ORBIT SIZES AND THE CENTRAL PRODUCT GROUP

by

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I. INTRODUCTION

In mathematics it is common practice to find bounds on a certain problem and then start to ask questions like: When do we have equality? What is the smallest case we have equality in? The next case after that? Dr. Keller and Dr. Yang started this process by finding a bound on the action of finite solvable groups on finite faithful completely reducible G-modules in [**9**] which we will later state as Theorem 1:

Let G be a finite solvable group and V a finite faithful completely reducible G-module, possibly of mixed characteristic. Let M be the largest orbit size in the action of G on V. Then

$$|G/G'| \leq M$$

More precisely, we have one of the following

1. $|G/G'| < M$
2. $|G/G'| = M$ and G is abelian; or
3. $|G/G'| = M$, G is nilpotent, and G has at least two different orbits of size M on V.

As stated, the next intuitive question to ask yourself is when can we push these bounds to the edge, in this case when part (c) is minimal and there are exactly two orbits. This conjecture was proven by Nathan Jones and Dr. Thomas Keller in [**7**].

Let G be a finite nonabelian group and V a finite faithful irreducible G-module. Suppose that $M = |G/G'|$ is the largest orbit size of G on V, and that there are exactly two orbits of size M on V. Then G is the dihedral group of order 8, and $|V| = 9$. Now the next intuitive question to ask ourselves is what happens when there are three orbits. In fact, at the end of the first smallest case, Dr. Keller hypothesizes at what the next smallest case is and offers it as the following conjecture:

Let G be a finite nonabelian group and V a finite faithful irreducible G-module. Suppose that $M = |G/G'|$ is the largest orbit size of G on V and that there are exactly three orbits of size M on V. Then, $G = EZ$, where $E = D_8$, $Z = Z(G)$ is cyclic of order 4, and $E \cap Z = Z(E)$ is cyclic of order 2, and $V = V(2,5)$ is of order 25.

The goal of this thesis is to prove the conjecture. The proof method follows very closely along to [**7**] and in the processes of reading the original paper some inaccuracies were found and corrected. Most notably is the last case of [**7**] which has earned its own section of this paper as it had to be corrected in order for this proof to continue smoothly.

The first section will cover definition, theorems and lemmas that will be necessary to read through the rest of the paper. Section two will prove to the reader that our group $D_8 \circ D_4$ does in fact meet the requirements and it will formally state the theorem we plan to prove. The following section will just give additional information to the reader about $D_8 \circ C_4$ and its subgroups, as well as all of their orbits of $V(2,5)$. We then have our section with serves as a correction to [**7**] and finally we prove the main result.

Definitions

This thesis will assume that the reader is familiar with the undergraduate concepts of a group; in this section we will establish the language used throughout this thesis. The definitions used in this paper have been taken from [**3**] and the notation is consistent with literature in finite group theory. To quickly review, a group is a set along with an associative binary operation that is closed contains both an identity element and inverses. This thesis is concerned with only finite groups. Therefore, we may say that the size (or order) of G, denoted $|G|$, is not infinite. Two examples of groups are the dihedral group of order 8, denoted D_8, and the cyclic group of order 4, denoted C_4. The group D_8 consists of the symmetries of a square, four rotations and four "flips." The group can be represented as $D_8 = \{1, r, r^2, r^3, s, rs, r^2, s, r^3 s\}$ where r is representing a 90 degree rotation of a square and s is representing a flip over a fixed axis of symmetry of a square. We can find smaller group structures within D_8 called subgroups. A subset of a group G is a subgroup if it is closed under G's multiplication and forms a group with respect to this multiplication. The statement that H is a subgroup of G will be denoted by $H \leq G$. In this paper subgroups will be used to allow for inductions on the group. In the case of D_8 we can look at the subset of rotations $\{1, r, r^2, r^3\}$ and see that it forms a subgroup. Specifically, the cyclic group of order four is a group of size four generated by one element, in this subgroup's case, r. We will now define more interesting structures and properties that a group can have.

Definition 1. The **center** of a group G, denoted $Z(G)$, is the set $Z(G) = \{y \in G | xy = yx \text{ for all } x \in G\}$.

Note that the center $Z(G)$ is a subgroup of G.

Definition 2. A **normal subgroup** is a subgroup $H \leq G$ such that for all $g \in G$ we have $g^{-1}Hg = H$. This is denoted by $H \trianglelefteq G$, and $H \triangleleft G$ if and only if $H \trianglelefteq G$ and $H \neq G$.

Notice that a normal subgroup is a group that is invariant under conjugation by elements in G. These groups are also known to be characterized as the kernel of some group homomorphism.

Definition 3. Let $H \leq G$ be a subgroup and let $g \in G$. The **right coset** of H determined by g is the set $Hg = \{hg \,|\, h \in H\}$.

Definition 4. Let $H \trianglelefteq G$. The **factor group** of G by H is the set $\{Hg | g \in G\}$ with identity element eH and multiplication defined as $(Hx)(Hy) = Hxy$ for all $x, y \in G$.

Definition 5. Let G and H be groups with $V \leq Z(G)$ and $W \leq Z(H)$ such that there exists isomorphism $\phi : V \to W$. The **central product** of G and H with respect to ϕ is the factor group of $(G \times H)/X$ where $X = \{(x, \phi(x)^{-1}) | x \in V\}$.

This paper will focus on the central product of D_8 and C_4. This group, denoted $G = D_8 \circ C_4$, is of order 16 and can be represented as a subgroup of $GL(2,5)$, the group of invertible 2×2 matrices over the field with 5 elements.

Definition 6. A group G is **solvable** if there exists a chain of subgroups $1 = H_0 \lhd H_1 \lhd \cdots \lhd H_n = G$ such that H_{i+1}/H_i is abelian for $i = 0, \ldots, n^{-1}$.

Notice this means that solvable groups are constructed by extensions of abelian groups. Solvable groups will be the focus of this paper.

Definition 7. The **Frattini subgroup**, $\Phi(G)$ is the intersection of all maximal proper subgroups of G.

The Frattini subgroup always exists in finite groups and possesses many useful properties. For example, the Frattini subgroup is always normal. In the case of $D_8 \circ C_4$ we have three maximal subgroups D_8 , $C_2 \times C_4$, and Q_8. The reader can find that, $\Phi(D_8 \circ C_4) = C_4$. One property of the Frattini subgroup we will benefit from is that it is nilpotent.

Definition 8. Let G be a finite group and p a prime. A **Sylow p-subgroup** of G is a subgroup $P \leq G$ such that $|P| = p^a$ is the full power of p dividing $|G|$. The set of all Sylow p-subgroups of G is denoted $\mathrm{Syl}_p(G)$

Definition 9. Let G be finite and solvable and let π be any set of prime numbers. Hall's Theorem guarantees a subgroup $H \leq G$ with order divisible only by primes in π with $|G|/|H|$ divisible by none of these primes. A subgroup $H \leq G$ satisfying these conditions is called a **Hall π-subgroup** of G.

The Sylow and Hall subgroups will provide us a natural way to 'split-up' groups into two subgroups using a free product.

Definition 10. A group G is called **nilpotent** if there exists a chain of subgroups $1 = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G$ such that $G_{i+1}/G_i \leq Z(G/G_i)$. Where $Z(G/G_i) = \{g \in G/G_i \,|\, gx = xg \text{ for all } x \in G/G_i\}$.

Note that a group G is nilpotent if and only if it can be written as the direct product of its Sylow subgroups for all primes p dividing $|G|$. Notice that because $D_8 \circ C_4$ is a finite p-group that it must be a nilpotent group.

Definition 11. The **Fitting Subgroup** of G, written as $F(G)$ is the unique largest normal nilpotent subgroup of G.

In the example of $D_8 \circ C_4$, we would have $F(D_8 \circ C_4) = D_8 \circ C_4$, because $D_8 \circ C_4$ is nilpotent. The Fitting subgroup will appear in relation to Gaschütz' Theorem which we will state in the next section.

Definition 12. Let H_1 and H_2 be subgroups of G. We define the **commutator** of these groups to be

$$[H_1, H_2] = \langle h_1^{-1} h_2^{-1} h_1 h_2 \,|\, h_1 \in H_1, h_2 \in H_2 \rangle.$$

The **commutator subgroup** of G is the group $[G, G]$ and denoted by G'.

The commutator subgroup is the smallest normal subgroup such that the quotient group of the group by its commutator is abelian. It turns out that $|G/G'|$ has a relation to the largest orbit size of a group action. In the case of $D_8 \circ C_4$ the commutator subgroup is $\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 4 & 0 \\ 0 & 4 \end{smallmatrix}\right)\}$.

Definition 13. Let X be a set and G be a group. We say that G **acts** on X if for every $x \in X$ and $g \in G$ there exists an element $x^g \in X$ such that $x^1 = x$ and $x^{g^h} = x^{gh}$ for all $g, h \in G$. If G acts on X, we call this a **right group action**.

Definition 14. Let G act on the set X and $x \in X$. The **orbit** containing x of this group action is the set $\{x^g \,|\, g \in G\}$

An orbit of an element x can be thought of informally as elements in the set that x can be taken to by an element in the group.

Definition 15. A group action is **faithful** if there is no $g \in G$ where $g \neq 1$ such that $x^g = x$ for all $x \in X$.

One will notice that a faithful action induces an injection from G to the symmetric group on X.

Definition 16. A group action of G on X is said to be **transitive** if for every two elements $x, y \in X$, there exists $g \in G$ with $x^g = y$. If this g is unique, we say that the action is **regular**.

Definition 17. Let G be a group and V a vector space over a field. Let G act on V such that $(a+b)^g = a^g + b^g$ for all $a, b \in V$ and $g \in G$. We call V a G**-module**.

Definition 18. Let D be a subgroup of G and V be a G-module. If we consider only the action of D on V, we get a D-module denoted V_D.

Definition 19. Let V, W be G-modules. We say $V \cong W$ as G-modules if and only if there exists a vector space isomorphism $\phi : V \to W$ such that $\phi(v^g) = \phi(v)^g$ for all $v \in V$ and $g \in G$.

Definition 20. A G-module V is **irreducible** if V has no proper non-zero G-submodules.

Definition 21. A G-module V is **completely reducible** if it can be represented as the direct sum of irreducible G-modules.

Definition 22. Let V be a completely reducible G-module. Then $V = V_1 \oplus V_2 \oplus \ldots \oplus V_n$ for irreducible G-modules V_i. One can write $V = W_1 \oplus \ldots \oplus W_m$ for some $m \leq n$

such that $V_i, V_j \leq W_k$ for some $i, j \in \{1, \dots, n\}$ and some $k \in \{1, \dots, m\}$ if and only if $V_i \cong V_j$ (as G-modules). Then the W_i are called the **homogeneous components** of V.

Definition 23. An irreducible G-module V is called **imprimitive** if V can be written as $V = V_1 \oplus \cdots \oplus V_n$ for $n > 1$ subspaces V_i that are permuted transitively by G. We say that V is **primitive** if V is not imprimitive. V is called **quasiprimitive** if V_N is homogeneous for all $N \lhd G$ (where V_N denotes V viewed as an N-module).

It is known that quasiprimitive is a weaker condition than primitive. That is primitive implies quasiprimitive; however, the reverse implication is not true. This concludes the formal definitions that will be required for the main result.

Useful Theorems and Lemmas

This section will contain some useful theorems and lemmas used throughout the thesis. The following results are well-known in the group theory community and appear in other papers. The original sources have been indicated for proofs. The first lemma will come in handy when we examine abelian subgroups in our main theorem.

Lemma 1. *[**9**] Let A be an abelian finite group and let V be a finite faithful completely reducible A-module. It is well-known that A has a regular orbit on V. Write $V = V_1 \oplus \cdots \oplus V_n$ for irreducible A-modules V_i. Suppose that A has exactly one regular orbit on V. Then $A/C_A(V_i)$ is cyclic of order $|V_i| - 1$ for all i and $A \cong \times_{i=1}^{n} A/C_A(V_i)$ is of order $\prod_{i=1}^{n}(|V_i| - 1)$.*

The following theorem is important because it is the main result of [**8**] and each part will be used throughout the paper. This paper is focused on expanding the third case, but the first two cases will be used within the proof to show our result. The following theorem will reference modules of mixed characteristic. Mixed characteristic means that if $V = V_1 \oplus \ldots \oplus V_n$ then V_i and V_j need not have the same characteristic for all $i, j \in \{1, \ldots, n\}$

Theorem 1. *[**9**] Let G be a finite solvable group and V a finite faithful completely reducible G-module, possibly of mixed characteristic. Let M be the largest orbit size in the action of G on V. Then*

$$|G/G'| \leq M$$

More precisely, we have one of the following

1. *$|G/G'| < M$*
2. *$|G/G'| = M$ and G is abelian; or*
3. *$|G/G'| = M$, G is nilpotent, and G has at least two different orbits of size M on V.*

The following Lemma is well-known, the reader may recognize this as a consequence of the isomorphism theorems.

Lemma 2. *[**9**] Let G be a finite group and $N \trianglelefteq G$. Then*

$$|G/G'| = |G/G'N| \cdot |N : N \cap G'|;$$

and

$$|G : G'| \quad \textit{divides} \quad |G/N : (G/N)'| \cdot |N : N'|.$$

Theorem 2 (Gaschütz' Theorem)**.** *[**11**] Let G be solvable. Then $F(G/\Phi(G)) = F(G)/\Phi(G)$ is a completely reducible and faithful $G/F(G)$-module (possibly of mixed characteristic). Furthermore, $G/\Phi(G)$ splits over $F(G)/\Phi(G)$.*

Lemma 3. *[**5**] Let G be a nilpotent group that acts faithfully and irreducibly on a finite vector space V, and assume that G' is cyclic. Then there exists a vector $v \in V$ such that $|\mathbf{C}_G(v)| \leq 2$*

II. EXAMPLES AND STATEMENT OF MAIN RESULT

In this section we will be stating the topic of this thesis. The following theorem is a corrected version of the conjecture from [**7**]. The theorem will expand upon the third case of Theorem 1. We will be considering the case where G is a finite nonabelian solvable group that has exactly three orbits of size $M = |G/G'|$ on V where V is a finite faithful irreducible $G - module$. Our claim is that this happens if and only if $G = D_8 \circ C_4$ and $V = V(2,5)$, the rank 2 module over the field of order 5. While the main proof is focused on showing that this is the only G and V that qualifies, we will quickly show that $G = D_8 \circ C_4$ and $V = V(2,5)$ fits the hypothesis. To define the group action of G on V we will start with defining G. G is isomorphic to a subgroup of the general linear group $GL(2,5)$ that is generated by the following matrices;

$$< \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix} >$$

This creates a group of 16 elements. The group action will be right multiplication of the matrices in G on the row vectors in $V(2,5)$. Due to this action being computation heavy, we will list the orbits created instead of computing a Cayley table.

$$\begin{bmatrix} 0 & 0 \end{bmatrix} G = \{\begin{bmatrix} 0 & 0 \end{bmatrix}\}$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} G = \{\begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 4 \end{bmatrix}, \begin{bmatrix} 4 & 0 \end{bmatrix}, \begin{bmatrix} 4 & 2 \end{bmatrix}\}$$

$$\begin{bmatrix} 0 & 1 \end{bmatrix} G = \{\begin{bmatrix} 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 2 \end{bmatrix}\}$$

$$\begin{bmatrix} 1 & 1 \end{bmatrix} G = \{\begin{bmatrix} 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 1 \end{bmatrix}\}$$

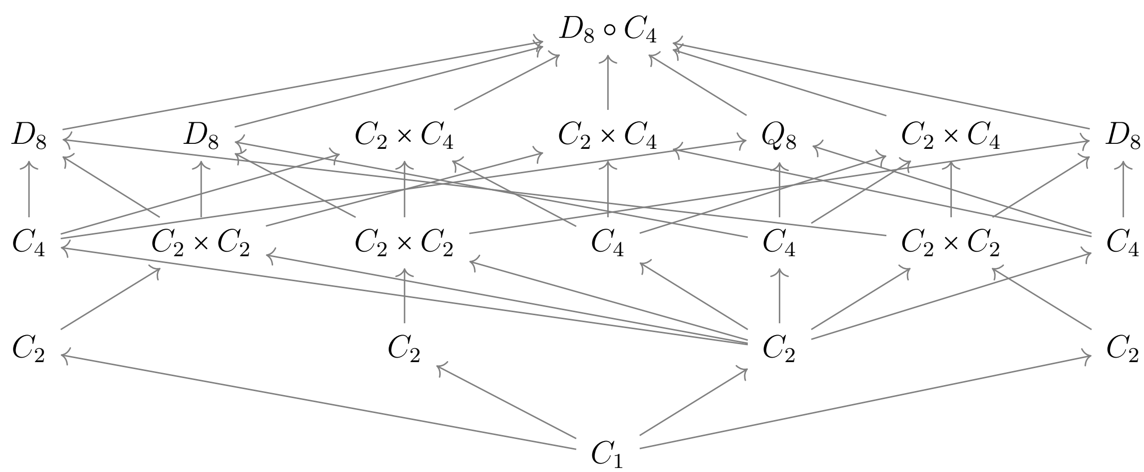
We see that $G = D_8 \circ C_4$ creates 3 orbits of size 8 and one fixed point. Let us recall that the size of $D_8 \circ C_4$ is 16 and the size of G' is 2, thus we see that $M =$

$|G/G'| = 8$. Therefore this action satisfies the conditions of the following theorem.

Theorem 3. *Let G be a finite nonabelian solvable group and V a finite faithful irreducible G-module. Suppose that $M = |G/G'|$ is the largest orbit size of G on V and that there are exactly three orbits of size M on V. Then $G = D_8 \circ C_4$, the central product of the dihedral group of order 8 and the cyclic group of order 4, and $V = V(2,5)$.*

III. THE STRUCTURE OF THE GROUP $D_8 \circ C_4$

This paper requires us to look at abnormal representations of the subgroups of $D_8 \circ C_4$. For example, naturally we would look at D_8 in $GL(2,3)$, but because our main group $D_8 \circ C_4$ is represented in $GL(2,5)$ and acts on $V(2,5)$, we must adapt to be able to see how D_8 acts on $V(2,5)$. While the computation on how to get these subgroups and how these subgroups act on $V(2,5)$ will not be shown, we will observe the lattice of subgroups for $D_8 \circ C_4$ as well as the orbits generated by each subgroup when acting on $V(2,5)$. This lattice is a recreating of the $D_4 \circ C_4$ lattice from GroupNames.org with the dihedral group of order 8 renamed from D_4 to D_8 to stay consistent with the rest of the paper.



In order to save space, we will list the subgroups by their generators in $GL(2,5)$ and then their orbits when taking the natural group action on $V(2,5)$. This list will go in order from top left to bottom right of the subgroup lattice chart starting with $D_8 \circ C_4$ for completeness.

Our Group $D_8 \circ C_4$ and its Orbits

$D_8 \circ C_4 = < \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix} >$

$\begin{bmatrix} 0 & 0 \end{bmatrix} D_8 \circ C_4 = \{ \begin{bmatrix} 0 & 0 \end{bmatrix} \}$

$$\begin{bmatrix}1 & 0\end{bmatrix} D_8 \circ C_4 = \{\begin{bmatrix}1 & 0\end{bmatrix}, \begin{bmatrix}1 & 3\end{bmatrix}, \begin{bmatrix}2 & 0\end{bmatrix}, \begin{bmatrix}2 & 1\end{bmatrix}, \begin{bmatrix}3 & 0\end{bmatrix}, \begin{bmatrix}3 & 4\end{bmatrix}, \begin{bmatrix}4 & 0\end{bmatrix}, \begin{bmatrix}4 & 2\end{bmatrix}\}$$

$$\begin{bmatrix}0 & 1\end{bmatrix} D_8 \circ C_4 = \{\begin{bmatrix}0 & 1\end{bmatrix}, \begin{bmatrix}1 & 4\end{bmatrix}, \begin{bmatrix}0 & 4\end{bmatrix}, \begin{bmatrix}0 & 2\end{bmatrix}, \begin{bmatrix}2 & 3\end{bmatrix}, \begin{bmatrix}0 & 3\end{bmatrix}, \begin{bmatrix}4 & 1\end{bmatrix}, \begin{bmatrix}3 & 2\end{bmatrix}\}$$

$$\begin{bmatrix}1 & 1\end{bmatrix} D_8 \circ C_4 = \{\begin{bmatrix}1 & 1\end{bmatrix}, \begin{bmatrix}2 & 4\end{bmatrix}, \begin{bmatrix}1 & 2\end{bmatrix}, \begin{bmatrix}2 & 2\end{bmatrix}, \begin{bmatrix}4 & 3\end{bmatrix}, \begin{bmatrix}4 & 4\end{bmatrix}, \begin{bmatrix}3 & 3\end{bmatrix}, \begin{bmatrix}3 & 1\end{bmatrix}\}$$

The First D_8 Subgroup and its Orbits

$$(D_8)_1 = < \begin{bmatrix}1 & 3\\0 & 4\end{bmatrix}, \begin{bmatrix}2 & 1\\0 & 3\end{bmatrix} >$$

$$\begin{bmatrix}0 & 0\end{bmatrix} (D_8)_1 = \{\begin{bmatrix}0 & 0\end{bmatrix}\}$$

$$\begin{bmatrix}1 & 0\end{bmatrix} (D_8)_1 = \{\begin{bmatrix}1 & 0\end{bmatrix}, \begin{bmatrix}1 & 3\end{bmatrix}, \begin{bmatrix}2 & 0\end{bmatrix}, \begin{bmatrix}2 & 1\end{bmatrix}, \begin{bmatrix}3 & 0\end{bmatrix}, \begin{bmatrix}3 & 4\end{bmatrix}, \begin{bmatrix}4 & 0\end{bmatrix}, \begin{bmatrix}4 & 2\end{bmatrix}\}$$

$$\begin{bmatrix}0 & 1\end{bmatrix} (D_8)_1 = \{\begin{bmatrix}0 & 1\end{bmatrix}, \begin{bmatrix}0 & 4\end{bmatrix}, \begin{bmatrix}2 & 3\end{bmatrix}, \begin{bmatrix}3 & 2\end{bmatrix}\}$$

$$\begin{bmatrix}0 & 2\end{bmatrix} (D_8)_1 = \{\begin{bmatrix}0 & 2\end{bmatrix}, \begin{bmatrix}1 & 4\end{bmatrix}, \begin{bmatrix}4 & 1\end{bmatrix}, \begin{bmatrix}0 & 3\end{bmatrix}\}$$

$$\begin{bmatrix}1 & 1\end{bmatrix} (D_8)_1 = \{\begin{bmatrix}1 & 1\end{bmatrix}, \begin{bmatrix}1 & 2\end{bmatrix}, \begin{bmatrix}4 & 3\end{bmatrix}, \begin{bmatrix}4 & 4\end{bmatrix}\}$$

$$\begin{bmatrix}2 & 2\end{bmatrix} (D_8)_1 = \{\begin{bmatrix}2 & 2\end{bmatrix}, \begin{bmatrix}2 & 4\end{bmatrix}, \begin{bmatrix}3 & 1\end{bmatrix}, \begin{bmatrix}3 & 3\end{bmatrix}\}$$

