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Existence and Multiplicity Results for Homoclinic Orbits of Hamiltonian Systems *

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Abstract

Homoclinic orbits play an important role in the study of qualitative behavior of dynamical systems. Such kinds of orbits have been studied since the time of Poincaré. In this paper, we discuss how to use variational methods to study the existence of homoclinic orbits of Hamiltonian systems.

Introduction

In the theory of differential equations, a trajectory which is asymptotic to a constant state as $|t| \to \infty$ is called a homoclinic orbit. Such kinds of orbits have been found in various models of dynamical systems and they frequently have tremendous effect to the dynamics of such nonlinear systems. The homoclinic orbits have been studied since the time of Poincaré but mainly by perturbation methods. It is only relatively recently that some new tools have been developed in the calculus of variations to show the existence of homoclinic solutions of nonlinear differential equations.

In this article, a class of second order Hamiltonian systems is considered:

$$\ddot{q} - L(t)q + V'(t,q) = 0 \tag{HS}$$

where $q \in \mathbb{R}^n$. The $n \times n$ matrix $L(t)$ is continuous, symmetric and satisfies the following condition:

(L) There are $\mu_1, \mu_2 \in (0,\infty)$ such that $\mu_1|y|^2 \le L(t)y\cdot y \le \mu_2|y|^2$ for all $t \in \mathbb{R}, y \in \mathbb{R}^n$.

The basic assumption for the function $V(t,y)$ is

(V1) $V \in C^2(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$, $V'(t,0) \equiv D_yV(t,0) = 0$, $D_y^2V(t,0) = 0$ and $\lim_{|y|\to 0} \frac{V'(t,y)}{|y|} = 0$ uniformly in $t \in \mathbb{R}$. For any $r > 0$, there is a $K = K(r)$ such that

$$\sup_{t\in\mathbb{R},\ |y|\le r} \|D_yV(t,y)\|_\infty + \|D_y^2V(t,y)\|_\infty \le K.$$

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The hypothesis (V1) implies that 0 is a constant solution of (HS). A non-constant solution of (HS) which tends to 0 as $|t| \to \infty$ will be called a homoclinic solution or a homoclinic orbit.

In [13], Rabinowitz proved existence of homoclinic orbits for (HS) when:

(P) L and V are T-periodic in t, and

(V2) there is a $\mu > 2$ such that $0 < \mu V(t,y) \leq y \cdot V'(t,y)$ for all $y \in \mathbb{R}^n \backslash \{0\}$ and $\inf_{|y|=1,\, t\in\mathbb{R}} V(t,y) > 0$.

Subsequently, Coti Zelati and Rabinowitz [5] obtained multibump homoclinic solutions for this system. For the first-order-time-periodic Hamiltonian system, the existence of multibump homoclinic solutions was proved by Séré [17]; there the one-bump solution had been obtained earlier in [3]. Subsequently there have been further extensions for using variational methods to study such problems in various directions [4, 6, 8, 9, 15, 18, 19]. The interested reader may consult [16] for more references.

In this paper we intend to investigate the existence of homoclinic orbits of (HS) when L and V are not necessarily periodic in t. The approach to (HS) will involve the use of variational methods of a mini-max nature. Let E be the space $W^{1,2}(\mathbb{R}, \mathbb{R}^n)$ under the norm

$$\left(\int_{-\infty}^{\infty} (|\dot{q}|^2 + |q|^2) dt \right)^{1/2} .$$

It is known that $E \subset C^0(\mathbb{R}, \mathbb{R}^n)$, the space of continuous functions q on $\mathbb{R}$ such that $q(t) \to 0$ as $|t| \to \infty$. The solutions of (HS) are the critical points of the functional J given by

$$\begin{aligned} J(q) &= \int_{-\infty}^{\infty} \left[\frac{1}{2}(|\dot{q}|^2 + L(t)q \cdot q) - V(t,q) \right] dt \\ &= \frac{1}{2}\|q\|^2 - \int_{-\infty}^{\infty} V(t,q)dt, \end{aligned}$$

where by assumption (L),

$$\|q\|^2 = \int_{-\infty}^{\infty} (|\dot{q}|^2 + L(t)q \cdot q)dt$$

is taken as an equivalent norm on E.

Note that assumption (V1) implies that $J \in C^1(E, \mathbb{R})$. Moreover, critical points of J are classical solutions of (HS), satisfying $\dot{q}(t) \to 0$ as $|t| \to \infty$. Thus q is a homoclinic solution of (HS). Let $I^\alpha = \{q \in E | J(q) \leq \alpha\}$. It is not difficult to see from (V1) and (V2) that $V(t,q) = o(|q|^2)$ as $|q| \to 0$ and $V(t,q)|q|^{-2} \to \infty$ as $|q| \to \infty$. Thus

$$J(q) = \frac{1}{2}\|q\|^2 + o(\|q\|^2) \quad \text{as } q \to 0 \tag{0.1}$$

and $I^0\backslash\{0\}$ is non-empty. By the Mountain Pass Theorem J would have a critical value $\beta > 0$ given by

$$\beta = \inf_{\gamma\in\Gamma} \max_{\theta\in[0,1]} J(\gamma(\theta))\,, \tag{MP}$$

where

$$\Gamma = \{\gamma \in C([0,1], E) | \gamma(0) = 0 \text{ and } \gamma(1) \in I^0\backslash\{0\}\},$$

provided that

(PS) whenever $J(u_m)$ is bounded and $J'(u_m) \to 0$ as $m \to \infty$, the sequence $\{u_m\}$ possesses a (strongly) convergent subsequence in E.

Unfortunately the Palais-Smale condition (PS) is not always satisfied. For example, let $q(t)$ be a homoclinic solution of (HS) obtained in [13]. Note that by (P)

$$J(\tau_m q) = J(q)$$

if $m \in \mathbb{Z}$ and

$$\tau_m q(t) = q(t - mT)\,.$$

Therefore, $J(\tau_m q) \to J(q) > 0$ and $J'(\tau_m q) \to 0$ as $m \to \infty$. Nevertheless, the sequence $\{\tau_m q\}$ does not possess a convergent subsequence in E. The same kind of difficulty occurs in dealing with homoclinic orbits of first order Hamiltonian systems (e.g. [3, 17]), there the authors used the idea of concentration-compactness to treat the Palais-Smale sequences.

Although the mini-max structure of (MP) cannot guarantee that there is a critical point $u \in E$ with $J(u) = \beta$, we find a way to justify whether β is a critical value of J. Our method is based on the following comparison argument.

