Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 109, pp. 1–16.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

CLASSICAL SOLUTIONS FOR DISCRETE POTENTIAL BOUNDARY VALUE PROBLEMS WITH GENERALIZED LERAY-LIONS TYPE OPERATOR AND VARIABLE EXPONENT

BILA ADOLPHE KYELEM, STANISLAS OUARO, MALICK ZOUNGRANA

Communicated by Vicentiu Radulescu

Abstract. In this article, we prove the existence of solutions for some discrete nonlinear difference equations subjected to a potential boundary type condition. We use a variational technique that relies on Szulkin's critical point theory, which ensures the existence of solutions by ground state and mountain pass methods.

1. Introduction

Let us consider the positive integer T and the discrete integer domain function $p : [|0, T|] \to (1, \infty)$. We study the existence of solutions for the following class of potential boundary value problems

$$
\begin{gathered}
-\Delta(a(k-1, \Delta u(k-1))) = f(k, u(k)) \quad \text{for } k \in [|1, T|], \\
(a(0, \Delta u(0)), -a(T, \Delta u(T)) \in \partial j(u(0), u(T+1)),
\end{gathered}
\tag{1.1}
$$

where $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator. The function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is Carathéodory and $a(k, \cdot)$: $\mathbb{R} \to \mathbb{R}$ is a continuous function for all $k \in [|0, T|]$. Also, $j : \mathbb{R} \times \mathbb{R} \to (-\infty; +\infty]$ is convex, proper (i.e., $D(j) := \{z \in \mathbb{R} \times \mathbb{R} : j(z) < +\infty\} \neq \emptyset$), lower semicontinuous (in short, l.s.c.) function and ∂j denotes the subdifferential of j. Recall that for $z \in \mathbb{R} \times \mathbb{R}$, the set $\partial j(z)$ is defined by

$$
\partial j(z) = \{m \in \mathbb{R} \times \mathbb{R} : j(t) - j(z) \geqslant \langle m; t - z\rangle; \forall t \in \mathbb{R} \times \mathbb{R}\}, \tag{1.2}
$$

where $\langle \cdot; \cdot \rangle$ stands for the usual inner product in $\mathbb{R} \times \mathbb{R}$. $u : X \to \mathbb{R}$ is a function, where the space X will be defined later.

Many potential boundary type condition problems arise from physical phenomena (see [13] and the references therein). In electrostatics, for example, in a system involving conductor electrodes, the potential is often specified on electrode surfaces and one is asked to find the potential in the space of the electrodes. Such problems are called potential boundary value problems.

It is usual to note that nonlinear multivalued boundary conditions include particular cases of classical boundary conditions; these are obtained by appropriate

2010 *Mathematics Subject Classification.* 47A75, 35B38.
Key words and phrases. Potential boundary type condition; variational method; critical point; ground state method; Palais-Smale condition; mountain pass theorem.

Submitted November 21, 2016. Published April 25, 2017.

choices of j (see, e.g., [9, Ch.2]). For other choices of j yielding various boundary conditions, we refer the reader to Gasinski and Papageorgiou [6] and Jebelean and Serban [10].

The study of boundary value problems with a discrete laplacian using variational approaches was developed a few years ago. Most of the papers deal with classical boundary conditions such as Dirichlet boundary conditions (see, e.g. Agarwal et al. [1], Cabada et al. [3]), Neumann boundary condition (see, e.g. Candito and D'agui [4], Tian and Ge [21]) and Periodic boundary conditions (see, e.g. He and Chen [8], Jebelean and Serban [10]). Recently, boundary value problems with the discrete laplacian subjected to Dirichlet, Neumann or Periodic boundary conditions have been studied by Molica Bisci and Repovš [16, 17], Galewski and Glab [5], Guiro et al. [7], Koné and Ouaro [11], Mashiyev et al. [12], Mihailescu et al. [14, 15]. In [2], Bereanu et al. have made use of variational approach to obtain ground state and mountain pass solutions for the following problem

$$\begin{gathered}-\Delta_{p(k-1)}\big(u(k-1)\big) = f(k,u(k)) \quad \text{for } k\in[|1,T|],\\ (h_{p(0)}(\Delta u(0)), -h_{p(T)}(\Delta u(T)) \in \partial j(u(0), u(T+1)),\end{gathered}$$

where $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator and $\Delta_{p(\cdot)}$ is a discrete $p(\cdot)$-Laplacian operator that is

$$-\Delta_{p(k-1)}(u(k-1)) := \Delta(h_{p(k-1)}(\Delta u(k-1))),$$

with $h_{p(k)} : \mathbb{R} \to \mathbb{R}$ is defined by $h_{p(k)}(u(k)) = |u(k)|^{p(k)-2}u(k)$.

In this paper, we consider a more general forward operator which involves a Leray-Lions type operator. Therefore, a variational approach underlying ground state and mountain pass techniques for problem (1.1) is essentially used. In this view, we use some ideas and technics originated by Radulescu and Repovš [18] (see also Molica Bisci and Repovš[16, 17] and Szulkin [20]) , and combined with specific tools, due to the discrete and anisotropic character of the problem. Our paper is organized as follows: the useful preliminary results are presented in Section 2. In Section 3, we deal with the existence of a solution to problem (1.1) using ground state methods. The last Section is devoted to the existence of non trivial solutions by using mountain pass techniques.

2. Preliminaries

Our approach for the boundary value problem (1.1) relies on the critical point theory developed by Szulkin [20]. We introduce the function $p : [|0,T|] \to (1,+\infty)$, where $[|0,T|] := \{0,1,2,\dots,T\}$ and the space of functions

$$X := \{u : u : [|0,T+1|] \to \mathbb{R}\}$$

which can be endowed with the Luxemburg norm

$$\|u\|_{\sigma,p(\cdot)} = \inf\Big\{\lambda > 0 : \sigma\sum_{k=1}^{T}\frac{1}{p(k)}|\frac{u(k)}{\lambda}|^{p(k)} \le 1\Big\}.$$

Let us denote

$$p^+ := \max_{k\in[|0,T|]} p(k) \quad\text{and}\quad p^- := \min_{k\in[|0,T|]} p(k).$$

For the function a, we assume the following.

(H1) There exists $A : [|0,T|]\times\mathbb{R} \to \mathbb{R}$ with $a(k,\xi) = \partial A(k,\xi)/\partial\xi$ for all $k \in [|0,T|]$ and $A(k,0) = 0$ for all $k \in [|0,T|]$.

(H2) There exists $C_1 > 0$ such that $|a(k,\xi)| \leqslant C_1(1+|\xi|^{p(k)-1})$ for all $k \in [|0,T|]$ and all $\xi \in \mathbb{R}$.
(H3) $(a(k,\xi) - a(k,\eta)) \cdot (\xi - \eta) > 0$ for all $(\xi,\eta) \in \mathbb{R}^2$ such that $\xi \neq \eta$.
(H4) $|\xi|^{p(k)} \leq a(k,\xi) \cdot \xi \leq p(k)A(k,\xi)$, with $p : [|0,T|] \to (1,+\infty)$.