The Second D_8 Subgroup and its Orbits

$$(D_8)_2 = < \begin{bmatrix}1 & 0\\1 & 4\end{bmatrix}, \begin{bmatrix}2 & 1\\0 & 3\end{bmatrix} >$$

$$\begin{bmatrix}0 & 0\end{bmatrix} (D_8)_2 = \{\begin{bmatrix}0 & 0\end{bmatrix}\}$$

$$\begin{bmatrix}1 & 0\end{bmatrix} (D_8)_2 = \{\begin{bmatrix}1 & 0\end{bmatrix}, \begin{bmatrix}2 & 1\end{bmatrix}, \begin{bmatrix}4 & 0\end{bmatrix}, \begin{bmatrix}3 & 4\end{bmatrix}\}$$

$$\begin{bmatrix}2 & 0\end{bmatrix} (D_8)_2 = \{\begin{bmatrix}2 & 0\end{bmatrix}, \begin{bmatrix}4 & 2\end{bmatrix}, \begin{bmatrix}3 & 0\end{bmatrix}, \begin{bmatrix}1 & 3\end{bmatrix}\}$$

$$\begin{bmatrix}0 & 1\end{bmatrix} (D_8)_2 = \{\begin{bmatrix}0 & 1\end{bmatrix}, \begin{bmatrix}0 & 3\end{bmatrix}, \begin{bmatrix}1 & 4\end{bmatrix}, \begin{bmatrix}0 & 4\end{bmatrix}, \begin{bmatrix}0 & 2\end{bmatrix}, \begin{bmatrix}2 & 3\end{bmatrix}, \begin{bmatrix}3 & 2\end{bmatrix}, \begin{bmatrix}4 & 1\end{bmatrix}\}$$

$$\begin{bmatrix}1 & 1\end{bmatrix} (D_8)_2 = \{\begin{bmatrix}1 & 1\end{bmatrix}, \begin{bmatrix}2 & 4\end{bmatrix}, \begin{bmatrix}4 & 4\end{bmatrix}, \begin{bmatrix}3 & 1\end{bmatrix}\}$$

$$\begin{bmatrix}1 & 2\end{bmatrix} (D_8)_2 = \{\begin{bmatrix}1 & 2\end{bmatrix}, \begin{bmatrix}2 & 2\end{bmatrix}, \begin{bmatrix}3 & 3\end{bmatrix}, \begin{bmatrix}4 & 2\end{bmatrix}\}$$

The First $C_2 \times C_4$ Subgroup and its Orbits

$$(C_2 \times C_4)_1 = < \begin{bmatrix}1 & 0\\1 & 4\end{bmatrix}, \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix} >$$

$$\begin{bmatrix}0 & 0\end{bmatrix} (C_2 \times C_4)_1 = \{\begin{bmatrix}0 & 0\end{bmatrix}\}$$

$$\begin{bmatrix}1 & 0\end{bmatrix} (C_2 \times C_4)_1 = \{\begin{bmatrix}1 & 0\end{bmatrix}, \begin{bmatrix}2 & 0\end{bmatrix}, \begin{bmatrix}4 & 0\end{bmatrix}, \begin{bmatrix}3 & 0\end{bmatrix}\}$$

$$\begin{bmatrix}1 & 3\end{bmatrix}(C_2 \times C_4)_1 = \{\begin{bmatrix}1 & 3\end{bmatrix}, \begin{bmatrix}4 & 2\end{bmatrix}, \begin{bmatrix}2 & 1\end{bmatrix}, \begin{bmatrix}3 & 4\end{bmatrix}\}$$

$$\begin{bmatrix}0 & 1\end{bmatrix}(C_2 \times C_4)_1 = \{\begin{bmatrix}0 & 1\end{bmatrix}, \begin{bmatrix}1 & 4\end{bmatrix}, \begin{bmatrix}0 & 2\end{bmatrix}, \begin{bmatrix}0 & 4\end{bmatrix}, \begin{bmatrix}0 & 3\end{bmatrix}, \begin{bmatrix}2 & 3\end{bmatrix}, \begin{bmatrix}3 & 2\end{bmatrix}, \begin{bmatrix}4 & 1\end{bmatrix}\}$$

$$\begin{bmatrix}1 & 1\end{bmatrix}(C_2 \times C_4)_1 = \{\begin{bmatrix}1 & 1\end{bmatrix}, \begin{bmatrix}2 & 4\end{bmatrix}, \begin{bmatrix}1 & 2\end{bmatrix}, \begin{bmatrix}2 & 2\end{bmatrix}, \begin{bmatrix}4 & 3\end{bmatrix}, \begin{bmatrix}4 & 4\end{bmatrix}, \begin{bmatrix}3 & 3\end{bmatrix}, \begin{bmatrix}3 & 1\end{bmatrix}\}$$

The Second $C_2 \times C_4$ Subgroup and its Orbits

$$(C_2 \times C_4)_2 = < \begin{bmatrix}2 & 1\\ 2 & 3\end{bmatrix}, \begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix} >$$

$$\begin{bmatrix}0 & 0\end{bmatrix}(C_2 \times C_4)_2 = \{\begin{bmatrix}0 & 0\end{bmatrix}\}$$

$$\begin{bmatrix}1 & 0\end{bmatrix}(C_2 \times C_4)_2 = \{\begin{bmatrix}1 & 0\end{bmatrix}, \begin{bmatrix}1 & 3\end{bmatrix}, \begin{bmatrix}2 & 0\end{bmatrix}, \begin{bmatrix}2 & 1\end{bmatrix}, \begin{bmatrix}3 & 0\end{bmatrix}, \begin{bmatrix}3 & 4\end{bmatrix}, \begin{bmatrix}4 & 0\end{bmatrix}, \begin{bmatrix}4 & 2\end{bmatrix}\}$$

$$\begin{bmatrix}0 & 1\end{bmatrix}(C_2 \times C_4)_2 = \{\begin{bmatrix}0 & 1\end{bmatrix}, \begin{bmatrix}0 & 4\end{bmatrix}, \begin{bmatrix}0 & 2\end{bmatrix}, \begin{bmatrix}0 & 3\end{bmatrix}\begin{bmatrix}1 & 4\end{bmatrix}, \begin{bmatrix}2 & 3\end{bmatrix}, \begin{bmatrix}3 & 2\end{bmatrix}, \begin{bmatrix}4 & 1\end{bmatrix}\}$$

$$\begin{bmatrix}1 & 1\end{bmatrix}(C_2 \times C_4)_2 = \{\begin{bmatrix}1 & 1\end{bmatrix}, \begin{bmatrix}2 & 2\end{bmatrix}, \begin{bmatrix}4 & 4\end{bmatrix}, \begin{bmatrix}3 & 3\end{bmatrix}\}$$

$$\begin{bmatrix}1 & 2\end{bmatrix}(C_2 \times C_4)_2 = \{\begin{bmatrix}1 & 2\end{bmatrix}, \begin{bmatrix}2 & 4\end{bmatrix}, \begin{bmatrix}4 & 3\end{bmatrix}, \begin{bmatrix}3 & 1\end{bmatrix}\}$$

The Only Q_8 Subgroup and its Orbits

$$Q_8 = < \begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix}, \begin{bmatrix}2 & 0\\ 2 & 3\end{bmatrix}, \begin{bmatrix}2 & 1\\ 0 & 3\end{bmatrix}, \begin{bmatrix}4 & 2\\ 4 & 1\end{bmatrix} >$$

$$\begin{bmatrix}0 & 0\end{bmatrix}Q_8 = \{\begin{bmatrix}0 & 0\end{bmatrix}\}$$

$$\begin{bmatrix}1 & 0\end{bmatrix}Q_8 = \{\begin{bmatrix}1 & 0\end{bmatrix}, \begin{bmatrix}1 & 3\end{bmatrix}, \begin{bmatrix}2 & 0\end{bmatrix}, \begin{bmatrix}2 & 1\end{bmatrix}, \begin{bmatrix}3 & 0\end{bmatrix}, \begin{bmatrix}3 & 4\end{bmatrix}, \begin{bmatrix}4 & 0\end{bmatrix}, \begin{bmatrix}4 & 2\end{bmatrix}\}$$

$$\begin{bmatrix}0 & 1\end{bmatrix}Q_8 = \{\begin{bmatrix}0 & 1\end{bmatrix}, \begin{bmatrix}1 & 4\end{bmatrix}, \begin{bmatrix}0 & 4\end{bmatrix}, \begin{bmatrix}0 & 2\end{bmatrix}, \begin{bmatrix}2 & 3\end{bmatrix}, \begin{bmatrix}0 & 3\end{bmatrix}, \begin{bmatrix}4 & 1\end{bmatrix}, \begin{bmatrix}3 & 2\end{bmatrix}\}$$

$$\begin{bmatrix}1 & 1\end{bmatrix}Q_8 = \{\begin{bmatrix}1 & 1\end{bmatrix}, \begin{bmatrix}2 & 4\end{bmatrix}, \begin{bmatrix}1 & 2\end{bmatrix}, \begin{bmatrix}2 & 2\end{bmatrix}, \begin{bmatrix}4 & 3\end{bmatrix}, \begin{bmatrix}4 & 4\end{bmatrix}, \begin{bmatrix}3 & 3\end{bmatrix}, \begin{bmatrix}3 & 1\end{bmatrix}\}$$

The Third $C_2 \times C_4$ Subgroup and its Orbits

$$(C_2 \times C_4)_3 = < \begin{bmatrix}1 & 3\\ 0 & 4\end{bmatrix}, \begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix} >$$

$$\begin{bmatrix}0 & 0\end{bmatrix}((C_2 \times C_4)_3 = \{\begin{bmatrix}0 & 0\end{bmatrix}\}$$

$$\begin{bmatrix}1 & 0\end{bmatrix}(C_2 \times C_4)_3 = \{\begin{bmatrix}1 & 0\end{bmatrix}, \begin{bmatrix}1 & 3\end{bmatrix}, \begin{bmatrix}2 & 0\end{bmatrix}, \begin{bmatrix}2 & 1\end{bmatrix}, \begin{bmatrix}3 & 0\end{bmatrix}, \begin{bmatrix}3 & 4\end{bmatrix}, \begin{bmatrix}4 & 0\end{bmatrix}, \begin{bmatrix}4 & 2\end{bmatrix}\}$$

$$\begin{bmatrix}0 & 1\end{bmatrix}(C_2 \times C_4)_3 = \{\begin{bmatrix}0 & 1\end{bmatrix}, \begin{bmatrix}0 & 4\end{bmatrix}, \begin{bmatrix}0 & 2\end{bmatrix}, \begin{bmatrix}0 & 3\end{bmatrix}\}$$

$$\begin{bmatrix}1 & 4\end{bmatrix}(C_2 \times C_4)_3 = \{\begin{bmatrix}1 & 4\end{bmatrix}, \begin{bmatrix}2 & 3\end{bmatrix}, \begin{bmatrix}3 & 2\end{bmatrix}, \begin{bmatrix}4 & 1\end{bmatrix}\}$$

$$\begin{bmatrix}1 & 1\end{bmatrix}(C_2 \times C_4)_3 = \{\begin{bmatrix}1 & 1\end{bmatrix}, \begin{bmatrix}2 & 4\end{bmatrix}, \begin{bmatrix}1 & 2\end{bmatrix}, \begin{bmatrix}2 & 2\end{bmatrix}, \begin{bmatrix}4 & 3\end{bmatrix}, \begin{bmatrix}4 & 4\end{bmatrix}, \begin{bmatrix}3 & 3\end{bmatrix}, \begin{bmatrix}3 & 1\end{bmatrix}\}$$

The Third D_8 Subgroup and its Orbits

$(D_8)_3 = < \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix} >$

$\begin{bmatrix} 0 & 0 \end{bmatrix} (D_8)_3 = \{ \begin{bmatrix} 0 & 0 \end{bmatrix} \}$

$\begin{bmatrix} 1 & 0 \end{bmatrix} (D_8)_3 = \{ \begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 0 \end{bmatrix}, \begin{bmatrix} 4 & 2 \end{bmatrix} \}$

$\begin{bmatrix} 2 & 0 \end{bmatrix} (D_8)_3 = \{ \begin{bmatrix} 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 4 \end{bmatrix} \}$

$\begin{bmatrix} 0 & 1 \end{bmatrix} (D_8)_3 = \{ \begin{bmatrix} 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 4 \end{bmatrix}, \begin{bmatrix} 4 & 1 \end{bmatrix} \}$

$\begin{bmatrix} 0 & 2 \end{bmatrix} (D_8)_3 = \{ \begin{bmatrix} 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 2 \end{bmatrix} \}$

$\begin{bmatrix} 1 & 1 \end{bmatrix} (D_8)_3 = \{ \begin{bmatrix} 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 1 \end{bmatrix} \}$

The First C_4 Subgroup and its Orbits

$(C_4)_1 = < \begin{bmatrix} 2 & 0 \\ 2 & 3 \end{bmatrix} >$

$\begin{bmatrix} 0 & 0 \end{bmatrix} (C_4)_1 = \{ \begin{bmatrix} 0 & 0 \end{bmatrix} \}$

$\begin{bmatrix} 1 & 0 \end{bmatrix} (C_4)_1 = \{ \begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \end{bmatrix}, \begin{bmatrix} 4 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 \end{bmatrix} \}$

$\begin{bmatrix} 1 & 3 \end{bmatrix} (C_4)_1 = \{ \begin{bmatrix} 1 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 4 \end{bmatrix}, \begin{bmatrix} 4 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \end{bmatrix} \}$

$\begin{bmatrix} 0 & 1 \end{bmatrix} (C_4)_1 = \{ \begin{bmatrix} 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 2 \end{bmatrix} \}$

$\begin{bmatrix} 0 & 2 \end{bmatrix} (C_4)_1 = \{ \begin{bmatrix} 0 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 4 \end{bmatrix} \}$

$\begin{bmatrix} 1 & 1 \end{bmatrix} (C_4)_1 = \{ \begin{bmatrix} 1 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 \end{bmatrix} \}$

$\begin{bmatrix} 2 & 2 \end{bmatrix} (C_4)_1 = \{ \begin{bmatrix} 2 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 4 \end{bmatrix} \}$

The First $C_2 \times C_2$ Subgroup and its Orbits

$(C_2 \times C_2)_1 = < \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} >$

$\begin{bmatrix} 0 & 0 \end{bmatrix} (C_2 \times C_2)_1 = \{ \begin{bmatrix} 0 & 0 \end{bmatrix} \}$

$\begin{bmatrix} 1 & 0 \end{bmatrix} (C_2 \times C_2)_1 = \{ \begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 4 \end{bmatrix} \}$

$\begin{bmatrix} 2 & 0 \end{bmatrix} (C_2 \times C_2)_1 = \{ \begin{bmatrix} 2 & 0 \end{bmatrix}, \begin{bmatrix} 4 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 3 \end{bmatrix} \}$

$\begin{bmatrix} 0 & 1 \end{bmatrix} (C_2 \times C_2)_1 = \{ \begin{bmatrix} 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 2 \end{bmatrix} \}$

$\begin{bmatrix} 0 & 2 \end{bmatrix} (C_2 \times C_2)_1 = \{ \begin{bmatrix} 0 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 4 \end{bmatrix} \}$

$\begin{bmatrix}1 & 1\end{bmatrix}(C_2 \times C_2)_1 = \{\begin{bmatrix}1 & 1\end{bmatrix}, \begin{bmatrix}4 & 4\end{bmatrix}\}$

$\begin{bmatrix}1 & 2\end{bmatrix}(C_2 \times C_2)_1 = \{\begin{bmatrix}1 & 2\end{bmatrix}, \begin{bmatrix}4 & 3\end{bmatrix}\}$

$\begin{bmatrix}2 & 2\end{bmatrix}(C_2 \times C_2)_1 = \{\begin{bmatrix}2 & 2\end{bmatrix}, \begin{bmatrix}3 & 3\end{bmatrix}\}$

$\begin{bmatrix}2 & 4\end{bmatrix}(C_2 \times C_2)_1 = \{\begin{bmatrix}2 & 4\end{bmatrix}, \begin{bmatrix}3 & 2\end{bmatrix}\}$

The Second $C_2 \times C_2$ Subgroup and its Orbits

$(C_2 \times C_2)_2 = < \begin{bmatrix}1 & 0\\ 1 & 4\end{bmatrix}, \begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix} >$

$\begin{bmatrix}0 & 0\end{bmatrix}(C_2 \times C_2)_2 = \{\begin{bmatrix}0 & 0\end{bmatrix}\}$

$\begin{bmatrix}1 & 0\end{bmatrix}(C_2 \times C_2)_2 = \{\begin{bmatrix}1 & 0\end{bmatrix}, \begin{bmatrix}4 & 0\end{bmatrix}\}$

$\begin{bmatrix}2 & 0\end{bmatrix}(C_2 \times C_2)_2 = \{\begin{bmatrix}2 & 0\end{bmatrix}, \begin{bmatrix}3 & 0\end{bmatrix}\}$

$\begin{bmatrix}1 & 3\end{bmatrix}(C_2 \times C_2)_2 = \{\begin{bmatrix}1 & 3\end{bmatrix}, \begin{bmatrix}4 & 2\end{bmatrix}\}$

$\begin{bmatrix}2 & 1\end{bmatrix}(C_2 \times C_2)_2 = \{\begin{bmatrix}2 & 1\end{bmatrix}, \begin{bmatrix}3 & 4\end{bmatrix}\}$

$\begin{bmatrix}0 & 1\end{bmatrix}(C_2 \times C_2)_2 = \{\begin{bmatrix}0 & 1\end{bmatrix}, \begin{bmatrix}1 & 4\end{bmatrix}, \begin{bmatrix}0 & 4\end{bmatrix}, \begin{bmatrix}4 & 1\end{bmatrix}\}$

$\begin{bmatrix}0 & 2\end{bmatrix}(C_2 \times C_2)_2 = \{\begin{bmatrix}0 & 2\end{bmatrix}, \begin{bmatrix}2 & 3\end{bmatrix}, \begin{bmatrix}0 & 3\end{bmatrix}, \begin{bmatrix}3 & 2\end{bmatrix}\}$

$\begin{bmatrix}1 & 1\end{bmatrix}(C_2 \times C_2)_2 = \{\begin{bmatrix}1 & 1\end{bmatrix}, \begin{bmatrix}2 & 4\end{bmatrix}, \begin{bmatrix}4 & 4\end{bmatrix}, \begin{bmatrix}3 & 1\end{bmatrix}\}$

$\begin{bmatrix}2 & 2\end{bmatrix}(C_2 \times C_2)_2 = \{\begin{bmatrix}2 & 2\end{bmatrix}, \begin{bmatrix}4 & 3\end{bmatrix}, \begin{bmatrix}3 & 3\end{bmatrix}, \begin{bmatrix}1 & 2\end{bmatrix}\}$

The Second C_4 Subgroup and its Orbits

$(C_4)_2 = < \begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix} >$

$\begin{bmatrix}0 & 0\end{bmatrix}(C_4)_2 = \{\begin{bmatrix}0 & 0\end{bmatrix}\}$

$\begin{bmatrix}1 & 0\end{bmatrix}(C_4)_2 = \{\begin{bmatrix}1 & 0\end{bmatrix}, \begin{bmatrix}2 & 0\end{bmatrix}, \begin{bmatrix}3 & 0\end{bmatrix}, \begin{bmatrix}4 & 0\end{bmatrix}\}$

$\begin{bmatrix}1 & 3\end{bmatrix}(C_4)_2 = \{\begin{bmatrix}1 & 3\end{bmatrix}, \begin{bmatrix}2 & 1\end{bmatrix}, \begin{bmatrix}4 & 2\end{bmatrix}, \begin{bmatrix}3 & 4\end{bmatrix}\}$

$\begin{bmatrix}0 & 1\end{bmatrix}(C_4)_2 = \{\begin{bmatrix}0 & 1\end{bmatrix}, \begin{bmatrix}0 & 4\end{bmatrix}, \begin{bmatrix}0 & 2\end{bmatrix}, \begin{bmatrix}0 & 3\end{bmatrix}\}$

$\begin{bmatrix}1 & 4\end{bmatrix}(C_4)_2 = \{\begin{bmatrix}1 & 4\end{bmatrix}, \begin{bmatrix}2 & 3\end{bmatrix}, \begin{bmatrix}4 & 1\end{bmatrix}, \begin{bmatrix}3 & 2\end{bmatrix}\}$

$\begin{bmatrix}1 & 1\end{bmatrix}(C_4)_2 = \{\begin{bmatrix}1 & 1\end{bmatrix}, \begin{bmatrix}2 & 2\end{bmatrix}, \begin{bmatrix}4 & 4\end{bmatrix}, \begin{bmatrix}3 & 3\end{bmatrix}\}$

$\begin{bmatrix}1 & 2\end{bmatrix}(C_4)_2 = \{\begin{bmatrix}1 & 2\end{bmatrix}, \begin{bmatrix}2 & 4\end{bmatrix}, \begin{bmatrix}4 & 3\end{bmatrix}, \begin{bmatrix}3 & 1\end{bmatrix}\}$

