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EXISTENCE OF POSITIVE PERIODIC SOLUTIONS TO THIRD-ORDER DELAY DIFFERENTIAL EQUATIONS

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Abstract. Using the continuation theorem of coincidence degree theory and analysis techniques, we establish criteria for the existence of periodic solutions to the following third-order neutral delay functional differential equation with deviating arguments

$$\dddot{x}(t) + a\ddot{x}(t) + g(\dot{x}(t - \tau(t))) + f(x(t - \tau(t))) = p(t).$$

Our results complement and extend known results and are illustrated with examples.

1. Introduction

The stability analysis of neutral delay differential systems has received considerable attention over the decades. In the literature, Lyapunov technique, characteristic equation method, or state solution approach have been utilized to derive sufficient conditions for asymptotic stability. In [4, 16], the stability of delay differential equations was considered. Also, boundedness of solutions was discussed in [16]. Later, many books and papers dealt with the delay differential equations and obtained many good results, for example, [1, 3, 5, 6, 7, 11, 12, 13], etc. However, the periodic solutions of third-order neutral delay functional differential equations with deviating arguments has been investigated only by a few researchers.

In recent years, the existence of periodic solutions for some types of second-order differential equation with deviating argument were studied; see [8, 9, 10, 15]. But the corresponding problem for third-order neutral delay functional differential with a deviating argument was studied far less often. In [14], Sadek obtained sufficient conditions to ensure the stability and the boundedness of system

$$\dddot{x}(t) + a\ddot{x}(t) + g(\dot{x}(t - \tau(t))) + f(x(t - \tau(t))) = p(t). \tag{1.1}$$

However, he did not research periodic solution of (1.1). The main purpose of this note is to establish criteria to guarantee the existence of positive periodic solutions to (1.1). By using the continuation theorem of Mawhin's coincidence degree theory [2], we obtain some new result which complement and extend the corresponding ones already known; see [8, 9, 10, 14, 15]. An example to illustrate the main result is given.

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2. Existence of periodic solutions

Throughout this paper, a is positive constant; the functions g, f, p are real continuous and defined on $\mathbb{R}$; $\tau(t)$ and $p(t)$ are periodic with common period ω. We point out that in this paper we allow $\int_0^\omega p(t)dt \neq 0$ (It is zero in [10, 14]). Meanwhile, we define $|p|_0 = \max_{t\in[0,\omega]} |p(t)|$, $|x|_i := \big(\int_0^{2\pi} |x(s)|^i ds\big)^{1/i}$, $i \geq 1$. Now we define $\mu(t) = t - \tau(t)$, then $\mu(t)$ has inverse function ν. set $b(t) = (1 - \dot{\tau}(\nu(t)))^{-1}$, $\dot{\tau} < 1$. The following is our main results.

Theorem 2.1. *Suppose that exist positive constants $\delta_1 \geqslant 0$, $\delta_2 \geqslant 0$, $K > 0$ and $M > 0$, such that*

- (H1) $|g(x)| \leqslant K + \delta_1|x|$ *for* $x \in \mathbb{R}$
- (H2) $xf(x) > 0$ *and* $|f(x)| > K + |p|_0 + \delta_1|x|$ *for* $|x| \geqslant D$
- (H3) $\lim_{x\to-\infty} \frac{f(x)}{x} \leqslant \delta_2$.

Then (1.1) *has at least one ω-periodic solution for* $a\omega+2\delta_1|b|_2\omega^{\frac{3}{2}}+2\omega^2(1+\omega)\delta_2 < 1$.

Theorem 2.2. *Suppose that there exist positive constants $\rho_1 \geqslant 0$, $\rho_2 \geqslant 0$, $K > 0$ and $M > 0$, such that*

- (S1) $|g(x)| \leqslant K + \rho_1|x|$ *for* $x \in \mathbb{R}$
- (S2) $xf(x) > 0$ *and* $|f(x)| > K + |p|_0 + \rho_1|x|$ *for* $|x| \geqslant D$
- (S3) $\lim_{x\to+\infty} \frac{f(x)}{x} \leqslant \rho_2$.

