A PROPOSAL FOR MAKING A DECISION FROM SAMPLE ELEMENTS OF ANY DISTRIBUTION FUNCTION

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TABLE OF CONTENTS

CHAPTER		Page
I.	INTRODUCTION	1
II.	PRELIMINARY DEFINITIONS AND THEOREMS	2
III.	DEFINITIONS AND THEOREMS CONCERNING DISTRIBUTION FUNCTIONS AND MOMENT GENERATING FUNCTIONS	11
IV.	USE OF THE BINOMIAL DISTRIBUTION FOR DECISION-MAKING CONCERNING SAMPLE POINTS OF ANY DISTRIBUTION	14
V.	THE PROPOSED APPROACH INTRODUCED IN N SPACE	38
BIBLIOGRAPHY		58

C H A P T E R I

INTRODUCTION

Statistical measures are involved in describing characteristics or in making decisions about groups of data. A statistical measure which is commonly used in decision-making concerning a particular group of data is the measure of central tendency called the arithmetic mean. In order to decide whether or not a sample is selected from a certain universe, a confidence interval may be set up for the arithmetic mean of the sample. However, in setting up a confidence interval for the arithmetic mean each of the sample points must be from a sequence of independently distributed chance variables, and the distribution function of the sample mean must be known. As a result of this needed information, a normal distribution of independent chance variables is assumed to set up the confidence interval for the sample mean.

The purpose of this paper is to propose an approach for deciding whether or not a sample is selected from a specific universe. The importance of this approach is the fact that a decision can be made about a particular sample from any distribution function, and the decision does not rely upon the independence of a sequence of chance variables. Therefore when the distribution function of the universe is known, the proposed approach may be used and no assumptions are made.

CHAPTER II

PRELIMINARY DEFINITIONS AND THEOREMS

DEFINITION 2.00. The statement that the set A is a subset of the set B, denoted $A \subseteq B$, means that A is a set such that each member of A is a member of B.

DEFINITION 2.01. The statement that C is the common part of the set A and the set B, or that $C = A \cap B$, means that C is the set such that p is a member of C if and only if p is a member of A and p is a member of B.

DEFINITION 2.02. The statement that C is the union of the set A and the set B, or that $C = A \cup B$, means that C is the set such that p is a member of C if and only if p is a member of A or p is a member of B.

NOTATION 2.00. $X \varepsilon A$ denotes the following phrase: X is an element of the set A.

NOTATION 2.01. The symbol ϕ denotes the null set.

DEFINITION 2.03. $A = B$ where A is a set and B is a set means that $A \subseteq B$ and $B \subseteq A$.

DEFINITION 2.04. The statement that D is a probability domain means that D is a collection of sets such that the following statements are true:

1) there is in D a set S such that if $A \in D$, then $A \subseteq S$;

2) there is in D a set ϕ such that if $A \in D$, then $\phi \subseteq A$;

3) if $A \in D$ and $B \in D$, then $A \cap B \in D$; and

4) if $A \in D$, then there is a set $A^c \in D$ such that $A \cap A^c = \phi$ and $A \cup A^c = S$. A^c is called the complement of A.

NOTE: In the definitions and theorems that follow in this paper if the symbol S is used, it will be understood that S is the set contained in the probability domain with the properties given above.

DEFINITION 2.05. An event is a set in a probability domain.

DEFINITION 2.06. A member of S is called a sample point.

DEFINITION 2.07. S is called a population or universe.

NOTATION 2.02. If r is a positive integer, then
$r! = r(r-1)(r-2)\cdot \cdots \cdot(r-r+1)$, and
$0! = 1$.

DEFINITION 2.08. Suppose $n > 0$ and r are integers; then

$$\binom{n}{r} = \begin{cases} 0 & \text{if } r < 0 \\ 1 & \text{if } r = 0 \\ \dfrac{n(n-1)(n-2)\cdot \cdots \cdot(n-r+1)}{r!} & \text{if } r > 0. \end{cases}$$

NOTATION 2.03. If n is a positive integer and r is a positive integer, then $\binom{n}{r}$ denotes the number of r-member subsets or combinations of an n-member set.

DEFINITION 2.09. Suppose D and R are sets. The statement that f is a function with domain D and range R means:

1) f is a collection of ordered pairs of real numbers;

2) no two pairs in f have the same first component;

3) if $(x,y) \in f$, then $x \in D$ and $y \in R$; and

4) if $x \in D$, then $(x,f(x)) \in f$.