Let $\varphi : X \to \mathbb{R}$ be defined by

$$\varphi(u) = \sum_{k=1}^{T+1} A(k-1, \Delta u(k-1)) \quad \text{for all } u \in X. \tag{2.1}$$

Using the functional j, we introduce the functional $J : X \to (-\infty,+\infty]$ given by

$$J(u) = j(u(0), u(T+1)) \quad \text{for all } u \in X. \tag{2.2}$$

Note that, as j is proper, convex and l.s.c, the same properties hold for J. Let us set

$$\psi = \varphi + J. \tag{2.3}$$

Let us also define

$$F(k,t) = \int_0^t f(k,\tau)d\tau \quad \text{for all } k \in [|1,T|] \text{ and all } t \in \mathbb{R}.$$

Let us now introduce

$$\Phi(u) := -\sum_{k=1}^{T} F(k, u(k)) \quad \text{for all } u \in X. \tag{2.4}$$

The energy functional associated with the problem (1.1) is

$$I = \Phi + \psi, \tag{2.5}$$

where $\Phi \in C^1(X,\mathbb{R})$ and $\psi : X \to (-\infty,+\infty]$ are convex, proper and lower semi-continuous.

Lemma 2.1. *Let $u \in X$ and $p^+ < +\infty$ then $\|u\|_{\sigma,p(\cdot)}$ is equivalent to the norm defined by*

$$\|u\|_{p(\cdot)} = \inf\Big\{\lambda > 0; \sum_{k=1}^{T} |\frac{u(k)}{\lambda}|^{p(k)} \leq 1\Big\}.$$

Proof. We have

$$\sigma \sum_{k=1}^{T} \frac{1}{p(k)} |\frac{u(k)}{\lambda}|^{p(k)} \geq \frac{\sigma}{p^+} \sum_{k=1}^{T} |\frac{u(k)}{\lambda}|^{p(k)};$$

thus,

$$\|u\|_{\sigma,p(\cdot)} \geq \kappa_1 \|u\|_{p(\cdot)},$$

$$\sigma \sum_{k=1}^{T} \frac{1}{p(k)} |\frac{u(k)}{\lambda}|^{p(k)} \leq \frac{\sigma}{p^-} \sum_{k=1}^{T} |\frac{u(k)}{\lambda}|^{p(k)};$$

hence,

$$\|u\|_{\sigma,p(\cdot)} \leq \kappa_2 \|u\|_{p(\cdot)}.$$

Therefore,

$$\kappa_1 \|u\|_{p(\cdot)} \leq \|u\|_{\sigma,p(\cdot)} \leq \kappa_2 \|u\|_{p(\cdot)}.$$

□

Now, let us present some basic properties of the general critical point theory. Let $I : X \to \mathbb{R} \cup \{+\infty\}$ be the functional satisfying the structural hypothesis

(H5) $I = \Phi + \psi$, with $\Phi : X \to \mathbb{R}$ is a C^1 function and $\psi : X \to \mathbb{R} \cup \{+\infty\}$ a is convex, lower semicontinuous and proper function.

Definition 2.2. An element $u \in X$ satisfying (H5) is called a critical point of the functional $I : X \to \mathbb{R} \cup \{+\infty\}$ if

$$\langle \Phi'(u); v - u\rangle + \psi(v) - \psi(u) \geq 0, \quad \text{for all } v \in X.$$

Definition 2.3. The functional $I : X \to \mathbb{R} \cup \{+\infty\}$ satisfying (H5) is said to satisfy the Palais-Smale (in short, (PS)) condition in the sense of Szulkin, if, every sequence $\{u_n\} \subset X$ for which $I(u_n) \to c \in \mathbb{R}$ and

$$\langle \Phi'(u_n); v - u_n\rangle + \psi(v) - \psi(u_n) \geq -\epsilon\|v - u_n\| \quad \text{for all } v \in X \tag{2.6}$$

where $\epsilon \to 0$, possesses a convergent subsequence.

Proposition 2.4 ([20, Proposition 1.1])**.** *If I satisfies* (H5) *then, each local minimum point of I is necessarily a critical point of I.*

Theorem 2.5 ([19, Theorem 23.2])**.** *Let f be a convex function and let x be a point where f is finite. Then x^* is a subgradient of f at x if and only if $f'(x, y) \geq \langle x^*; y\rangle$ for all $y \in X$. In fact, the closure of $f'(x, y)$ as a convex function of y is the support function of the closed convex set $\partial f(x)$.*

Theorem 2.6 ([20, Theorem 3.2])**.** *Assume that I satisfies* (H5)*, the (PS) condition and*

(i) *$I(0) = 0$ and there exist α, $\rho > 0$ such that $I(u) \geq \alpha$ if $\|u\| = \rho$,*
(ii) *$I(e) \leq 0$ for some $e \in X$ with $\|e\| \geq \rho$.*

Then, I has a critical value $c \geq \alpha$ which can be characterized by

$$c = \inf_{f\in\Gamma} \sup_{t\in[0,1]} I(f(t)),$$

where $\Gamma = \{f \in C([0, 1], X) : f(0) = 0, f(1) = e\}$.

Proposition 2.7. *Assume that* (H1)–(H3) *hold. Then*

(i) *φ is convex and is in $C^1(X;\mathbb{R})$;*
(ii) *J is proper, convex and l.s.c;*
(iii) *ψ is proper, convex and l.s.c;*
(iv) *$\Phi \in C^1(X;\mathbb{R})$.*

Proof. (i) φ is well defined since, according to (H1) and (H2),

$$\begin{aligned}
|\varphi(u)| &= |\sum_{k=1}^{T+1} A(k-1, \Delta u(k-1))| \\
&\leq \sum_{k=1}^{T+1} |A(k-1, \Delta u(k-1))| \\
&\leq C \sum_{k=1}^{T+1} \|\Delta u(k-1)\| < +\infty.
\end{aligned}$$

A is convex with respect to the second variable according to (H1) and (H3). Let $\lambda \in [0,1]$. For all $u, v \in X$,

$$\varphi((1-\lambda)u + \lambda v) = \sum_{k=1}^{T+1} A(k-1, (1-\lambda)\Delta u(k-1) + \lambda \Delta v(k-1)).$$

Then

$$\begin{aligned}
\varphi((1-\lambda)u + \lambda v) &\leq \sum_{k=1}^{T+1} (1-\lambda) A(k-1, \Delta u(k-1)) + \lambda A(k-1, \Delta v(k-1)) \\
&\leq (1-\lambda)\varphi(u) + \lambda \varphi(v).
\end{aligned}$$

