)^T \in X$. Then

$$\ker L = \{(z_1, z_2)^T \in X | (z_1, z_2)^T = (h_1, h_2)^T \in \mathbb{R}^2, t \in \mathbb{T}\},$$
$$\operatorname{Im} L = \{(z_1, z_2)^T \in Z | \int_\kappa^{\kappa+\omega} z_1(t)\Delta(t) = 0, \quad \int_\kappa^{\kappa+\omega} z_2(t)\Delta(t) = 0\},$$
$$\dim\ker L = 2 = \operatorname{codim}\operatorname{Im} L.$$

Since $\operatorname{Im} L$ is closed in Z , then L is a Fredholm mapping of index zero. It is easy to show that P and Q are continuous projectors such that

$$\operatorname{Im} P = \ker L, \ker Q = \operatorname{Im} L = \operatorname{Im}(I - Q).$$

Furthermore, the generalized inverse (of L) $K_p : \operatorname{Im} L \to \ker P \cap \operatorname{Dom} L$ exists and is given by

$$K_p\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \int_\kappa^t z_1(s)\Delta s - \frac{1}{\omega}\int_\kappa^{\kappa+\omega}\int_\kappa^t z_1(s)\Delta s\Delta t \\ \int_\kappa^t z_2(s)\Delta s - \frac{1}{\omega}\int_\kappa^{\kappa+\omega}\int_\kappa^t z_2(s)\Delta s\Delta t \end{pmatrix}.$$

Thus

$$QN\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\omega}\int_\kappa^{\kappa+\omega}(r(t) - b(t)\exp\{\theta z_1(t-\tau_1(t))\} - \frac{\alpha(t)\exp\{(p-1)z_1(t)\}}{1+m\exp\{pz_1(t)\}}\exp\{z_2(t-\sigma(t))\})\Delta t \\ \frac{1}{\omega}\int_\kappa^{\kappa+\omega}(-d(t) - a(t)\exp\{z_2(t-\tau_2(t))\} + \frac{\beta(t)\exp\{pz_1(t-\tau_3(t))\}}{1+m\exp\{p(t-\tau_3(t))\}})\Delta t \end{pmatrix},$$

$$K_p(I-Q)N\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \int_\kappa^t z_1(s)\Delta s - \frac{1}{\omega}\int_\kappa^{\kappa+\omega}\int_\kappa^t z_1(s)\Delta s\Delta t - (t - \kappa - \frac{1}{\omega}\int_\kappa^{\kappa+\omega}(t-\kappa)\Delta t)\bar{z}_1 \\ \int_\kappa^t z_2(s)\Delta s - \frac{1}{\omega}\int_\kappa^{\kappa+\omega}\int_\kappa^t z_2(s)\Delta s\Delta t - (t - \kappa - \frac{1}{\omega}\int_\kappa^{\kappa+\omega}(t-\kappa)\Delta t)\bar{z}_2 \end{pmatrix}.$$

Obviously, QN and $K_p(I-Q)N$ are continuous. According to Arela-Ascoli theorem, it is easy to show that $K_p(I-Q)N(\bar{\Omega})$ is compact for any open bounded set $\Omega \in X$ and $QN(\bar{\Omega})$ is bounded. Thus, N is L-compact on Ω.

Now,we shall search an appropriate open bounded subset Ω for the application of the continuation theorem. For the operator equation $Lx = \lambda Nx$, $\lambda \in (0, 1)$, we have

$$\begin{aligned} z_1^{\Delta}(t) &= \lambda\Big[r(t) - b(t)\exp\{\theta z_1(t-\tau_1(t))\} \\ &\quad - \frac{\alpha(t)\exp\{(p-1)z_1(t)\}}{1+m\exp\{pz_1(t)\}}\exp\{z_2(t-\sigma(t))\}\Big], \end{aligned} \tag{3.2}$$
$$z_2^{\Delta}(t) = \lambda\Big[-d(t) - a(t)\exp\{z_2(t-\tau_2(t))\} + \frac{\beta(t)\exp\{pz_1(t-\tau_3(t))\}}{1+m\exp\{pz_1(t-\tau_3(t))\}}\Big],$$

where $t \in \mathbb{T}$. Assume $(z_1(t), z_2(t))^T$ is a solution of (3.2). Integrating (3.2), we get

$$\begin{aligned} &\int_\kappa^{\kappa+\omega} b(t)\exp\{\theta z_1(t-\tau_1)\}\Delta t \\ &+ \int_\kappa^{\kappa+\omega} \frac{\alpha(t)\exp\{(p-1)z_1(t)\}\exp\{z_2(t-\sigma(t))\}}{1+m\exp\{pz_1(t)\}}\Delta t = \bar{r}\omega, \end{aligned} \tag{3.3}$$

$$\int_\kappa^{\kappa+\omega} \frac{\beta(t)\exp\{pz_1(t-\tau_3)\}}{1+m\exp\{pz_1(t-\tau_3)\}}\Delta t - \int_\kappa^{\kappa+\omega} a(t)\exp\{z_2(t-\tau_2)\}\Delta t = \bar{d}\omega, \tag{3.4}$$