Let $\Omega_k = \mathbb{R}\backslash[-k, k]$, and $E_k = W_o^{1,2}(\Omega_k, \mathbb{R}^n)$ with the norm

$$\|u\|_k = \left(\int_{\Omega_k} (L(t)u \cdot u + |\dot{u}|^2) dt\right)^{1/2}.$$

Since $w \in E_{k+1}$ can be identified with an element of E_k by extending w to be zero on $\Omega_k\backslash\Omega_{k+1}$, the inclusions

$$E_{k+1} \subset E_k \subset \cdots E \tag{0.2}$$

will be used without mentioned explicitly, and J_k will be the restriction of J to E_k.

Definition A sequence $\{u_m\} \subset E$ is called a $(\mathrm{PS})_\mathrm{c}$ sequence if $J(u_m) \to c$ and $J'(u_m) \to 0$ as $m \to \infty$.

Let Λ be the set of positive numbers c such that there exists a $(\mathrm{PS})_\mathrm{c}$ sequence. The set Λ in particular contains all the positive critical values of J. Let δ be the infimum of Λ.

It will be shown that Λ is a nonempty set and δ is a positive number. On the restriction J_k, we define the set Λ_k and its infimum δ_k similarly.

Theorem 1 *There exists a homoclinic solution q of (HS) with $J(q) = \delta$, provided that $\delta \notin \Lambda_k$ for some $k \in \mathbb{N}$.*

Remark 1 *We may take $\Omega_k = \mathbb{R}\backslash[a_k, b_k]$, where $\{a_k\}$ is a decreasing sequence with $\lim_{k\to\infty} a_k = -\infty$ and $\{b_k\}$ is an increasing sequence with $\lim_{k\to\infty} b_k = \infty$.*

When $\beta > \delta$, (HS) may possess more than one homoclinic orbit.

Theorem 2 *There exist a homoclinic solution q with $J(q) = \delta$ and a homoclinic solution $\hat{q}$ with $J(\hat{q}) = \beta$, if $\delta < \beta < \delta_k$ for some $k \in \mathbb{N}$.*

The proofs of Theorems 1 and 2 will be given in section 2. Section 1 contains several preliminary results such as a detailed analysis of Palais-Smale sequences. A sufficient condition for $\delta = \beta$ will be given in section 3. In section 4, some applications of the above theorems will be discussed, including the investigation of perturbations of time periodic Hamiltonian systems.

1 Preliminaries

This section contains several technical results such as various smoothness and qualitative properties of J. As mentioned in the introduction, the Mountain Pass Theorem cannot be directly applied to obtain the existence of homoclinic solutions of (HS), since verification of (PS) may not be possible. An alternative approach is to analyze the behavior of Palais-Smale sequences. In doing so, we begin with the Fréchet differentiability of J. A detailed proof of Proposition 1 can be found in [5].

Proposition 1 *If V satisfies (V1), then $J \in C^1(E, \mathbb{R})$.*

Next we prove the boundedness of Palais-Smale sequences.

Lemma 1 *If $\{u_m\}$ is a $(PS)_c$ sequence then there is a constant A such that*

$$\|u_m\| \leq A \tag{1.1}$$

and

$$\|u_m\|_{L^\infty} \leq \sqrt{2}\, A\,. \tag{1.2}$$

Proof. Since $J'(u_m) \to 0$ as $m \to \infty$, if m is large then

$$\|u_m\|^2 - \int_{-\infty}^{\infty} V'(t, u_m) \cdot u_m dt = J'(u_m)u_m = o(1) \cdot \|u_m\|\,. \tag{1.3}$$

Hence

$$\begin{aligned} c &= J(u_m) + o(1) \\ &= J(u_m) - \frac{1}{2}J'(u_m)u_m + o(1) \cdot (1 + \|u_m\|) \\ &= \int_{-\infty}^{\infty} [\frac{1}{2}V'(t, u_m) \cdot u_m - V(t, u_m)]dt + o(1) \cdot (1 + \|u_m\|) \\ &\geq (\frac{1}{2} - \frac{1}{\mu}) \int_{-\infty}^{\infty} V'(t, u_m) \cdot u_m dt + o(1) \cdot (1 + \|u_m\|)\,, \end{aligned} \tag{1.4}$$

where the last inequality follows from (V2). Substituting (1.3) into (1.4) yields

$$c \geq (\frac{1}{2} - \frac{1}{\mu})\|u_m\|^2 + o(1) \cdot (1 + \|u_m\|), \tag{1.5}$$

which completes the proof of (1.1). Then (1.2) follows from the inequality $\|u\|_{L^\infty} \leq \sqrt{2}\|u\|$.

Corollary 1 *If $\{u_m\}$ is a $(PS)_c$ sequence then*

$$\lim_{m\to\infty} \sup \|u_m\| \leq \left(\frac{2\mu c}{\mu - 2}\right)^{1/2}.$$

The proof of the above corollary follows directly from (1.5) and Lemma 1.

Corollary 2 *If $q \in E$, and $J'(q) = 0$ then*

$$J(q) \geq \frac{\mu - 2}{2\mu}\|q\|^2. \tag{1.6}$$

Proof. Note that (1.6) is trivially satisfied when $q \equiv 0$. If $q \not\equiv 0$, (1.6) follows from (1.5) by setting $u_m = q$ for all m.

Lemma 2 *There exists a $(PS)_\beta$ sequence, where β is the number defined in (MP).*

This lemma is proved using deformation theory, as Theorem A4 is proven in [14]. From (MP) and (0.1), it is clear that $\beta > 0$. Thus Lemma 2 shows that Λ is non-empty.

Proposition 2 *If (V1), (V2) and (L) are satisfied then $\delta > 0$.*

Proof. Choose $\rho > 0$ such that

$$|y \cdot V'(t, y)| \leq \frac{\mu_1}{2}|y|^2$$

if $|y| \leq \rho$. Let $\{u_m\}$ be a $(PS)_c$ sequence. Pick $\bar{c} > 0$ such that $(3\mu\bar{c}/(\mu - 2))^{1/2} = \rho/\sqrt{2}$. By Corollary 1

$$\|u_m\| < \left(\frac{3\mu\bar{c}}{\mu - 2}\right)^{1/2}$$

if $c \in (0, \bar{c})$ and m is large. Since $\|u_m\|_{L^\infty} \leq \sqrt{2}\|u_m\| < \rho$,

$$J'(u_m)\frac{u_m}{\|u_m\|} \geq \|u_m\|^{-1}\left(\|u_m\|^2 - \frac{\mu_1}{2}\int_{-\infty}^{\infty} |u_m|^2 dt\right) \geq \frac{1}{2}\|u_m\|$$

which implies $\|u_m\| \to 0$ and consequently $J(u_m) \to 0$ as $m \to \infty$. This violates the fact that $\lim_{n\to\infty} J(u_m) = c > 0$. Therefore there is no $(PS)_c$ sequence if $c \in (0, \bar{c})$. So $\delta \geq \bar{c} > 0$.