Then (1.1) *has at least one ω-periodic solution for* $a\omega+2\rho_1|b|_2\omega^{\frac{3}{2}}+2\omega^2(1+\omega)\rho_2 < 1$.

To prove our results, we need the notion of the continuation theorem of coincidence degree theory formulated in [2].

Lemma 2.3. *Let X and Z be two Banach space. Consider an operator equation*

$$Lx = \lambda N(x, \lambda), \tag{2.1}$$

where $L : \operatorname{Dom} L \cap X \to Z$ is a Fredholm operator of index zero, $\lambda \in [0, 1]$ is a parameter. Let P and Q denote two projectors such that

$$P : X \to \ker L, \quad \text{and} \quad Q : Z \to Z/ImL.$$

Assume that $N : \bar{\Omega} \times [0, 1] \to Z$ is L-compact on $\bar{\Omega} \times [0, 1]$, where Ω is open bounded in X. Furthermore, suppose that

- (a) *For each $\lambda \in (0, 1)$ and $x \in \partial\Omega \cap \operatorname{Dom} L$, $Lx \neq \lambda N(x, \lambda)$*
- (b) *For each $x \in \partial\Omega \cap \ker L$, $QNx \neq 0$,*
- (c) $\deg\{QN, \Omega \cap \ker L, 0\} \neq 0$.

Then $Lx = N(x, 1)$ has at least one solution in $\bar{\Omega}$.

Next we present the proof for Theorem 2.1. Since the proof of Theorem 2.2 is similar we omit it.

Proof of Theorem 2.1. To use Lemma 2.3 for (1.1), we take $X = \{x \in C^3(R, R) : x(t + \omega) = x(t)$ for all $t \in \mathbb{R}\}$ and $Z = \{z \in C(R, \mathbb{R}) : z(t + \omega) = z(t)$ for all $t \in \mathbb{R}\}$ and denote $|x|_0 = \max_{t\in[0,\omega]} |x(t)|$ and $\|x\| = \max\{|x|_0, |\dot{x}|_0, |\ddot{x}|_0\}$. Then X and Z are Banach spaces, for $x \in X$ and $z \in Z$, endowed with the norms $\|\cdot\|$ and $|\cdot|_0$, respectively. Set

$$Lx(t) = \dddot{x}, \quad x \in X,\ t \in \mathbb{R};$$
$$N(x(t), \lambda) = -a\ddot{x}(t) - \lambda g(\dot{x}(t-\tau(t))) - f(x(t-\tau(t))) + \lambda p(t), \quad x \in X,\ t \in \mathbb{R};$$
$$Px(t) = \frac{1}{\omega}\int_0^\omega x(t)dt, \quad Qz(t) = \frac{1}{\omega}\int_0^\omega z(t)dt, \quad x \in X,\ t \in \mathbb{R};$$

where $x \in X$, $z \in Z$, $t \in \mathbb{R}$, $\lambda \in [0,1]$.

It is easy to prove that L is a Fredholm mapping of index 0, that $P : X \to \ker L$ and $Q \to Z/\operatorname{Im} L$ are projectors, and that N is L-compact on $\bar{\Omega}$ for any given open and bounded subset Ω in X.

The corresponding differential equation for the operator $Lx = \lambda N(x, \lambda)$, $\lambda \in (0,1)$, takes the form

$$\dddot{x}(t) + \lambda a\ddot{x}(t) + \lambda^2 g(\dot{x}(t-\tau(t))) + \lambda f(x(t-\tau(t))) = \lambda^2 p(t). \tag{2.2}$$

Let $x \in X$ be a solution of (2.2) for a certain $\lambda \in (0,1)$. Integrating (2.2) over $[0,\omega]$, we obtain

$$\int_0^\omega \left[\lambda g(\dot{x}(t-\tau(t))) + f(x(t-\tau(t))) - \lambda p(t)\right] dt = 0. \tag{2.3}$$