Therefore φ is convex. For $u, v \in X$ we have

$$\begin{aligned}
&\lim_{\delta \to 0^+} \frac{\varphi(u+\delta v) - \varphi(u)}{\delta} \\
&= \lim_{\delta \to 0^+} \sum_{k=1}^{T+1} \frac{A(k-1, \Delta u(k-1) + \delta \Delta v(k-1)) - A(k-1, \Delta u(k-1))}{\delta} \\
&= \sum_{k=1}^{T+1} \lim_{\delta \to 0^+} \frac{A(k-1, \Delta u(k-1) + \delta \Delta v(k-1)) - A(k-1, \Delta u(k-1))}{\delta} \\
&= \sum_{k=1}^{T+1} a(k-1, \Delta u(k-1)) \Delta v(k-1).
\end{aligned}$$

Therefore,

$$\langle \varphi'(u); v \rangle = \sum_{k=1}^{T+1} a(k-1, \Delta u(k-1)) \Delta v(k-1).$$

Every limit is finite because of the continuity of $A(k, \cdot)$. The continuity of the derivative comes from the continuity of $a(k, \cdot)$. Hence, φ is in $C^1(X; \mathbb{R})$.

(ii) Note that as j is proper, convex and l.s.c, the same properties hold for J.

(iii) Since φ and J are convex then, ψ is convex. Suppose that ψ can take the value $-\infty$; then, in this case, $J = \psi - \varphi$ can take the value $-\infty$, which is not possible. Therefore, ψ cannot take the value $-\infty$. Hence, ψ is proper. Also,

$$J(u) \leq \lim_{y \to u} \inf J(y).$$

Then

$$\varphi(u) + J(u) \leq \lim_{y \to u} \inf J(y) + \varphi(u) \leq \lim_{y \to u} \inf J(y) + \lim_{y \to u} \inf \varphi(y) \leq \lim_{y \to u} \inf \psi(y).$$

Therefore $\psi(u) \leq \lim_{y \to u} \inf \psi(y)$. Hence, ψ is l.s.c.

(iv) $|\Phi(u)| = |\sum_{k=1}^{T} F(k, u(k))| < \infty$ since F is continuous. Then Φ is well defined. By definition, Φ is derivable and its derivative is continuous; hence $\Phi \in C^1(X; \mathbb{R})$. Moreover,

$$\begin{aligned}
\langle \Phi'(u); y \rangle &= \lim_{\delta \to 0^+} \frac{\phi(u+\delta y) - \phi(u)}{\delta} \\
&= -\lim_{\delta \to 0^+} \sum_{k=1}^{T} \frac{F(k, u(k) + \delta y(k)) - F(k, u(k))}{\delta}
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{k=1}^{T} \lim_{\delta\to 0^+} \frac{F(k,u(k)+\delta y(k)) - F(k,u(k))}{\delta} \\
&= -\sum_{k=1}^{T} f(k,u(k))y(k) \quad \text{for all } u,y \in X.
\end{aligned}
$$

□

Now, let us claim the following important result.

Proposition 2.8. *If $u \in X$ is a critical point of the functional I in the sense that*

$$\langle \Phi'(u); y-u\rangle + \psi(y) - \psi(u) \geqslant 0, \quad \textit{for all } y \in X; \tag{2.7}$$

then u is a classical solution of (1.1).

Proof. Since $\langle \Phi'(u); y-u\rangle + \psi(y) - \psi(u) \geqslant 0$, we can take $y = u + sw$ for all $s > 0$ in (2.7). Dividing (2.7) by s and letting $s \to 0^+$, we obtain

$$\langle \Phi'(u); w\rangle + \langle \varphi'(u); w\rangle + J'(u;w) \geq 0 \quad \forall w \in X, \tag{2.8}$$

where $J'(u;w)$ is the directional derivative of the convex function J at u in the direction of w. Since

$$J(u) = j(u(0), u(T+1)),$$

we obtain from (2.8),

$$\langle \Phi'(u); w\rangle + \langle \varphi'(u); w\rangle + j'((u(0),u(T+1));(w(0),w(T+1))) \geqslant 0, \quad \text{for all } w \in X.$$

Since

$$\langle \Phi'(u); w\rangle = -\sum_{k=1}^{T} f(k,u(k))w(k) \quad \text{for all } u,w \in X$$

and

$$\langle \varphi'(u); w\rangle = \sum_{k=1}^{T+1} a(k-1,\Delta u(k-1))\Delta w(k-1) \quad \text{for all } u,w \in X,$$

it follows that

$$
\begin{aligned}
&-\sum_{k=1}^{T} f(k,u(k))w(k) + \sum_{k=1}^{T+1} a(k-1,\Delta u(k-1))\Delta w(k-1) \\
&+ j'((u(0),u(T+1));(w(0),w(T+1))) \geqslant 0.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&-\sum_{k=1}^{T} f(k,u(k))w(k) + \sum_{k=1}^{T+1} a(k-1,\Delta u(k-1))[w(k)-w(k-1)] \\
&+ j'((u(0),u(T+1));(w(0),w(T+1))) \geqslant 0.
\end{aligned}
$$

Then

$$
\begin{aligned}
&-\sum_{k=1}^{T} f(k,u(k))w(k) + \sum_{k=1}^{T+1} a(k-1,\Delta u(k-1))w(k) \\
&-\sum_{k=1}^{T+1} a(k-1,\Delta u(k-1))w(k-1) + j'((u(0),u(T+1));(w(0),w(T+1))) \geqslant 0
\end{aligned}
$$

such that

$$\begin{aligned}
&-\sum_{k=1}^{T} f(k,u(k))w(k) + a(T,\Delta u(T))w(T+1) + \sum_{k=1}^{T} a(k-1,\Delta u(k-1))w(k) \\
&-\sum_{k=0}^{T} a(k,\Delta u(k))w(k) + j'((u(0),u(T+1));(w(0),w(T+1))) \geqslant 0,
\end{aligned}$$

which leads to

$$\begin{aligned}
&-\sum_{k=1}^{T} f(k,u(k))w(k) + a(T,\Delta u(T))w(T+1) - a(0,\Delta u(0))w(0) \\
&+\sum_{k=1}^{T} a(k-1,\Delta u(k-1))w(k) \\
&-\sum_{k=1}^{T} a(k,\Delta u(k))w(k) + j'((u(0),u(T+1));(w(0),w(T+1))) \geqslant 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&-\sum_{k=1}^{T} f(k,u(k))w(k) + a(T,\Delta u(T))w(T+1) - a(0,\Delta u(0))w(0) \\
&-\sum_{k=1}^{T} [a(k,\Delta u(k)) - a(k-1,\Delta u(k-1))]w(k) \\
&+ j'((u(0),u(T+1));(w(0),w(T+1))) \geqslant 0.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&-\sum_{k=1}^{T} f(k,u(k))w(k) - \sum_{k=1}^{T} \Delta a(k-1,\Delta u(k-1))w(k) \\
&+ a(T,\Delta u(T))w(T+1) - a(0,\Delta u(0))w(0) \\
&+ j'((u(0),u(T+1));(w(0),w(T+1))) \geqslant 0.
\end{aligned}$$