Let $\xi : \mathbb{R} \to [0, 1]$ be a C^∞-function which satisfies

$$\xi(t) = \begin{cases} 0 & \text{if } t \in [-(k+1), k+1] \\ 1 & \text{if } t \notin [-(k+2), k+2]. \end{cases} \tag{1.7}$$

Lemma 3 *Let $\{u_m\}$ be a $(PS)_c$ sequence. Assume there is an increasing sequence $\{t_m\}$ such that $t_m \to \infty$ and*

$$\int_{-t_m}^{t_m} |u_m|^2 dt \to 0 \tag{1.8}$$

as $m \to \infty$. Let w_m be the restriction of ξu_m on Ω_k. Then $w_m \in E_k$ and $J_k(w_m) \to c$ and $J_k'(w_m) \to 0$ as $m \to \infty$.

Proof. It suffices to show that

$$\lim_{m\to\infty} |J_k(w_m) - J(u_m)| = 0 \tag{1.9}$$

and

$$\lim_{m\to\infty} \sup_{\|\phi\|_k \le 1} |J_k'(w_m)\phi - J'(u_m)\phi| = 0\,. \tag{1.10}$$

By a direct computation,

$$\begin{aligned} J_k'(w_m)\phi - J'(u_m)\phi \;&=\; \int_{t\ge|k|} [L(t)(\xi(t)-1)u_m\cdot\phi + (\xi(t)-1)\dot u_m\dot\phi \\ &\qquad + u_m\dot\xi\dot\phi + (V'(t,w_m) - V'(t,u_m))\cdot\phi]dt \end{aligned} \tag{1.11}$$

Let $Q = [-(k+2), -k] \cup [k, k+2]$. Applying Schwarz inequality yields

$$\left|\int_{t\ge|k|} L(t)(\xi(t)-1)u_m\cdot\phi dt\right| \;\le\; \left(\mu_2\int_Q |u_m|^2 dx\right)^{1/2}, \tag{1.12}$$

$$\left|\int_{t\ge|k|} u_m\dot\xi\dot\phi dt\right| \;\le\; \|\dot\xi\|_{L^\infty}\left(\int_Q |u_m|^2 dx\right)^{1/2}, \tag{1.13}$$

$$\left|\int_{t\ge|k|} (\xi(t)-1)\dot u_m\dot\phi dt\right| \;\le\; \left(\int_Q |\dot u_m|^2 dt\right)^{1/2}, \tag{1.14}$$

and

$$\begin{aligned} &\left|\int_{t\ge|k|} (V'(t,w_m) - V'(t,u_m))\cdot\phi dt\right| \\ &\le\; 2A_1\int_Q |u_m||\phi| dt \qquad (1.15)\\ &\le\; 2A_1\left(\int_Q |\phi|^2 dt\right)^{1/2}\left(\int_Q |u_m|^2 dt\right)^{1/2}, \end{aligned}$$

where $A_1 = \max\{\left|D_y^2 V(t,y)\right| : |y| \le \sqrt{2}\,A,\ t\in Q\}$, by making use of Lemma 1. Since $Q \subset [-t_m, t_m]$ if m is large, (1.10) follows from (1.11)-(1.15) and (1.8), provided that

$$\int_Q |\dot u_m|^2 dt \to 0 \quad \text{as } m\to\infty. \tag{1.16}$$

We now prove (1.16). Note that by passing to a subsequence if necessary we may assume $t_{m+1} - t_m > 2$. Let ξ_m be a $C_0^\infty(\mathbb{R})$ function which satisfies $0 \le \xi_m \le 1$, $|\dot{\xi}_m| \le 1$ and

$$\xi_m(t) = \begin{cases} 1 & \text{if } t \in [-t_m, t_m] \\ 0 & \text{if } t \notin (-t_{m+1}, t_{m+1}) . \end{cases}$$

Then by Lemma 1 there is a $C_1 > 0$ such that

$$\|\xi_m u_m\|^2 \le \int_{-\infty}^{\infty} \mu_2 |u_m|^2 dt + 2\int_{-\infty}^{\infty} |u_m|^2 |\dot{\xi}_m|^2 dt + 2\int_{-\infty}^{\infty} \xi_m^2 |\dot{u}_m|^2 dt \le C_1 .$$

If m is large then

$$\begin{aligned} &\int_{-t_m}^{t_m} L(t) u_m \cdot \xi_m u_m dt + \int_{-t_m}^{t_m} \dot{\xi}_m u_m \cdot \dot{u}_m dt - \int_{-t_m}^{t_m} V'(t, u_m) \xi_m u_m dt \\ &+ \int_{-t_m}^{t_m} \xi_m |\dot{u}_m|^2 dt = J'(u_m)\xi_m u_m = o(1). \end{aligned} \tag{1.17}$$

Arguing like above and using (1.8), we conclude that the first three integrals of (1.17) tend to zero as $m \to \infty$ and consequently

$$\int_Q |\dot{u}_m|^2 dt \le \int_{-t_{m-1}}^{t_{m-1}} |\dot{u}_m|^2 dt \le \int_{-t_m}^{t_m} \xi_m |\dot{u}_m|^2 dt \to 0 \text{ as } m \to \infty.$$

Observe that

$$\begin{aligned} J_k(w_m) - J(u_m) &= \int_Q \frac{1}{2}[(\xi^2 - 1)(L(t)u_m \cdot u_m + |\dot{u}_m|^2) \\ &\quad + |\dot{\xi}|^2 |u_m|^2 + 2\xi\dot{\xi} u_m \cdot \dot{u}_m] + [V(t, u_m) - V(t, w_m)] dt \\ &\quad - \int_{-k}^{k} [\frac{1}{2}(L(t)u_m \cdot u_m + |\dot{u}_m|^2) - V(t, u_m)] dt. \end{aligned}$$

Thus (1.9) follows from several estimates which are similar to the above.