Thus, there is a point $\xi \in [0,\omega]$, such that

$$\lambda g(\dot{x}(\xi-\tau(\xi))) + f(x(\xi-\tau(\xi))) - \lambda p(\xi) = 0$$

Thus in view of condition (H1),

$$\begin{aligned} |f(x(\xi-\tau(\xi)))| &\leqslant |g(\dot{x}(\xi-\tau(\xi)))| + |p(\xi)| \\ &\leqslant K + \delta_1|\dot{x}(\xi-\tau(\xi))| + |p|_0 \leqslant K + |p|_0 + \delta_1|\dot{x}|_0\,. \end{aligned} \tag{2.4}$$

In what follows, we will prove that there is a point $t_0 \in [0,\omega]$ such that

$$|x(t_0) < |\dot{x}|_0 + D. \tag{2.5}$$

Case 1: $\delta_1 = 0$**.** If $|x(\xi-\tau(\xi))| > D$, (H1), (H2) and (2.4) ensure $K + |p|_0 < |f(x(\xi-\tau(\xi)))| \leqslant K + |p|_0$, which is a contradiction. So

$$|x(\xi-\tau(\xi))| \leqslant D. \tag{2.6}$$

Case 2: $r_1 > 0$**.** If $|x(\xi-\tau(\xi))| > D$, then $K + |p|_0 + \delta_1|x(\xi-\tau(\xi))| < |f(x(\xi-\tau(\xi)))| \leqslant K + |p|_0 + \delta_1|\dot{x}|_0$. So that

$$|x(\xi-\tau(\xi))| \leqslant |\dot{x}|_0. \tag{2.7}$$

Hence from (2.6) and (2.7), we see in either case 1 or case 2 that

$$|x(\xi-\tau(\xi))| \leqslant |\dot{x}|_0 + D.$$

Let $\xi - \tau(\xi) = 2k\pi + t_0$, where k is an integer and $t_0 \in [0,\omega]$. Then

$$|x(t_0)| = |x(\xi-\tau(\xi))| < |\dot{x}|_0 + D.$$

So (2.5) holds, and then

$$|x|_0 \leqslant |x(t_0)| + \int_0^\omega |\dot{x}(s)|ds < (\omega+1)|\dot{x}|_0 + D. \tag{2.8}$$

Let $G(\theta) = a\omega + 2\delta_1|b|_2\omega^{\frac{3}{2}} + 2\omega^2(1+\omega)(\delta_2+\theta), \theta \in [0,+\infty)$. ¿From the assumption $G(0) = a\omega + 2\delta_1|b|_2\omega^{\frac{3}{2}} + 2\omega^2(1+\omega)\delta_2 < 1$ and $G(\theta)$ is continuous on $[0,+\infty)$, we know that there must be a small constant $\theta_0 > 0$ such that $G(\theta) = a\omega + 2\delta_1|b|_2\omega^{\frac{3}{2}} + 2\omega^2(1+\omega)(\delta_2+\theta) < 1$, $\theta \in (0,\theta_0]$. Set $\varepsilon = \theta_0/2$, one can easily obtain that $a\omega + 2\delta_1|b|_2\omega^{\frac{3}{2}} + 2\omega^2(1+\omega)(\delta_2+\varepsilon) < 1$ For such a small $\varepsilon > 0$, in view of assumption (H_3), we find that there must be a constant $\rho > D$, which is independent of λ and x, such that

$$\frac{f(x)}{x} < (\delta_2+\varepsilon), \quad \text{for } x < -\rho. \tag{2.9}$$

Thus letting $\Delta_1 = \{t : t \in [0,\omega], x(t-\tau(t)) > \rho\}$, $\Delta_2 = \{t : t \in [0,\omega], x(t-\tau(t)) < -\rho\}$, $\Delta_3 = \{t : t \in [0,\omega], |x(t-\tau(t))| \leqslant \rho\}$ and $f_\rho = \sup_{|x|\leqslant\rho} f(x)$, we have

$$\int_{\Delta_2} |f(t-\tau(t))|dt < \omega(\delta_2+\varepsilon)|x|_0, \quad \int_{\Delta_3} |f(t-\tau(t))|dt \leqslant \omega f_\rho.$$