Furthermore, if D is a collection of sets, then f is called a set function.

DEFINITION 2.10. Suppose D is a probability domain and f is a set function with domain D. The statement that f is additive means that if $A \in D$ and $B \in D$, then

$$f(A) + f(B) = f(A \cup B) + f(A \cap B).$$

DEFINITION 2.11. To say that $A_1, A_2, A_3, \ldots$ is a sequence means that there is a function A whose domain is the set of positive integers. Moreover, A_i denotes the second component of the pair in A whose first component is i, $i = 1,2,\ldots$. $(i,A_i) \in A$.

DEFINITION 2.12. To say that $A_1, A_2, \ldots, A_n$ is a finite sequence means that n is a positive integer and there is a function A with domain $\{1,2,\ldots,n\}$. A_i denotes the second component of the pair in A with first component i, $i = 1,2, \ldots,n$. $(i, A_i) \in A$.

DEFINITION 2.13. The statement that $X = X_1, X_2, \ldots, X_m$ is an m-term increasing sequence from the first n positive integers means that X is an m-term sequence from the first n positive integers and $X_k < X_{k+1}$ for $k \in \{1,2,\ldots,m-1\}$.

THEOREM 2.00. If f is an additive set function and $A_1, A_2, \ldots, A_n$ is a sequence of n sets each in the domain D of f and D is a probability domain, then

$$f(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{i=1}^{n} f(A_i) - \sum_{1 \le i < j \le n} f(A_i \cap A_j) + \sum_{1 \le i < j < k \le n} f(A_i \cap A_j \cap A_k) + \ldots - (-1)^n f(A_1 \cap A_2 \cap \ldots \cap A_n).$$

DEFINITION 2.14. The statement that D is a complete probability domain means that D is a probability domain such that if

1) $A_1, A_2, A_3, \ldots$ is a sequence of members of D,
2) $A_{n+1} \subseteq A_n$ for $n = 1,2,3,\ldots,$ and
3) $A_1 \cap A_2 \cap A_3 \cap \ldots = A$,

then $A \in D$.

DEFINITION 2.15. The statement that (D,P) is a probability distribution means that

1) D is a complete probability domain, and
2) P is a function with domain D such that
   a) if $A \in D$, then $P(A) \ge 0$,
   b) P is additive,
   c) $P(S) = 1$ and $P(\phi) = 0$,
   d) if $A_1, A_2, A_3, \ldots$ is a sequence of members of D such that $A_{n+1} \subseteq A_n$ for $n \in \{1,2,3,\ldots\}$, $A = A_1 \cap A_2 \cap A_3 \cap \ldots$, and $\varepsilon > 0$, then there is a positive integer N such that if $n \ge N$, then $|P(A_n) - P(A)| < \varepsilon$.

If $X \in D$, then $P(X)$ is the probability of X.

DEFINITION 2.16. To say that τ is a chance variable, a stockastic variable, or a random variable means that there is a probability domain D, and τ is a function with domain $S \varepsilon D$ and range the real numbers. Moreover, if t is a real number and $(\tau \leq t)$ denotes the set such that $s \varepsilon (\tau \leq t)$ if and only if $s \varepsilon S$ and $\tau(s) \leq t$, then $(\tau \leq t) \varepsilon D$.

DEFINITION 2.17. Suppose τ is a chance variable and (D,P) is a probability distribution. The statement that F is the distribution function for τ means that if t is a real number, then $F(t) = P(\tau \leq t)$. P is called the probability function of τ.

DEFINITION 2.18. The statement that $\{A_1, A_2, \ldots, A_n\}$ is a partition of the set R means that $A_1, A_2, \ldots, A_n$ is an n-term sequence of sets such that $\bigcup_{i=1}^{n} A_i = R$ and $A_i \cap A_j = \phi$ if $i \neq j$, $j = 1,2,\ldots,n$.

NOTATION 2.04. Suppose a and b are real numbers such that $a < b$.

i. $[a,b] = \{X \mid a \leq X \leq b\}$.

ii. $(a,b) = \{X \mid a < X < b\}$.

DEFINITION 2.19. Suppose that f is a function whose domain contains an interval [a,b], and that t is a number between a and b.