The Third C_4 Subgroup and its Orbits

$(C_4)_3 = < \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} >$

$\begin{bmatrix} 0 & 0 \end{bmatrix} (C_4)_3 = \{\begin{bmatrix} 0 & 0 \end{bmatrix}\}$

$\begin{bmatrix} 1 & 0 \end{bmatrix} (C_4)_3 = \{\begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 4 \end{bmatrix}, \begin{bmatrix} 4 & 0 \end{bmatrix}\}$

$\begin{bmatrix} 1 & 3 \end{bmatrix} (C_4)_3 = \{\begin{bmatrix} 1 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 \end{bmatrix}, \begin{bmatrix} 4 & 2 \end{bmatrix}\}$

$\begin{bmatrix} 0 & 1 \end{bmatrix} (C_4)_3 = \{\begin{bmatrix} 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 \end{bmatrix}\}$

$\begin{bmatrix} 1 & 4 \end{bmatrix} (C_4)_3 = \{\begin{bmatrix} 1 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 2 \end{bmatrix}\}$

$\begin{bmatrix} 1 & 1 \end{bmatrix} (C_4)_3 = \{\begin{bmatrix} 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 4 \end{bmatrix}, \begin{bmatrix} 4 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 1 \end{bmatrix}\}$

$\begin{bmatrix} 1 & 2 \end{bmatrix} (C_4)_3 = \{\begin{bmatrix} 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 3 \end{bmatrix}\}$

The Third $C_2 \times C_2$ Subgroup and its Orbits

$(C_2 \times C_2)_3 = < \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} >$

$\begin{bmatrix} 0 & 0 \end{bmatrix} (C_2 \times C_2)_3 = \{\begin{bmatrix} 0 & 0 \end{bmatrix}\}$

$\begin{bmatrix} 1 & 0 \end{bmatrix} (C_2 \times C_2)_3 = \{\begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 0 \end{bmatrix}, \begin{bmatrix} 4 & 2 \end{bmatrix}\}$

$\begin{bmatrix} 2 & 0 \end{bmatrix} (C_2 \times C_2)_3 = \{\begin{bmatrix} 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 4 \end{bmatrix}\}$

$\begin{bmatrix} 0 & 1 \end{bmatrix} (C_2 \times C_2)_3 = \{\begin{bmatrix} 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 4 \end{bmatrix}\}$

$\begin{bmatrix} 1 & 4 \end{bmatrix} (C_2 \times C_2)_3 = \{\begin{bmatrix} 1 & 4 \end{bmatrix}, \begin{bmatrix} 4 & 1 \end{bmatrix}\}$

$\begin{bmatrix} 0 & 2 \end{bmatrix} (C_2 \times C_2)_3 = \{\begin{bmatrix} 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \end{bmatrix}\}$

$\begin{bmatrix} 2 & 3 \end{bmatrix} (C_2 \times C_2)_3 = \{\begin{bmatrix} 2 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 2 \end{bmatrix}\}$

$\begin{bmatrix} 1 & 1 \end{bmatrix} (C_2 \times C_2)_3 = \{\begin{bmatrix} 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 4 \end{bmatrix}, \begin{bmatrix} 4 & 3 \end{bmatrix}\}$

$\begin{bmatrix} 2 & 2 \end{bmatrix} (C_2 \times C_2)_3 = \{\begin{bmatrix} 2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 1 \end{bmatrix}\}$

The Fourth C_4 Subgroup and its Orbits

$(C_4)_4 = < \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} >$

$\begin{bmatrix} 0 & 0 \end{bmatrix} (C_4)_4 = \{\begin{bmatrix} 0 & 0 \end{bmatrix}\}$

$\begin{bmatrix} 1 & 0 \end{bmatrix} (C_4)_4 = \{\begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 4 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 0 \end{bmatrix}\}$

$\begin{bmatrix}2 & 0\end{bmatrix}(C_4)_4 = \{\begin{bmatrix}2 & 0\end{bmatrix}, \begin{bmatrix}3 & 4\end{bmatrix}, \begin{bmatrix}3 & 0\end{bmatrix}, \begin{bmatrix}2 & 1\end{bmatrix}\}$

$\begin{bmatrix}0 & 1\end{bmatrix}(C_4)_4 = \{\begin{bmatrix}0 & 1\end{bmatrix}, \begin{bmatrix}0 & 4\end{bmatrix}, \begin{bmatrix}4 & 1\end{bmatrix}, \begin{bmatrix}1 & 4\end{bmatrix}\}$

$\begin{bmatrix}0 & 2\end{bmatrix}(C_4)_4 = \{\begin{bmatrix}0 & 2\end{bmatrix}, \begin{bmatrix}3 & 2\end{bmatrix}, \begin{bmatrix}0 & 3\end{bmatrix}, \begin{bmatrix}2 & 3\end{bmatrix}\}$

$\begin{bmatrix}1 & 1\end{bmatrix}(C_4)_4 = \{\begin{bmatrix}1 & 1\end{bmatrix}, \begin{bmatrix}3 & 3\end{bmatrix}, \begin{bmatrix}4 & 4\end{bmatrix}, \begin{bmatrix}2 & 2\end{bmatrix}\}$

$\begin{bmatrix}1 & 2\end{bmatrix}(C_4)_4 = \{\begin{bmatrix}1 & 2\end{bmatrix}, \begin{bmatrix}2 & 4\end{bmatrix}, \begin{bmatrix}4 & 3\end{bmatrix}, \begin{bmatrix}3 & 1\end{bmatrix}\}$

The First C_2 Subgroup and its Orbits

$(C_2)_1 = <\begin{bmatrix}2 & 1\\ 2 & 3\end{bmatrix}>$

$\begin{bmatrix}0 & 0\end{bmatrix}(C_2)_1 = \{\begin{bmatrix}0 & 0\end{bmatrix}\}$

$\begin{bmatrix}1 & 0\end{bmatrix}(C_2)_1 = \{\begin{bmatrix}1 & 0\end{bmatrix}, \begin{bmatrix}2 & 1\end{bmatrix}\}$

$\begin{bmatrix}2 & 0\end{bmatrix}(C_2)_1 = \{\begin{bmatrix}2 & 0\end{bmatrix}, \begin{bmatrix}4 & 2\end{bmatrix}\}$

$\begin{bmatrix}3 & 0\end{bmatrix}(C_2)_1 = \{\begin{bmatrix}3 & 0\end{bmatrix}, \begin{bmatrix}1 & 3\end{bmatrix}\}$

$\begin{bmatrix}4 & 0\end{bmatrix}(C_2)_1 = \{\begin{bmatrix}4 & 0\end{bmatrix}, \begin{bmatrix}3 & 4\end{bmatrix}\}$

$\begin{bmatrix}0 & 1\end{bmatrix}(C_2)_1 = \{\begin{bmatrix}0 & 1\end{bmatrix}, \begin{bmatrix}2 & 3\end{bmatrix}\}$

$\begin{bmatrix}0 & 2\end{bmatrix}(C_2)_1 = \{\begin{bmatrix}0 & 2\end{bmatrix}, \begin{bmatrix}4 & 1\end{bmatrix}\}$

$\begin{bmatrix}0 & 3\end{bmatrix}(C_2)_1 = \{\begin{bmatrix}0 & 3\end{bmatrix}, \begin{bmatrix}1 & 4\end{bmatrix}\}$

$\begin{bmatrix}0 & 4\end{bmatrix}(C_2)_1 = \{\begin{bmatrix}0 & 4\end{bmatrix}, \begin{bmatrix}3 & 2\end{bmatrix}\}$

$\begin{bmatrix}1 & 1\end{bmatrix}(C_2)_1 = \{\begin{bmatrix}1 & 1\end{bmatrix}, \begin{bmatrix}4 & 4\end{bmatrix}\}$

$\begin{bmatrix}2 & 2\end{bmatrix}(C_2)_1 = \{\begin{bmatrix}2 & 2\end{bmatrix}, \begin{bmatrix}3 & 3\end{bmatrix}\}$

$\begin{bmatrix}1 & 2\end{bmatrix}(C_2)_1 = \{\begin{bmatrix}1 & 2\end{bmatrix}\}$

$\begin{bmatrix}2 & 4\end{bmatrix}(C_2)_1 = \{\begin{bmatrix}2 & 4\end{bmatrix}\}$

$\begin{bmatrix}4 & 3\end{bmatrix}(C_2)_1 = \{\begin{bmatrix}4 & 3\end{bmatrix}\}$

$\begin{bmatrix}3 & 1\end{bmatrix}(C_2)_1 = \{\begin{bmatrix}3 & 1\end{bmatrix}\}$

The Second C_2 Subgroup and its Orbits

$(C_2)_2 = <\begin{bmatrix}1 & 0\\ 1 & 4\end{bmatrix}>$

$\begin{bmatrix}0 & 0\end{bmatrix}(C_2)_2 = \{\begin{bmatrix}0 & 0\end{bmatrix}\}$

$$\begin{bmatrix}1 & 0\end{bmatrix}(C_2)_2 = \{\begin{bmatrix}1 & 0\end{bmatrix}\}$$

$$\begin{bmatrix}2 & 0\end{bmatrix}(C_2)_2 = \{\begin{bmatrix}2 & 0\end{bmatrix}\}$$

$$\begin{bmatrix}3 & 0\end{bmatrix}(C_2)_2 = \{\begin{bmatrix}3 & 0\end{bmatrix}\}$$

$$\begin{bmatrix}4 & 0\end{bmatrix}(C_2)_2 = \{\begin{bmatrix}4 & 0\end{bmatrix}\}$$

$$\begin{bmatrix}1 & 3\end{bmatrix}(C_2)_2 = \{\begin{bmatrix}1 & 3\end{bmatrix}, \begin{bmatrix}4 & 2\end{bmatrix}\}$$

$$\begin{bmatrix}2 & 1\end{bmatrix}(C_2)_2 = \{\begin{bmatrix}2 & 1\end{bmatrix}, \begin{bmatrix}3 & 4\end{bmatrix}\}$$

$$\begin{bmatrix}0 & 1\end{bmatrix}(C_2)_2 = \{\begin{bmatrix}0 & 1\end{bmatrix}, \begin{bmatrix}1 & 4\end{bmatrix}\}$$

$$\begin{bmatrix}0 & 2\end{bmatrix}(C_2)_2 = \{\begin{bmatrix}0 & 2\end{bmatrix}, \begin{bmatrix}2 & 3\end{bmatrix}\}$$

$$\begin{bmatrix}0 & 3\end{bmatrix}(C_2)_2 = \{\begin{bmatrix}0 & 3\end{bmatrix}, \begin{bmatrix}3 & 2\end{bmatrix}\}$$

$$\begin{bmatrix}0 & 4\end{bmatrix}(C_2)_2 = \{\begin{bmatrix}0 & 4\end{bmatrix}, \begin{bmatrix}4 & 1\end{bmatrix}\}$$

$$\begin{bmatrix}1 & 1\end{bmatrix}(C_2)_2 = \{\begin{bmatrix}1 & 1\end{bmatrix}, \begin{bmatrix}2 & 4\end{bmatrix}\}$$

$$\begin{bmatrix}2 & 2\end{bmatrix}(C_2)_2 = \{\begin{bmatrix}2 & 2\end{bmatrix}, \begin{bmatrix}4 & 3\end{bmatrix}\}$$

$$\begin{bmatrix}3 & 3\end{bmatrix}(C_2)_2 = \{\begin{bmatrix}3 & 3\end{bmatrix}, \begin{bmatrix}1 & 2\end{bmatrix}\}$$

$$\begin{bmatrix}4 & 4\end{bmatrix}(C_2)_2 = \{\begin{bmatrix}4 & 4\end{bmatrix}, \begin{bmatrix}3 & 1\end{bmatrix}\}$$

The Third C_2 Subgroup and its Orbits

$$(C_2)_3 = < \begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix} >$$

$$\begin{bmatrix}0 & 0\end{bmatrix}(C_2)_3 = \{\begin{bmatrix}0 & 0\end{bmatrix}\}$$

$$\begin{bmatrix}1 & 0\end{bmatrix}(C_2)_3 = \{\begin{bmatrix}1 & 0\end{bmatrix}, \begin{bmatrix}4 & 0\end{bmatrix}\}$$

$$\begin{bmatrix}2 & 0\end{bmatrix}(C_2)_3 = \{\begin{bmatrix}2 & 0\end{bmatrix}, \begin{bmatrix}3 & 0\end{bmatrix}\}$$

$$\begin{bmatrix}1 & 3\end{bmatrix}(C_2)_3 = \{\begin{bmatrix}\text{‘} & 3\end{bmatrix}, \begin{bmatrix}4 & 2\end{bmatrix}\}$$

$$\begin{bmatrix}2 & 1\end{bmatrix}(C_2)_3 = \{\begin{bmatrix}2 & 1\end{bmatrix}, \begin{bmatrix}3 & 4\end{bmatrix}\}$$

$$\begin{bmatrix}0 & 1\end{bmatrix}(C_2)_3 = \{\begin{bmatrix}0 & 1\end{bmatrix}, \begin{bmatrix}0 & 4\end{bmatrix}\}$$

$$\begin{bmatrix}0 & 2\end{bmatrix}(C_2)_3 = \{\begin{bmatrix}0 & 2\end{bmatrix}, \begin{bmatrix}0 & 3\end{bmatrix}\}$$

$$\begin{bmatrix}1 & 4\end{bmatrix}(C_2)_3 = \{\begin{bmatrix}1 & 4\end{bmatrix}, \begin{bmatrix}4 & 1\end{bmatrix}\}$$

$$\begin{bmatrix}2 & 3\end{bmatrix}(C_2)_3 = \{\begin{bmatrix}2 & 3\end{bmatrix}, \begin{bmatrix}3 & 2\end{bmatrix}\}$$

$$\begin{bmatrix}1 & 1\end{bmatrix}(C_2)_3 = \{\begin{bmatrix}1 & 1\end{bmatrix}, \begin{bmatrix}4 & 4\end{bmatrix}\}$$

$\begin{bmatrix}2 & 2\end{bmatrix}(C_2)_3 = \{\begin{bmatrix}2 & 2\end{bmatrix}, \begin{bmatrix}3 & 3\end{bmatrix}\}$

$\begin{bmatrix}1 & 2\end{bmatrix}(C_2)_3 = \{\begin{bmatrix}1 & 2\end{bmatrix}, \begin{bmatrix}4 & 3\end{bmatrix}\}$

$\begin{bmatrix}2 & 4\end{bmatrix}(C_2)_3 = \{\begin{bmatrix}2 & 4\end{bmatrix}, \begin{bmatrix}3 & 1\end{bmatrix}\}$

The Fourth C_2 Subgroup and its Orbits

$(C_2)_4 = < \begin{bmatrix}1 & 3\\ 0 & 4\end{bmatrix} >$

$\begin{bmatrix}0 & 0\end{bmatrix}(C_2)_4 = \{\begin{bmatrix}0 & 0\end{bmatrix}\}$

$\begin{bmatrix}1 & 0\end{bmatrix}(C_2)_4 = \{\begin{bmatrix}1 & 0\end{bmatrix}, \begin{bmatrix}1 & 3\end{bmatrix}\}$

$\begin{bmatrix}2 & 0\end{bmatrix}(C_2)_4 = \{\begin{bmatrix}2 & 0\end{bmatrix}, \begin{bmatrix}2 & 1\end{bmatrix}\}$

$\begin{bmatrix}3 & 0\end{bmatrix}(C_2)_4 = \{\begin{bmatrix}3 & 0\end{bmatrix}, \begin{bmatrix}3 & 4\end{bmatrix}\}$

$\begin{bmatrix}4 & 0\end{bmatrix}(C_2)_4 = \{\begin{bmatrix}4 & 0\end{bmatrix}, \begin{bmatrix}4 & 2\end{bmatrix}\}$

$\begin{bmatrix}0 & 1\end{bmatrix}(C_2)_4 = \{\begin{bmatrix}0 & 1\end{bmatrix}, \begin{bmatrix}0 & 4\end{bmatrix}\}$

$\begin{bmatrix}0 & 2\end{bmatrix}(C_2)_4 = \{\begin{bmatrix}0 & 2\end{bmatrix}, \begin{bmatrix}0 & 3\end{bmatrix}\}$

$\begin{bmatrix}1 & 4\end{bmatrix}(C_2)_4 = \{\begin{bmatrix}1 & 4\end{bmatrix}\}$

$\begin{bmatrix}2 & 3\end{bmatrix}(C_2)_4 = \{\begin{bmatrix}2 & 3\end{bmatrix}\}$

$\begin{bmatrix}4 & 1\end{bmatrix}(C_2)_4 = \{\begin{bmatrix}4 & 1\end{bmatrix}\}$

$\begin{bmatrix}3 & 2\end{bmatrix}(C_2)_4 = \{\begin{bmatrix}3 & 2\end{bmatrix}\}$

$\begin{bmatrix}1 & 1\end{bmatrix}(C_2)_4 = \{\begin{bmatrix}1 & 1\end{bmatrix}, \begin{bmatrix}1 & 2\end{bmatrix}\}$

$\begin{bmatrix}2 & 2\end{bmatrix}(C_2)_4 = \{\begin{bmatrix}2 & 2\end{bmatrix}, \begin{bmatrix}2 & 4\end{bmatrix}\}$

$\begin{bmatrix}3 & 3\end{bmatrix}(C_2)_4 = \{\begin{bmatrix}3 & 3\end{bmatrix}, \begin{bmatrix}3 & 1\end{bmatrix}\}$

$\begin{bmatrix}4 & 4\end{bmatrix}(C_2)_4 = \{\begin{bmatrix}4 & 4\end{bmatrix}, \begin{bmatrix}4 & 3\end{bmatrix}\}$

The final subgroup is the trivial subgroup which results in 25 fixed points when acting naturally on $V(2,5)$.

IV. CORRECTIONS TO A PREVIOUS PAPER

This section serves as a correction to case 2.4 in [**7**]. An issue was found during the creation of this paper and as a result Dr. Keller wrote a corrected version to the section with the following proof. This same proof will later be adapted to fit the current paper.

Case 2.4: There is "<" in the first and "=" in the second inequality.

Suppose we have equality in (4) and strict inequality in (3). That is

$$M \geq M_1 M_2 \geq p|D : D'C_D(V_1)||C_D(V_1) : C_D(V_1)'| \geq p|D/D'| = |G : G'|.$$

Because $|G : G'| = M$ we have equality everywhere, and $M = M_1 M_2, M_1 = p|D : D'C_D(V_1)| > |D : D'C_D(V_1)|, M_2 = |C_D(V_1) : C_D(V_1)'|$. Again let M_D denote the largest orbit size of D on V, then $M_D \geq M_1 M_2$ so $M_D = M$. By Theorem 1.1 $C_D(V_1)$ is abelian or has at least two orbits of size M_2 on W_1. We consider again some subcases.

Case 2.4.1 $C_D(V_1)$ has at least two orbits of size M_2 on W_1.

Let $w_1, w_2 \in W_1$ be representatives of such orbits.

Assume that $D/C_D(V_1)$ has at least two orbits of size M_1 on V_1. Because $M = M_1 M_2$ we have that $(v_1 + w_1)^D, (v_1 + w_2)^D$, $(v_2 + w_2)^D$ and $(v_2 + w_1)^D$ are all distinct orbits of size $M_D = M$, contradicting there being only two orbits of size M. Therefore $D/C_D(V_1)$ has exactly one orbit of size M_1 on V_1. Let $v_1 \in V_1$ be a representative of this orbit.

Now let w_1, w_2 be representatives of two distinct orbits of size M_2 of $C_D(V_1)$ on W_1, then $(v_1 + w_1)^D$ and $(v_1 + w_2)^D$ are two distinct $D-$orbits of size M, and if $C_D(V_1)$ had a third orbit of size M_2 on W_1, similarly we would get a third orbit of G of size M, a contradiction. Thus $C_D(V_1)$ exactly two orbits of size M_2 on W_1.

Now write $W_1 = \bigoplus_{i=1}^{k} X_i$ for a suitable $k \in \{1, \dots, n\}$ and irreducible $C_D(V_1)$-modules X_i $(i = 1, \dots, k)$. We may assume that $X_1 \le V_2$. Then the intersection of all the $C_{C_D(V_1)}(X_i)$ is trivial, and hence

$$C_D(V_1) \lesssim C_D(V_1)/C_{C_D(V_1)}(X_1) \times \cdots \times C_D(V_1)/C_{C_D(V_1)}(X_k) \quad (+)$$

Moreover, if we put $N_0 = C_D(V_1)$, $Z_0 = W_1$ and recursively for $i \ge 1$ let $Y_i \le Z_{i-1}$ be an irreducible N_{i-1}-module, $N_i = C_{N_{i-1}}(Y_i)$, and Z_i be a $C_D(V_1)$-invariant complement of Y_i in Z_{i-1}, and put $t = i - 1$ and stop the process as soon as $Z_i = 0$ and $N_i = 1$, then we have that $\bigcap_{i=0}^{t} N_i = 1$ and $W_1 = \bigoplus_{i=0}^{t} Y_i$. Also, $R_{i-1} := N_{i-1}/N_i$ acts faithfully and irreducibly on Y_{i-1} for $i = 1, \dots, t$. Write M_{i-1}^* for the largest orbit size of N_{i-1}/N_i on Y_{i-1} for $i = 1, \dots, t$. Then by repeated use of Lemma 2.1 we see that

$$M_2 = |C_D(V_1) : C_D(V_1)'| \le \prod_{i=1}^{t} |R_i : R_i{}'| \le \prod_{i=1}^{t} M_i^* \le M_2, \quad (++)$$

the last inequality easily following by considering the sum of representatives of orbits of size M_{i-1}^* of N_{i-1}/N_i on Y_{i-1}. Thus we have equality everywhere, and it follows that $|R_i : R_i{}'| = M_i^*$ for $i = 1, \dots, t$. It also follows that the elements of every orbit of $C_D(V_1)$ on W_1 of size M_2 have the form $y_1 + \dots, y_t$ for some $y_i \in Y_i$ $(i = 1, \dots, t)$ which lies in an orbit of size M_i^* of N_i/N_{i+1} on Y_i $\quad (+++)$.