As $w \in X$ is arbitrarily chosen, we can take $w(0)=w(T+1)=0$ to obtain

$$\sum_{k=1}^{T} \Big(-\Delta a(k-1,\Delta u(k-1))\Big) w(k) = \sum_{k=1}^{T} f(k,u(k))w(k).$$

Hence, it follows that

$$-\Delta a(k-1,\Delta u(k-1)) = f(k,u(k)) \quad \text{for all } k \in [|1,T|]. \tag{2.9}$$

It remains to show that $(a(0,\Delta u(0)), -a(T,\Delta u(T))) \in \partial j(u(0),u(T+1))$. One has

$$\begin{aligned}
&-\sum_{k=1}^{T} f(k,u(k))w(k) - \sum_{k=1}^{T} \Delta a(k-1,\Delta u(k-1))w(k) + a(T,\Delta u(T))w(T+1) \\
&- a(0,\Delta u(0))w(0) + j'((u(0),u(T+1)),(w(0),w(T+1))) \geqslant 0
\end{aligned}$$

and

$$\sum_{k=1}^{T}\Big(-\Delta a(k-1,\Delta u(k-1))\Big)=\sum_{k=1}^{T}f(k,u(k)).$$

Therefore,

$$j'((u(0),u(T+1));(w(0),w(T+1)))\geqslant -a(T,\Delta u(T))w(T+1)+a(0,\Delta u(0))w(0),$$

for all $w\in X$. Taking $w\in X$ with $w(0)=p$ and $w(T+1)=q$, where $p,q\in\mathbb{R}$ are arbitrarily chosen, it follows that

$$j'((u(0),u(T+1));(p,q))\geqslant -a(T,\Delta u(T))q+a(0,\Delta u(0))p,\quad \text{for all } (p,q)\in\mathbb{R}^2$$

which by Theorem 2.5 implies

$$(a(0,\Delta u(0)),-a(T,\Delta u(T))\in\partial j(u(0);u(T+1))).$$

□

Lemma 2.9. *Let $u\in X$ and $p^+<+\infty$. Then, the following properties hold:*

(i) $\|u\|_{\sigma,p(\cdot)}<1$ *implies*

$$\|u\|_{\sigma,p(\cdot)}^{p^+}\leq\sigma\sum_{k=1}^{T}\frac{|u(k)|^{p(k)}}{p(k)}\leq\|u\|_{\sigma,p(\cdot)}^{p^-};$$

(ii) $\|u\|_{\sigma,p(\cdot)}>1$ *implies*

$$\|u\|_{\sigma,p(\cdot)}^{p^-}\leq\sigma\sum_{k=1}^{T}\frac{|u(k)|^{p(k)}}{p(k)}\leq\|u\|_{\sigma,p(\cdot)}^{p^+}.$$

Proof. Suppose that $\|u\|_{\sigma,p(\cdot)}>1$. Then

$$\begin{aligned}\sigma\sum_{k=1}^{T}\frac{|u(k)|^{p(k)}}{p(k)}&=\sigma\sum_{k=1}^{T}\frac{1}{p(k)}\Big|\frac{u(k)}{\|u\|_{\sigma,p(\cdot)}}\|u\|_{\sigma,p(\cdot)}\Big|^{p(k)}\\&\leq\sigma\sum_{k=1}^{T}\frac{1}{p(k)}\Big|\frac{u(k)}{\|u\|_{\sigma,p(\cdot)}}\Big|^{p(k)}\|u\|_{\sigma,p(\cdot)}^{p^+}.\end{aligned}$$

Therefore,

$$\frac{\sigma}{\|u\|_{\sigma,p(\cdot)}^{p^+}}\sum_{k=1}^{T}\frac{|u(k|^{p(k)}}{p(k)}\leq\sigma\sum_{k=1}^{T}\frac{1}{p(k)}\Big|\frac{u(k)}{|\|u\|_{\sigma,p(\cdot)}}\Big|^{p(k)}.$$

Using the same arguments, one has

$$\sigma\sum_{k=1}^{T}\frac{|u(k)|^{p(k)}}{p(k)}\geq\sigma\sum_{k=1}^{T}\frac{1}{p(k)}\Big|\frac{u(k)}{\|u\|_{\sigma,p(\cdot)}}\Big|^{p(k)}\|u\|_{\sigma,p(\cdot)}^{p^-}.$$

Then

$$\frac{\sigma}{\|u\|_{\sigma,p(\cdot)}^{p^-}}\sum_{k=1}^{T}\frac{|u(k)|^{p(k)}}{p(k)}\geq\sigma\sum_{k=1}^{T}\frac{1}{p(k)}\Big|\frac{u(k)}{\|u\|_{\sigma,p(\cdot)}}\Big|^{p(k)}.$$

Finally,

$$\frac{\sigma}{\|u\|_{\sigma,p(\cdot)}^{p^+}}\sum_{k=1}^{T}\frac{|u(k)|^{p(k)}}{p(k)}\leq\sigma\sum_{k=1}^{T}\frac{1}{p(k)}\Big|\frac{u(k)}{\|u\|_{\sigma,p(\cdot)}}\Big|^{p(k)}\leq\frac{\sigma}{\|u\|_{\sigma,p(\cdot)}^{p^-}}\sum_{k=1}^{T}\frac{|u(k)|^{p(k)}}{p(k)},$$

which is equivalent to

$$\frac{\sigma}{\|u\|_{\sigma,p(\cdot)}^{p^+}}\sum_{k=1}^{T}\frac{|u(k)|^{p(k)}}{p(k)}\le 1\le \frac{\sigma}{\|u\|_{\sigma,p(\cdot)}^{p^-}}\sum_{k=1}^{T}\frac{|u(k)|^{p(k)}}{p(k)}.$$

Thus,

$$\|u\|_{\sigma,p(\cdot)}^{p^-}\le \sigma\sum_{k=1}^{T}\frac{|u(k)|^{p(k)}}{p(k)}\le \|u\|_{\sigma,p(\cdot)}^{p^+}.$$

Suppose now that $\|u\|_{\sigma,p(\cdot)}<1$. Then

$$\begin{aligned}\sigma\sum_{k=1}^{T}\frac{|u(k)|^{p(k)}}{p(k)}&=\sigma\sum_{k=1}^{T}\frac{1}{p(k)}\Big|\frac{u(k)}{\|u\|_{\sigma,p(\cdot)}}\|u\|_{\sigma,p(\cdot)}\Big|^{p(k)}\\&\le \sigma\sum_{k=1}^{T}\frac{1}{p(k)}\Big|\frac{u(k)}{\|u\|_{\sigma,p(\cdot)}}\Big|^{p(k)}\|u\|_{\sigma,p(\cdot)}^{p^-}.\end{aligned}$$