The next two lemmas indicate the relationship between Palais-Smale sequences and critical points of J. We refer to [5] for detailed proofs.

Lemma 4 *Let $\{u_m\}$ be a $(PS)_c$ sequence. Then there exists a subsequence $\{u_{m_j}\}$ such that*

$$u_{m_j} \to u \text{ weakly in } E \text{ and strongly in } L^\infty_{loc}(\mathbb{R}, \mathbb{R}^n) \tag{1.18}$$

for some $u \in E$ which satisfies $J'(u) = 0$ and $J(u) \le c$.

Lemma 5 *Let $\{u_m\}$ be a $(PS)_c$ sequence. Assume $\{u_m\}$ converges to $u \in E$ both weakly in E and strongly in $L^\infty_{loc}(\mathbb{R}, \mathbb{R}^n)$. If $v_m = u_m - u$, then*

$$\lim_{m\to\infty} J'(v_m) = 0 \quad \text{and} \quad \lim_{m\to\infty} J(v_m) = c - J(u) .$$

2 Existence results

We now prove the existence of homoclinic orbits of (HS).

Theorem 3 *Suppose there exists a $(PS)_c$ sequence such that $c > 0$ and*

$$c \notin \Lambda_k \tag{2.1}$$

for some $k \in \mathbb{N}$, then there is a homoclinic solution q of (HS) and

$$c \geq J(q) \geq \delta. \tag{2.2}$$

Proof Let $\{u_m\}$ be a $(PS)_c$ sequence. By Lemma 4, there exist a $q \in E$ and a subsequence, still denoted by $\{u_m\}$, such that

$$u_m \to q \quad \text{weakly in } E \text{ and strongly in } L^\infty_{loc}(\mathbb{R}, \mathbb{R}^n),$$
$$J'(q) = 0, \text{ and} \tag{2.3}$$
$$J(q) \leq c\,. \tag{2.4}$$

We claim that $q \not\equiv 0$. This is true if there exist $k, l \in \mathbb{N}$ and $b > 0$ such that if $m \geq l$ then

$$\int_{-k}^{k} |u_m|^2 dt \geq b. \tag{2.5}$$

Suppose (2.5) is false. Then there exist a sequence $\{t_m\}$ with $\lim_{m\to\infty} t_m = \infty$, and a subsequence, still denoted by $\{u_m\}$, such that

$$\lim_{m\to\infty} \int_{-t_m}^{t_m} |u_m|^2 dt = 0\,.$$

Let ξ be defined as in (1.7) and w_m be the restriction of ξu_m to Ω_k. Invoking Lemma 3 yields

$$c \in \Lambda_k$$

which contradicts (2.1). Therefore (2.5) holds and $q \not\equiv 0$. This together with (2.3) and Corollary 2 shows that $J(q) > 0$. Moreover (2.2) follows from the definition of δ and (2.4).

Since $E \subset C^0(\mathbb{R}, \mathbb{R}^n), q(t) \to 0$ as $|t| \to \infty$. This together with (HS) and (V1) shows that $\ddot{q} \in L^2(\mathbb{R}, \mathbb{R}^n)$. Then $q \in W^{2,2}(\mathbb{R}, \mathbb{R}^n)$ implies $\dot{q} \in C^0(\mathbb{R}, \mathbb{R}^n)$. Thus q is a homoclinic solution of (HS).

Proof of Theorem 1. By the definition of Λ and Proposition 2, there is a $(PS)_\delta$ sequence. Applying Theorem 3 gives a homoclinic solution q of (HS) with $J(q) = \delta$.

Proof of Theorem 2. Since $\delta \notin \Lambda_k$, by Theorem 1 there is a homoclinic solution q of (HS) with $J(q) = \delta$. Invoking Lemma 2 and Lemma 3, we get a $(PS)_\beta$ sequence $\{u_m\}$ which converges to $\hat{q}$ both weakly in E and strongly in $L^\infty_{loc}(\mathbb{R}, \mathbb{R}^n)$. Moreover $J'(\hat{q}) = 0$ and $J(\hat{q}) \leq \beta$. Since $\beta \notin \Lambda_k$, it follows from the same reasoning as the proof of Theorem 3 that $\hat{q}$ is a homoclinic solution of (HS). Suppose $\hat{q} = q$. Setting $v_m = u_m - q$, we see from Lemma 5 that $\{v_m\}$ is a $(PS)_{\beta-\delta}$ sequence. Since $0 < \beta - \delta < \delta_k$, repeating the above arguments shows that $\{v_m\}$ converges weakly to some $\bar{v} \in E\backslash\{0\}$. This contradicts the fact that u_m converges weakly to q. So $\hat{q} \neq q$.

Remark 2 *This proof shows that Theorem 2 still holds if $\delta < \beta$ and $\beta \notin \Lambda_k$, $\delta \notin \Lambda_j$, $\beta - \delta \notin \Lambda_i$ for some $i, j, k \in \mathbb{N}$.*

3 A sufficient condition for $\delta = \beta$

Although it has been shown that $\delta > 0$, in general it seems to be difficult to obtain an optimal lower bound for δ. Here we illustrate a condition which ensures $\delta = \beta$:

(V3) For all $t \in \mathbb{R}, |y| = 1$, $\rho^{-1}V'(t, \rho y) \cdot y$ is an increasing function of ρ if $\rho \in (0, \infty)$.

Proposition 3 *If (V1)-(V3) and (L) are satisfied then $\delta = \beta$.*

To Prove Proposition 3, we need the following proposition.

Proposition 4 *If (V1)-(V3) and (L) are satisfied then*

$$\beta = \inf_{\substack{u \in E \\ u \neq 0}} \max_{\theta \in [0,\infty)} J(\theta u). \tag{3.1}$$

Since the proof of this proposition is similar to that of Proposition 2.14 in [7], we omit it.

Corollary 3 *If (V1)-(V3) and (L) are satisfied, then $\beta \leq \beta_k \leq \beta_{k+1}$, where*

$$\beta_k = \inf_{\substack{u \in E_k \\ u \neq 0}} \max_{\theta \in [0,\infty)} J_k(\theta u)\,.$$

The proof of this corollary follows easily from (0.2).