From (2.3), we have

$$\begin{aligned}\int_0^\omega f(x(t-\tau(t)))dt &= \Big(\int_{E_1} + \int_{E_2} + \int_{E_3}\Big) f(x(t-\tau(t)))dt \\ &\leqslant \int_0^\omega |g(\dot{x}(t-\tau(t)))|dt + \int_0^\omega |p(t)|dt.\end{aligned} \tag{2.10}$$

That is

$$\begin{aligned}\int_{E_1} |f(x(t-\tau(t)))|dt \leqslant \int_{E_2} |f(x(t-\tau(t)))|dt + \int_{E_3} |f(x(t-\tau(t)))|dt \\ + \int_0^\omega |g(\dot{x}(t-\tau(t)))|dt + \omega|p|_0.\end{aligned} \tag{2.11}$$

Using condition (H1), we have

$$\begin{aligned}\int_0^\omega |g(\dot{x}(t-\tau(t)))|dt &= \int_{-\tau(0)}^{\omega-\tau(\omega)} \frac{1}{1-\dot{\tau}(\nu(s))}|g(\dot{x}(s))|ds \\ &= \int_0^\omega \frac{1}{1-\dot{\tau}(\nu(s))}|g(\dot{x}(s)|ds \\ &\leqslant \int_0^\omega \frac{\delta_1}{1-\dot{\tau}(\nu(s))}|\dot{x}(s)|ds + \int_0^\omega \frac{K}{1-\dot{\tau}(\nu(s))}ds \\ &\leqslant \delta_1|b|_2\Big(\int_0^\omega |\dot{x}(s)|ds\Big)^{1/2} + |b|_2K\sqrt{\omega}.\end{aligned} \tag{2.12}$$

Thus, by (2.11) and (2.12), we have

$$\begin{aligned}\int_0^\omega |\ddot{x}(s)|ds &\leqslant a\int_0^\omega |\ddot{x}(s)|ds + \int_0^\omega |g(\dot{x}(t-\tau(t)))|dt \\ &\quad + \int_0^\omega |f(x(t-\tau(t)))|dt + \omega|p|_0 \\ &= a\int_0^\omega |\ddot{x}(s)|ds + \int_0^\omega |g(\dot{x}(t-\tau(t)))|dt \\ &\quad + \Big(\int_{\Delta_1} + \int_{\Delta_2} + \int_{\Delta_3}\Big)|f(x(t-\tau(t)))|dt + \omega|p|_0 \\ &\leqslant a\sqrt{\omega}\Big(\int_0^\omega |\ddot{x}(s)|^2ds\Big)^{1/2} + 2\delta_1|b|_2\Big(\int_0^\omega |\dot{x}(s)|^2ds\Big)^{1/2} \\ &\quad + 2\omega(\delta_2+\varepsilon)|x|_0 + 2K\sqrt{\omega}|b|_2 + 2\omega f_\rho + 2|p|_0.\end{aligned} \tag{2.13}$$

Since $x(0)=x(\omega)$, there exists $t_1\in[0,\omega]$, such that $\dot x(t_1)=0$, Hence for $t\in[0,\omega]$,

$$|\dot x|_0\leqslant\int_0^\omega|\ddot x(t)|dt\leqslant\sqrt{\omega}\Big(\int_0^\omega|\ddot x(s)|^2ds\Big)^{1/2},\tag{2.14}$$

$$\Big(\int_0^\omega|\dot x(s)|^2ds\Big)^{1/2}\leqslant\sqrt{\omega}\max_{t\in[0,\omega]}|\dot x(t)|\leqslant\omega\Big(\int_0^\omega|\ddot x(s)|^2ds\Big)^{1/2}.\tag{2.15}$$

Since $x(t)$ is periodic function, for $t\in[0,\omega]$, we have

$$|\ddot x(t)|\leqslant\int_0^\omega|\dddot x(t)|dt,\tag{2.16}$$

$$\Big(\int_0^\omega|\ddot x(s)|^2ds\Big)^{1/2}\leqslant\sqrt{\omega}\max_{t\in[0,\omega]}|\ddot x(t)|\leqslant\sqrt{\omega}\int_0^\omega|\dddot x(t)|dt.\tag{2.17}$$