Therefore,

$$\frac{\sigma}{\|u\|_{\sigma,p(\cdot)}^{p^-}}\sum_{k=1}^{T}\frac{|u(k)|^{p(k)}}{p(k)}\le \sigma\sum_{k=1}^{T}\frac{1}{p(k)}\Big|\frac{u(k)}{\|u\|_{\sigma,p(\cdot)}}\Big|^{p(k)}.$$

One also has

$$\sigma\sum_{k=1}^{T}\frac{|u(k)|^{p(k)}}{p(k)}\ge \sigma\sum_{k=1}^{T}\frac{1}{p(k)}\Big|\frac{u(k)}{\|u\|_{\sigma,p(\cdot)}}\Big|^{p(k)}\|u\|_{\sigma,p(\cdot)}^{p^+}.$$

Thus,

$$\frac{\sigma}{\|u\|_{\sigma,p(\cdot)}^{p^+}}\sum_{k=1}^{T}\frac{|u(k)|^{p(k)}}{p(k)}\ge \sigma\sum_{k=1}^{T}\frac{1}{p(k)}\Big|\frac{u(k)}{\|u\|_{\sigma,p(\cdot)}}\Big|^{p(k)}.$$

So,

$$\frac{\sigma}{\|u\|_{\sigma,p(\cdot)}^{p^-}}\sum_{k=1}^{T}\frac{|u(k)|^{p(k)}}{p(k)}\le \sigma\sum_{k=1}^{T}\frac{1}{p(k)}\Big|\frac{u(k)}{\|u\|_{\sigma,p(\cdot)}}\Big|^{p(k)}\le \frac{\sigma}{\|u\|_{\sigma,p(\cdot)}^{p^+}}\sum_{k=1}^{T}\frac{|u(k)|^{p(k)}}{p(k)},$$

which is equivalent to

$$\frac{\sigma}{\|u\|_{\sigma,p(\cdot)}^{p^-}}\sum_{k=1}^{T}\frac{|u(k)|^{p(k)}}{p(k)}\le 1\le \frac{\sigma}{\|u\|_{\sigma,p(\cdot)}^{p^+}}\sum_{k=1}^{T}\frac{|u(k)|^{p(k)}}{p(k)}.$$

We conclude that

$$\|u\|_{\sigma,p(\cdot)}^{p^+}\le \sigma\sum_{k=1}^{T}\frac{|u(k)|^{p(k)}}{p(k)}\le \|u\|_{\sigma,p(\cdot)}^{p^-}.$$

$\square$

3. Proof of the existence of classical solutions by ground state methods

We begin with a result which states that the energy functional I has a minimum point in X provided that the potential of the nonlinearity f lies asymptotically on the left of the first eigenvalue like constant

$$\lambda_1 := \inf\Big\{\Big(\sum_{k=1}^{T+1}\frac{1}{p(k-1)}|\Delta u(k-1)|^{p(k-1)}\Big)/\Big(\sum_{k=1}^{T}\frac{1}{p(k)}|u(k)|^{p(k)}\Big): \\ u\in X-\{0\} \text{ and } \big(u(0),u(T+1)\big)\in D(j)\Big\}.$$

The existence result will be obtained under the assumption that $\lambda_1 > 0$.

Theorem 3.1. *Assume that* (H1)–(H4) *hold and* $\lambda_1 > 0$. *Also assume that*

$$\lim_{|t|\to\infty}\sup\frac{p(k)F(k,t)}{|t|^{p(k)}} < \lambda_1, \quad \text{for all } k\in[|1,T|]. \tag{3.1}$$

Then, problem (1.1) *has at least one classical solution which minimizes I on X.*

Proof. **Step 1:** We first show that I is sequentially lower semicontinuous on X. Indeed, from Proposition 2.7, the functional ψ is lower semicontinuous and the function Φ is C^1 on X. Therefore, the functional I is sequentially lower semicontinuous on X.

Step 2: We prove that I is bounded from below and coercive on X. One the one hand, using (3.1), one obtains the existence of some constants $\alpha > 0$ and $\rho > 1$ such that

$$F(k,t) \le \frac{\lambda_1-\alpha}{p(k)}|t|^{p(k)} \quad \text{for all } k\in[|1,T|] \text{ and all } t\in\mathbb{R} \text{ with } |t|>\rho.$$

On the other hand, by the continuity of $F(k,.)$ over $[-\rho,\rho]$, there is a constant $M_\rho > 0$ such that

$$|F(k,t)| \le M_\rho, \quad \text{for all } k\in[|1,T|] \text{ and all } t\in[-\rho,\rho].$$

Hence, we infer that

$$F(k,t) \le M_\rho + \frac{\lambda_1-\alpha}{p(k)}|t|^{p(k)}, \quad \text{for all } (k,t)\in[|1,T|]\times\mathbb{R}.$$

To prove the coercivity of I, we use the above inequality to obtain for all $(k,t)\in [1;T]\times\mathbb{R}$,

$$-\sum_{k=1}^{T}F(k,u(t)) \ge -M_\rho T - (\lambda_1-\alpha)\sum_{k=1}^{T}\frac{|u(k)|^{p(k)}}{p(k)}.$$

It follows that

$$\begin{aligned}
I(u) &\ge \varphi(u) - M_\rho T - (\lambda_1-\alpha)\sum_{k=1}^{T}\frac{|u(k)|^{p(k)}}{p(k)} + J(u)\\
&\ge \varphi(u) - M_\rho T - \lambda_1\sum_{k=1}^{T}\frac{|u(k)|^{p(k)}}{p(k)} + \alpha\sum_{k=1}^{T}\frac{|u(k)|^{p(k)}}{p(k)} + J(u)\\
&\ge \varphi(u) - M_\rho T - \sum_{k=1}^{T+1}\frac{1}{p(k-1)}|\Delta u(k-1)|^{p(k-1)} + \alpha\sum_{k=1}^{T}\frac{|u(k)|^{p(k)}}{p(k)} + J(u).
\end{aligned}$$

From hypothesis (H4),

$$A(k-1,\Delta u(k-1)) \geq \frac{1}{p(k-1)}|\Delta u(k-1)|^{p(k-1)},$$

which leads to

$$\varphi(u) \geq \sum_{k=1}^{T+1} \frac{1}{p(k-1)}|\Delta u(k-1)|^{p(k-1)}. \tag{3.2}$$

Hence, if $\|u\|_{\sigma,p(\cdot)} > 1$, one makes use of Lemma 2.9 and (3.2) to obtain

$$\begin{aligned} I(u) &\geq \sum_{k=1}^{T+1} \frac{1}{p(k-1)}|\Delta u(k-1)|^{p(k-1)} - M_\rho T \\ &\quad - \sum_{k=1}^{T+1} \frac{1}{p(k-1)}|\Delta u(k-1)|^{p(k-1)} + \sigma \sum_{k=1}^{T} \frac{|u(k)|^{p(k)}}{p(k)} + J(u) \\ &\geq -M_\rho T + \sigma \sum_{k=1}^{T} \frac{|u(k)|^{p(k)}}{p(k)} + J(u) \\ &\geq -M_\rho T + \|u\|_{\sigma,p(\cdot)}^{p^-} + J(u). \end{aligned}$$