Proof of Proposition 3. It suffices to show that $\delta \geq \beta$, since the reversed inequality is always true. Let $\{u_m\}$ be a $(PS)_c$ sequence with $c > 0$. Then there is an $\epsilon_1 > 0$ such that for large m

$$\|u_m\| \geq \epsilon_1. \tag{3.2}$$

For $u_m \not\equiv 0$, we set

$$g_m(\rho) = J(\rho u_m). \tag{3.3}$$

It is clear that $g_m(0) = 0$. Since

$$g'_m(\rho) = \rho\|u_m\|^2 - \int_{-\infty}^{\infty} V'(t, \rho u_m) \cdot u_m dt, \tag{3.4}$$

it follows from (V1) that $g'_m(\rho) > 0$ if ρ is a sufficiently small positive number. Moreover, we know from (V2) that

$$\lim_{\rho\to\infty} g_m(\rho) = -\infty.$$

Hence there is a $\rho_m \in (0, \infty)$ such that

$$g'_m(\rho_m) = 0 \tag{3.5}$$

and

$$g_m(\rho_m) = \max_{\rho\in[0,\infty)} g_m(\rho)\,. \tag{3.6}$$

By Proposition 4

$$\beta \le g_m(\rho_m)\,. \tag{3.7}$$

Let $R(z) = \{t \in \mathbb{R} \| t - z| \le \frac{1}{2}\}$. We claim there exist a sequence $\{z_m\} \subset \mathbb{Z}$ and an $\epsilon_2 > 0$ such that

$$\|u_m\|_{L^\infty(R(z_m))} \ge \epsilon_2\,. \tag{3.8}$$

Suppose (3.8) is false. Then there is a subsequence, still denoted by $\{u_m\}$, such that

$$\sup_{z\in\mathbb{Z}} \|u_m\|_{L^\infty(R(z))} \to 0 \quad \text{ as } m \to \infty\,. \tag{3.9}$$

Given $\epsilon_3 > 0$, by (V1) there is an $s > 0$ such that

$$|V'(t, y) \cdot y| \le \epsilon_3 |y|^2 \quad \text{ for } |y| \le s\,. \tag{3.10}$$

If m is large, (3.9) and (3.10) imply that

$$\left|\int_{-\infty}^{\infty} V'(t, u_m) \cdot u_m dt\right| \le \epsilon_3 \int_{-\infty}^{\infty} |u_m|^2 dt.$$

Since

$$\|u_m\|^2 = J'(u_m)u_m + \int_{-\infty}^{\infty} V'(t, u_m) \cdot u_m dt \le o(1)\|u_m\| + \epsilon_3\|u_m\|^2,$$

it follows from Lemma 1 that

$$\|u_m\|^2 \le \frac{1}{4}\epsilon_1^2 + A^2\epsilon_3\,.$$

Choosing $\epsilon_3 < \epsilon_1^2/(4A^2)$ yields

$$\|u_m\|^2 < \frac{1}{2}\epsilon_1^2 \tag{3.11}$$

which contradicts (3.2). Therefore (3.8) must hold.

Let $v_m(t) = u_m(t+z_m)$. Since $\|v_m\|$ is bounded, there is a subsequence, still denoted by $\{v_m\}$, such that

$$v_m \to \bar{v} \text{ in } L^\infty_{loc}(\mathbb{R}, \mathbb{R}^n)$$

and

$$\|\bar{v}\|_{L^\infty[-1/2,1/2]} \geq \epsilon_2 \, .$$

Since v_m and $\bar{v}$ are continuous on $[-1/2, 1/2]$, there is a subinterval $[a,b]$ of $[-1/2,1/2]$ such that

$$\min_{t\in[a,b]} |v_m(t)| \geq \epsilon_4 \text{ for all large m,} \tag{3.12}$$

where $\epsilon_4 = \epsilon_2/2$. By (3.4) and (3.5)

$$\|u_m\|^2 = \frac{1}{\rho_m}\int_{-\infty}^{\infty} V'(t, \rho_m u_m)\cdot u_m dt.$$

Let $D_m = [a+z_m, b+z_m]$. Since (V2) implies that

$$\frac{y\cdot V'(t,y)}{|y|^2} \geq \frac{\mu V(t,y)}{|y|^2} \to \infty \quad \text{ uniformly in } t \text{ as } |y| \to \infty,$$

We get

$$\begin{aligned}
\|u_m\|^2 &\geq \frac{1}{\rho_m}\int_{D_m} V'(t, \rho_m u_m(t))\cdot u_m(t)dt \\
&= \int_{D_m} \rho_m^{-2}|u_m(t)|^{-2} V'\left(t, \rho_m|u_m(t)|\frac{u_m(t)}{|u_m(t)|}\right) \times \\
&\quad \rho_m|u_m(t)|\frac{u_m(t)}{|u_m(t)|}|u_m(t)|^2 dt \\
&\geq \epsilon_4^2 \int_{D_m} \rho_m^{-2}|u_m(t)|^{-2} V'\left(t, \rho_m|u_m(t)|\frac{u_m(t)}{|u_m(t)|}\right) \times \\
&\quad \rho_m|u_m(t)|\frac{u_m(t)}{|u_m(t)|} dt \to \infty \text{ as } \rho_m \to \infty.
\end{aligned} \tag{3.13}$$

It follows from Lemma 1 that $\{\rho_m\}$ is bounded. Let

$$h(\rho) = \frac{1}{2}\rho^2 \int_{-\infty}^{\infty} V'(t,u_m)\cdot u_m dt - \int_{-\infty}^{\infty} V(t, \rho u_m)dt.$$

Since

$$h'(\rho) = \rho\int_{-\infty}^{\infty} V'(t,u_m)\cdot u_m dt - \int_{-\infty}^{\infty} V'(t,\rho u_m)\cdot u_m dt,$$

it follows from (V3) that $h'(\rho) > 0$ if $\rho \in (0,1)$ and $h'(\rho) < 0$ if $\rho \in (1,\infty)$. Therefore,

$$h(1) = \max_{\rho\in[0,\infty)} h(\rho).$$

Since

$$\begin{aligned} g_m(\rho_m) &= \frac{1}{2}\rho_m^2 J'(u_m)u_m + \frac{1}{2}\rho_m^2 \int_{-\infty}^{\infty} V'(t, u_m) \cdot u_m dt \\ &\quad - \int_{-\infty}^{\infty} V(t, \rho_m u_m) dt = h(\rho_m) + o(1), \end{aligned}$$

we have

$$\begin{aligned} \beta &\leq \lim_{m\to\infty} \inf g_m(\rho_m) \\ &\leq \lim_{m\to\infty} \inf \int_{-\infty}^{\infty} [\frac{1}{2}V'(t, u_m) \cdot u_m - V(t, u_m)]dt \\ &= \lim_{m\to\infty} \inf[\frac{1}{2}\|u_m\|^2 - \int_{-\infty}^{\infty} V(t, u_m)dt - \frac{1}{2}J'(u_m)u_m] \\ &= \lim_{m\to\infty} J(u_m) = c. \end{aligned}$$

Since c is arbitrary, it follows that $\beta \leq \delta$. This completes the proof.