Substituting (2.17) in (2.14), we obtain

$$|\dot x|_0\leqslant\omega\int_0^\omega|\dddot x(t)|dt.\tag{2.18}$$

Substituting (2.18) in (2.8),

$$|x|_0\leqslant D+\omega(1+\omega)\int_0^\omega|\dddot x(t)|dt.\tag{2.19}$$

Substituting (2.15),(2.17) and (2.19) in (2.13), and using inequality (2.16), we obtain

$$|\ddot x|_0\leqslant\int_0^\omega|\dddot x(t)|dt\leqslant\frac{2K\sqrt{\omega}|b|_2+2\omega f_\rho+2\omega|p|_0+2\omega(\delta_2+\varepsilon)D}{1-a\omega-2\delta_1|b|_2\omega^{\frac32}-2\omega^2(1+\omega)(\delta_2+\varepsilon)}\triangleq A_3.\tag{2.20}$$

Substituting (2.20) in (2.18) and (2.19), we obtain

$$|x|_0\leqslant D+\omega(1+\omega)A_3\triangleq A_1,\quad |\dot x|_0\leqslant\omega A_3\triangleq A_2.\tag{2.21}$$

Let $A_0=\max\{A_1,A_2,A_3\}$ and take $\Omega=\{x\in X:\|x\|\leqslant A_0\}$. The above priori estimates show that condition (a) of Lemma 2.3 is satisfied. If $x\in\partial\Omega\cap\ker L=\partial\Omega\cap\mathbb{R}$, then x is a constant with $x(t)=A_0$ or $x(t)=-A_0$. Then

$$\begin{aligned}QN(x,0)&=\frac1\omega\int_0^\omega\big[-a\ddot x(t)-f(x(t-\tau(t))\big]dt\\&=\frac1\omega\int_0^\omega-f(x)dt=\frac1\omega\int_0^\omega-f(\pm A_0)dt\neq0\end{aligned}$$

Finally, consider the mapping

$$H(x,\mu)=\mu x+\frac{1-\mu}{\omega}\int_0^\omega f(x)dt,\quad \mu\in[0,1].$$

Since for every $\mu\in[0,1]$ and x in the intersection of $\ker L$ and $\partial\Omega$, we have

$$xH(x,\mu)=\mu x^2+\frac{1-\mu}{\omega}\int_0^\omega xf(x)dt>0,$$

thus $H(x,\mu)$ is a homotopy. This shows that

$$\begin{aligned}\deg\{QN(x,0),\Omega\cap\ker L,0\}&=\deg\{-f(x),\Omega\cap\ker L,0\}\\&=\deg\{-x,\Omega\cap\ker L,0\}\\&=\deg\{-x,\Omega\cap R,0\}\neq0.\end{aligned}$$

All conditions in Lemma 2.3 are satisfied; therefore, (1.1) has at least one solution in Ω. $\square$

Example. Consider the equation

$$\dddot{x}(t) + \frac{1}{2\pi}\ddot{x}(t) + \frac{7}{3\pi^2}\dot{x}(t - \cos 8t)) + \frac{3}{2}e^{-(\dot{x}(t-\cos 8t))^2} + f(x(t - \cos 8t)) = \frac{1 + \sin 8t}{4}$$

where $p(t) = (1 + \sin 8t)/4$, $\tau(t) = \cos 8t$, $g(u) = \frac{7}{3\pi^2}u + \frac{3}{2}e^{-u^2}$ and

$$f(u) = \begin{cases} \frac{7}{3\pi^2}u + \frac{3}{2} + \arctan u, & \text{for } u > D, \\ \big(\frac{7}{3\pi^2} + \frac{3}{2} + \frac{\pi}{4}\big), & \text{for } |u| \leqslant D, \\ \frac{7}{3\pi^2}u - \frac{3}{2} + \arctan u, & \text{for } u < -D. \end{cases}$$

So we can chose $\delta_1 = \delta_2 = 7/(3\pi^2)$, $D = 1$, $K = 1$, $|p|_0 = 1/2$, $|b|_2 < \sqrt{\omega}$, $\omega = \pi/4$. It is easy to verify that all the assumptions in Theorem 2.1 are satisfied. Thus this equation has a periodic solution with period $\pi/4$.

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