Since j is convex and l.s.c, it is bounded from below by an affine functional. Therefore, using $J(u) = j(u(0), u(T))$, there are constants m_1, m_2, $m_3 \geq 0$ such that

$$\begin{aligned} I(u) &\geq -M_\rho T + \|u\|_{\sigma,p(\cdot)}^{p^-} - m_1|u(0)| - m_2|u(T+1)| - m_3 \\ &\geq \|u\|_{\sigma,p(\cdot)}^{p^-} - m_1|u(0)| - m_2|u(T+1)| - C_1, \quad \text{where } C_1 = M_\rho T + m_3 \\ &\geq \|u\|_{\sigma,p(\cdot)}^{p^-} - C_2\|u\|_\infty - C_1, \quad \text{where } C_2 = m_1 + m_2. \end{aligned}$$

Also, any norm on X is equivalent to $\|\cdot\|_{\sigma,p(\cdot)}$. Then, there exists $C_3 > 0$ such that

$$I(u) \geq \|u\|_{\sigma,p(\cdot)}^{p^-} - C_3\|u\|_{\sigma,p(\cdot)} - C_1.$$

Consequently, $I(u) \to +\infty$ as $\|u\|_{\sigma,p(\cdot)} \to \infty$. Therefore, I is coercive on X.

Step 3: We now show that the functional I is bounded from below. For that, let $\|u\|_{\sigma,p(\cdot)} < 1$. By (H4) and Lemma 2.9, we obtain

$$\begin{aligned} I(u) &\geq \varphi(u) - M_\rho T + J(u) \\ &\geq -M_\rho T + \|u\|_{\sigma,p(\cdot)}^{p^+} + J(u) \\ &\geq -M_\rho T + \|u\|_{\sigma,p(\cdot)}^{p^+} - m_1|u(0)| - m_2|u(T+1)| - m_3 \\ &\geq \|u\|_{\sigma,p(\cdot)}^{p^+} - K_1\|u\|_\infty - K', \end{aligned}$$

where $K' = M_\rho T + m_3$. Since any norm on X is equivalent to $\|\cdot\|_{\sigma,p(\cdot)}$, there exists K'' such that

$$\begin{aligned} I(u) &\geq \|u\|_{\sigma,p(\cdot)}^{p^+} - K''\|u\|_{\sigma,p(\cdot)} - K' \\ &\geq -K''\|u\|_{\sigma,p(\cdot)} - K' \\ &\geq -K'' - K' > -\infty. \end{aligned}$$

Therefore, I is bounded from below. Finally, we conclude that I is sequentially lower semicontinuous, bounded from below and coercive on the real Banach space

X. Thus, I attains its infimum at some $u \in X$. Using now Proposition 2.4 and the Proposition 2.8, on obtains that the problem (1.1) has at least one solution on X. □

4. Existence of classical solutions using mountain pass methods

In this section we are concerned with the existence of non trivial solutions to problem (1.1). The main tool in obtaining such results will be [20, Theorem 3.2].

Theorem 4.1. *Assume* (H1)–(H5) *hold. Also assume that* $\lambda_1 > 0$ *and that there exist constants* $\theta > p^+$, K, $M > 0$ *such that*

- (H6) $j(0,0) = 0$;
- (H7) $j'(z,z) \le \theta j(z) + K, \forall z \in D(j)$;
- (H8) $\lim_{|t|\to 0} \sup \frac{p(k)F(k,t)}{|t|^{p(k)}} < \lambda_1$, *for all* $k \in [|1,T|]$;
- (H9) $0 < \theta F(k,t) \le tf(k,t)$ *for all* $k \in [|1,T|]$ *with* $|t| > M$.

Then, there exists a non trivial solution $u \in X$ *to problem* (1.1).

Proof. **Step 1:** We show that the functional I satisfying (H5) satisfies the (PS) condition in the sense of Szulkin on $(X, \|\cdot\|_{\sigma,p(\cdot)})$. So, let $\{u_n\} \subset X$ be a sequence for which, $I(u_n) \to c \in \mathbb{R}$ and (2.6) hold, with $\epsilon_n \to 0$. For this purpose, since X is a finite dimensional space, it is sufficient to prove that $\{u_n\}$ is bounded. We may assume that $\{u_n\} \subset D(I) = D(J)$ and $\|u_n\|_{\sigma,p(\cdot)} > 1$ for all $n \in \mathbb{N}$. By (H7) and (2.2), it follows that

$$J(v) - \frac{1}{\theta}J'(v;v) \ge -K_1, \quad \text{for all } v \in D(J), \tag{4.1}$$

with $K_1 = \frac{K}{\theta}$. Using the relation (H9), one deduces that for all $n \in \mathbb{N}$,

$$\sum_{k=1,|u(k)|>M}^{T} [\theta F(k,u_n(k)) - u_n(k)f(k,u_n(k))] \le 0.$$

Consequently,

$$\begin{aligned}
-\Phi(u_n) + \frac{1}{\theta}\langle \Phi'(u_n); u_n\rangle &= \frac{1}{\theta}\sum_{k=1}^{T}[\theta F(k,u_n(k)) - u_n(k)f(k,u_n(k))] \\
&= \frac{1}{\theta}\sum_{k=1,|u_n(k)|>M}^{T}[\theta F(k,u_n(k)) - u_n(k)f(k,u_n(k))] \\
&\quad + \frac{1}{\theta}\sum_{k=1,|u_n(k)|\le M}^{T}[\theta F(k,u_n(k)) - u_n(k)f(k,u_n(k))] \\
&\le \frac{1}{\theta}\sum_{k=1,|u_n(k)|\le M}^{T}[\theta F(k,u_n(k)) - u_n(k)f(k,u_n(k))] \\
&\le \frac{1}{\theta}\sum_{k=1}^{T}\max_{|x|\le M}|\theta F(k,x) - xf(k;x)| =: C_3,
\end{aligned}$$

where C_3 is a positive constant. Therefore, one can write

$$-\Phi(u_n) + \frac{1}{\theta}\langle \Phi'(u_n); u_n\rangle \le C_3. \tag{4.2}$$

Since the real sequence $(I(u_n))_{n\in\mathbb{N}}$ converges to the real number c, it is clear that, there is a constant $C_4 > 0$, such that

$$|I(u_n)| \leq C_4, \quad \text{for all } n \in \mathbb{N}. \tag{4.3}$$

Furthermore, setting $v = u_n + su_n$ in (2.6), dividing by $s > 0$ and letting $s \to 0^+$, one obtains

$$\langle \Phi'(u_n); u_n\rangle + \langle \varphi'(u_n); u_n\rangle + J'(u_n; u_n) \geq -\epsilon_n \|u_n\| \quad \text{for all } n \in \mathbb{N}. \tag{4.4}$$