Case 2.4.1.1 $C_D(V_1)$ is not abelian.

Put $C = C_D(V_1) \cap C_D(X_1) = C_{C_D(V_1)}(X_1)$. Then by $(+)$ we may assume that $C_D(V_1)/C$ is nonabelian, and it also acts faithfully and irreducibly on X_1. We also clearly may assume that $Y_1 = X_1$ and hence with $(++)$ and $(+++)$ conclude that $C_D(V_1)/C$ has exactly two orbits of size of its abelian quotient on X_1. Hence we may apply induction and, in particular, get $p = 2$, $|X_1| = 9$ and $C_D(V_1)/C \cong D_8$. Moreover, since $C_D(V_1)$ has exactly two orbits of size M_2 on W_1, then from

$(+++)$ it follows that R_{i-1} has exactly one orbit of size M^*_{i-1} on Y_{i-1} for $i = 2, \ldots, t$. This forces, for $i = 2, \ldots, t$, that R_{i-1} is cyclic of order 2, $|Y_{i-1}| = 3$, and hence $C_{C_D(V_1)}(X_1)$ is elementary abelian of order $p^{\dim W_1 - 2}$. Note that $W_1 = V_2$ since $p = 2$.

Assume that $k \geq 2$, so $t \geq 3$ (since the X_i all have dimension 2). Then we may assume that $X_2 = Y_1 \oplus Y_2$, and from the above we know that $C/C_C(X_2)$ is elementary abelian of order 4.

Now consider the action of $C_D(V_1)$ on X_1. We know that $C_D(V_1)$ is isomorphic to a subgroup of a direct product of k copies of D_8, and $C_D(V_1)/C$ is isomorphic to D_8 and has four noncentral involutions. If all of them have inverse images in $C_D(V_1)$ which act trivially on $X_2 \oplus \cdots \oplus X_k$, then $C_D(V_1)$ has a D_8 as a subgroup which acts trivially on $X_2 \oplus \cdots \oplus X_k$, and since the X_i are transitively permuted by D, it follows that $C_D(V_1)$ is isomorphic to a direct product of k copies of D_8; in particular, then $C/C_C(X_2) \cong C_D(V_1)/C_{C_D(V_1)}(X_2) \cong D_8$, contradicting the above observation that $C/C_C(X_2)$ is elementary abelian of order 4. Hence there exists an element $c \in C_D(V_1)$ such that $c \notin C$, $c^2 \in C$, and c acts nontrivially on at least one X_i for some $i \in \{2, \ldots, k\}$, so without loss we may assume that c acts nontrivially on X_2. Now there is a $0 \neq x \in V_1$ such that c centralizes x. Since $c \notin C$ and $C/C_C(X_2)$ is elementary abelian of order 4, this shows that $C_D(x)/C_{C_D(x)}(X_2)$ has order divisible by 8, and thus $C_D(x)/C_{C_D(x)}(X_2)$ is isomorphic to D_8 and therefore has two orbits of size 4 on X_2. This allows us in an obvious way to construct two different orbits of size $M_2 = 4^k$ of $C_D(V_1)$ on $V_2 = W_1$ having representatives with x in their X_1-component; in addition to another orbit of size M_2 having a representative in the X_1-component from the second orbit of size 4 of $C_D(V_1)/C$ on X_1, givng us in total three distinct orbits of $C_D(V_1)$ on V_2, contradicting the current fact that $C_D(V_1)$ has exactly two orbits of size M_2 on V_2.

Hence our assumption that $k \geq 2$ was wrong, and we now have $k = 1$. So $W_1 = V_2 = X_1$ is of order 9, and $C_D(V_1) \cong D_8$ acts irreducibly on it and has two orbits of size $M_2 = 4$ on it. Hence $D_8 \times D_8 \cong C_D(V_2) \times C_D(V_1)$ is a normal subgroup of G. Now since $D/C_D(V_1)$ has exactly one orbit of size M_1 on V_1 (as we saw above), it follows that $M_1 = 8$ and $D/C_D(V_1)$ must be at least of order 16, and thus $D/C_D(V_1)$ is a full Sylow 2-subgroup of $\mathrm{GL}(2,3)$, i.e., a semidihedral group of order 16. Moreover, $|G : G'| = M = M_1 M_2 = 8 \cdot 4 = 2^5$ and $|G| = |G/D|\,|D/C_D(V_1)|\,|C_D(V_1)| = 2 \cdot 16 \cdot 8 = 2^8$. Therefore $|G'| = 2^3$. Now let $Z = C_D(V_1)' \times C_D(V_2)'$. Then $Z \leq D'$ is a Klein 4-group and $G'/Z = (G/Z)'$. Working in G/Z, we notice that $(C_D(V_1) \times C_D(V_2)')/Z$ is elementary abelian of order 2^4, and if $g \in G - D$, then gZ interchanges the two subgroups $C_D(V_i)Z/Z \cong C_D(V_i)/C_D(V_i)'$ $(i = 1, 2)$. Looking at the elements $[gZ, xZ] \in (G/Z)'$ for $x \in C_D(V_1)$ shows us that $|(G/Z)'| \geq |C_D(V_1)Z/Z| = 4$ so that altogether $2^3 = |G'| = |G'/Z||Z| \geq 4 \cdot 4 = 2^4$, which is a contradiction. This completes Case 2.4.1.1.

Case 2.4.1.2 $C_D(V_1)$ is abelian.

Then $C_D(V_1)$ has regular orbits on W_1, and thus $M_2 = |C_D(V_1)|$, so $C_D(V_1)$ has exactly two regular orbits on W_1.

Note that $M_2 = |C_D(V_1)|$ and so

$$\begin{aligned} M &= M_1 M_2 = M_1 |C_D(V_1)| = |G/G'| = p|D/D'| \\ &= p|D : D'C_D(V_1)||D'C_D(V_1) : D'| \\ &= M_1 |D'C_D(V_1) : D'| \\ &= M_1 |C_D(V_1) : (D' \cap C_D(V_1))| \end{aligned}$$

This forces $D' \cap C_D(V_1) = 1$. So if $x \in D$ and $c \in C_D(V_1)$, then $[x, c] \in D' \cap C_D(V_1) = 1$. This shows that $C_D(V_1) \leq Z(D)$ and hence $C_D(V_i) \leq Z(D)$ for $i = 1, \ldots p$.

Now we consider the k in (+).

First suppose that $k = 1$, then $W_1 = X_1$, but since $W_1 = V_2 \oplus \cdots \oplus V_p$, we see that $X_1 = V_2$ and $p = 2$. In particular, V_2 is an irreducible faithful $C_D(V_1)$-module, so $C_D(V_1)$ is cyclic and has only regular orbits on $V_2 - \{0\}$. So there are exactly two such orbits, which shows that $(|V_2| - 1)/2 = |C_D(V_1)|$. Since $|C_D(V_1)|$ is a power of 2, by an elementary result from number theory (see [**12**]) it follows that - if we write q for the characteristic of V - either V_2 is of dimension 1 and $|V_2|$ is a Fermat prime, or $|V_2| = 9$ and $|C_D(V_1)| = 4$. In the former case we get that $D/C_D(V_1)$ is abelian and hence D is abelian, and so G is abelian (since $G' = D'$), a contradiction. In the latter case we get that $D/C_D(V_1)$ must be at least of order 8 (since it has exactly one maximal orbit (of size M_1) on V_1, and it must be isomorphic to a subgroup the the semidehedral group SD_{16}, as Sylow 2-subgroup of $\mathrm{GL}(2,3)$. However, all such subgroups have center of order 2, contradicting the fact that $|C_D(V_1)| = |C_D(V_2)| = 4$ and $C_D(V_1) \leq Z(D)$. This concludes the case that $k = 1$.

So let $k > 1$. Then define $X_0 = 0$ and $L_i = C_{C_D(V_1)}(X_0 \oplus \cdots \oplus X_i)/C_{C_D(V_1)}(X_0 \oplus \cdots \oplus X_{i+1})$ for $i = 0, \ldots, k-1$. As $k > 1$, we see that L_0 has exactly one regular orbit on X_1, because otherwise also L_1 would have at least two regular orbits on X_2 which ultimately would lead to $C_D(V_1)$ having at least four regular orbits on W_1, a contradiction. Since all orbits of L_0 on X_1 must be regular, we thus conclude that $|L_0| = |X_1| - 1$. Since $C_D(V_1)$ has exactly two regular orbits on W_1, it follows that there is exaclty one $l \in \{1, \ldots, k\}$ such that L_{l-1} has exactly two regular orbits on X_l, whereas all the other L_i's have exactly one regular orbit on X_{i+1}. However, since L_{l-1} only has regular orbits on $X_l - \{0\}$, it is clear that the single regular orbit of size $|X_l| - 1$ of $C_D(V_1)/C_{C_D(V_1)}(X_l)$ on X_l splits into at least p regular orbits for L_{l-1} on X_L. This shows that $p = 2$. Hence $C_D(V_1)C_D(V_2) = C_D(V_1) \times C_D(V_2) \leq Z(G)$, and since $D/C_D(V_1)$ acts faithfully and irreducibly on V_1, we see that $C_D(V_1) \cong C_D(V_1)C_D(V_2)/C_D(V_2)$ is cyclic and thus has only regular orbits on $V_2 - \{0\}$. So there are exactly two such orbits and we now can arrive at a contradiction just as in the case that $k = 1$

This concludes Case 2.4.1.2 and thus Case 2.4.1 is completed.

Case 2.4.2 $C_D(V_1)$ has exactly one orbit of size M_2 on W_1.

Then by Theorem 1.1 $C_D(V_1)$ is abelian and hence has regular orbits on W_1, so $M_2 = |C_D(V_1)|$ and the same argument as at the beginning of Case 2.4.1.2 shows that $C_D(V_1) \leq Z(D)$ and hence $C_D(V_i) \leq Z(D)$ for $i = 1, \dots p$.

Assume that $X_1 < V_2$ (where X_1 is as in (+)). Since $C_D(V_1)$ has exactly one regular orbit on W_1, it also has exactly one regular orbit on V_2, and since V_2 is not irreducible as $C_D(V_1)$-module, by Lemma 2.2 it is clear that $C_D(V_1)/C_{C_D(V_1)}(V_2)$ is not cyclic. But since $C_D(V_1) \leq Z(D)$, we see that

$$C_D(V_1)/C_{C_D(V_1)}(V_2) = C_D(V_1)/C_D(V_1 \oplus V_2) = C_D(V_1)/(C_D(V_1) \cap C_D(V_2))$$

$$\cong C_D(V_1)C_D(V_2)/C_D(V_2)$$

is a noncyclic central subgroup of $D/C_D(V_2)$. But on the other hand, $D/C_D(V_2)$ acts faithfully and irreducibly on V_2 and hence has a cyclic center, and we have a contradiction. This shows that $X_1 = V_2$, so V_2 is an irreducible $C_D(V_1)$-module and $C_D(V_1)$ has exactly two orbits (one of them being the trivial orbit) on V_2. Therefore again by [**12**]) it follows that - if we write q for the characteristic of V - either

- $p = 2$, V_2 is of dimension 1 and $|V_2|$ is a Fermat prime; or
- $q = 2$ and $|C_D(V_1)/C_D(V_1 \oplus V_2)| = p$ is a Mersenne prime; or
- $p = 2$, $q = 3$, $|V_2| = 9$ and $|C_D(V_1)| = 8$.

In the first case we get (as earlier) that $D/C_D(V_1)$ is abelian and thus D is abelian, a contradiction. In the second case, since D is a p-group, with [**12**] we see that $D/C_D(V_1)$ cyclic of order p and thus abelian, making D abelian, a contradiction. So we are left with the third case. Here we have that $D/C_D(V_1)$ is a subgroup of the semidihedral group of order 16, so $|G| \leq 2^9$, and $|G| \leq 2^8$ unless $D \cong \mathrm{SD}_{16} \times \mathrm{SD}_{16}$. Moreover, since any $g \in G - D$ interchanges $C_D(V_1)$ and $C_D(V_2)$, by taking commutators of elements in $C_D(V_1)$ with g we easily see that $|D'| \geq 8$ and so $|G'| \geq 8$. Now D has an orbit of size $\geq 2^6$ on V (from the regular orbit of $C_D(V_1) \times C_D(V_2)$. So if $|G| \leq 2^8$, we get $2^5 < 2^6 \leq M = |G/G'| \leq 2^8/2^3 = 2^5$, a contradiction. This leaves us with $|G| = 2^9$, and $D \cong \mathrm{SD}_{16} \times \mathrm{SD}_{16}$, but in this case for similar reasons as above we see that $|D'| \geq 2^4$ and thus get the contradiction $2^5 < 2^6 \leq M = |G/G'| \leq 2^9/2^4 = 2^5$.

This final contradiction concludes the proof of the theorem. $\diamond$

V. PROOF OF MAIN RESULT

The goal of this section is to prove the main result of this paper. First, we restate the result,

Theorem 3. *Let G be a finite nonabelian solvable group and V a finite faithful irreducible G-module. Suppose that $M = |G/G'|$ is the largest orbit size of G on V and that there are exactly three orbits of size M on V. Then $G = D_8 \circ C_4$, the central product of the dihedral group of order 8 and the cyclic group of order 4, and $V = V(2,5)$.*

Proof. Assume that the result is not true and let G, V be a counterexample such that $|GV|$ is minimal. We will note that since $|G/G'| = M$, then by Theorem 1 we know that G is nilpotent. A proof of this can be found in both [**9**] and [**7**]. We will start by making the reduction to p-groups.

Step 1: A Reduction to p-groups

Assume that G is not a p–group, then $|G|$ is divisible by at least two distinct primes, call them p and q. Let $P \in \mathrm{Syl}_p(G)$ and $H \in \mathrm{Hall}_q(G)$. Since G is nilpotent, we know that $P \triangleleft G$ [**3**]. Furthermore, we know that $G = P \times H$ [**3**]. By the hypothesis we know V is a finite G–module over a field, call it K. By [**14**], there exists a field extension L of K such that if U is an irreducible summand of V viewed as an $LG - module$, then the permutation actions of G on V and U are permutation isomorphic. This allows us to consider the action of G on U instead of V. Through relabeling we can assume V is absolutely irreducible. Using [**1**] we may assume that $V = X_1 \otimes X_2$ where X_1 is a faithfully irreducible P–module and X_2 is a faithfully irreducible H–module. We can pick an $x_1 \in X_1$ and a $x_2 \in X_2$ such that $|x_1^P|$ is the largest orbit size of P on X_1, $|x_2^H|$ is the largest orbit size of H on X_2, and we have

$$|P/P'| \leq |x_1^P|,$$

$$|H/H'| \leq |x_2^H|.$$

If $g \in P$ and $g \in H$ such that $gh \in C_G(x_1 \otimes x_2)$ then $x1g = \alpha x_1$ and $x_2h = \beta x_2$ where α, β are scalars in the field with $\alpha\beta = 1$ [**11**]. Now g and h have coprime orders so we have that $\alpha = \beta = 1$ which gives

$$C_G(x_1 \otimes x_2) = C_P(x_1) \times C_H(x_2).$$

We now have the following

$$\begin{aligned} |G/G'| = M \geq |(x_1 \otimes x_2)^G| &= |G : C_p(x_1) \times C_H(x_2)| \\ &= |P : C_P(x_1)||H : C_H(x_2)| \\ = |x_1^P||x_2^H| &\geq |P/P'||H/H'| = |G/G'|. \end{aligned} \tag{1}$$

Since we have inequality everywhere in (1), we have $|x_1^P| = |P/P'|$ and $|x_2^H| = |H/H'|$. Therefore $|P/P'|$ is the largest orbit size of P on X_1 and similarly $|H/H'|$ is the largest orbit size of H on X_2. From now on let $M_1 = |P/P'|$ and $M_2 = |H/H'|$. We can now break this into 7 cases.

Case 1: P and H have exactly one maximal orbit of size M_1 and M_2 respectively. Then by theorem 1, both P and H are abelian. But if P and H are both abelian then $G = P \times H$ is abelian. A contradiction to G being nonabelian.

Case 2: P and H have exactly two maximal orbits of size M_1 and M_2 respectively. Let $y_1 \in X_1$ be a representative of the second orbit of size M_1 on X_1 and $y_2 \in X_2$ be a representative of the second orbit of size M_2 on X_2. Then using (1),

$$|G/G'| = |(x_1 \otimes x_2)^G| = |(x_1 \otimes y_2)^G| = |(y_1 \otimes x_2)^G| = |(y_1 \otimes y_2)^G| = M$$

we have four orbits of size M in the action of G on V. A contradiction to having exactly three orbits of size M.

Case 3: P and H have exactly three maximal orbits of size M_1 and M_2 respectively. Let $y_1, y_3 \in X_1$ be the representatives of the other two orbits of size M_1 of P on X_1 and $y_2, y_4 \in X_2$ be the representatives of the other two orbits of size M_2 of H on X_2. Then using (1),

$$|G/G'| = |(x_1 \otimes x_2)^G| = |(x_1 \otimes y_2)^G| = |(x_1 \otimes y_4)^G| = |(y_1 \otimes x_2)^G| = |(y_1 \otimes y_2)^G| = |(y_1 \otimes y_4)^G|$$

$$= |(y_3 \otimes x_2)^G| = |(y_3 \otimes y_2)^G| = |(y_3 \otimes y_4)^G| = M$$

we have nine orbits of size M in the action of G on V. A contradiction to having exactly 3 orbits of size M.

Case 4: one of the following holds, P has more than three orbits of size M_1 on X_1 or H has more than three orbits of size M_2 on X_2. First suppose P has four orbits of maximal size on X_1. Let $y_1, y_3, y_5 \in X_1$ be the representatives of the remaining three orbits. Then using 1,

$$|G/G'| = |(x_1 \otimes x_2)^G| = |(y_1 \otimes x_2)^G| = |(y_3 \otimes x_2)^G| = |(y_5 \otimes x_2)^G| = M$$

we have four orbits of size M when G acts on V. A contradiction of our hypothesis having exactly three orbits of size M. Replace P with H to show that H can not have more than three orbits of size M_2. Clearly, P and H can not both have more than three orbits of size M_1 and M_2 respectively.

Case 5: P has exactly three orbits of size M_1 on X_1 and H has exactly two orbits of size M_2 on X_2 or the opposite, H has exactly three orbits of size M_2 on X_2 and P has exactly two orbits of size M_1 on X_1. Consider the first option. Let $y_1, y_3 \in X_1$ be representatives the other two orbits of size M_1 on X_1 and let $y_2 \in X_1$

be representatives the other orbit of size M_2 on X_2. Then using 1,

$$|G/G'| = |(x_1 \otimes x_2)^G| = |(x_1 \otimes y_2)^G| = |(y_1 \otimes x_2)^G| = |(y_1 \otimes y_2)^G| = |(y_3 \otimes x_2)^G|$$

$$= |(y_3 \otimes y_2)^G| = M$$

we have six orbits of size M in the action of G on V. A contradiction to having exactly 3 orbits of size M. To see the other option, just recreate this proof with $y_1 \in X_1$ as a representative of the other orbit of size M_1 on X_1 and $y_2, y_4 \in X_2$ as representatives of the other two orbits of size M_2 on X_2.

Case 6: P has exactly two orbits of size M_1 on X_1 and H has exactly one maximal orbit on X_2 or the opposite, H has exactly two orbits of size M_2 on X_2 and P has exactly one maximal orbit on X_1. Consider the first option. Suppose P has exactly two orbits of size M_1 on X_1 and H has exactly one maximal orbit on X_2. Let $y_1 \in X_1$ represent the second orbit of P on X_1. Then $(x_1 \otimes x_2)^G$ and $(y_1 \otimes x_2)^G$ are two distinct orbits of size M of G on V. Since H has exactly one maximal orbit, by Theorem 1 H must be abelian. Therefore $|H/H'| = |H| = |x_2^H|$ and by Lemma 1 $H \cong H/C_H(X_2)$ and H is a cyclic group of order $|X_2| - 1$. We know P can not be abelian or else $G = P \times H$ would be abelian. Thus we can use [**7**] to get $P = D_8$ and $X_1 = V(2,3)$. This makes $\text{char}(X_1) = 3$ and $\text{char}(X_2) = 3$, leaving us with $|H| = 3^n - 1$ for some $n \in \mathbb{N}$. This makes $|H|$ even and contradicts that $P \in \text{Syl}_2(G)$. Now consider when H has exactly two orbits of size M_2 on X_2 and P has exactly one maximal orbit on X_1. By Theorem 1 P is an abelian group and by Lemma 1 P is a cyclic group of order $|X_1| - 1$. We know H can not be abelian or else G is, so we may use induction to see that $H = D_8$ and $X_2 = V(2,3)$. This means that $|H| = 8$ and $\text{char}(X_1) = \text{char}(X_2) = 3$ and $|P|$ is even, contradicting that $\gcd(|H|, |G|) = 1$.

Case 7: P has exactly three orbits of size M_1 on X_1 and H has exactly one maximal orbit on X_2 or the opposite, H has exactly three orbits of size M_2 on X_2 and P has exactly one maximal orbit on X_1. Consider the first option. Suppose P has exactly three orbits of size M_1 on X_1 and H has exactly one maximal orbit on X_2. Let $y_1, y_2 \in X_1$ represent the other two orbits of P on X_1. Then $(x_1 \otimes x_2)^G$, $(y_1 \otimes x_2)^G$ and $(y_2 \otimes x_2)^G$ are three distinct orbits of size M of G on V. Since H has exactly one maximal orbit, by Theorem 1 H must be abelian. Therefore $|H/H'| = |H| = |x_2^H|$ and by Lemma 1 $H \cong H/C_H(X_2)$ and H is a cyclic group of order $|X_2|-1$. We know P can not be abelian or else $G = P \times H$ would be abelian. Thus we can use induction to get $P = D_8 \circ C_4$ and $X_1 = V(2,5)$. This makes $\mathrm{char}(X_1) = 5$ and $\mathrm{char}(X_2) = 5$, leaving us with $|H| = 5^n - 1$ for some $n \in \mathbb{N}$. This makes $|H|$ even and contradicts that $P \in \mathrm{Syl}_2(G)$. Now consider when H has exactly three orbits of size M_2 on X_2 and P has exactly one maximal orbit on X_1. By Theorem 1 P is an abelian group and by Lemma 1 P is a cyclic group of order $|X_1| - 1$. We know H can not be abelian or else G is, so we may use induction to see that $H = D_8 \circ C_4$ and $X_2 = V(2,5)$. This means that $|H| = 16$ and $\mathrm{char}(X_1) = \mathrm{char}(X_2) = 5$ and $|P|$ is even, contradicting that $\gcd(|H|, |G|) = 1$.