1) The statement that $f(t^-) = c_1$ means that c_1 is a number and that if $\varepsilon > 0$, then there is in [a,b] an interval [p,t] such that $|f(s) - c_1| < \varepsilon$ if s is between p and t.

2) The statement that $f(t^+) = c_2$ means that c_2 is a number and that if $\varepsilon > 0$, then there is an interval $[t,q]$ in $[a,b]$ such that $|f(s) - c_2| < \varepsilon$ if s is between t and q.

DEFINITION 2.20. Suppose that $\tau_1, \tau_2, \ldots, \tau_n$ is an n-term sequence of chance variables and (D,P) is a probability distribution such that if $t_1, t_2, \ldots, t_n$ is an n-term real-number sequence, then each of the sets $(\tau_k \le t_k)$, $k = 1,2,\ldots,n$, is a member of D. The statement that F is the distribution function for $\tau_1, \tau_2, \ldots, \tau_n$ means that F is the function such that if $t_1, t_2, \ldots, t_n$ is an n-term sequence of real numbers, then

$$F(t_1,t_2,\ldots,t_n) = P[(\tau_1 \le t_1) \cap (\tau_2 \le t_2) \cap \ldots \cap (\tau_n \le t_n)].$$

THEOREM 2.01. Suppose that τ is a chance variable and that F is the distribution function for τ; then each of the following statements is true:

1) F is nondecreasing; that is, if $t_1 \le t_2$, t_1, t_2 are real numbers, then $F(t_1) \le F(t_2)$;
2) if t is a real number, then $0 \le F(t) \le 1$;
3) if $\varepsilon > 0$, then there is a real number X such that $F(X) < \varepsilon$;
4) if $\varepsilon > 0$, then there is a real number Y such that $F(Y) > 1 - \varepsilon$;
5) F in continuous from the right; that is, if t is a real number, then $F(t^+) = F(t)$.

DEFINITION 2.21. Suppose $[a,b]$ is a number interval. The statement that D is a subdivision of $[a,b]$ means that D is a finite set of intervals,
$\{[t_0,t_1], [t_1,t_2], \ldots, [t_{n-1},t_n]\}$ such that $a = t_0 < t_1 < t_2 < \ldots < t_n = b$.

DEFINITION 2.22. Suppose D is a subdivision of an interval $[a,b]$. The statement that E is a refinement of D means that E is a subdivision of $[a,b]$ such that if $[p,q]$ is an interval in D, then there is a subset of E which is a subdivision of $[p,q]$.

DEFINITION 2.23. Suppose X is a function, Y is a function, and $[a,b]$ is an interval. The statement that $\int_a^b Y(t)\, d\, X(t)$ exists means that $\int_a^b Y(t)\, d\, X(t)$ is a number, and for every real number $\varepsilon > 0$ there is a subdivision D of $[a,b]$ such that if E is a refinement of D, then

$$\left|\sum_E \frac{1}{2}[Y(t_i) + Y(t_{i+1})][X(t_{i+1}) - X(t_i)] - \int_a^b Y(t)\, d\, X(t)\right| < \varepsilon$$

where $[t_i,t_{i+1}] \varepsilon E$.

THEOREM 2.02. Suppose X and Y are functions such that $\int_a^b Y(t)\, d\, X(t)$ exists; then if k is a number, the following statements are true:

1) $\int_a^b kY(t)\, d\, X(t) = k\int_a^b Y(t)\, d\, X(t)$;
2) $\int_a^b Y(t)\, d\, kX(t) = k\int_a^b Y(t)\, d\, X(t)$; and
3) $\int_a^b Y(t)\, d\, [X(t) + k] = \int_a^b Y(t)\, d\, X(t)$.

DEFINITION 2.24. The statement that $\int_{-\infty}^{\infty} Y(t)\, d\, X(t) = c$ means that c is a number, and if $\varepsilon > 0$, then there is an interval [A,B] such that if $[A,B] \subseteq [a,b]$, then $\left|\int_a^b Y(t)\, d\, X(t) - c\right| < \varepsilon$.

DEFINITION 2.25. Suppose that Y is a function whose range is the set of real numbers. The statement that the expected value of Y is u or that E(Y) = u means that

$$\int_{-\infty}^{\infty} Y(t)\, d\, F(t) = u$$

where τ is a chance variable with distribution function F. The number E(t), if there is such a number, is said to be the mean of τ. The number $E(t-u)^2$, if there is such a number, is denoted by σ^2 and is said to be the variance of τ. The non-negative square root of the variance, σ, is the standard deviation of τ.