Using (4.3) and (4.4), we deduce that

$$\begin{aligned}
&C_4 + \frac{\epsilon_n}{\theta}\|u_n\|_{\sigma,p(\cdot)} \\
&\geq \Phi(u_n) + \varphi(u_n) + J(u_n) + \frac{\epsilon_n}{\theta}\|u_n\|_{\sigma,p(\cdot)} \\
&\geq \Phi(u_n) - \frac{1}{\theta}\langle \Phi'(u_n); u_n\rangle + \varphi(u_n) - \frac{1}{\theta}\langle \varphi'(u_n); u_n\rangle + J(u_n) - \frac{1}{\theta}J'(u_n; u_n)
\end{aligned}$$

and by (4.1), (4.2) and (4.3), it follows that

$$K_1 + C_3 + C_4 + \frac{\epsilon_n}{\theta}\|u_n\|_{\sigma,p(\cdot)} \geq \varphi(u_n) - \frac{1}{\theta}\langle \varphi'(u_n); u_n\rangle,$$

while

$$\begin{aligned}
&\varphi(u_n) - \frac{1}{\theta}\langle \varphi'(u_n); u_n\rangle \\
&= \sum_{k=1}^{T+1} A(k-1, \Delta u_n(k-1)) - \frac{1}{\theta}\sum_{k=1}^{T+1} a(k-1, \Delta u_n(k-1))\Delta u_n(k-1).
\end{aligned}$$

We use (H4) to obtain

$$\varphi(u_n) - \frac{1}{\theta}\langle \varphi'(u_n); u_n\rangle \geq \Big(\frac{-1}{\theta} + \frac{1}{p^+}\Big)\sum_{k=1}^{T+1} |\Delta u_n(k-1)|^{p(k-1)}.$$

So,

$$K_1 + C_3 + C_4 + \frac{\epsilon_n}{\theta}\|u_n\|_{\sigma,p} \geq (\frac{-1}{\theta} + \frac{1}{p^+})\sum_{k=1}^{T+1} |\Delta u_n(k-1)|^{p(k-1)}. \tag{4.5}$$

Since

$$\lambda_1 \leq \frac{\sum_{k=1}^{T+1} \frac{1}{p(k-1)}|\Delta u(k-1)|^{p(k-1)}}{\sum_{k=1}^{T} \frac{1}{p(k)}|u(k)|^{p(k)}},$$

we have

$$\lambda_1 \sum_{k=1}^{T} \frac{1}{p(k)}|u(k)|^{p(k)} \leq \sum_{k=1}^{T+1} \frac{1}{p(k-1)}|\Delta u(k-1)|^{p(k-1)}.$$

Hence,

$$K_1 + C_3 + C_4 + \frac{\epsilon_n}{\theta}\|u_n\|_{\sigma,p} \geq (\frac{-1}{\theta} + \frac{1}{p^+})\lambda_1 \sum_{k=1}^{T} \frac{1}{p(k)}|u(k)|^{p(k)}$$

and, from Lemma 2.9, we deduce that

$$K_1 + C_3 + C_4 + \frac{\epsilon_n}{\theta}\|u_n\|_{\sigma,p(\cdot)} \geq \Big(\frac{-1}{\theta} + \frac{1}{p^+}\Big)\frac{\lambda_1}{\sigma}\|u\|_{\sigma,p(\cdot)}^{p^-}.$$

Moreover, $\theta > p^+$. Then, we infer that the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded.

Step 2: We show that I has a "mountain pass geometry". From (H6), it is clear that

$$I(0) = \Phi(0) + \varphi(0) + J(0) = 0.$$

Using (H8), we have

$$\lim_{|u|\to 0} \sup \frac{p(k)F(k,u(k))}{|u(k)|^{p(k)}} < \lambda_1.$$

This leads to the existence of $\epsilon, \beta > 0$ such that

$$F(k,t) < \frac{\lambda_1 - \epsilon}{p(k)} |t|^{p(k)} \quad \text{with } |t| < \beta.$$

Consequently,

$$\Phi(u) \geq -(\lambda_1 - \epsilon) \sum_{k=1}^{T} \frac{1}{p(k)} |u(k)|^{p(k)}, \tag{4.6}$$

for all $u \in X - \{0\}$, $u(0) = u(T+1)$ and $|u| < \beta$. Using again hypothesis (H4) and (4.6), we can write

$$\begin{aligned} \Phi(u) + \varphi(u) &\geq -(\lambda_1 - \epsilon) \sum_{k=1}^{T} \frac{1}{p(k)} |u(k)|^{p(k)} + \sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\Delta u(k-1)|^{p(k-1)} \\ &\geq \epsilon \sum_{k=1}^{T} \frac{1}{p(k)} |u(k)|^{p(k)}. \end{aligned}$$

According to (4.6) and (H6) we have $J(u) = j(u(0), u(T+1)) = j(0,0) = 0$. Therefore, for $\beta < 1$,

$$\Phi(u) + \varphi(u) + J(u) \geq \epsilon \sum_{k=1}^{T} \frac{1}{p(k)} |u(k)|^{p(k)} \geq \frac{\epsilon}{\sigma} \|u\|_{\sigma,p(\cdot)}^{p^+}.$$

Hence, choosing $\|u\|_{\sigma,p(\cdot)}^{p^+} = \beta$ which is equivalent to $\|u\|_{\sigma,p(\cdot)} = \beta^{\frac{1}{p^+}}$, then $I(u) \geq \alpha$ with $\alpha = \frac{\epsilon}{\sigma}\beta$.

Coming back to relation (H9) and taking $|u|$ big enough, we have

$$\frac{f(k,u(k))}{F(k,u(k))} \geq \frac{\theta}{u}.$$

So, $F(k,u(k)) \geq cu^\theta$ for $|u|$ big enough. Thus, $F(k,u(k)) \geq cu^\theta - K$, for all $u > 0$.