Remark 3 *Under the hypothesis of Proposition 3, it follows that*

$$\delta_k = \beta_k\,. \tag{3.14}$$

and

$$\delta_k \leq \delta_{k+1}\,. \tag{3.15}$$

In this case, Theorem 1 can be recast as follows.

Theorem 4 *There exists a homoclinic solution of (HS) if*

$$\beta < \lim_{k\to\infty} \delta_k. \tag{3.16}$$

4 Examples

In this section some existence and nonexistence results for the homoclinic orbits of (HS) will be discussed. For the time-periodic Hamiltonian system of the form (HS), it has been shown (e.g. [13, 5]) that there exists at least one homoclinic orbit.

Proposition 5 *If (L) (P) (V1) and (V2) are satisfied, then (HS) possesses a homoclinic orbit* $q \in W^{1,2}(\mathbb{R}, \mathbb{R}^n)$ *such that* $J(q) = \delta$.

Remark 4 *If (V3) is also satisfied, then by Proposition 3 there is a homoclinic orbit* q *of (HS) with* $J(q) = \beta$.

Lemma 6 *If (L),(P) and (V1)-(V3) are satisfied then* $\beta_k = \beta$ *for all* k.

Proof. As noted in Remark 1, we may take $\Omega_k = \mathbb{R}\backslash[-kT, kT]$. Let q be a homoclinic orbit of (HS) with $J(q) = \beta$. If

$$u_m(t) = \begin{cases} q(t-mT) & \text{for } t \in [(k+1)T, \infty) \\ \frac{t-kT}{T} q((k-m+1)T) & \text{for } t \in [kT, (k+1)T) \\ 0 & \text{for } t \in (-\infty, -kT], \end{cases}$$

then by a direct computation

$$\lim_{m\to\infty} \max_{\theta\in[0,\infty)} J_k(\theta u_m) = \beta. \tag{4.1}$$

Since $\lim_{\theta\to\infty} J_k(\theta u_m) = -\infty$, (4.1) implies

$$\beta_k \le \beta. \tag{4.2}$$

On the other hand, $W_0^{1,2}(\Omega_k, \mathbb{R}^n) \subset W^{1,2}(\mathbb{R}, \mathbb{R}^n)$ yields

$$\beta_k \ge \beta. \tag{4.3}$$

Combining (4.2) with (4.3) gives $\beta_k = \beta$.

In the use of comparison arguments in what follows, we let

$$\ddot{q} - \widetilde{L}(t)q + \widetilde{V}'(t, q) = 0 \tag*{$(HS)^\sim$}$$

be a Hamiltonian system having the same form as (HS). The corresponding functional associated with (HS)$^\sim$ is

$$\widetilde{J}(q) = \int_{-\infty}^{\infty} \left[\frac{1}{2}\left(|\dot{q}|^2 + \widetilde{L}(t)q\cdot q\right) - \widetilde{V}(t, q)\right] dt.$$

Here $\widetilde{V}$ and $\widetilde{L}$ as well as V and L are assumed to satisfy (V1), (V2) and (L). In the same way as β is defined in (MP), let $\widetilde{\beta}$ be the mountain pass minimax value of $\widetilde{J}$. Similarly, we define $\widetilde{\delta}$, $\widetilde{J}_k$, $\widetilde{\beta}_k$, and $\widetilde{\delta}_k$ by the same manner. It is clear from (MP) that

$$\widetilde{\beta} \le \beta \tag{4.4}$$

if

$$(L(t) - \widetilde{L}(t))y \cdot y \ge 0 \tag{4.5}$$

and

$$\widetilde{V}(t, y) \ge V(t, y) \tag{4.6}$$

for all $t \in \mathbb{R}$ and $y \in \mathbb{R}^n$.

We now investigate the existence of homoclinic orbits of (HS) when a time-periodic Hamiltonian system is perturbed. Let $A = \{y | \alpha < |y| < \widehat{\alpha}\}$, where $0 \le \alpha < \widehat{\alpha}$. Set $B = (a, b) \times A$, where $-\infty < a < b < \infty$. In the next two examples, it is assumed that L and V satisfy (P) and (V3), $\widetilde{L} = L$,

$$\widetilde{V}(t, y) > V(t, y) \quad \text{for } (t, y) \in B \tag{4.7}$$

$$\widetilde{V}(t, y) = V(t, y) \quad \text{for } (t, y) \in \mathbb{R}^{n+1}\backslash B \tag{4.8}$$

Example 1. The perturbed system (HS)$^\sim$ possesses at least one homoclinic solution $\widetilde{q}$. Indeed, by Proposition 5 and Remark 4 there is a homoclinic solution q_1 of (HS) with $J(q_1) = \beta$. If the set $\mathcal{L} = \{(t, q_1(t))|t \in \mathbb{R}\}$ does not intersect B then q_1 is also a homoclinic solution of (HS)$^\sim$. Suppose $\mathcal{L} \cap B \neq \emptyset$. Then by (4.7) and (4.8),

$$\widetilde{J}(q_1) < J(q_1)$$

and

$$\widetilde{J}(\theta q_1) \leq J(\theta q_1) \quad \text{for } \theta \in [0, \infty).$$

It follows that $\max_{\theta\in[0,\infty)} \widetilde{J}(\theta q_1) < \beta$. This together with $\lim_{\theta\to\infty} \widetilde{J}(\theta q_1) = -\infty$ and (MP) implies that $\widetilde{\beta} < \beta$. On the other hand, it is clear from Lemma 6 and (3.14) that $\beta = \beta_k = \widetilde{\beta}_k = \widetilde{\delta}_k$ for large k. Hence (HS)$^\sim$ possesses a homoclinic solution $\widetilde{q}$ with $\widetilde{J}(\widetilde{q}) = \widetilde{\beta}$.