Step 2: A Reduction to the Case that V is Imprimitive

The next step is to show that V is imprimitive. We will recreate the proof from [**7**] for the sake of completeness. Assume that V is quasiprimitive. Using the proof of Theorem 3.3 in [**12**] we can write $G = S \times T$ where S is a 2-group and T is a cyclic group of odd order and by Corollary 1.3 in [**12**] G is cyclic, quaternion, dihedral, or semi-dihedral and $G \ncong D_8$. It is well-known that the derived subgroup of the quaternion, dihedral, and semi-dihedral 2-groups have index 4 [**IS**]. We also know there exists a $U \lhd G$ where U is cyclic, $|G:U| \leq 2$ and U has a regular orbit on V. This gives the inequality $M \geq |U| \geq |G|/2$. Since G is a nonabelian p–group, we have $|T| = 1$ and $G = S$, and $p = 2$. This makes G a nonabelian 2–group so $|G| > 4$

and $8||G|$. All together we have

$$|G/G'| = 4 \leq |G|/2 \leq M$$

We have equality here so $|G'| = 2$ and $|G| = 8$ making G the quaternion group. It is known that the quaternion group has a regular orbit on V [**12**] contradicting $M = 4$. Therefore V can not be quasiprimitive. In fact, V is imprimitive. Thus completes the recreation of the proof that V is imprimitive in [**7**].

Step 3: The Case Where V is Imprimitive

Since we know V is imprimitive now, then there exists a $D \trianglelefteq G$ with $|G : D| = p$ where p is prime and $V_D = V_1 \oplus \cdots \oplus V_p$ for irreducible $D-$modules V_i of V_D. There are two cases to consider, $D' < G'$ and $D' = G'$.

Step 3.1: The Case Where $D' < G'$

Suppose $D' < G'$, that is, $p|D'| \leq |G'|$. Then by using Theorem 1 we have the following inequality:

$$M \geq |D : D'| = \frac{|D|}{|D'|} = \frac{p|D|}{p|D'|} \geq \frac{|G|}{|G'|} = M.$$

Therefore $|D : D'| = M$. Since $\cap_{i=1}^{p} C_G(V_i) = 1$, we have that D is isomorphic to a subgroup of $D/C_D(V_1) \times \ldots \times D/C_D(V_p)$ [**9**]. From now on we will use the symbol $H \lessapprox G$ to denote the fact that H is isomorphic to a subgroup of G. Therefore

$$D \lessapprox D/C_D(V_1) \times \ldots \times D/C_D(V_p) = \bigtimes_{i=1}^{p} D/C_D(V_i) =: T. \tag{2}$$

The above equation tells us that if $D/C_D(V_1)$ is abelian for any $i = 1, \ldots, p$ then $D/C_D(V_i)$ is abelian for all $i = 1, \ldots, p$ and D will follow. This is an important fact that we will reference in the following arguments. We know from Theorem 1 that since $|D : D'| = M$ then one of the following is true: D is abelian, D has 2 orbits of size M on V_D or D has 3 orbits of size M on V_D. We know that D can not have more than 3 orbits of size M on V_D or else G would have more than three orbits of size M on V or an orbit larger than size M on V. Recall that $V_D = V_1 \oplus V_2 \oplus \cdots \oplus V_p$, with each V_i being irreducible faithful D-modules. Let $W_i = \oplus_{j=1, j\neq i}^{p} V_j$. Write M_1 for the largest orbit size of D on V_1 and M_2 for the largest orbit size of $C_D(V_1)$ on W_1. Also let M_D be the largest orbit size of D on V. Let $x \in V_D$ be in a largest orbit of D on V_D. Write $x = x_1 + x_2$ for some $x_1 \in V_1$ and $x_2 \in W_1$. Observe that

$$M_D = |D : C_D(x)| = |D : C_D(x_1) \cap C_D(x_2)| =$$

$$|D : C_D(x_1)||C_D(x_1) : C_D(x_1) \cap C_D(x_2)| = |x_1^D||x_2^{C_D(x_1)}|.$$

If $|x_1^D| < M_1$, then the same calculation would show that if $y_1 \in V_1$ with $|y_1^D| = M_1$, then $|D : C_D(y_1 + x_2)| > M_D$, contradicting the definition of M_D. Thus we have $|x_1^D| = M_1$. Moreover, $|x_2^{C_D(x_1)}| \geq |x_2^{C_D(V_1)}|$. We also can conclude that $|x_2^{C_D(x_1)}| \geq M_2$, because if $|x_2^{C_D(x_1)}| < M_2$, then let $y_2 \in W_1$ such that $|y_2^{C_D(V_1)}| = M_2$, and then

$$M_D = |(x_1 + y_2)^D| = |D : C_D(x_1) \cap C_D(y_2)| = |D : C_D(x_1)||C_D(x_1) : C_D(x_1) \cap C_D(y_2)|$$

$$M_1|y_2^{C_D(x_1)}| = M_1M_2 > M_1|x_2^{C_D(x_1)}| = |D : C_D(x)| = M_D,$$

a contradiction. Thus altogether we get $M \geq M_D \geq M_1M_2$.Then

$$M \geq |x^D| = |D : C_D(x)| = |D : C_D(x_1) \cap C_D(x_2)|$$

$$= |D : C_D(x_1)||C_D(x_1) : C_D(x_1) \cap C_D(x_2)| = M_1|x_2^{C_D(x_1)}| \geq M_1|x_2^{C_D(W_1)}| = M_1M_2.$$

By applying Theorem 2 $|D : D'|$ divides $|D/C_D(V_1) : D/C_D(V_1)'||C_D(V_1)/C_D(V_1)'|$

and we have

$$M \geq M_1M_2 \geq |D/C_D(V_1) : (D/C_D(V_1)')||C_D(V_1) : C_D(V_1)'| \geq |D : D'| = M.$$

This shows $M_1M_2 = M_D = M$ and it follows that $M_1 = |D/C_D(V_1) : (D/C_D(V_1))'|$ and $M_2 = |C_D(V_1) : C_D(V_1)'|$. Suppose that there exists $y_1, z_1, a_1 \in V_1$ where y_1, z_1, a_1 are representatives of 3 new orbits of size M_1 of $D/C_D(V_1)$ acting on V_1. Then as we have seen previously, $x_1 + x_2, y_1 + x_2, z_1 + x_2$, and $a_1 + x_2$ are four representatives of orbits of size M of D on V_D. This contradicts our hypothesis. So we know $D/C_D(V_1)$ has at most 3 orbits of size M_1 on V_1. We also know that if $D/C_D(V_1)$ has exactly one orbit of size M_1 on V_1 then $D/C_D(V_1)$ is abelian by Theorem 1 and by (2) D would also be abelian. Therefore we can split the case into D being abelian and D being nonabelian. We will look at the case that D is nonabelien first.

Step 3.1.1: The Case Where D is Nonabelian

When D is nonabelian then we know $D/C_D(V_1)$ must have either two or three orbits of size M_1 on V_1. We know by (2) that $D/C_D(V_1)$ is not abelian or else D would be abelian as a result. We also know that $C_D(V_1)$ must have exactly one orbit of size M_2 on W_1 or else G would have too many orbits of size M as a result, this is a similar argument as made in the reduction to p–groups section. Thus $D/C_D(V_1)$ has either exactly two orbits or exactly three orbits.

Step 3.1.1.1: The Case Where $D/C_D(V_1)$ has Exactly Two Orbits of Size M_1

When $D/C_D(V_1)$ has exactly two orbits of size M_1, by [**7**], $D/C_D(V_1) = D_8$, $V_1 = V(2,3)$, and $p = 2$ so $|W_1| = |V_1| = |V_2|$. We have $D \lessapprox D/C_D(V_1) \times D/C_D(V_2) \cong D_8 \times D_8$ making $C_D(V_1) \lessapprox D_8$. If $C_D(V_1) \cong D_8$ then $C_D(V_1)$ has two orbits of size

M_2 on V_2, contradicting $C_D(V_1)$ having exactly one orbit of size M_2 on $W_1 = V_2$. Therefore $C_D(V_1)$ is of order one, two or four. This makes $C_D(V_1)$ abelian which gives it at least two regular orbits (i.e, two orbits of size $M_2 = |C_D(V_1)|$) unless $C_D(V_1)$ is the Klein four-group. Since $C_D(V_1)$ has only one orbit of size M_2 on V_2, we conclude that $C_D(V_1)$ is the Klein four-group. Since $C_D(V_1)$ and $C_D(V_2)$ are conjugate under the action of G, then $C_D(V_2)$ is also a Klein four-group. Thus $C = C_D(V_1)C_D(V_2)$ is elementary abelian of order 16. Let $d \in D - C$, then $[d, C_D(V_1)] \leq C_D(V_1)$ is cyclic of order 2, $[d, C_D(V_2)] \leq C_D(V_2)$ is cyclic of order 2 and $C_D(V_1) \cap C_D(V_1) = 1$. It follows that $|D'| \geq 4$, so $16 = M = |D/D'| \leq 32/4 = 8$, a contradiction.

Step 3.1.1.2: The Case Where $D/C_D(V_1)$ has Exactly Three Orbits of Size M_1

When $D/C_D(V_1)$ has exactly three orbits of size M_1, by induction, we have $D/C_D(V_1) = D_8 \circ C_4$, $V_1 = V(2,5)$, and $p = 2$ thus $|W_1| = |V_1| = |V_2|$. We have $D \lnapprox D/C_D(V_1) \times D/C_D(V_2) \cong (D_8 \circ C_4) \times (D_8 \circ C_4)$ making $C_D(V_1) \lnapprox D_8 \circ C_4$. But, if $C_D(V_1) \cong D_8 \circ C_4$ then $C_D(V_1)$ has three orbits of size M_2 on V_2; a contradiction to $C_D(V_1)$ having exactly one orbit of size M_2 on $W_1 = V_2$. Therefore $C_D(V_1)$ must be isomorphic to a proper subgroup of $D_8 \circ C_4$ (i.e. $C_D(V_1)$ is one of the following: C_2, C_4, V_4, $C_2 \times C_4$, Q_8, D_8). All proper subgroups of $C_D(V_1)$ have at least two regular orbits (an orbit of size $M_2 = |C_D(V_1)|$), unless $C_D(V_1)$ is the dihedral group of order 8. Therefore we can conclude $C_D(V_1)$ is D_8. Since $C_D(V_1)$ and $C_D(V_2)$ are conjugate under the action of G, then $C_D(V_2)$ is also a dihedral group of order 8. Let $v_1 \in V_1$ be in an orbit of size $M_1 = 8$ of $D/C_D(V_1)$ on V_1, such that v_1 is in a regular orbit of $C_D(V_2)$ on V_1. We claim that $|C_D(v_1)| = 16$. Note that $D_8 \cong C_D(V_1) \leq C_D(v_1)$ so $|C_D(v_1)| \geq 8$. Assume $|C_D(v_1)| = |C_D(V_1)| = 8$.Then $|v_1^D| = |D : C_D(v_1)| = 16$ and $v_1^D \leq V_1$ so D has an orbit of size 16 on V_1, a contradiction to $M_1 = 8$. Thus $|C(v_1)|$ must be greater than 8 and as a p-group, it must be of order at least 16. Since $|C_D(v_1) \cap C_D(V_2)| = 1$, then $|C_D(v_1)C_D(V_2)| \geq 16 \cdot 8 = |D|$.

Combined with $C_(v_1)C_D(V_2) \leq D$ and $|D_D(v_1)C_D(V_2)| = |C_D(v_1)||C_D(V_2)|$, gives us $|C_D(v_1)| = 16$. Now $|C_D(v_1)| = 16$ and $C_D(v_1) \cap C_D(V_2) = 1$, therefore

$$C_D(v_1) \cong C_D(v_1)/(C_D(v_1) \cap C_D(V_2)) \cong C_D(v_1)C_D(V_2)/C_D(V_2) \cong D/C_D(V_2) \cong D_8 \circ C_4$$

where the third isomorphism comes from the isomorphism theorems. Hence we have $C_D(v_1) \cong D_8 \circ C_4$. Therefore $C_D(v_1)$ acts faithfully on V_2 and if $z_1, z_2, z_3 \in V_2$ are representatives of the three orbits of size eight in the action of $C_D(v_1)$ on V_2, then $v_1 + z_1$, $v_1 + z_2$, and $v_1 + z_3$ are representatives of three orbits of size $64 = M_D = M$ of D on V_D. Now let $w_1 \in V_1$ be in an orbit of D of size $M_1 = 8$ such that w_1 is not in a regular orbit of $C_D(V_2)$ on V_1. Let $z_4 \in V_2$ be in the (unique) regular orbit of $C_D(V_1)$ on V_2 (so it is of size $M_2 = 8$). Then clearly $w_1 + z_4$ is a representative of an orbit of size $64 = M$ of D on V that is different from the orbits containing $v_1 + z_1$, $v_1 + z_2$, and $v_1 + z_3$. Thus we have found four orbits of size $64 = M$ of D on V which contradicts our hypothesis. This concludes the case where D is nonabelian.

Step 3.1.2 The Case Where D is Abelian

When D is abelian, there are three possibilities; D has exactly one orbit of size M on V, D has exactly two orbits of size M on V, and D has exactly three orbits of size M on V. Clearly D can not have four orbits of size M or else G has four orbits of size M on V or an orbit larger than M on V. Recall that $|G/G'| = M = M_D = |D| = |G|/p$, thus $|G'| = p$. We may use Theorem 3.2 from [**5**] to state that $p = 2$ and there exists a $v \in V$ such that $|C_G(v)| \leq 2$, in particular $|C_G(v)| = 2$ in our case or else G would have a regular orbit making $|G| = M = |D|$ which contradicts $p|D| = |G|$. This allows us to improve on the proof given by [**7**].

Step 3.1.2.1: The Case Where D has Exactly One Orbit of Size M

Since D is abelian, D has regular orbits, so $M = |D|$. Therefore D must have exactly one regular orbit. Then by Lemma 1, we have $D = \bigtimes_{i=1}^{p} C_D(W_i)$. We can note that G/D cycles these direct factors around, therefore $|G : G'| = p|C_D(W_1)|$ (see [**7**]). Additionally, $|G/G'| = |D| = |C_D(W_1)|^p$ which implies that $|C_D(W_1)|^{p-1} = p$. Since $p = 2$, then $|C_D(W_1)| = 2$. It follows that $|D| = |C_D(W_1)|^p = 4$, D is elementary abelian, $|G| = 8$, and G is nonabelian. Since $M = |D| = 4$ and $|G| = 8$, G does not have a regular orbit on V making G the dihedral group of order 8. Since D is elementary abelian and has exactly one regular orbit on V, we can conclude that $|V| = 9$. We know by [**7**] that $G = D_8$ has exactly 2 orbits of size M on $V = V(2,3)$. This contradicts that G has exactly three orbits of size M on V in the hypothesis. This concludes the case where D has exactly one orbit of size M on V.

Step 3.1.2.2: The Case Where D has Exactly Two Orbits of Size M

Assume that D has exactly two orbits of size M on V, we know that these are regular orbits as D is abelian. To keep consistent, we will denote the largest orbit size of the action of $D/C_D(V_1)$ on V_1 as M_1, and M_2 will denote the largest orbit size of $C_D(V_1)$ acting on $W_1 = V_2$. Recall that $D/C_D(V_1)$ has at most two orbits of size M_1 on V_1 if D has only 2 orbits. So for $i = 1, 2$, $D/C_D(V_i)$ are isomorphic and $D/C_D(V_i)$ has either one or two orbits of size M_1 acting on V_i. Suppose we have two orbits of size M_1 in the action of $D/C_D(V_1)$ on V_1, then as argued previously $C_D(V_1)$ must have exactly one orbit of size M_2 on V_2 or else G would have too many orbits of size M on V. Since $D/C_D(V_1)$ has two regular orbits on V_1, then $D/C_D(V_2)$ will have at least two regular orbits on V_2, which immediately implies that $C_D(V_1)$ has two regular orbits on V_2, contradiction. Therefore we know that $D/C_D(V_1)$ has only one orbit of size M_1 in the action on V_1 and $C_D(V_1)$ has exactly two orbits of size M_2 on V_2. Thus we have $|D/C_D(V_1)| = |V_1| - 1$, and $D/C_D(V_1)$ is cyclic. Now we have $D \lneqq D/C_D(V_i) =: T$ and T has exactly one regular orbit on V. Every regular orbit of T on V splits into $\frac{|T|}{|D|}$ regular orbits of D. Since D has no more than two orbits of size $M = M_D = |D|$, we see that $\frac{|T|}{|D|} \leq 2$. If $\frac{|T|}{|D|} = 1$, then $T = D$ and so $C_D(V_1) \cong \bigtimes_{i=2}^{p} D/C_D(V_i)$ has only one regular orbit on W_1, a contradiction.Therefore we know that $\frac{|T|}{|D|} = 2$. Then from Lemma 1 we have that $|T| = (|V_i| - 1)^p$, thus

$$(|V_1| - 1)^p = 2|D| = 2p^k \tag{3}$$

for appropriate k. We wish to show that $|C_D(V_1)| = 2$. Observe that $C_D(V_1) \cap C_D(V_2) = 1$ and so $C_D(V_1) \times C_D(V_2) = C_D(V_1)C_D(V_2) \leq D$. Let $g \in G - D$ and $(1,a) \in C_D(V_1) \times C_D(V_2)$, then G' contains the element

$$[(1,a), g] = (1,a)^{-1}g^{-1}(1,a)g = (1,a)^{-1}(1,a)^g = (1,a^{-1})(a^*,1) = (a^*,a^{-1})$$

for suitable $a^* \in C_D(V_1)$. If there are more than two choices for $a \in C_D(V_2)$ then $|G'| \geq 3$. On the other hand, $\frac{|G|}{|G'|} = |D|$ and $\frac{|G|}{|D|} = 2$ so $|G'| = |G|/|D| = 2$. We conclude that $|C_D(V_1)| \leq 2$. If $|C_D(V_1)| = 1$ then $D = D/C_D(V_1)$, but we know that D has exactly two orbits of size M_1 on V_1 while $D/C_D(V_1)$ has only one orbit of size M_1 on V_1. Therefore we can say $|C_D(V_1)| = 2$. Therefore we can say $|C_D(V_1)| = 2$. We can now determine $|D|$ using $\frac{|T|}{|D|} = \frac{|D/C_D(V_1)||D/C_D(V_2)|}{|D|} = 2$. So $2|D| = \frac{|D|}{2}\frac{|D|}{2} = \frac{|D|^2}{4}$, or $|D| = 8$ and $|G| = 16$. From Lemma 1 $|D/C_D(V_1)| = 4 = |V_1| - 1$ giving us $|V_1| = 5, i = 1,2$, and thus $V = \mathrm{GF}(5)^2$. Then $\{(1,0),(2,0),(3,0),(4,0)\}$ and $\{(0,1),(0,2),(0,3),(0,4)\}$ are both orbits of D on V (as $D/C_D(V_1)$ has an orbit of size 4 on V_i), and their union is an orbit of size 8 of G on V. Moreover, if $a,b \in \mathrm{GF}(5) - \{0\}$, then $(a,b)^D$ will contain $(a,-b)$ as $C_D(V_1)$ acts as $x \to -x$ on V_2. Since $D/C_D(V_1)$ has an orbit of size 4 on V_1, we also see that $(a,b)^D$ will contain elements of the form $(1,*),(2,*),(3,*)$, and $(4,*)$. Altogether we see that $|(a,b)^D| = 8 = M$. Putting this together shows that, since $8 = M$, G has three orbits of size 8 on V. We know that G must be a subset of the Sylow 2-group of $GL(2,5)$. There are three subgroups the the Sylow 2-group of $GL(2,5)$ that G can possibly be: $C_4 \times C_4$, $M_4(2)$ (the maximum modular cyclic group $M_n(2)$,a semidirect product $C_2^{n-1} \rtimes C_2$ where C_2 acts on C_2^{n-1} by $x \mapsto x^{2^{n-2}+1}$), and our group $D_8 \circ C_4$. We know that $G \neq C_4 \times C_4$ as $C_4 \times C_4$ is abelian and G is not. By calculating the orbits of $M_4(2)$ on $V(2,5)$ by matrix multiplication, we receive one orbit of size 8 and one orbit of size 16. But we know that our group has 3 orbits of size $M = 8$. This leaves only $G = D_8 \circ C_4$, a second verification that $D_8 \circ C_4$ satisfies our hypothesis. This concludes the case that D has exactly two orbits of size M.