By the first equation of (3.2) and (3.3), we get

$$\begin{aligned} \int_\kappa^{\kappa+\omega} |z_1^{\Delta}(t)|\Delta t &\le \int_\kappa^{\kappa+\omega} |r(t)|\Delta t + \int_\kappa^{\kappa+\omega} \Big[b(t)\exp\{\theta z_1(t-\tau_1(t))\} \\ &\quad + \frac{\alpha(t)\exp\{(p-1)z_1(t)\}\exp\{z_2(t-\sigma(t))\}}{1+m\exp\{pz_1(t)\}}\Big]\Delta t \\ &\le (\bar{R}+\bar{r})\omega. \end{aligned}$$

By the second equation of (3.2) and (3.4), we have

$$\begin{aligned} &\int_\kappa^{\kappa+\omega} |z_2^{\Delta}(t)|\Delta t \\ &\le \int_\kappa^{\kappa+\omega} |d(t)|\Delta t + \int_\kappa^{\kappa+\omega} \Big[\frac{\beta(t)\exp\{pz_1(t-\tau_3(t))\}}{1+m\exp\{pz_1(t-\tau_3(t))\}} + a(t)\exp\{z_2(t-\tau_2(t))\Big]\Delta t \\ &\le (\bar{D}+\bar{d})\omega. \end{aligned}$$

By (3.1) and (3.4), we obtain

$$\begin{aligned} \bar{a}\omega\exp\{z_2\{\xi_2\}\} &\le \int_\kappa^{\kappa+\omega} a(t)\exp\{z_2(t-\tau_2(t))\}\Delta t \\ &= \int_\kappa^{\kappa+\omega} \frac{\beta(t)\exp\{pz_1(t-\tau_3(t))\}}{1+m\exp\{pz_1(t-\tau_3(t))\}}\Delta t - \bar{d}\omega \le \frac{\bar{\beta}\omega}{m}; \end{aligned}$$

that is,

$$z_2(\xi_2) \le \ln\{\frac{\bar{\beta}}{m\bar{a}}\} := L_2,$$

hence

$$z_2(t) \le z_2(\xi_2) + \int_\kappa^{\kappa+\omega} |z_2^{\Delta}(t)|\Delta t \le \ln\{\frac{\bar{\beta}}{m\bar{a}}\} + (\bar{D}+\bar{d})\omega := H_3. \tag{3.5}$$

From (3.1) and (3.3), we have

$$\bar{r}\omega \ge \int_\kappa^{\kappa+\omega} b(t)\exp\{\theta z_1(t-\tau_1(t))\}\Delta t \ge \bar{b}\omega\exp\{\theta z_1(\xi_1)\};$$

that is

$$z_1(\xi_1) \le \frac{1}{\theta}\ln\{\frac{\bar r}{\bar b}\} := L_1,$$

then

$$z_1(t) \le z_1(\xi_1) + \int_\kappa^{\kappa+\omega} |z_1^\Delta(t)|\Delta t \le \frac{1}{\theta}\ln\{\frac{\bar r}{\bar b}\} + (\bar R + \bar r)\omega := H_1. \tag{3.6}$$

By (3.1), (3.3), (3.6), lemma 2.5 and under the assumptions of theorem 3.1, we have

$$\begin{aligned}
\bar b\omega \exp\{\theta z_1(\eta_1)\} &\ge \int_\kappa^{\kappa+\omega} b(t)\exp\{\theta z_1(\eta_1)\}\Delta t\\
&= \bar r\omega - \int_\kappa^{\kappa+\omega} \frac{\alpha(t)\exp\{(p-1)z_1(t)\}\exp\{z_2(t-\sigma(t))\}}{1+m\exp\{pz_1(t)\}}\\
&\ge \bar r\omega - \frac{\bar\alpha\bar\beta}{m\bar a}\exp\{(\bar D + \bar d)\omega\},
\end{aligned}$$

thus

$$z_1(\eta_1) \ge \frac{1}{\theta}\ln(\frac{m\bar a\bar r - \max\{\frac{1}{m},1\}\bar\alpha\bar\beta\exp\{(\bar D+\bar d)\omega\}}{m\bar a\bar b}) := l_1.$$

We also can get that

$$\begin{aligned}
z_1(t) &\ge z_1(\eta_1) - \int_\kappa^{\kappa+\omega} |z_1^\Delta(t)|\Delta t\\
&\ge \frac{1}{\theta}\ln(\frac{m\bar a\bar r - \max\{\frac{1}{m},1\}\bar\alpha\bar\beta\exp\{(\bar D+\bar d)\omega\}}{m\bar a\bar b}) - (\bar R + \bar r)\omega := H_2.
\end{aligned} \tag{3.7}$$