Example 2 In Example 1, if α is a sufficiently small positive number and $\hat{\alpha}$ is sufficiently large, then (HS)$^\sim$ possesses two homoclinic solutions q and $\widetilde{q}$ such that $\widetilde{J}(\widetilde{q}) = \widetilde{\beta} < \beta = \widetilde{J}(q)$. Note that $q_1(t) \to 0$ as $|t| \to \infty$. Letting $q(t) = q_1(t+lT)$, we see that if l is sufficiently large then $|q(t)| < \alpha$ for $t \in (a, b)$. Thus q is a homoclinic solution of (HS)$^\sim$. On the other hand, by choosing a suitable j, the set $\{(t, q_1(t + jT))|t \in \mathbb{R}\}$ must intersect B, since, by the basic existence and uniqueness theorem of ordinary differential equations, the point t at which $|q(t)| = 0$ is isolated if such a point exists. We may proceed as in Example 1 to obtain a homoclinic solution $\widetilde{q}$ for (HS)$^\sim$ with $\widetilde{J}(\widetilde{q}) = \widetilde{\beta}$.

Remark 5 *We may proceed as in Example 2 to perturb a time-periodic Hamiltonian system (HS) in a suitable way to obtain any finite number of homoclinic solutions for the perturbed system (HS)$^\sim$, and all these solutions have different critical values for $\widetilde{J}$.*

In the next two examples, we investigate the homoclinic orbits of (HS)$^\sim$ for the case where $\widetilde{L} = L$,

$$\widetilde{V}(t, y) < V(t, y) \quad \text{for } (t, y) \in \widetilde{B} \tag{4.9}$$

$$\widetilde{V}(t, y) = V(t, y) \quad \text{for } (t, y) \in \mathbb{R}^{n+1}\backslash\widetilde{B}. \tag{4.10}$$

Here $\widetilde{B}$ is not necessarily a connected set. Also, L and V are assumed to satisfy (P) and (V3).

Example 3. Let $\widetilde{B} = B$, $\alpha = 0$ and $\widehat{\alpha} = \infty$. Assume that $\widetilde{V}$ satisfies (V3). Then there is no homoclinic solution $\widetilde{q}$ of (HS)$^\sim$ with $\widetilde{J}(\widetilde{q}) = \widetilde{\beta}$. Note that

$$\widetilde{\beta} \geq \beta \tag{4.11}$$

by (4.9) and (4.10). Moreover, if q is a homoclinic solution of (HS) with $J(q) = \beta$, letting $u_m(t) = q(t - mT)$ yields

$$\lim_{m\to\infty} \max_{\theta\in[0,\infty)} \widetilde{J}(\theta u_m) = \beta, \tag{4.12}$$

by making use of the fact that

$$\lim_{m\to\infty} |u_m(t)| = 0 \quad \text{if } t \in [a,b].$$

Since $\lim_{\theta\to\infty} \widetilde{J}(\theta u_m) = -\infty$, (4.12) implies $\widetilde{\beta} \le \beta$. This together with (4.11) yields

$$\widetilde{\beta} = \beta\,. \tag{4.13}$$

Suppose there were a homoclinic solution $\widetilde{q}$ of (HS)$^\sim$ with $\widetilde{J}(\widetilde{q}) = \widetilde{\beta}$. Arguing as in Example 1, we would obtain $\widetilde{\beta} > \beta$ which contradicts (4.13).

Example 4. When $\widetilde{V}$ satisfies (4.9) and (4.10), the perturbed system (HS)$^\sim$ may possess a homoclinic solution $\widetilde{q}$ with $\widetilde{J}(\widetilde{q}) > \widetilde{\beta}$. Here we construct an example as follows. Let $W(t,y)$ be a function which satisfies

$$W(t,y) < V(t,y) \quad \text{for } (t,y) \in B\,, \tag{4.14}$$

and

$$W(t,y) = V(t,y) \quad \text{for } (t,y) \in ([0,T] \times \mathbb{R}^n)\backslash B\,, \tag{4.15}$$

where $B = (a,b) \times A$, $A = \{y | \alpha < |y| < \infty\}$, $0 < \alpha$ and $0 < a < b < T$. Moreover, let W satisfy (P), (V1), (V2) and (V3). By Proposition 5 and Remark 4, (HS) possesses a homoclinic solution q with $J(q) = \beta$, and the Hamiltonian system

$$\ddot{q} - L(t)q + W'(t,q) = 0$$

possesses a homoclinic solution $\widetilde{q}$. If α, a and $T-b$ are sufficiently small, the set $\mathcal{L}_1 = \{(t,q(t)) | t \in \mathbb{R}\}$ intersects B_l for some $l \in \mathbb{Z}$, where $B_l = \{(t + lT, y) | (t,y) \in B\}$. Since $\alpha > 0$ and $\lim_{|t|\to\infty} |\widetilde{q}(t)| = 0$, the set $\mathcal{L}_2 = \{(t, \widetilde{q}(t)) | t \in \mathbb{R}\}$ intersects only a finite number of B_l.

Let $\widetilde{B}$ be the union of B_l for which $B_l \cap \mathcal{L}_2 \neq \emptyset$. Define $\widetilde{V}(t,y)$ by

$$\widetilde{V}(t,y) = \begin{cases} W(t,y) & \text{if } (t,y) \in \widetilde{B}. \\ V(t,y) & \text{if } (t,y) \in \mathbb{R}^{n+1}\backslash \widetilde{B}. \end{cases}$$

Clearly, $\widetilde{q}$ is a homoclinic solution of (HS)$^\sim$. Moreover, q_1 is also a homoclinic solution of (HS)$^\sim$ if $q_1(t) = q(t + jT)$ and j is sufficiently large. Consequently $\widetilde{\beta} = \beta$. Finally the fact that $\mathcal{L}_2$ intersects B_l shows that $\widetilde{J}(\widetilde{q}) > \beta = \widetilde{\beta}$.

Remark 6 *In Examples 1-4, all these perturbations can be made arbitrarily "small".*

Example 5. If V satisfies (V3), and there exist $t_0 \in \mathbb{R}$ and $T > 0$ such that for all $y \in \mathbb{R}^n$

$$V(t,y) \ge V(t+T, y), \qquad (L(t) - L(t+T))y \cdot y \le 0 \quad \text{if } t \ge t_0 \tag{4.16}$$

$$V(t,y) \ge V(t-T, y), \qquad (L(t) - L(t-T))y \cdot y \le 0 \quad \text{if } t < t_0 + T, \tag{4.17}$$

then (HS) possesses a homoclinic solution q with $J(q) = \beta$.