THEOREM 2.03. If Y is a function whose range is the set of real numbers and Y(t) = 1 for each real number t, then E(y) = 1.

DEFINITION 2.26. Suppose F is a distribution function and $\theta_1, \theta_2, \ldots, \theta_n$ is an n-term sequence of real numbers that characterize F; then θ_i, i = 1,2,...,n, is called a parameter of F.

DEFINITION 2.27. The statement that [a,b] is the α% confidence interval for the chance variable τ with distribution function F means

1) [a,b] is a number interval;

2) $P(a<\tau\leq b) = \dfrac{\alpha}{100}$; and

3) $P(\tau\leq a) = P(\tau>b)$.

THEOREM 2.04. If x and y are real numbers and n is a positive integer, n > 2, then

$$(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}.$$

CHAPTER III

DEFINITIONS AND THEOREMS CONCERNING DISTRIBUTION FUNCTIONS AND MOMENT GENERATING FUNCTIONS

DEFINITION 3.00. The statement that Λ is the moment generating function for the distribution function F of a chance variable τ means if t is a real number, then $\Lambda(z) = \int_{-\infty}^{\infty} e^{-zt}\, d\, F(t)$ for each number z such that $\Lambda(z)$ exists.

THEOREM 3.00. If Λ is the moment generating function for the distribution function F and Ω is the moment generating function for the distribution function G and $\Lambda = \Omega$, then $F = G$.

DEFINITION 3.01. The statement that $\tau_1, \tau_2, \ldots, \tau_n$ is a sequence of n independent chance variables means that n is a positive integer greater than or equal to 2, and if $t_1, t_2, \ldots, t_n$ is an n-term sequence of real numbers, then $P[(\tau_1 \leq t_1) \cap (\tau_2 \leq t_2) \cap \ldots \cap (\tau_n \leq t_n)] = \prod_{i=1}^{n} P(\tau_i \leq t_i)$.

THEOREM 3.01. If $\tau_1, \tau_2, \ldots, \tau_n$ is a sequence of n independently distributed chance variables having moment generating functions $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ respectively and

$\tau = \tau_1 + \tau_2 + \ldots + \tau_n$, then τ has moment generating function Λ such that

$$\Lambda(z) = \prod_{i=1}^{n} \Lambda_i(z)$$

for each number z in the domain of Λ_i, $i = 1,2,\ldots,n$.

DEFINITION 3.02. The statement that the chance variable τ is binomially distributed with parameters n, p or that τ has the binomial distribution means that

$$P(\tau=x) = \begin{cases} \binom{n}{x} p^x(1 - p)^{n-x} & \text{for each integer } x, \\ 0 & \text{otherwise.} \end{cases}$$

NOTE. If D is the probability domain for P, then (D,P) is said to be a binomial distribution.

THEOREM 3.02. If τ, a chance variable, has the binomial distribution with parameters n and p such that $P(\tau=x) = \binom{n}{x} p^x(1 - p)^{n-x}$, then $P(\tau\geq k) = \sum_{i=k}^{n} \binom{n}{i} p^i (1 - p)^{n-i}$ where $0 \leq k \leq n$, and k is an integer.

NOTATION 3.00. If τ is a chance variable which has the binomial distribution with parameters n and p and x is a non-negative integer, then

$$P(\tau\geq x|n,p) = \sum_{i=x}^{n} \binom{n}{i} p^i(1 - p)^{n-i}, \text{ and}$$

$$P(\tau<x|n,p) = \sum_{i=0}^{x-1} \binom{n}{i} p^i(1 - p)^{n-i}.$$

THEOREM 3.03. Suppose τ is a chance variable and t is a real number, then $P(\tau\geq t) = 1 - P(\tau<t)$.

THEOREM 3.04. Suppose τ is a chance variable which has the binomial distribution with parameters n and p. If x is a non-negative integer, then

$P(\tau \geq x | n,p) = 1 - P(\tau \geq n-x+1 | n,1-p)$.

DEFINITION 3.03. The statement that the chance variable τ has the normal probability distribution with mean u and variance σ^2 means that

$$P(\tau \leq t) = \frac{1}{\sqrt{2\pi}\ \sigma} \int_{-\infty}^{t} e^{-\frac{1}{2}\left(\frac{x - u}{\sigma}\right)^2} d\ x.$$

If τ has the normal distribution, then P is a continuous function such that $P(\tau=t) = 0$.