Step 3.1.2.3: The Case Where D has Exactly Three Orbits of Size M

Assume that D has exactly three orbits of size M on V, we know that these are regular orbits as D is abelian. We will denote the largest orbit size of the action of $D/C_D(V_1)$ on V_1 as M_1, and M_2 will denote the largest orbit size of $C_D(V_1)$ acting on W_1. Recall that $D/C_D(V_1)$ has at most three orbits of size M_1 on V_1. So the $D/C_D(V_i)$ for $i = 1, \dots, p$ are all isomorphic and $D/C_D(V_i)$ has either one, two, or three orbits of size M_1 acting on V_i. Suppose we have three orbits of size M_1 in the action of $D/C_D(V_1)$ on V_1, then as argued previously $C_D(V_1)$ must have exactly one orbit of size M_2 on W_1 or else we have too many orbits as they do not combine. Suppose we have two orbits of size M_1 in the action of $D/C_D(V_1)$ on V_1, then if $C_D(V_1)$ has two or three orbits of maximal size M_2 then we have too many orbits and if $C_D(V_1)$ has one orbit of maximal size then we do not have enough. Therefore we know that $D/C_D(V_1)$ has only one orbit of size M_1 in the action of $D/C_D(V_1)$ on V_1 and $C_D(V_1)$ has exactly three orbits of size M_2 on W_1. Thus we have $|D/C_D(V_1)| = |V_1| - 1$, and $D/C_D(V_1)$ is cyclic. Now we have $D \lessapprox \bigtimes_{i=1}^{p} D/C_D(V_i) =: T$ and T has exactly one regular orbit on V. Every regular orbit of T on V splits into $\frac{|T|}{|D|}$ regular orbits of D. Since D has no more than three orbits of size

$M = M_D = |D|$, we see that $\frac{|T|}{|D|} \leq 3$. If $\frac{|T|}{|D|} = 1$, then $T = D$ and so $C_D(V_1) \cong \bigtimes_{i=1}^{p} D/C_D(V_i)$ has only one regular orbit on W_1, a contradiction. Suppose $\frac{|T|}{|D|} = 2$. Then from Lemma 1 we have that $|T| = (|V_i| - 1)^p$, thus

$$(|V_1| - 1)^p = 2|D| = 2p^k \tag{4}$$

for appropriate k. Since $p = 2$, we have $V = V_1 \oplus V_2$ and $W_2 = V_2 \oplus .. \oplus V_p = V_2$. We know that $|C_D(V_1)| < |D/C_D(V_2)|$ and that $C_D(V_1) < D/C_D(V_2)$. So the regular orbit of $D/C_D(V_2)$ would split into $\frac{|D/C_D(V_2)|}{|C_D(V_1)|}$ regular orbits. This is impossible as $C_D(V_1)$ has 3 regular orbits. Therefore we are left with $\frac{|T|}{|D|} = 3$, then $p = 3$. But as stated previously in step 3.1.2, by Theorem 3.2 in [**5**], $p = 2$, a contradiction. This concludes the case where D has exactly three orbits and in turn concludes the broader cases where D is abelian and $D' < G'$.

Step 3.2: The Case Where $D' = G'$

We will consider the action of $D/C_D(V_1)$ on V_1 and $C_D(V_1)$ acting on W_1. Using Theorem 1, we have the following inequalities

$$|D : D'C_D(V_1)| \leq M_1 \tag{5}$$

$$|C_D(V_1) : C_D(V_1)'| \leq M_2 \tag{6}$$

where M_1 is the largest orbit size of $D/C_D(V_1)$ on V_1 and M_2 is the largest orbit size of $C_D(V_1)$ on W_1. There are now four cases to consider; strict inequality in (5) and (6), equality in (5) and strict inequality in (6), equality in both (5) and (6), and equality in (6) and strict inequality in (5).

Step 3.2.1: The Case Where We have Strict Inequality in (5) and (6)

First we consider the case where we have strict inequality in (5) and (6). Because G is a p-group we know that $p|D : D'C_D(V_1)| \leq M_1$ and $p|C_D(V_1) : C_D(V_1)'| \leq M_2$. Therefore

$$M \geq M_1 M_2 \geq p^2 |D : D'C_D(V_1)||C_D(V_1) : C_D(V_1)'|.$$

Recall $|G : D| = p$ and notice that $|D : D'| \leq M_D \leq pM_1M_2$ so

$$p^2|D : D'C_D(V_1)||C_D(V_1) : C_D(V_1)'| \geq p|G : D||D : D'| = p|G : D'| = p|G : G'| > |G : G'|.$$

Putting the above equations together we have $|G : G'| < M$. This contradicts our hypothesis that $|G : G'| = M$, and therefore either (5) or (6) must be an equality.

Step 3.2.2: The Case Where We have Equality in (5) and Strict Inequality in (6)

Consider the case of equality in (5), that is $|D : D'C_D(V_1)| = M_1$. If $D/C_D(V_1)$ is abelian then by (2) we have D is abelian. If D is abelian then $1 = D' = G'$ and G is abelian, a contradiction. Therefore, we note that $D/C_D(V_1)$ cannot be abelian for the rest of the paper. By Theorem 1 we have that $D/C_D(V_1)$ has at least two orbits of size M_1 on V_1. Let $v_1, v_2 \in V_1$ be representatives of two different orbits of size M_1 in the action of $D/C_D(V_1)$ on V_1. Let $w \in W_1$ be in an orbit of size M_2 in the action fo $C_D(V_1)$ on W_1. Because (6) is strict, we have $p|C_D(V_1) : C_D(V_1)'| \leq M_2$. This gives us the following,

$$M \geq |(v_i + w)^G| \geq |v_i^{D/C_D(V_1)}||w^{C_D(V_1)}| = M_1 M_2$$

$$\geq p|D : D'C_D(V_1)||C_D(V_1) : C_D(V_1)'| \geq |G : G'| = M$$

for $i = 1, 2$. This gives equality everywhere. Since G has exactly three orbits of size M on V, $D/C_D(V_1)$ has either two or three orbits of size M_1 on V_1. Suppose $D/C_D(V_1)$ has two orbits of size M_1 on V_1, then by [**7**] $D/C_D(V_1) \cong D_8$, $|V_1| = 9$ and $p = 2$. Thus $M_1 = 4$ and $M_2 \leq 4$. This gives us $C_D(V_1) \leq D_8$. If $C_D(V_1) \cong D_8$ then $|C_D(V_1) : C_D(V_1)'| = 4$ contradicting (6) is strict. Therefore $C_D(V_1)$ is size one, two, or four. This makes $C_D(V_1)$ abelian. That means $M_2 = |C_D(V_1)| = |C_D(V_1) : C_D(V_1)'|$ which contradicts (6) is strict. Thus $D/C_D(V_1)$ must have three orbits of size M_1 on V_1 and by induction $D/C_D(V_1) \cong D_8 \circ C_4$, $|V_1| = 25$ and $p = 2$. Thus $M_1 = 8$ and $M_2 \leq 8$. This gives us $C_D(V_1) \leq D_8 \circ C_4$. If $C_D(V_1) \cong D_8 \circ C_4$ then $|C_D(V_1) : C_D(V_1)'| = 8$ contradicting (6) is strict. Therefore $C_D(V_1)$ is size one, two, four, or eight. If $C_D(V_1)$ is of size one, two, or four then $C_D(V_1)$ is abelain and that means $M_2 = |C_D(V_1)| = |C_D(V_1) : C_D(V_1)'|$ which contradicts (6) is strict. So, $C_D(V_1)$ must be of size eight and thus be D_8, $C_2 \times C_4$, or Q_8 as it is a subgroup of $D_8 \circ C_4$. If $C_D(V_1)$ is $C_2 \times C_4$ or Q_8, then (6) can not be strict because each have more than one maximal orbit. Which leaves $C_D(V_1)$ to be D_8. Then the action $D/C_D(V_1)$ on V_1 has three orbits of order eight and the action $C_D(V_1)$ on W_1 has one orbit of order 8 and four orbits of order four. Following the argument from Step 3.1.1.2. Let $v_1 \in V_1$ be in an orbit of size $M_1 = 8$ of $D/C_D(V_1)$ on V_1, such that v_1 is in a regular orbit of $C_D(V_2)$ on V_1. We claim that $|C_D(v_1)| = 16$. Note that $D_8 \cong C_D(V_1) \leq C_D(v_1)$ so $|C_D(v_1)| \geq 8$. Assume $|C_D(v_1)| = |C_D(V_1)| = 8$.Then $|v_1^D| = |D : C_D(v_1)| = 16$ and $v_1^D \leq V_1$ so D has an orbit of size 16 on V_1, a contradiction to $M_1 = 8$. Thus $|C(v_1)|$ must be greater than 8 and as a p-group, it must be of order at least 16. Since $|C_D(v_1) \cap C_D(V_2)| = 1$, then $|C_D(v_1)C_D(V_2)| \geq 16 \cdot 8 = |D|$. Combined with $C_(v_1)C_D(V_2) \leq D$ and $|D_D(v_1)C_D(V_2)| = |C_D(v_1)||C_D(V_2)|$, gives us

$|C_D(v_1)| = 16$. Now $|C_D(v_1)| = 16$ and $C_D(v_1) \cap C_D(V_2) = 1$, therefore

$$C_D(v_1) \cong C_D(v_1)/(C_D(v_1) \cap C_D(V_2)) \cong C_D(v_1)C_D(V_2)/C_D(V_2) \cong D/C_D(V_2) \cong D_8 \circ C_4$$

where the third isomorphism comes from the isomorphism theorems. Hence we have $C_D(v_1) \cong D_8 \circ C_4$. Therefore $C_D(v_1)$ acts faithfully on V_2 and if $z_1, z_2, z_3 \in V_2$ are representatives of the three orbits of size eight in the action of $C_D(v_1)$ on V_2, then v_1+z_1, v_1+z_2, and v_1+z_3 are representatives of three orbits of size $64 = M_D = M$ of D on V_D. Now let $w_1 \in V_1$ be in an orbit of D of size $M_1 = 8$ such that w_1 is not in a regular orbit of $C_D(V_2)$ on V_1. Let $z_4 \in V_2$ be in the (unique) regular orbit of $C_D(V_1)$ on V_2 (so it is of size $M_2 = 8$). Then clearly $w_1 + z_4$ is a representative of an orbit of size $64 = M$ of D on V that is different from the orbits containing $v_1 + z_1$, $v_1 + z_2$, and $v_1 + z_3$. Thus we have found four orbits of size $64 = M$ of D on V which contradicts our hypothesis.

Step 3.2.3: The Case Where We have Equality in (5) and (6)

We now consider the case that (5) and (6) are equalities. That is $|D : D'C_D(V_1)| = M_1$ and $|C_D(V_1) : C_D(V_1)'| = M_2$. Then

$$M = |G : G'| = p|D|/|D'| = p|C_D(V_1) : D'C_D(V_1)|C_D(V_1) : C_D(V_1) \cap D'| \leq pM_1M_2$$

also, $M \geq M_D \geq M_1M_2$ so

$$(\dagger) \quad M_1M_2 \leq M_D \leq M \leq pM_1M_2.$$

We know that exactly one of these inequalities is strict because $|D|/|D'| < p|D|/|D'| = |G|/|G'| = M$. We now have three cases to consider.

In all of these cases we know that $D/C_D(V_1)$ has at least two orbits of size M_1 on V_1, otherwise $D/C_D(V_1)$ would be abelian, making $D' \leq (\bigtimes_{i=1}^{p} D/C_D(V_i))' =$

1. Then we would have $D' = G' = 1$, contradicting that G is nonabelian. Throughout the following arguments we will let $v_1, v_2 \in V_1$ be representatives of two orbits of size M_1 on V_1.

Step 3.2.3.1: The Case Where the Last Inequality in † is Strict

That is, $M_1M_2 = M_D = M < pM_1M_2$. Assume that $v_3, v_4 \in V_1$ are a third and fourth orbit of size M_1 in the action of $D/C_D(V_1)$ on V_1. Then let $w_1 \in W_1$ be a representative of an orbit of size M_2 in the action of $C_D(V_1)$ on W_1. This gives us $(v_1 + w_1)^D, (v_2 + w_1)^D$, $(v_3 + w_1)^D$, and $(v_4 + w_1)^D$, four orbits of size $M_D = M$ on V. Thus we have four orbits of size M in the action of G on V, a contradiction. Therefore $D/C_D(V_1)$ can only have two or three orbits of size M_1 on V_1. Let $w_1, w_2 \in W_1$ be representatives of two distinct orbits of size M_2 in the action of $C_D(V_1)$ on W_1. We see that $(v_1 + w_1)^D$, $(v_1 + w_2)^D, (v_2 + w_1)^D$, and $(v_2 + w_2)^D$ are four orbits of size $M_D = M$ on V, a contradiction. Therefore we know that $C_D(V_1)$ has exactly one orbit of size M_2 on W_1 and by Theorem 1 we see that $C_D(V_1)$ is abelian.

Suppose $D/C_D(V_1)$ has two orbits of size M_1 on V_1. By [**7**] we have that $D/C_D(V_1) \cong D_8, p = 2$, and $V_1 = V_2 = V(2,3)$. Therefore we have $C_D(V_1) \times C_D(V_2) \lessapprox D \lessapprox D/C_D(V_1) \times D/C_D(V_2)$ and $C_D(V_1) \le D/C_D(V_1) \cong D_8$. If $|C_D(V_1)| = 8$, then $C_D(V_1) \cong D_8$, contradicting $C_D(V_1)$ is abelian. If $|C_D(V_1)| = 4$, then $C_D(V_1)$ must be the Klein-4, since it has only one orbit of size 4 on V_2, which is shown to be a contradiction in [**7**] corresponding case. If $|C_D(V_1)| = 2$, then $C_D(V_1)$ is Z_2, the cyclic group of order two. If we calculate the orbits, we see that all subgroups of D_8 of order two have at least three orbits of size two in the action on V_2, a contradiction. If $|C_D(V_1)| = 1$, then $D/C_D(V_1)$ has two orbits of size four in the action of $D/C_D(V_1)$ on V_1 and $D/C_D(V_2)$ has two orbits of size four in the action of $D/C_D(V_2)$ on V_2. Therefore D has either four orbits of size four or an orbit of size eight. This contradicts that G has exactly three orbits of size $M = M_1M_2 = 4$ in the action of G on V.

Suppose $D/C_D(V_1)$ has three orbits of size M_1 on V_1. By induction we have that $D/C_D(V_1) \cong D_8 \circ C_4, p = 2$, and $V_1 = V_2 = V(2,5)$. Therefore we have $C_D(V_1) \times C_D(V_2) \lessapprox D \lessapprox D/C_D(V_1) \times D/C_D(V_2)$ and $C_D(V_1) \leq D/C_D(V_1) \cong D_8 \circ C_4$. If $|C_D(V_1)| = 16$, then $C_D(V_1) \cong D_8 \circ C_4$, contradicting $C_D(V_1)$ is abelian. If $|C_D(V_1)| = 8$, then $C_D(V_1)$ is either D_8, Q_8, or $C_2 \times C_4$. If $C_D(V_1) \cong D_8$ or Q_8 then we contradict $C_D(V_1)$ is abelian. If $C_D(V_1) = C_2 \times C_4$ then $C_D(V_1)$ has two orbits of size M_2 on W_1 contradicting the statement above that $C_D(V_1)$ has exactly one orbit of size M_2 on W_1. If $|C_D(V_1)| = 4$, by calculating the orbits of all subgroup of $D_8 \circ C_4$ that are of order 4, we see that they all ahve at least four orbits of size four, a contradiction. If $|C_D(V_1)| = 2$, then $C_D(V_1)$ is Z_2, the cyclic group of order two. We know that all subgroups of $D_8 \circ C_4$ of order two have at least ten orbits of size two in the action on V_2, a contradiction. If $|C_D(V_1)| = 1$, then $D/C_D(V_1)$ has three orbits of size eight in the action of $D/C_D(V_1)$ on V_1 and $D/C_D(V_2)$ has three orbits of size eight in the action of $D/C_D(V_2)$ on V_2. Therefore D has either six orbits of size eight or at least one orbit of size sixteen. This contradicts that G has exactly three orbits of size $M = M_1M_2 = 8$ in the action of G on V. This concludes the case where $M_1M_2 = M_D = M$.

Step 3.2.3.2: The Case Where the First Inequality in † is Strict

That is, $M_1M_2 < M_D = M = pM_1M_2$. We know $|D|/|D'| < M_D$ and $D/C_D(V_1)$ has at least two orbits of size M_1 on V_1. We claim that $D/C_D(V_1)$ has either two or three orbits of size M_1 on V_1 and that $C_D(V_1)$ has at least two orbits of size M_2 on W_1. This proof can be found in the corresponding argument in [**7**]; however, the argument has a few gaps and assumptions that have been corrected here. Assume $C_D(V_1)$ has exactly one orbit of size M_2 on W_1. Since we have inequality in (6) , by [**9**] we know that $C_D(V_1)$ must be abelian and thus the largest orbit of $C_D(V_1)$ on W_1 is a regular orbit of size $M_2 = |C_D(V_1)|$. It follows by [**9**] that $M_2 = \prod_{i=2}^{p}(|V_i| - 1)$. But then $M_1 = |V_1| - 1$ and thus the largest orbit of size M_D on V is M_1M_2 as the corresponding orbit is $\{x_1+\cdots+x_p | x_i \in V_i-\{0\}$ for $i=1,\ldots,p\}$ and there cannot be a larger orbit. So $M_D = M_1M_2$, a contradiction to the fact that $M_D = pM_1M_2$ in this case. Therefore $C_D(V_1)$ has at least two orbits of size M_2 on W_1.Now, let w_1, w_2 be representatives of two different orbits of size M_1 of $C_D(V_1)$ on W_1. We now aim to show $D/C_D(V_1)$ can not have more than three orbits. Let $v_i \in V_1$ where $i = 1,2,3,4$ be representative of four different orbits of size M_1 on V_1. Fix $i \in \{1,2,3,4\}$. Consider $z_j = v_i + w_j$ for $j = 1,2$. We can see that $|(z_j)^D| \geq M_1M_2$ for $j = 1,2$. Since $M = pM_1M_2$, we have $|(Z_j)^D| \in \{M_1M_2, M\}$ for $j = 1,2$. If $|(z_1)^D| = M_1M_2$ then $|(z_2)^M| > M_1M_2$ since any $g \in G-D$ cannot fix both orbits. So, $|(z_2)^D| = M$. Since this is true for all $i \in \{1,2,3,4\}$, G has four orbits of size M on V a contradiction to our hypothesis. Thus $D/C_D(V_1)$ has either two or three orbits of size M_1 on V_1.

If there are exactly two orbits of size M_1 on V_1 then by [**7**] and we have

$$D/C_D(V_1) \cong D_8, p = 2, V_1 = V_2 = V(2,3).$$

We also know $C_D(V_1) \times C_D(V_2) \lessapprox D \lessapprox D/C_D(V_1) \times D/C_D(V_1) \cong D_8 \times D_8$, and $C_D(V_1) \leq D_8$. This tells us $|C_D(V_1)| \in \{1,2,4,8\}$. In order to have the complete proof, the following cases have been taken from [**7**].

If $|C_D(V_1)| = 8$, then $C_D(V_1) \cong D_8$. This means $D \cong D_8 \times D_8$, $|D| = 64$, and $|G| = 128$. By Lemma 2.8 [**12**] we have $G \leq D_8 \wr Z_2$. Because $|D_8 \wr Z_2| = 128$ we have that $G = D_8 \wr Z_2$, which is known to not be metabelian by Satz 3.15.3 [**2**], that is $G'' \neq 1$. However we have that $G' = D' = (D_8 \times D_8)'$ which is size four. This makes G' abelian and $G'' = 1$, a contradiction.

If $|C_D(V_1)| = 4$ we have that $|D| < |D/C_D(V_1) \times D/C_D(V_1)| = |D|^2/16 = 64$. That is $|D| = 32$. We also have $D' \leq (D_8 \times D_8)'$ so $|D'| \leq 4$. Suppose $|D'| = 4$ then $\frac{|D|}{|D'|} = \frac{32}{4} = 8$, so $M_D = 16 = M_1M_2$ a contradiction. Suppose that $|D'| = 2$. Notice $C_D(V_1) \cap C_D(V_2) = 1$ so $C_D(V_1) \times C_D(V_2) = 1$ and $C_D(V_1) \times C_D(V_2) = C_D(V_1)C_D(V_2)$.

Let $g \in G - D$ and $(1,a) \in C_D(V_1) \times C_D(V_2)$. Then

$$[(1,a),g] = (1,a)^{-1}g^{-1}(1,a)g = (1,a)^{-1}(1,a)^g = (1,a^{-1})(a^*,1) = (a^*,a^{-1})$$

for some $a^* \in C_D(V_1)$, and we have four choices for $a \in C_D(V_1)$. Thus $2 = |D'| = |G'| \geq 4$, a contradiction.

If $|C_D(V_1)| = 2$, then $\frac{|D|^2}{4} = 64$ or $|D| = 16$. As before, we know that $|D'| \in \{2,4\}$. Suppose $|D'| = 4$, then $\frac{|D|}{|D'|} = \frac{16}{4} = 4$, and $M_D = 8$. We know $(v_1,0)$ and $(v_2,0)$ are both in D-orbits of size four on V. Let $w \in V_2$ be a regular orbit. Then $(v_1 + w)$ would be in an orbit of size eight, contradicting $M_1M_2 < M_D$. Suppose that $|D'| = 2$, then $\frac{|D|}{|D'|} = 8$ and $M_D = 16$. Thus $C_D(V_1)$ must have four orbits of size two on V_2. Let $w_i \in V_2$ for $i = 1,2,3,4$ be representatives of these four orbits. Then we know that $C_D(V_1) = \{1,r\}$ as all other subgroups of D_8 of size two have only three orbits of size two on V_2. We also know there must exist a $d_i \in D, i = 1,2,3,4$ where $w_1^{d_1} = w_2, w_2^{d_2} = w_3, w_3^{d_3} = w_4, w_4^{d_4} = w_1$. Without loss let $w_1 = (1,0)$ and $w_2 = (1,-1)$. Then there exists a $d \in D$ with $(1,0)^d = (1,-1)$ in the action of D on V_2, a contradiction.

If $|C_D(V_1)| = 1$ then $D \cong D_8$, but $D_8/D_8' = 4 = M_D$, a contradiction. This concludes the case from [**7**].

Therefore there are exactly three orbits of size M_1 on V_1 thus by induction, we have $D/C_D(V_1) \cong D_8 \circ C_4, p = 2, V_1 = V_2 = V(2,5)$. We also know $C_D(V_1) \times C_D(V_2) \lnapprox D \lnapprox D/C_D(V_1) \times D/C_D(V_1) \cong (D_8 \circ C_4) \times (D_8 \circ C_4)$, and $C_D(V_1) \leq D_8 \circ C_4$. This tells us $|C_D(V_1)| \in \{1,2,4,8,16\}$.