One can use (H1) to say that

$$A(k,\xi) = \int_0^\xi a(k,\lambda) d\lambda.$$

Using (H2), we have the existence of a real $C_1 > 0$ such that

$$|a(k,\xi)| \leqslant C_1(1 + |\xi|^{p(k)-1}) \quad \text{for all } k \in [|0,T|] \text{ and all } \xi \in \mathbb{R}.$$

Therefore,

$$\begin{aligned} \int_0^\xi |a(k,\lambda)| d\lambda &\leq C_1 \int_0^\xi (1 + |\lambda|^{p(k)-1}) d\lambda \\ &\leq C_1[\lambda]_0^\xi + C_1[\frac{\lambda^{p(k)}}{p(k)}]_0^\xi \end{aligned}$$

$$\le C_1|\xi| + C_1\frac{|\xi|^{p(k)}}{p(k)}.$$

One deduces that

$$\begin{aligned}\varphi(u) &= \sum_{k=1}^{T+1} A(k-1, \Delta u(k-1))\\ &\le C_1 \sum_{k=1}^{T+1} |\Delta u(k-1)| + C_1 \sum_{k=1}^{T+1} \frac{|\Delta u(k-1)|^{p(k)}}{p(k)}.\end{aligned}$$

Let $u_0 \in X - \{0\}$ be such that $u_0(0) = u_0(T+1) = 0$ and $\|u_0\|_{\sigma;p(\cdot)} > 1$. From (H6), we have that $J(su_0) = 0$ for all $s \in \mathbb{R}$. Then

$$\begin{aligned}I(su_0) &= \Phi(su_0) + \varphi(su_0) + J(su_0)\\ &= -\sum_{k=1}^{T} F(k, su_0(k)) + \sum_{k=1}^{T+1} A(k-1, \Delta su_0(k-1)) + 0\\ &\le \sum_{k=1}^{T}(K - c|su_0(k)|^\theta) + \sum_{k=1}^{T+1} C_1\Big(|s\Delta u_0(k-1)| + \frac{|s\Delta u_0(k-1)|^{p(k)}}{p(k)}\Big)\\ &= TK + \sum_{k=1}^{T}\Big[C_1'|su_0(k)| - c|su_0(k)|^\theta\Big] + \frac{C_2'}{\sigma}\|su_0\|_{\sigma,p(\cdot)}^{p^+}\\ &\le TK + C_1''\|su_0\|_\infty - cs^\theta\|u_0\|_\infty^\theta + C_2'' s^{p+}\|u_0\|_\infty \to -\infty\end{aligned}$$

as $s \to +\infty$ because $\theta > p^+$. Hence, we can choose s large enough such that $I(su_0) \le 0$ and $\|su_0\|_{\sigma,p(\cdot)} > \beta$. Using [20, Theorem 3.2] deduce that problem (1.1) has at least one non trivial solution. □

References

[1] Agarwal, R. P.; Perera, K.; O'Regan, D.; *Multiple positive solutions of singular discrete p-Laplacian problems via variational methods*, Adv. Differ. Equ., **(2)** (2005), 93-99.

[2] Bereanu, C.; Jebelean, P.; Serban, C.; *Ground state and mountain pass solutions for discrete $p(\cdot)$-Laplacian.* Bound. Value Probl., **104** (2012), pp. 1-13.

[3] Cabada, A.; Iannizzotto, A.; Tersian, S.; *Multiple solutions for discrete boundary value problems.* J. Math. Anal. Appl. **356** (2009), 418-428.

[4] Candito, P.; D'Agui, G.; *Three solutions for a discrete nonlinear Neumann problem involving p-Laplacian.* Adv. Differ.Equ. **2010** (2010), 862016.

[5] Galewski, M.; Glab, S.; *On the discrete boundary value problem for anisotropic equation*, J. Math. Anal. Appl., **386** (2012), 956-965.

[6] Gasinski, L.; Papageorgiou, N.S.; *Nonlinear second-order multivalued boundary value problems.* Proc. Indian Acad. Sci. Math. Sci., **113** (2003), 293-319.

[7] Guiro, A.; Nyanquini, I.; Ouaro, S.; *On the solvability of discrete nonlinear Neumann problems involving the $p(x)$-Laplacian.* Adv. Differ. Equ. **2011**, 32 (2011).

[8] He, T.; Chen, W.; *Periodic solutions of second order discrete convex systems involving the p-Laplacian.* Appl. Math. Comput. **206** (2008), 124-132.

[9] Hokkanen, V.-M.; Morosanu, G.; *Functional Methods in Differential Equations*, Chapmann and Hall/CRC, 2002.

[10] Jebelean, P.; Serban, C; *Ground state periodic solutions for difference equations with discrete p-Laplacian.* Appl. Math. Comput., **217** (2011), 9820-9827.

[11] Koné, B.; Ouaro, S.; *Weak solutions for anisotropic discrete boundary value problems.* J. Differ. Equ. Appl. **16** (2) (2010), 1-11.

[12] Mashiyev, R.; Yucedag, Z.; Ogras, S.; *Existence and multiplicity of solutions for a Dirichlet problem involving the discrete $p(x)$-Laplacian operator*. Electron. J. Qual. Theory Differ. Equ. **67** (2011), 1-10.
[13] Mihailescu, M,; Radulescu, V.; Repovš, D.; *On a non-homogeneous eigenvalue problem involving a potential: an Orlicz-Sobolev space setting*, J. Math. Pures Appl., **93** (9) (2010), no. 2, 132-148.
[14] Mihailescu, M,; Radulescu, V.; Tersian, S.; *Eigenvalue problems for anisotropic discrete boundary value problems*. J. Differ. Equ. Appl. **15** (6) (2009), 557-567.
[15] Mihailescu, M,; Radulescu, V.; Tersian, S.; *Homoclinic solutions of difference equations with variable exponents*. Topol. Methods Nonlinear Anal., **38** (2011), 277-289.
[16] Molica Bisci, G.; Repovš, D.; *Existence of solutions for p-Laplacian discrete equations*, Appl. Math. Comput., **242** (2014), 454-461.
[17] Molica Bisci, G.; Repovš, D.; *On sequences of solutions for discrete anisotropic equations*, Expo. Math. **32** (2014), no. 3, 284-295.
[18] Radulescu, V.; Repovš, D.; *Partial differential equations with variable exponents. Variational methods and qualitative analysis*, Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 2015.
[19] Rockafellar, R. T.; *Convex Analysis*, Princeton University Press, 1972.
[20] Szulkin, A.; *Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems*. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **3** (2) (1986), 77-109.
[21] Tian, Y.; Ge, W.; *The existence of solutions for a second-order discrete Neumann problem with a p-Laplacian*. J. Appl.Math. Comput. **26** (2008), 333-340.

Bila Adolphe Kyelem
Université Ouaga I-Professeur Joseph Ki-Zerbo, Unité de Formation et de Recherche en Sciences Exactes et Appliquées, Département de Mathématiques, B.P. 7021 Ouagadougou 03, Burkina Faso
E-mail address: kyeleadoc@yahoo.fr

Stanislas Ouaro
Université Ouaga I-Professeur Joseph Ki-Zerbo, Unité de Formation et de Recherche en Sciences Exactes et Appliquées, Département de Mathématiques, B.P. 7021 Ouagadougou 03, Burkina Faso
E-mail address: ouaro@yahoo.fr, souaro@univ-ouaga.bf

Malick Zoungrana
Université Ouaga I-Professeur Joseph Ki-Zerbo, Unité de Formation et de Recherche en Sciences Exactes et Appliquées, Département de Mathématiques, B.P. 7021 Ouagadougou 03, Burkina Faso
E-mail address: zoungmalick@yahoo.fr