THEOREM 3.05. If τ is normally distributed with mean u and variance σ^2 then the moment generating function for τ is the function Λ such that

$$\Lambda(z) = e^{-uz}\ e^{\frac{\sigma^2 z^2}{2}}.$$

THEOREM 3.06. If τ, a chance variable, is normally distributed and t is a real number, then $P(\tau<-t) = 1 = P(\tau \geq t)$.

CHAPTER IV

USE OF THE BINOMIAL DISTRIBUTION FOR DECISION-MAKING CONCERNING SAMPLE POINTS OF ANY DISTRIBUTION

THEOREM 4.00. Suppose τ_1 and τ_2 are chance variables such that for each real number pair $(u,t>0)$,

$P(u-t<\tau_1\leq u+t) = P(u-t<\tau_2\leq u+t)$,

then τ_1 and τ_2 have the same distribution function.

Proof:

1. Let F_1 be the distribution function for the chance variable τ_1 and F_2 be the distribution function for τ_2. To say that $F_1 = F_2$ means that if $\varepsilon > 0$, then $|F_1 - F_2| < \varepsilon$.
2. Let $\varepsilon > 0$; then $\frac{\varepsilon}{2} > 0$.
3. Since $\frac{\varepsilon}{2} > 0$, then by Theorem 2.01 there is a real number u_1 such that $F_1(u_1) < \frac{\varepsilon}{2}$;

   and there is a real number u_2 such that $F_2(u_2) < \frac{\varepsilon}{2}$.

   Let $t > 0$ be a real number.
4. Let u be the minimum of u_1, u_2, $t - 1$.

   Let $\delta = t - u > 0$.

5. $|F_1(t) - F_2(t)|$

$= |P(\tau_1 \leq t) - P(\tau_2 \leq t)|$

$= |P(\tau_1 \leq u+\delta) - P(\tau_2 \leq u+\delta)|$

$= |P(\tau_1 \leq u+\delta) - P(\tau_1 \leq u-\delta) + P(\tau_1 \leq u-\delta) - P(\tau_2 \leq u+\delta)|$

$= |P(u-\delta<\tau_1<u+\delta) + P(\tau_1<u-\delta) - P(\tau_2<u+\delta)|$

$= |P(u-\delta<\tau_2 \leq u+\delta) + P(\tau_1 \leq u-\delta) - P(\tau_2 \leq u+\delta)|$ by the hypothesis

$= |P(\tau_2 \leq u+\delta) - P(\tau_2 \leq u-\delta) + P(\tau_1 \leq u-\delta) - P(\tau_2 \leq u+\delta)|$

$= |P(\tau_1 \leq u-\delta) - P(\tau_2 \leq u-\delta)|$

$\leq |P(\tau_1 \leq u-\delta)| + |P(\tau_2 \leq u-\delta)|$

$= P(\tau_1 \leq u-\delta) + P(\tau_2 \leq u-\delta)$

$\leq P(\tau_1 \leq u) + P(\tau_2 \leq u)$ Since $\delta > 0$, then $u - \delta < u$.

$= F_1(u) + F_2(u)$

$\leq F_1(u_1) + F_2(u_2)$ Since $u \leq u_1$ and $u \leq u_2$, then $F_1(u) \leq F_1(u_1)$ and $F_2(u) \leq F_2(u_2)$.

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$ since $F_1(u_1) < \frac{\varepsilon}{2}$ and $F_2(u_2) < \frac{\varepsilon}{2}$ from step 3.

$= \varepsilon$.

Therefore, $|F_1 - F_2| < \varepsilon$, and it follows that $F_1 = F_2$.

THEOREM 4.01. Suppose τ_1 and τ_2 are chance variables with means μ_1 and μ_2. Suppose $\mu_1 \leq \mu_2$ and

$P(\mu_2-t<\tau_2\leq\mu_2+t)P(\mu_2-t<\tau_2\leq\mu_1+t) = P(\mu_1-t<\tau_1\leq\mu_2+t)P(\mu_1-t<\tau_1\leq\mu_1+t)$

for each real number $t > 0$, and $P(\mu_1-t<\tau_1\leq\mu_1+t) \neq 0$, then $\mu_1 = \mu_2$.