If $|C_D(V_1)| = 16$, then $C_D(V_1) \cong D_8 \circ C_4$. This means $D \cong (D_8 \circ C_4) \times (D_8 \circ C_4)$, $|D| = 256$, and $|G| = 512$. By Lemma 2.8 [**12**] we have $G \leq (D_8 \circ C_4) \wr Z_2$. Because $|(D_8 \circ C_4) \wr Z_2| = 512$ we have that $G = (D_8 \circ C_4) \wr Z_2$, which is not metabelian. However we have that $G' = D' = ((D_8 \circ C_4) \times (D_8 \circ C_4))'$ which is of order four. This makes G' abelian and $G'' = 1$, a contradiction.

If $|C_D(V_1)| = 8$, then $C_D(V_1)$ is isomorphic to one of the following: D_8, Q_8 or $C_4 \times C_2$. The case where $C_D(V_1) \cong D_8$ can be seen in the case that $D/C_D(V_1)$ has exactly two orbits. Suppose $C_D(V_1) \cong Q_8$. Since $C_D(V_1)$ acts faithfully on V_2 there exists a regular orbit of size 8. Therefore $M_2 = 8$, but $M_2 = |C_D(V_1) : C_D(V_1)'| = 8/2 = 4$, a contradiction. Suppose $C_D(V_1) \cong C_4 \times C_2$. We know that $C_D(V_2)$ is also isomorphic to $C_4 \times C_2$ and therefore has 2 regular orbits. Let $v_1, v_2 \in V_1$ be representatives of regular orbits of $C_D(V_2)$ acting on V_1 and in orbits of size M_1 in the action of $D/C_D(V_1)$ on V_1. Consider $C_D(v_1)$, $|C_{D/C_D(V_1)}(v_1)| = 2$ thus $|C_D(v_1)| = 16$. Let $w_1, w_2, w_3 \in V_2$ be representatives of the three maximal orbits of $D/C_D(V_1)$ on V_2. Now consider the action of G on $V = V_1 \oplus V_2$ of the form $(v_i + w_j)^G$ where $i \in 1, 2$ and $j \in 1, 2, 3$. Since elements in $G - D$ swap the orbits from V_1 to V_2, there exists a $g \in G - D$ such that in the action of $C_D(V_2)$ on V_1 the orbit is $(v_1)^D$ and in the action of $C_D(V_1)$ on V_2 the orbit is $(v_1^g)^D$. We have chose g such that v_1^D has to be exactly one of the following, w_1, w_2, w_3. Without loss of generality we may assume $v_1^g = w_1$. The orbit $(v_1 + w_1)^G = (v_1 + v_1^g)^G$ is fixed and will be of order $M_2 = 64$, but we have two orbits $(v_1 + w_2)^G$ and $(v_1 + w_3)^G$ of maximal size $M = 128$. We can repeat this method with v_2 in place of v_1 to receive two more orbits of size $M = 128$, a contradiction to our hypothesis.

The cases $|C_D(V_1)| = 4$ and $|C_D(v_1)| = 2$ are the same as when $D/C_D(V_1)$ has exactly two orbits. If $|C_D(V_1)| = 1$, then $D \cong D_8 \circ C_4$, but $(D_8 \circ C_4)/(D_8 \circ C_4)' = 8 = M_D$, a contradiction.

Step 3.2.3.3: The Case Where the Second Inequality in † is Strict

That is, $M_1M_2 = M_D < M = pM_1M_2$. Since by Theorem 1 $|D/D'| \leq M_D$, and since $M_1M_2 = M_D < M = pM_1M_2$ and $p|D/D'| = |G/G'| = M$ we see that $|D/D'| = M_1M_2 = M_D < M = pM_1M_2$. We know that $D/C_D(V_1)$ has at least two orbits of size M_1 in the action of V_1. We now must find how many orbits $C_D(V_1)$

has of size M_2 on W_1.

Step 3.2.3.3.1: The Case Where $C_D(V_1)$ has Exactly One Orbit of Size M_2 on W_1

We note that this case can be handled by following word for word the corresponding argument in the proof of Theorem 1 in [**9**]; however, there was a small error in the argument presented there. A corrected version of this case, that replaces the corresponding argument in [**9**], can be found in [**7**] and has been recreated here for completeness.

Let us first assume that $C_D(V_1)$ has exactly one orbit of size M_2 on W_1. By Theorem 1, $C_D(V_1)$ is abelian. Therefore $C_D(V_i)$ is abelian for all $i = 1, \ldots, p$, and it follows that for each i that $C_D(W_i)$ is abelian. We further claim that for each i, $C_D(W_i)$ has exactly one regular orbit of V_i. To see this, observe that $C_D(V_1)$ has exactly one regular orbit on W_1. Hence $C_D(V_1)/(C_D(V_1) \cap C_D(V_2))$ has exactly one regular orbit on V_2, and $(C_D(V_1) \cap C_D(V_2))/(C_D(V_1) \cap C_D(V_2) \cap C_D(V_3))$ has exactly one regular orbit on V_3. We can repeat this until we finally get

$$\left(\bigcap_{j=1}^{p-1} C_D(V_j)\right) \Bigg/ \left(\bigcap_{j=1}^{p} C_D(V_i)\right) \cong C_D(W_p)$$

has exactly one regular orbit on V_p. Since the actions of $C_D(W_i)$ on V_i are equivalent for all i, the claim is true. Let $A := \prod_{i=1}^{p} C_D(W_i)$. We see that $A \trianglelefteq G$, and $A = \bigtimes_{i=1}^{p} C_D(W_i)$ is an internal direct product because $C_D(W_i) \cap \prod_{j \in \{1,\ldots,p\}-\{i\}} C_D(W_j)) = 1$, for $i = 1, \ldots, p$ (all elements in $\prod_{j \in \{1,\ldots,p\}-\{i\}} C_D(W_j))$ act trivially on V_i). Applying Lemma 2.2 to the action of $C_D(W_i)$ on V_i for all i. Putting this together thus shows that if we write $V_A = X_1 \oplus \ldots \oplus X_m$ for some $M \in \mathbb{N}$ and irreducible A-modules such that $V_1 = X_1 \oplus \ldots \oplus X_k$ for some $k \in \mathbb{N}$, then $m = kp$ and

$$|A| = \prod_{i=1}^{m} (|X_i| - 1) = (|X_1| - 1)^m \;\; \leq M$$

and

$$M_1 \geq \prod_{i=1}^{k}(|X_i| - 1) = (|X_1| - 1)^k.$$

Now recall that $v_1, v_2 \in V_1$ are representatives of two orbits of size M_1 of $D/C_D(V_1)$ on V_1. Write $v_i = x_{i1} + \ldots + x_{ik}$ with $x_{ij} \in X_j$ for $j = 1, \ldots, k$, $i = 1, 2$. We may assume that v_1 is in a regular orbit of $A/C_A(V_1)$, and thus $x_{1j} \neq 0$ for $j = 1, \ldots, k$. But then $v_1^D = v_1^A = \{y_1 + \ldots + y_k | 0 \neq y_i \in X_i \text{ for } i = 1, \ldots, k\}$, and this forces that $x_{2j} = 0$ for at least one $j \in \{1, \ldots, k\}$. If we let $g \in G - D$ and put $z_i = v_i + \sum_{j=1}^{p-1} v_1^{g^j}$ for $i = 1, 2$, then it is clear that both z_1 and z_2 are in different orbits of size greater than or equal to $|A|$ of G. Hence $|G/G'| \geq |A|$.

Now let q be the characteristic of V and write $|X_1| = q^s$. Write $|A/C_A(X_1)| = p^t$. Then $p^t = q^s - 1$ and hence with [**12**] we know that either $s = 1$, $p = 2$, and q is a Fermat prime; or $t = 1$, $q = 2$, and p is a Mersenne prime; or $s = 2$, $t = 3$, $p = 2$, and $q = 3$. Moreover, by [**12**] we know that $N_G(X_1)/C_G(X_1)\Gamma(X_1)$ and since G is a p-group, altogether we conclude that

$$N_G(X_1)/C_G(X_1) \cong A/C_A(X_1)$$

unless possibly in the third case, when $|V_1| = 9$ and $N_G(X_1)/C_G(X_1) \cong \Gamma(3^2)$ is possible (in the first case this is clear, in the second it follows by Fermat's Little Theorem). For the moment suppose that $N_G(X_1)/C_G(X_1) \cong A/C_A(X_1)$. Because of the size of A with [**12**] we conclude that $G \cong A/C_A(X_1) \wr G/A$ where G/A transitively and faithfully permutes the X_i $(i = 1, \ldots, m)$. Now with arguments similar to the one in the proof of [**4**] we see that

$$|[A, G]| \geq |A/C_A(X_1)|^{m-1} = (|X_1| - 1)^{m-1}.$$

Moreover, by [**1**] we have $|G : G'A| = |G/A : (G/A)'| \leq p^{m/p}$. Hence altogether we have

$$\begin{aligned}(|X_1|-1)^m = |A| \le |G/G'| &= |G:G'A||G'A:G'| \\ &\le p^{m/p}|A:A\cap G'| \\ &\le p^{m/p}|A:[A,G]| \\ &= p^{m/p}(|X_1|-1)\end{aligned}$$

Now clearly $|X_1|-1 \ge p$, and so it follows that

$$p^{m-1} \le (|X_1|-1)^{m-1} \le p^{\frac{m}{p}}.$$

So $m-1 \le m/p$, and since $m \ge p$, we get that $p = m = 2$, $|X_1| = 3$, $k = 1$, $V_1 = X_1$, $|V| = 9$, $M = |G/G'| = 4$ and thus G is dihedral of order 8 acting on the nine elements of V. But then D is abelian, and since $G' = D'$, G is also abelian. This is a contradiction.

In the exceptional case $s = 2$, $t = 3$, $p = 2$, $q = 3$ above we have that the kernel K/A of the permutation action of G/A on the X_i is of order at most 2^m. So we see that $G/\Omega_2(A)$ has A as an abelian normal subgroup, and so similarly as above

$$\begin{aligned}|G/G'| &\le |G:G'K||G'K:G'| \\ &\le 2^{\frac{m}{2}}|K:K\cap G'| \\ &\le 2^{\frac{m}{2}}|K/\Omega_1(A):(K\cap G')\Omega_1(A)/\Omega_1(A)|\cdot|\Omega_1(A)|\end{aligned}$$

Now $|[K/\Omega_1(A), G/\Omega_1(A)]| \ge 4^{m-1}$, and thus altogether

$$\begin{aligned}2^{3m} = 8^m = (|X_1|-1)^m &= |A| \\ &\le |G/G'|\end{aligned}$$

$$
\begin{aligned}
&\leq 2^{\frac{m}{2}}|K/\Omega_1(A) : [K/\Omega_1(A), G/\Omega_1(A)]| \cdot 2^m \\
&\leq 2^{\frac{m}{2}} \cdot \frac{8^m}{4^{m-1}} \cdot 2^m \\
&= 2^{\frac{3}{2}m+m-1} \cdot 8 \ = \ 2^{\frac{5}{2}m+2}
\end{aligned}
$$

Hence $3m \leq \frac{5}{2}m + 2$ or, equivalently, $m \leq 4$. Since $m = kp = 2k$, we have that $m = 2$ or $m = 4$.

If $m = 2$, then $k = 1$ and thus $X_i = V_i$ for $i = 1, 2$. But then $M_1 = 8 = |V_1| - 1$, and $D/C_D(V_1)$ has exactly one orbit of size M_1 on V_1, contradicting our observation above that $D/C_D(V_1)$ has at least two orbits of size M_1 on V_1.

If $m = 4$, then $k = 2$ and $|V_1| = 3^4$. Hence G is isomorphic to a subgroup of $\mathrm{GL}(8,3)$ and thus $|G| \leq 2^{19}$. As above, we know that $|G'| \geq |[A,G]| \geq (|X_1| - 1)^{m-1} = 8^3 = 2^9$, and hence $|G/G'| \leq 2^{10} < 2^{12} = |A| \leq M$, contradicting $|G/G'| = M$.

Therefore we now know that $C_D(V_1)$ has at least two orbits of size M_2 on W_1.

Step 3.2.3.3.2: The Case Where $C_D(V_1)$ has at Least Two Orbits of Size M_2 on W_1

The set up for this argument is the same as in [**7**] and deviates in the sub-cases that follow. Let w_1 and w_2 be representatives of such orbits. If there exists a $d \in C_D(v_1)$ such that $w_1^d = w_2$ we see that

$$
M \geq M_D \geq M_1 p M_2 \geq p|D : D'| = |G/G'|
$$

contradicting that $M_D < M$. Therefore, no such d exists. This tells us that $(v_1 + w_1)$ and $(v_1 + w_2)$ lie in different D-orbits on V. Similarly $v_2 + w_1$ and $v_2 + w_1$ are in different D-orbits on V.

Now identify D with a subgroup of $\bigtimes_{i=1}^{p} D/C_D(V_i)$. Also let $g \in G - D$ and put

$$L_i = \sum_{j=0}^{p-1} (v_i^{g^j})^D := \{\sum_{j=0}^{p-1} x_j | x_j \in (v_i^{g^j})^D \text{ for } j = 0, \ldots, p-1\} \subset V$$

for $i = 1, 2$. The L_i are clearly G-invariant subsets, and $L_1 \cap L_2 = \varnothing$. For any $x \in V_2 \oplus \ldots \oplus V_p$ it follows that if the orbit $(v_i + x)^D \subset L_i$ $(i \in \{1, 2\})$.

We now have four cases to consider. The D-orbits $(v_1 + w_1)^D$ and $(v_1 + w_2)^D$ are both G–invariant. The D-orbits $(v_2 + w_1)^D$ and $(v_2 + w_2)^D$ are both G-invariant. Lastly, one of the D-orbits $(v_1 + w_1)^D$ or $(v_1 + w_2)^D$ is not G-invariant, and at least one of the orbits D-orbits $(v_2 + w_1)^D$ or $(v_2 + w_2)^D$ is not G-invariant.

Step 3.2.3.3.2.1: The Case Where the D–orbits $(v_1 + w_1)^D$ and $(v_1 + w_1)^D$ are Both G–invariant

Then $v_1 + w_1 \in L_1$ and $v_1 + w_2 \in L_1$, and thus $v_2 + w_1 \notin L_2$ and $v_2 + w_2 \notin L_2$, that is $(v_2 + w_1)^D$ and $(v_2 + w_2)^D$ are not G-invariant, so that

$$M \geq |(v_2 + w_i)^G| \geq p|(v_2 + w_i)^D| \geq pM_1M_2 \geq p|D/D'| = |G/G'| = M$$

for $i = 1, 2$. This tells us that $C_D(V_1)$ has either two or three orbits of size M_2 in the action on W_1, because otherwise the above argument shows that we would get more than three orbits of size M.

The following argument is a corrected version of the corresponding part in [**7**] and should serve as a replacement proof. Suppose that $D/C_D(V_1)$ has three orbits of size M_1 on V_1, let v_3 be a representative of a third such orbit. Arguing just as for v_2, we know that $(v_3 + w_1)^D$ and $(v_3 + w_2)^D$ are not G–invariant, and (also as above for v_2) we get $|(v_3 + w_i)^G| = M$ for $i = 1, 2$. But this gives us four orbits of size M_D and we can only have two or three, therefore at least two must be the same. There are six possible combinations as order does not matter.

Suppose $(v_2 + w_1)^D$ and $(v_3 + w_1)^D$ are G–conjugate, then there exists a $x \in G$ such that $(v_3 + w_1) = (v_2 + w_1)^x = v_2^x + w_1^x$. If $x \in D$ then $v_2^x \in V_1$ and $v_2^x = v_3$, but then $v_2^D = v_3^D$ a contradiction to them being representatives of different orbits. Thus $x \in G - D$. Since $w_1 \in \sum_{i=1}^{p-1}(v_1^{g^i})^D$, w_1^x has a component in V_1 and that component is in v_1^D so $v_3 \in v_1^D$ and thus $v_3^D = v_1^D$ a contradiction to them being representatives of different orbits. Therefore $(v_2 + w_1)^D$ and $(v_3 + w_1)^D$ are not G−conjugate. The same contradiction can be found for $(v_2 + w_2)^D$ and $(v_3 + w_2)^D$ being G−conjugate by changing the index as they also have the same element in W.

Suppose $(v_2+w_1)^D$ and $(v_3+w_2)^D$ are G−conjugate, then there exists a $x \in G$ such that $(v_3 + w_2) = (v_1 + w_1)^x = v_2^x + w_1^x$. If $x \in D$ then $v_2^x = v_3$ and $v_2^D = v_3^D$ a contradiction to them being representatives of different orbits. Thus $x \in G - D$. Since $w_2 \in \sum_{i=1}^{p-1}(v_1^{g^i})^D$, w_2^x has a component in V_1 and that component is in v_1^D. Therefore $v_3 \in v_1^D$ and thus $v_3^D = v_1^D$ a contradiction to them being representatives of different orbits. Therefore $(v_2 + w_1)^D$ and $(v_3 + w_2)^D$ can not be G−conjugate. The same contradiction can be found for $(v_2 + w_2)^D$ and $(v_2 + w_1)^D$ being G−conjugate.

If $(v_2 + w_1)^D$ and $(v_2 + w_2)^D$ are G−conjugate then they are D−conjugate, otherwise if $x \in G - D$ then $v_2 + w_2 = (v_2 + w_1)^x = v_2^x + w_1^x$ then since $w_2 \in \sum_{i=1}^{p-1}(v_1^{g^i})^D$, w_1^x has a component in V_1 and that component is in v_1^D so $v_2 \in v_1^D$ thus $v_2^D = v_1^D$ a contradiction to them representing different orbits. Thus if $(v_2 + w_1)^D$ and $(v_2 + w_2)^D$ are G−conjugate then they are D−conjugate. Since they are D−conjugate, there exists a $d \in D$ such that $v_2^d = v_2$ and $w_1^d = w_2$. We know $d \in C_D(v_2)$ but $d \notin C_D(V_1)$ and thus w_1, w_2 are in some $C_D(v_2)$−orbits. So the largest orbit size of $C_D(v_2)$ on W_1 is greater than or equal to pM_1. Then $|(v_2 + w_1)^D| = M_1 p M_2 = M_D$ a contradiction to M_D being strictly less than pM_1M_2.

Thus there can not be a third orbit of $D/C_D(V_1)$ on W_1 of size M_2 and by [**7**] $D/C_D(V_1) \cong D_8$, $p = 2$, and $V_1 = V(2,3)$. We know that $C_D(V_1) \times C_D(V_2) \leq D \leq D/C_D(V_1) \times D/C_D(V_2) \cong D_8 \times D_8$ and $C_D(V_1) \leq D/C_D(V_1) \cong D_8$. Then $|C_D(V_1)| \in \{2, 4, 8\}$.

If $|C_D(V_1)| = 8$ then $D = D_8 \times D_8$, so $|D| = 64$, $|D'| = 4 = |G'|$ and $|G| = 128$.

By [**12**] we have $G \leq D_8 \wr Z_2$, so since $|G| = 128$, we have $G = D_8 \wr Z_2$ contradicting $|G'| = 4$.

If $|C_D(V_1)| = 4$, then $C_D(V_1) \cong Z_4$ (because $C_D(V_1)$ has two orbits of size M_2 on V_2, therefore it is not the Klein-4). Thus $Z_4 \times Z_4 \leq D \leq D_8 \times D_8$, $|D| = |D/C_D(V_1)|\,|C_D(V_1)| = 8 \cdot 4 = 32$.Because $D' \leq (D_8 \times D_8)'$ we have that $|D'| \in \{2, 4\}$. If $|D'| = 4$ then $M_D = \frac{|D|}{|D'|} = \frac{32}{4} = 8$, contradicting that $Z_4 \times Z_4$ has a regular orbit (size 16) on V. Therefore $|D'| = |G'| = 2$; but $C_D(V_1) = 4$ which - using again [**5**] as we did before -makes $|G'| \geq 4$, a contradiction.

If $|C_D(V_1)| = 2$, then $C_D(V_1) \cong Z_2$. This means that $C_D(V_1)$ has at least three orbits of size $M_2 = 2$ on V_2, a contradiction.

Step 3.2.3.3.2.2: The Case Where the D–orbits $(v_2 + w_1)^D$ and $(v_2 + w_2)^D$ are Both G–invariant

This case will follow the same proof as in Step 3.2.3.3.2.1 if we replace v_1 with v_2.

Step 3.2.3.3.2.3: The Case Where the at Least One of the D-orbits $(v_1 + w_1)^D$ or $(v_1 + w_2)^D$ is Not G-invariant, and at Least One of the Orbits D-orbits $(v_2 + w_1)^D$ or $(v_2 + w_2)^D$ is Not G-invariant.

Without loss of generality, we may assume that $(v_1+w_1)^D$ is not G–invariant. If $(v_2 + w_1)^D$ is also not G–invariant, we see that $(v_1 + w_1)^G$ and $(v_1 + w_1)^G$ are two distinct orbits of size M, because

$$M \geq (v_i + w_1)^G \geq pM_1M_2 = |G/G'| = M$$

for $i = 1, 2$ and if we write $w_1 = (x_2, \ldots, x_p) \in V_2 \oplus \cdots \oplus V_p$ then $v_1 + w_1 = (v_1, x_2, \ldots, x_p$ and $v_2 + w_2 = (v_2, x_2, \ldots, x_p)$ have a different number of components in the corresponding component of L_1 and cannot be conjugate in G.

Now we show that $D/C_D(V_1)$ cannot have a fourth orbit of size M_1 on V_1.