To show this, we let $\widetilde{V}(t,y) = V(t,y)$, $\widetilde{L}(t) = L(t)$ for t$\in [t_0, t_0+T]$, $y \in \mathbb{R}$. Let $\widetilde{V}$ and $\widetilde{L}$ satisfy (P). By Proposition 5 and Remark 4, (HS)$^\sim$ possesses a homoclinic solution $\widetilde{q}$ with $\widetilde{J}(\widetilde{q}) = \widetilde{\beta}$. Let

$$\mathcal{L}_3 = \{(t,y)|\widetilde{V}(t,y) > V(t,y) \text{ or } (\widetilde{L}(t) - L(t))y \cdot y < 0)\}\,.$$

If the set $\mathcal{L}_4 = \{(t, \widetilde{q}(t))|t \in \mathbb{R}\}$ does not intersect $\mathcal{L}_3$ then $\widetilde{q}$ is also a homoclinic solution of (HS). Otherwise, we may proceed as in Example 1 to get $\beta < \widetilde{\beta}$. Then $\delta_k > \beta$ for some k and consequently (HS) possesses a homoclinic solution q with $J(q) = \beta$, provided that

$$\delta_k \geq \widetilde{\beta}. \tag{4.18}$$

Indeed, by Lemma 6,

$$\widetilde{\beta}_k = \widetilde{\beta}. \tag{4.19}$$

for all k. Moreover it follows from (3.14) that

$$\delta_k = \beta_k. \tag{4.20}$$

Since (4.16) and (4.17) imply that

$$\beta_k \geq \widetilde{\beta}_k, \tag{4.21}$$

putting (4.19)-(4.21) together gives (4.18).

Remark 7 *The results of Example 5 still hold if instead of (V3), V satisfies*

(V3)' There is a $t_1 > 0$ such that, for all $t \in (-\infty, -t_1] \cup [0,T] \cup [t_1, \infty)$ and $|y| = 1$, $\rho^{-1}V'(t, \rho y) \cdot y$ is an increasing function of ρ if $\rho \in (0, \infty)$.

Example 6. Consider (HS) where L satisfies (P) and there is a $t_0 \in \mathbb{R}$ such that for all $y \in \mathbb{R}^n$

$$\begin{aligned} V(t,y) &= V(t-T,y) \quad \text{if } t \leq t_0 \\ V(t,y) &= V(t+T,y) \quad \text{if } t \geq t_0\,. \end{aligned} \tag{4.22}$$

Set $\widetilde{L} = L$. Without loss of generality, we may assume that $t_0 = 0$. Let $\beta_+ = \widetilde{\beta}$, when

$$\widetilde{V} \text{ satisfies (P) and } \widetilde{V}(t,y) = V(t,y) \text{ for } t \geq 0. \tag{4.23}$$

Let $\beta_- = \widetilde{\beta}$, when

$$\widetilde{V} \text{ satisfies (P) and } \widetilde{V}(t,y) = V(t,y) \text{ for } t \leq 0 \tag{4.24}$$

If $\beta_- \leq \beta_+$, V satisfies (V3) and there is an $\alpha_1 > 0$ such that

$$V(t,y) > V(t-T,y) \qquad \text{for } t \in (0,T) \text{ and } |y| < \alpha_1, \tag{4.25}$$

then (HS) possesses a homoclinic solution q. This can be done by proving $\beta < \beta_{\_}$, since $\delta_k = \beta_{\_}$ if k is large. When (4.24) holds, let $\widetilde{q}$ be a homoclinic solution of $(\mathrm{HS})^{\sim}$ with $\widetilde{J}(\widetilde{q}) = \beta_{\_}$. Let $\theta_0 > 0$ be fixed such that

$$\widetilde{J}(\theta_0 \widetilde{q}) < 0.$$

If t is sufficiently large then $\theta_0|\widetilde{q}(t)| < \alpha_1$. Let $q_1(t) = \widetilde{q}(t + lT)$ and choose l large enough such that $\theta_0|q_1(t)| = \theta_0|\widetilde{q}(t + lT)| < \alpha_1$ for $t \geq 0$. Then invoking (4.25) yields

$$J(\theta q_1) < \widetilde{J}(\theta q_1) \quad \text{for } \theta \in [0, \theta_0]\,.$$

Hence

$$\max_{\theta\in[0,\theta_0]} J(\theta q_1) < \max_{\theta\in[0,\theta_0]} \widetilde{J}(\theta q_1) = \beta_{\_}$$

and

$$J(\theta_0 q_1) < \widetilde{J}(\theta_0 q_1) < 0\,.$$

It follows from (MP) that $\beta < \beta_{\_}$.

Remark 8 *(a) It is easy to see that in Example 6 (HS) still possesses a homoclinic solution if (4.25) is replaced by*

$$V(t, y) \geq V(t - T, y) \text{ for } t \in [0, T] \text{ and } |y| < \alpha_1.$$

(b) As a similar case to Example 6, we may consider

$$\begin{aligned} L(t) &= L(t - T), \qquad V(t, y) = V(t - T, y) \text{ if } t \leq t_0 \\ L(t) &= L(t + T_1), \qquad V(t, y) = V(t + T_1, y) \text{ if } t \geq t_0, \end{aligned}$$

where $T_1 > 0$.

As illustrated in the above examples, Proposition 3 has been applied as a good way to obtain an optimal lower bound for δ_k if condition (V3) is satisfied. In the next example, our aim is to find estimates of δ_k to fulfill the requirement of Theorem 1 in the situation where (V3) does not hold.

Example 7. Let $V(t, y) = H(t)\hat{V}(y)$, where $\inf_{t\in\mathbb{R}} H(t) > 0$. Let β_0 be the mountain pass minimax value of J on the subspace $W_0^{1,2}((-k, k), \mathbb{R}^n)$ of E. Clearly

$$\beta_0 \geq \beta\,.$$

In view of the proof of Proposition 2, $\bar{c} = (\mu - 2)\rho^2/(6\mu)$ is a lower bound of δ_k, where ρ can be chosen as large as possible; so that, for $|t| \geq k$, if $|y| \leq \rho$ then

$$|H(t)y \cdot \hat{V}'(y)| \leq \frac{\mu_1}{2}|y|^2\,.$$

Since this inequality holds when $\sup_{|t|\geq k} H(t) \leq \lambda$ and

$$\frac{|y \cdot \hat{V}'(y)|}{|y|^2} \leq \frac{\mu_1}{2\lambda}\,,$$

we see that $\bar{c} > \beta_0$ and consequently $\delta_k > \beta$ if λ is small enough.

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