Proof:

Assume that the conclusion is false; that is, assume that $\mu_1 < \mu_2$; then there exists $t > 0$ such that $P(\mu_2-t<\tau_2\leq\mu_1+t) = 0$ and $P(\mu_2-t<\tau_2\leq\mu_2+t) \neq 0$. A possible value for t such that the above statement is true is $t = \frac{\mu_1 + \mu_2}{2}$. Since by the hypothesis $P(\mu_1 - t < \tau_1 \leq \mu_1 + t) \neq 0$, then certainly $P(\mu_1 - t < \tau_1 \leq \mu_2 + t) \neq 0$. Therefore,

$P(\mu_1 - t < \tau_1 \leq \mu_2 + t)\ P(\mu_1 - t < \tau_1 \leq \mu_1 + t) \neq 0$.

Since $P(\mu_2 - t < \tau_2 \leq \mu_1 + t) = 0$, then

$P(\mu_2 - t < \tau_2 \leq \mu_1 + t)\ P(\mu_2 - t < \tau_2 \leq \mu_2 + t) = 0$.

Consequently,

$P(\mu_2 - t < \tau_2 \leq \mu_2 + t)\ P(\mu_2 - t < \tau_2 \leq \mu_1 + t) \neq P(\mu_1 - t < \tau_1 \leq \mu_2 + t)\ P(\mu_1 - t < \tau_1 \leq \mu_1 + t)$.

A contradiction has been reached, and it follows that $\mu_1 = \mu_2$.

An example for Theorem 4.01 is given. With the preceding hypothesis in mind, notice that it is possible for chance variables of different distribution functions to have the same mean.

EXAMPLE 4.00. Suppose F is a distribution function for the chance variable τ_1 with mass function f over the interval [0,20] such that

$P(\tau_1 = 0) = \frac{1}{5}$

$P(\tau_1 = 5) = \frac{1}{5}$

$P(\tau_1 = 10) = \frac{1}{5}$

$P(\tau_1 = 15) = \frac{1}{5}$

$P(\tau_1 = 20) = \frac{1}{5}$

Now $\mu_1 = \Sigma x f(x) = (0 + 5 + 10 + 15 + 20)\frac{1}{5} = 10$.

Also suppose that G is a distribution function for the chance variable τ_2 with mass function g over the interval [0,20] such that

$$P(\tau_2=0) = \frac{1}{3}$$

$$P(\tau_2=10) = \frac{1}{3}$$

$$P(\tau_2=20) = \frac{1}{3}$$

$$\mu_2 = \Sigma x\ g(x) = (0 + 10 + 20)\frac{1}{3} = 10.$$

Therefore, $\mu_1 = \mu_2$.

Note that for each real number $t > 0$

$$[F(\mu_2+t) - F(\mu_1-t)][F(\mu_1+t) - F(\mu_1-t)]$$
$$= [G(\mu_2+t) - G(\mu_2-t)][G(\mu_1+t) - G(\mu_2-t)].$$

For example: Suppose $t = 10$.

$$[F(\mu_2+10) - F(\mu_1-10)][F(\mu_1+10) - F(\mu_1-10)]$$
$$= [F(10+10) - F(10-10)][F(10+10) - F(10-10)]$$
$$= [F(20) - F(0)][F(20) - F(0)]$$
$$= \left[\frac{1}{5} - \frac{1}{5}\right]\left[\frac{1}{5} - \frac{1}{5}\right]$$
$$= 0$$
$$= \left[\frac{1}{3} - \frac{1}{3}\right]\left[\frac{1}{3} - \frac{1}{3}\right]$$
$$= [G(20) - G(0)][G(20) - G(0)]$$
$$= [G(10+10) - G(10-10)][G(10+10) - G(10-10)]$$
$$= [G(\mu_2+10) - G(\mu_2-10)][G(\mu_2+10) - G(\mu_2-10)].$$

DEFINITION 4.00. The statement that H is an n-term sample from a chance variable τ means that

1) $n \geq 1$ is a positive integer;

2) H is an n-term sequence from τ; and

3) $\bar{f}$ is a function with domain the collection of statements

$$\{s \in (\tau \leq a) \mid (s,\tau(s)) \in H \text{ and } a \text{ is real}\}$$
$$\cup \{s \notin (\tau \leq a) \mid (s,\tau(s)) \in H \text{ and } a \text{ is real}\}$$

such that $\bar{f}[s \in (\tau \leq a)] = P(\tau \leq a)$ and $\bar{f}[s \notin (\tau \leq a)] = P(\tau > a)$ where (D,P) is a probability distribution, and $(\tau \leq a) \in D$.