Let $v_3, v_4 \in V_1$ be a third and fourth such orbit. If $(v_3+w_1)^D$ and $(v_3+w_2)^D$ are both G−invariant then we can follow the proof in Step 3.2.3.2.1 by replacing v_1 with v_3 to find a contradiction. Similarly, we can replace v_1 with v_4 in the same argument to show $(v_4 + w_1)^D$ and $(v_4 + w_2)^D$ cannot both be G−invariant. If $(v_3 + w_1)^D$ and $(v_4 + w_1)^D$ are not G−invariant then $(v_3 + w_1)^G$ and $(v_4 + w_1)^G$ are a third and fourth orbit of size M on V, a contradiction. If both $(v_3 + w_1)^D$ and $(v_4 + w_1)^D$ are G−invariant then $(v_3 + w_2)^D$ and $(v_4 + w_2)^D$ must not be G−invariant. Thus $|(v_2 + w_2)^G| = |(v_4 + w_2)^G| = M$ and since there are only three orbits of size M, at least two orbits must be equal. Without loss, assume $(v_4 + w_2)^G = (v_j + w_1)^G$ for some $j \in \{1, 2, 3\}$. Then there exists a $g \in G - D$ with $(v_4 + w_2)^g = v_j + w_1$. Then $(v_4 + w_2)^G \subseteq L_4$ and $(v_j + w_1)^G \subseteq L_j$, a contradiction to $L_4 \cap L_j = \varnothing$. If one of $(v_3 + w_1)^D$ and $(v_4 + w_1)^D$ is not G−invariant then we have three orbits and can use induction to tell us $D/C_D(V_1) \cong D_8 \circ C_4$, $V_1 \cong V(2, 5)$, $p = 2$ and $V = V_1 \oplus V_2$, which is known to be a contradiction as seen in Step 3.2.3.2.1. Therefore there can not be a fourth orbit of size M_1 when $D/C_D(V_1)$ acts on V_1.

If there is exactly three orbits of size M_1 of $D/C_D(V_1)$ then we can use induction to find $D/C_D(V_1) \cong D_8 \circ C_4$, $V_1 = V(2, 5)$, $p = 2$ and $V = V_1 \oplus V_2$, a contradiction. If there is exactly two orbits of size M_1 of $D/C_D(V_1)$ then we can use [**7**] to find $D/C_D(V_1) \cong D_8$, $V_1 = V(2, 3)$, $p = 2$ and $V = V_1 \oplus V_2$, a contradiction by step 3.2.3.2.1.

Therefore $(v_2 + w_1)^D$ is G-invariant and thus $(v_2 + w_2)^D$ is not G-invariant. A similar argument to the one above follows and we can again use induction and [**7**] to find $D/C_D(V_1) \cong D_8 \circ C_4$ and $D/C_D(V_1) \cong D_8$, respectively, a contradiction.

Thus $(v_1 + w_1)^D$ and $(v_2 + w_2)^D$ are both G-invariant. If $(v_1 + w_2)^D$ and $(v_2 + w_2)^D$ are G-conjugate, then $v_2 + w_1 \in L_2$ and $v_1 + w_2 \in L_1$. This means that $v_1 + w_1$ and $v_2 + w_2$ can only be conjugate in G if $p = 2$. We can now let $p = 2$.

Suppose there are four orbits of size M_2 in the action $C_D(V_1)$ on V_2. Let

$w_3, w_4 \in W_1 = V_2$ be representative of such orbits. As above it follows that $v_1 + w_1$, $v_1 + w_2$, $v_1 + w_3$ and $v_1 + w_4$ all lie in different D–orbits on V and so do $v_2 + w_1$, $v_2 + w_2$, $v_2 + w_3$ and $v_2 + w_4$, and as above, with w_3 and w_4 in place of w_2 we see the following must be true.

At least one of the orbits $(v_1 + w_1)^D$ or $(v_1 + w_3)^D$ is not G–invariant, and at least one of the orbits $(v_2 + w_1)^D$ or $(v_2 + w_3)^D$ is not G–invariant. We already know that $(v_2 + w_1)^D$ is G–invariant, it follows that $(v_2 + w_3)^D$ is not G–invariant. The argument from earlier shows that $(v_2 + w_2)^G$ and $(v_2 + w_3)^G$ are different G–orbits. Assume there is a third orbit of size M_1 in the action of $D/C_D(V_1)$ on V_1, let v_3 be a representative of this orbit. Then, at least two of the three orbits $(v_3 + w_1)^D$, $(v_3 + w_2)^D$ and $(v_3 + w_3)^D$ are not G–invariant. If $(v_3 + w_1)^D$ is G–invariant, then $(v_3 + w_2)^G$, $(v_1 + w_1)^G$, $(v_3 + w_3)^G$ and $(v_2 + w_1)^G$ are four G–orbits of size M, a contradiction. If $(v_3 + w_2)^D$ is G–invariant, then $(v_3 + w_1)^G$, $(v_1 + w_2)^G$, $(v_3 + w_3)^G$ and $(v_2 + w_2)^G$ are four G–orbits of size M, a contradiction. Lastly, if $(v_3 + w_3)^D$ is G–invariant, then $(v_3 + w_1)^G$, $(v_1 + w_3)^G$, $(v_3 + w_2)^G$ and $(v_2 + w_3)^G$ are four G–orbits of size M, a contradiction. Thus we see that $D/C_D(V_1)$ has exactly two orbits of size M_1 on V_1. By [**7**], we have that $D/C_D(V_1) \cong D_8$, a contradiction as proven above. This concludes section 2.3 entirely.

Step 3.2.4: The Case Where We Have Strict Inequality in (5) and Equality in (6)

Suppose we have equality in (4) and strict inequality in (3). That is

$$M \geq M_1 M_2 \geq p|D : D'C_D(V_1)||C_D(V_1) : C_D(V_1)'| \geq p|D/D'| = |G : G'|.$$

Because $|G : G'| = M$ we have equality everywhere, and $M = M_1 M_2, M_1 = p|D : D'C_D(V_1)| > |D : D'C_D(V_1)|, M_2 = |C_D(V_1) : C_D(V_1)'|$. Again let M_D denote the largest orbit size of D on V, then $M_D \geq M_1 M_2$ so $M_D = M$. By Theorem 1.1

$C_D(V_1)$ is abelian or has at least two orbits of size M_2 on W_1. We consider again some subcases.

Step 3.2.4.1: The Case Where $C_D(V_1)$ has at Least Two Orbits of Size M_2 on W_1

Let $w_1, w_2 \in W_1$ be representatives of such orbits.

Assume that $D/C_D(V_1)$ has at least two orbits of size M_1 on V_1. Because $M = M_1 M_2$ we have that $(v_1 + w_1)^D, (v_1 + w_2)^D$, $(v_2 + w_2)^D$ and $(v_2 + w_1)^D$ are all distinct orbits of size $M_D = M$, contradicting there being only three orbits of size M. Therefore $D/C_D(V_1)$ has exactly one orbit of size M_1 on V_1. Let $v_1 \in V_1$ be a representative of this orbit.

Now let w_1, w_2 be representatives of two distinct orbits of size M_2 of $C_D(V_1)$ on W_1, then $(v_1 + w_1)^D$ and $(v_1 + w_2)^D$ are two distinct D–orbits of size M, and if $C_D(V_1)$ had a third and fourth orbit of size M_2 on W_1, similarly we would get a third and fourth orbit on G of size M, a contradiction. Thus $C_D(V_1)$ has two or three orbits of size M_2 on W_1.

Now write $W_1 = \bigoplus_{i=1}^{k} X_i$ for a suitable $k \in \{1, \ldots, n\}$ and irreducible $C_D(V_1)$-modules X_i $(i = 1, \ldots, k)$. We may assume that $X_1 \leq V_2$. Then the intersection of all the $C_{C_D(V_1)}(X_i)$ is trivial, and hence

$$C_D(V_1) \lesssim C_D(V_1)/C_{C_D(V_1)}(X_1) \times \cdots \times C_D(V_1)/C_{C_D(V_1)}(X_k) \quad (+)$$

Moreover, if we put $N_0 = C_D(V_1)$, $Z_0 = W_1$ and recursively for $i \geq 1$ let $Y_i \leq Z_{i-1}$ be an irreducible N_{i-1}-module, $N_i = C_{N_{i-1}}(Y_i)$, and Z_i be a $C_D(V_1)$-invariant complement of Y_i in Z_{i-1}, and put $t = i - 1$ and stop the process as soon as $Z_i = 0$ and $N_i = 1$, then we have that $\bigcap_{i=0}^{t} N_i = 1$ and $W_1 = \bigoplus_{i=0}^{t} Y_i$. Also, $R_{i-1} := N_{i-1}/N_i$ acts faithfully and irreducibly on Y_{i-1} for $i = 1, \ldots, t$. Write M_{i-1}^* for the largest orbit size of N_{i-1}/N_i on Y_{i-1} for $i = 1, \ldots, t$. Then by repeated use of Lemma 2.1 we see that

$$M_2 = |C_D(V_1) : C_D(V_1)'| \leq \prod_{i=1}^{t} |R_i : R_i'| \leq \prod_{i=1}^{t} M_i^* \leq M_2, \quad (++)$$

the last inequality easily follows by considering the sum of representatives of orbits of size M_{i-1}^* of N_{i-1}/N_i on Y_{i-1}. Thus we have equality everywhere, and it follows that $|R_i : R_i'| = M_i^*$ for $i = 1, \ldots, t$. It also follows that the elements of every orbit of $C_D(V_1)$ on W_1 of size M_2 have the form $y_1 + \ldots, y_t$ for some $y_i \in Y_i$ $(i = 1, \ldots, t)$ which lies in an orbit of size M_i^* of N_i/N_{i+1} on Y_i $\quad (+++)$. We now split into two cases $C_D(V_1)$ is not abelian and $C_D(V_1)$ is abelian.

Step 3.2.4.1.1: The Case Where $C_D(V_1)$ is not Abelian

Put $C = C_D(V_1) \cap C_D(X_1) = C_{C_D(V_1)}(X_1)$. Then by $(+)$ we may assume that $C_D(V_1)/C$ is nonabelian, and it also acts faithfully and irreducibly on X_1. We also clearly may assume that $Y_1 = X_1$ and hence with $(++)$ and $(+++)$ conclude that $C_D(V_1)/C$ has either two or three orbits of size of its abelian quotient on X_1. Hence we may apply induction and, in particular, get $p = 2$, $|X_1| = 25$ and $C_D(V_1)/C \cong D_8 \circ C_4$. Moreover, since $C_D(V_1)$ has either two or three orbits of size M_2 on W_1, then from $(+++)$ it follows that R_{i-1} has exactly one orbit of size M_{i-1}^* on Y_{i-1} for $i = 2, \ldots, t$. This forces, for $i = 2, \ldots, t$, that R_{i-1} is cyclic of order 2, $|Y_{i-1}| = 3$, and hence $C_{C_D(V_1)}(X_1)$ is elementary abelian of order $p^{\dim W_1 - 2}$. Note that $W_1 = V_2$ since $p = 2$.

Assume that $k \geq 2$, so $t \geq 3$ (since the X_i all have dimension 2). Then we may assume that $X_2 = Y_1 \oplus Y_2$, and from the above we know that $C/C_C(X_2)$ is the dihedral group of order eight, since it is a subgroup of $D_8 \circ C_4$ that has only one regular orbit.

Now consider the action of $C_D(V_1)$ on X_1. We know that $C_D(V_1)$ is isomorphic to a subgroup of a direct product of k copies of $D_8 \circ C_4$, and $C_D(V_1)/C$ is isomorphic to $D_8 \circ C_4$ and has six noncentral involutions. If all of them have inverse images in $C_D(V_1)$ which act trivially on $X_2 \oplus \cdots \oplus X_k$, then $C_D(V_1)$ has a $D_8 \circ C_4$ as a subgroup which acts trivially on $X_2 \oplus \cdots \oplus X_k$, and since the X_i are transitively permuted by D, it follows that $C_D(V_1)$ is isomorphic to a direct product of k copies of $D_8 \circ C_4$; in particular, then $C/C_C(X_2) \cong C_D(V_1)/C_{C_D(V_1)}(X_2) \cong D_8 \circ C_4$, contradicting the above observation that $C/C_C(X_2)$ is the dihedral group of order eight. Hence there exists an element $c \in C_D(V_1)$ such that $c \notin C$, $c^2 \in C$, and c acts nontrivially on at least one X_i for some $i \in \{2, \ldots, k\}$, so without loss we may assume that c acts nontrivially on X_2. Now there is a $0 \neq x \in V_1$ such that c centralizes x. Since $c \notin C$ and $C/C_C(X_2)$ is dihedral of order eight, this shows that $C_D(x)/C_{C_D(x)}(X_2)$ has order divisible by 16,and thus $C_D(x)/C_{C_D(x)}(X_2)$ is isomorphic to $D_8 \circ C_4$ and therefore has three orbits of size 8 on X_2. This allows us in an obvious way to construct three different orbits of size $M_2 = 8^k$ of $C_D(V_1)$ on $V_2 = W_1$ having representatives with x in their X_1-component; in addition to another orbit of size M_2 having a representative in the X_1-component from the second orbit of size 8 of $C_D(V_1)/C$ on X_1, giving us in total four distinct orbits of $C_D(V_1)$ on V_2, contradicting the current fact that $C_D(V_1)$ has exactly three orbits of size M_2 on V_2.

Hence our assumption that $k \geq 2$ was wrong, and we now have $k = 1$. So $W_1 = V_2 = X_1$ is of order 25, and $C_D(V_1) \cong D_8 \circ C_4$ acts irreducibly on it and has three orbits of size $M_2 = 8$ on it. Hence $(D_8 \circ C_4) \times (D_8 \circ C_4) \cong C_D(V_2) \times C_D(V_1)$ is a normal subgroup of G. Now since $D/C_D(V_1)$ has exactly one orbit of size M_1 on V_1 (as we saw above), it follows that $M_1 = 16$ and $D/C_D(V_1)$ must be at least of order 32, and thus $D/C_D(V_1)$ is a full Sylow 2-subgroup of $\mathrm{GL}(2,5)$, i.e., a semidihedral group of order 32. Moreover, $|G : G'| = M = M_1M_2 = 16 \cdot 8 = 2^7$ and $|G| = |G/D|\ |D/C_D(V_1)|\ |C_D(V_1)| = 2 \cdot 32 \cdot 16 = 2^{10}$. Therefore $|G'| = 2^3$. Now let $Z = C_D(V_1)' \times C_D(V_2)'$. Then $Z \leq D'$ is a Klein 4-group and $G'/Z = (G/Z)'$. Working in G/Z, we notice that $(C_D(V_1) \times C_D(V_2)')/Z$ is elementary abelian of order 2^4, and if

$g \in G - D$, then gZ interchanges the two subgroups $C_D(V_i)Z/Z \cong C_D(V_i)/C_D(V_i)'$ $(i = 1, 2)$. Looking at the elements $[gZ, xZ] \in (G/Z)'$ for $x \in C_D(V_1)$ shows us that $|(G/Z)'| \geq |C_D(V_1)Z/Z| = 4$ so that altogether $2^3 = |G'| = |G'/Z||Z| \geq 4 \cdot 4 = 2^4$, which is a contradiction. This completes Case 2.4.1.1 where $C_D(V_1)$ is not abelian.

Step 3.2.4.1.2: The Case Where $C_D(V_1)$ is Abelian

Then $C_D(V_1)$ has regular orbits on W_1, and thus $M_2 = |C_D(V_1)|$, so $C_D(V_1)$ has either two or three regular orbits on W_1.

Note that $M_2 = |C_D(V_1)|$ and so

$$
\begin{aligned}
M &= M_1 M_2 = M_1 |C_D(V_1)| = |G/G'| = p|D/D'| \\
&= p|D : D'C_D(V_1)||D'C_D(V_1) : D'| \\
&= M_1 |D'C_D(V_1) : D'| \\
&= M_1 |C_D(V_1) : (D' \cap C_D(V_1))|
\end{aligned}
$$

This forces $D' \cap C_D(V_1) = 1$. So if $x \in D$ and $c \in C_D(V_1)$, then $[x, c] \in D' \cap C_D(V_1) = 1$. This shows that $C_D(V_1) \leq Z(D)$ and hence $C_D(V_i) \leq Z(D)$ for $i = 1, \ldots p$.

Now we consider the k in $(+)$.

First suppose that $k = 1$, then $W_1 = X_1$, but since $W_1 = V_2 \oplus \cdots \oplus V_p$, we see that $X_1 = V_2$ and $p = 2$. In particular, V_2 is an irreducible faithful $C_D(V_1)$-module, so $C_D(V_1)$ is cyclic and has only regular orbits on $V_2 - \{0\}$. So there are exactly two such orbits, which shows that $(|V_2| - 1)/2 = |C_D(V_1)|$. Since $|C_D(V_1)|$ is a power of 2, by an elementary result from number theory (see [**12**]) it follows that - if we write q for the characteristic of V - either V_2 is of dimension 1 and $|V_2|$ is a Fermat prime, or $|V_2| = 9$ and $|C_D(V_1)| = 4$.

In the former case we get that $D/C_D(V_1)$ is abelian and hence D is abelian, and so G is abelian (since $G' = D'$), a contradiction. In the latter case we get that

$D/C_D(V_1)$ must be at least of order 8 (since it has exactly one maximal orbit (of size M_1) on V_1, and it must be isomorphic to a subgroup of the semidihedral group SD_{16}, as Sylow 2-subgroup of $\mathrm{GL}(2,3)$. However, all such subgroups have center of order 2, contradicting the fact that $|C_D(V_1)| = |C_D(V_2)| = 4$ and $C_D(V_1) \leq Z(D)$. This concludes the case that $k = 1$.

So let $k > 1$. Then define $X_0 = 0$ and $L_i = C_{C_D(V_1)}(X_0 \oplus \cdots \oplus X_i)/C_{C_D(V_1)}(X_0 \oplus \cdots \oplus X_{i+1})$ for $i = 0, \ldots, k-1$. As $k > 1$, we see that L_0 has exactly one regular orbit on X_1, because otherwise also L_1 would have at least two regular orbits on X_2 which ultimately would lead to $C_D(V_1)$ having at least four regular orbits on W_1, a contradiction. Since all orbits of L_0 on X_1 must be regular, we thus conclude that $|L_0| = |X_1| - 1$. Since $C_D(V_1)$ has exactly two regular orbits on W_1, it follows that there is exactly one $l \in \{1, \ldots, k\}$ such that L_{l-1} has exactly two regular orbits on X_l, whereas all the other L_i's have exactly one regular orbit on X_{i+1}. However, since L_{l-1} only has regular orbits on $X_l - \{0\}$, it is clear that the single regular orbit of size $|X_l| - 1$ of $C_D(V_1)/C_{C_D(V_1)}(X_l)$ on X_l splits into at least p regular orbits for L_{l-1} on X_L. This shows that $p = 2$. Hence $C_D(V_1)C_D(V_2) = C_D(V_1) \times C_D(V_2) \leq Z(G)$, and since $D/C_D(V_1)$ acts faithfully and irreducibly on V_1, we see that $C_D(V_1) \cong C_D(V_1)C_D(V_2)/C_D(V_2)$ is cyclic and thus has only regular orbits on $V_2 - \{0\}$. So there are exactly two such orbits and we now can arrive at a contradiction just as in the case that $k = 1$

This concludes Case 2.4.1.2 and thus Case 2.4.1 is completed and it is left to show that $C_D(V_1)$ can not have exactly one orbit of size M_2 on W_1.

Step 3.2.4.2: The Case Where $C_D(V_1)$ has exactly one orbit of size M_2 on W_1

Then by Theorem 1.1 $C_D(V_1)$ is abelian and hence has regular orbits on W_1, so $M_2 = |C_D(V_1)|$ and the same argument as at the beginning of Case 2.4.1.2 shows that $C_D(V_1) \leq Z(D)$ and hence $C_D(V_i) \leq Z(D)$ for $i = 1, \ldots p$.

Assume that $X_1 < V_2$ (where X_1 is as in (+)). Since $C_D(V_1)$ has exactly one regular orbit on W_1, it also has exactly one regular orbit on V_2, and since V_2 is not irreducible as $C_D(V_1)$-module, by Lemma 2.2 it is clear that $C_D(V_1)/C_{C_D(V_1)}(V_2)$ is not cyclic. But since $C_D(V_1) \leq Z(D)$, we see that

$$C_D(V_1)/C_{C_D(V_1)}(V_2) = C_D(V_1)/C_D(V_1 \oplus V_2) = C_D(V_1)/(C_D(V_1) \cap C_D(V_2))$$

$$\cong C_D(V_1)C_D(V_2)/C_D(V_2)$$

is a non-cyclic central subgroup of $D/C_D(V_2)$. But on the other hand, $D/C_D(V_2)$ acts faithfully and irreducibly on V_2 and hence has a cyclic center, and we have a contradiction. This shows that $X_1 = V_2$, so V_2 is an irreducible $C_D(V_1)$-module and $C_D(V_1)$ has either two or three (one of them being the trivial orbit) on V_2. Therefore again by [**12**]) it follows that - if we write q for the characteristic of V - either

- $p = 2$, V_2 is of dimension 1 and $|V_2|$ is a Fermat prime; or
- $q = 2$ and $|C_D(V_1)/C_D(V_1 \oplus V_2)| = p$ is a Mersenne prime; or
- $p = 2$, $q = 3$, $|V_2| = 9$ and $|C_D(V_1)| = 8$.

In the first case we get (as earlier) that $D/C_D(V_1)$ is abelian and thus D is abelian, a contradiction. In the second case, since D is a p-group, with [**12**] we see that $D/C_D(V_1)$ cyclic of order p and thus abelian, making D abelian, a contradiction. So we are left with the third case. Here we have that $D/C_D(V_1)$ is a subgroup of the semidihedral group of order 16, so $|G| \leq 2^9$, and $|G| \leq 2^8$ unless $D \cong \mathrm{SD}_{16} \times \mathrm{SD}_{16}$. Moreover, since any $g \in G - D$ interchanges $C_D(V_1)$ and $C_D(V_2)$, by taking commutators of elements in $C_D(V_1)$ with g we easily see that $|D'| \geq 8$ and so $|G'| \geq 8$. Now D has an orbit of size $\geq 2^6$ on V (from the regular orbit of $C_D(V_1) \times C_D(V_2)$. So if $|G| \leq 2^8$, we get $2^5 < 2^6 \leq M = |G/G'| \leq 2^8/2^3 = 2^5$, a contradiction. This leaves us with $|G| = 2^9$, and $D \cong \mathrm{SD}_{16} \times \mathrm{SD}_{16}$, but in this case for similar reasons as above we see that $|D'| \geq 2^4$ and thus get the contradiction

$2^5 < 2^6 \leq M = |G/G'| \leq 2^9/2^4 = 2^5.$

This final contradiction concludes the proof of the theorem.

$\square$

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