By (3.6) and (3.7), we have

$$\max_{t\in[0,\omega]} |z_1(t)| \le \max\{|H_1|, |H_2|\} := H_5. \tag{3.8}$$

Now we are in a position to estimate $z_2(\eta_2)$. From (3.1), (3.4) and (3.7), we get

$$\begin{aligned}
\bar a\omega\exp\{z_2(\eta_2)\} &\ge \int_\kappa^{\kappa+\omega} a(t)\exp\{z_2(t-\tau_2)\}\Delta t\\
&= \int_\kappa^{\kappa+\omega} \frac{\beta(t)\exp\{pz_1(t-\tau_3(t))\}}{1+m\exp\{pz_1(t-\tau_3(t))\}} - \bar d\omega\\
&\ge \frac{\bar\beta\omega exp\{pH_2\}}{1+m\exp\{pH_2\}} - \bar d\omega,
\end{aligned}$$

thus

$$z_2(\eta_2) \ge \ln\{\frac{\frac{\bar\beta\exp\{pH_2\}}{1+m\exp\{pH_2\}} - \bar d}{\bar a}\} := l_2,$$

we have also

$$z_2(t) \ge z_(\eta_2) - \int_\kappa^{\kappa+\omega} |z_2^\Delta|\Delta t \ge \ln\{\frac{\frac{\bar\beta\exp\{pH_2\}}{1+m\exp\{pH_2\}} - \bar d}{\bar a}\} - (\bar D + \bar d)\omega := H_4. \tag{3.9}$$

By (3.5) and (3.9), we get

$$\max_{t\in[0,\omega]} |z_2(t)| \le \max\{|H_3|, |H_4|\} := H_6,$$

clearly, H_5, H_6 are dependent on λ. Let $H_8 = H_5 + H_6 + H_7$, where H_7 is large enough, such that $H_8 \geq |l_1| + |L_1| + |l_2| + |L_2|$. Next, for $(z_1, z_2)^T \in \mathbb{R}^2$, $\mu \in [0,1]$, we shall consider the following algebraic equations:

$$\begin{gathered} \bar{b}\exp\{\theta z_1\} + \mu\frac{\bar{\alpha}\exp\{(p-1)z_1\}\exp\{z_2\}}{1+m\exp\{pz_1\}} - \bar{r} = 0, \\ \frac{\bar{\beta}\exp\{pz_1\}}{1+m\exp\{pz_1\}} - \bar{a}\exp\{z_2\} - \bar{d} = 0. \end{gathered} \tag{3.10}$$

Similar to the above discussion, we can easily check that, every solution $(z_1^*, z_2^*)^T$ of (3.10) satisfies

$$l_1 \leq z_1^* \leq L_1, l_2 \leq z_2^* \leq L_2.$$

Take $\Omega = \{(z_1(t), z_2(t))^T \in z : \|(z_1, z_2)^T\| < H_8\}$. Obviously, Ω satisfies the condition (a) of lemma 2.2. When $z \in \partial\Omega \cap \ker L$, $(z_1, z_2)^T$ is a constant vector in $\mathbb{R}^2$, and $\|(z_1, z_2)^T\| = H_8$. So we have

$$QNz = \begin{pmatrix} \bar{b}\exp\{\theta z_1\} + \frac{\bar{\alpha}\exp\{(p-1)z_1\}\exp\{z_2\}}{1+m\exp\{pz_1\}} - \bar{r} \\ \frac{\bar{\beta}\exp\{pz_1\}}{1+m\exp\{pz_1\}} - \bar{a}\exp\{z_2\} - \bar{d} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

To calculate the Brouwer degree, we consider the homotopy:

$$H_\mu(z_1, z_2) = \mu QN(z_1, z_2) + (1-\mu)G(z_1, z_2), \mu \in (0,1],$$

where

$$G\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \bar{b}\exp\{\theta z_1\} - \bar{r} \\ \frac{\bar{\beta}\exp\{pz_1\}}{1+m\exp\{pz_1\}} - \bar{a}\exp\{z_2\} - \bar{d} \end{pmatrix}.$$

It is easy to show that $0 \notin H_\mu(\partial \cap \ker L, 0)$, for $\mu \in (0,1]$. Moreover, by lemma 2.4, algebraic equation $G(z_1, z_2) = 0$ has a unique solution in $\mathbb{R}^2$. Because of the invariance property of homotopy, we have

$$\deg\{JQN, \Omega \cap \ker L, 0\} = \deg\{QN, \Omega \cap \ker L, 0\} = \deg\{G, \Omega \cap \ker L, 0\} \neq 0.$$

We have proved that Ω satisfies all requirements of lemma 2.2. Thus, in $\bar{\Omega}$, system (1.2) has at least one ω-periodic solution. The proof is complete. □

Remark 3.2. Obviously, (1.1) in [17] is the special case of (1.2). So our result is general than that of [17]. Moreover, few papers discuss on the general functional response, such as Gillpin model we concern in this paper.