NOTATION 4.00. If $E_1, E_2, \ldots, E_n$ is a sequence of n statements, then $E_1 \wedge E_2 \wedge \ldots \wedge E_k$, $2 \leq k \leq n$, means E_1 is true, E_2 is true, . . . , and E_k is true.

DEFINITION 4.01. The statement that A is a random sample of size n from a chance variable τ means that

1) A is an n-term sample from a chance variable τ;

2) if $(s,\tau(s))$ is a term in A and $[t_1,t_2]$ is a number interval, then

$$[s \in (t_1 < \tau \leq t_2)]' = [s \notin (t_1 < \tau \leq t_2)]$$
$$= [s \in ((\tau \leq t_1) \cup (\tau > t_2))]; \text{ and}$$

3) if k is a positive integer and A_j, $j = 1,2,\ldots,k$, represents either of $[s \in (t_1 < \tau \leq t_2)]$ or $[s \in (t_1 < \tau \leq t_2)]'$, then $\bar{f}(A_1 \wedge A_2 \wedge \ldots \wedge A_k) = \sum_{i=1}^{k} \bar{f}(A_i)$.

NOTATION4.01. Given a chance variable τ and an integer $n > 0$, then $W_{\tau,n}$ denotes the set such that $x \varepsilon W_{\tau,n}$ if and only if x is an n-term sequence from τ.

DEFINITION 4.02. The statement that $(s_1,\tau(s_1))$ has property L_{t_1,t_2} means that $[t_1,t_2]$ is a number interval and $t_1 < \tau(s_1) \leq t_2$.

NOTATION 4.02. Let ξ be a function with domain $W_{\tau,n}$ such that if $x \varepsilon W_{\tau,n}$, then $\xi(x)$ is the number of terms in x that have property L_{t_1,t_2}.

THEOREM 4.02. For each $i \varepsilon \{0,1,\ldots,n\}$ let $P_i \subseteq W_{\tau,n}$ such that $s \varepsilon P_i$ if and only if $\xi(s) = i$, then $\bigcup\limits_{i=o}^{n} P_i = W_{\tau,n}$. Moreover, if $i \neq j$ and $s \varepsilon P_i$, then $s \not\varepsilon P_j$.

Proof:

Let $i,j = 0,1,\ldots,n$, and let $s \varepsilon P_i$; then $\xi(s) = i$. If $i \neq j$, then $\xi(s) \neq j$ and $s \not\varepsilon P_j$.

Now, it must be shown that $W_{\tau,n} = \bigcup\limits_{i=0}^{n} P_i$.

Let $s \varepsilon \bigcup\limits_{i=0}^{n} P_i$; then there is an integer k, $0 \leq k \leq n$, such that $s \varepsilon P_k$. Since $P_k \subseteq W_{\tau,n}$, then $s \varepsilon W_{\tau,n}$ and $\bigcup\limits_{i=0}^{n} P_i \subseteq W_{\tau,n}$.

Conversely, let $s \varepsilon W_{\tau,n}$; then there is an integer q, $0 \leq q \leq n$, such that $s \varepsilon P_q$. Since $s \varepsilon P_q$ and q is an integer, $0 \leq q \leq n$, then $s \varepsilon \bigcup\limits_{i=0}^{n} P_i$ and $W_{\tau,n} \subseteq \bigcup\limits_{i=0}^{n} P_i$.

Since $\bigcup\limits_{i=o}^{n} P_i \subseteq W_{\tau,n}$ amd $W_{\tau,n} \subseteq \bigcup\limits_{i=0}^{n} P_i$, then $\bigcup\limits_{i=0}^{n} P_i = W_{\tau,n}$.

NOTE. In the remainder of this chapter P_i, $i = 0,1,\ldots,n$, will denote the set described in the above theorem.