4. An example

Consider the system

$$\begin{aligned} x_1^\Delta(t) &= \frac{1}{5} - \frac{1}{20}(1+\sin t)\exp\{x_1(t-0.5)\} - \frac{\exp\{x_1(t)+x_2(t)\}}{15(1+3\exp\{2x_1(t)\}}, \\ x_2^\Delta(t) &= -\frac{1}{16}(1-\sin t) - 2\exp\{x_2(t-0.3)\} + \frac{3\exp\{2x_1(t-0.8)\}}{1+3\exp\{2x_1(t-0.8)\}}, \end{aligned} \tag{4.1}$$

where $a(t) = 2$, $b(t) = \frac{1}{20}(1+\sin t)$, $r(t) = \frac{1}{5}$, $d(t) = \frac{1}{16}(1-\sin t)$, $\alpha(t) = \frac{1}{15}$, $\beta(t) = 3$, $\tau_1(t) = 0.5$, $\tau_2(t) = 0.3$, $\sigma(t) = 0$, and $\tau_3(t) = 0.8$ are $2\pi-$period functions.

If $\mathbb{T}=\mathbb{R}$, then (4.1) reduces to the differential system

$$\begin{aligned} x_1'(t) &= \frac{1}{5} - \frac{1}{20}(1+\sin t)\exp\{x_1(t-0.5)\} - \frac{\exp\{x_1(t)+x_2(t)\}}{15(1+3\exp\{2x_1(t)\}}, \\ x_2'(t) &= -\frac{1}{16}(1-\sin t) - 2\exp\{x_2(t-0.3)\} + \frac{3\exp\{2x_1(t-0.8)\}}{1+3\exp\{2x_1(t-0.8)\}}, \end{aligned} \tag{4.2}$$

Obviously, $m=3$, $p=2$, $\theta=1$ and $\omega=2\pi$. It is easy to show that $\bar{a}=2$, $\bar{b}=\frac{1}{20}$, $\bar{r}=\bar{R}=\frac{1}{5}$, $\bar{d}=\bar{D}=\frac{1}{16}$, $\bar{\alpha}=\frac{1}{15}$ and $\bar{\beta}=3$. By some calculations, we get

$$m\bar{a}\bar{r} - \max\{\frac{1}{m},1\}\bar{\alpha}\bar{\beta}\exp\{(\bar{D}+\bar{d})\omega\} = 0.7613 > 0,$$

and

$$\frac{\bar{\beta}\exp\{pH_2\}}{1+m\exp\{pH_2\}} - \bar{d} = 0.05 > 0.$$

According to theorem 3.1, it is easy to see that (4.2) has at least one 2π-periodic solution. Numerical simulations of solution for (4.2) and the solution tends to the 2π-periodic solution see Figure 1a and Figure 1b, respectively. The simulation is performed using MATLAB software.

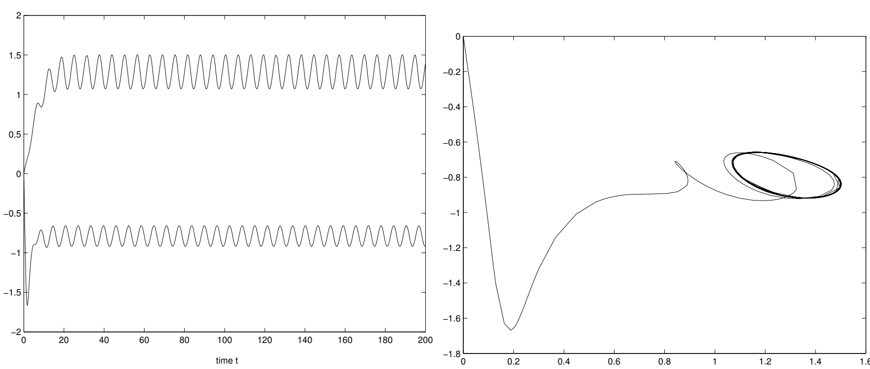

FIGURE 1. (a) Numerical solution $x_1(t)$, $x_2(t)$ of system (4.2), where $x_1(s)=x_2(s)=0$ for $s\in[-0.8,0]$. (b) Phase trajectories of system (4.2), where $x_1(s)=x_2(s)=0$ for $s\in[-0.8,0]$.

Numerical simulations of solution for (4.2) and the solution tends to the 2π-periodic solution; see Fig. 1.

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