NOTATION 4.03. Let n be a positive integer, $i \varepsilon \{0,1,\ldots,n\}$, and $\{r_1, r_2, \ldots, r_{\binom{n}{i}}\}$ be the collection of i-term increasing sequences from $\{1,2,\ldots,n\}$. For each $j \varepsilon \{1,2,\ldots,\binom{n}{i}\}$ let $Q_j \subseteq P_i$, such that $x \varepsilon Q_j$ if and only if the terms in x specified by r_j have property L_{t_1,t_2}.

THEOREM 4.03. If $i \varepsilon \{0,1,\ldots,n\}$, then $\{Q_1, Q_2, \ldots, Q_{\binom{n}{i}}\}$ is a partition of P_i.

Proof:

Let $i \varepsilon \{0,1,\ldots,n\}$. Since for $Q_j \subseteq P_i$, $j = 1,2,\ldots,\binom{n}{i}$, x is specified by an i-term increasing sequence, then $x \notin Q_k$ if $k \neq j$ and $j,k = 1,2,\ldots,\binom{n}{i}$. Therefore, Q_k and Q_j are disjoint if $k \neq j$ and $Q_k \cap Q_j = \phi$. It can be shown that $\bigcup_{j=1}^{\binom{n}{i}} Q_j = P_i$ by a method similar to the proof to Theorem 4.02.

DEFINITION 4.03. Let $0 \leq i \leq n$ where i is an integer. Given a chance variable τ define $\overline{P}_i$ as the additive function with domain the partition $\{Q_1, Q_2, \ldots, Q_{\binom{n}{i}}\}$ of P_i such that $\overline{P}_i(Q_j) = \beta^i(1-\beta)^{n-i}$, $j = 1,2,\ldots,\binom{n}{i}$, where $\beta = P(t_1<\tau\leq t_2)$ and t_1 and t_2 are real numbers.

THEOREM 4.04. Suppose A is a random sample of size n from a chance variable τ. If $t_1,t_2 > 0$ are real numbers such that $P(t_1<\tau\leq t_2) = \beta$ where P is the probability function for τ, and $\hat{P}$ is an additive function with domain $\hat{D}$, the collection of subsets of $\{\{A\varepsilon P_0\},\{A\varepsilon P_1\}, \ . \ . \ . \ , \{A\varepsilon P_n\}\}$, such that $\hat{P}(\phi) = 0$ and $\hat{P}(\{A\varepsilon P_i\}) = \sum_{j=1}^{\binom{n}{i}} \bar{P}_i(Q_j)$, $i = 0,1,\ldots,n$, then $\hat{P}(\{A\varepsilon P_i\}) = \binom{n}{i} \beta^i(1 - \beta)^{n-i}$. Moreover, $(\hat{D},\hat{P})$ is a probability distribution.

Proof:

1. Let $t_1,t_2 > 0$ be real numbers such that $P(t_1<\tau\leq t_2) = \beta$, where P is the probability function for τ, and let $\hat{P}$ be an additive function with domain the collection of subsets of $\{\{A\varepsilon P_0\},\{A\varepsilon P_1\}, \ . \ . \ . \ , \{A\varepsilon P_n\}\}$ such that

$$\hat{P}(\{A\varepsilon P_i\}) = \sum_{j=1}^{\binom{n}{i}} \bar{P}_i(Q_j), \ i = 0,1,\ldots,n.$$

2. $$\hat{P}(\{A\varepsilon P_i\}) = \sum_{j=1}^{\binom{n}{i}} \bar{P}_i(Q_j)$$
$$= \sum_{j=1}^{\binom{n}{i}} \beta^i(1 - \beta)^{n-i} \text{ by Definition 4.03}$$
$$= \binom{n}{i} \beta^i(1 - \beta)^{n-i}.$$

3. $(\hat{D},\hat{P})$ is a probability distribution:

   A. $\hat{D}$ is a probability domain:

      1) $S = \{\{A\varepsilon P_0\},\{A\varepsilon P_1\}, \ . \ . \ . \ ,\{A\varepsilon P_n\}\}$;

      2) Since $\phi \subseteq S$ and $\hat{D}$ is a collection of the subsets of S, then $\phi \ \varepsilon \ \hat{D}$, and if $B \ \varepsilon \ \hat{D}$, then $\phi \subseteq B$;

      3) Let $C \ \varepsilon \ \hat{D}$ and $B \ \varepsilon \ \hat{D}$; then $C \cap B \ \varepsilon \ \hat{D}$, since $C \subseteq S$ and $B \subseteq S$ and $C \cap B \subseteq S$; and

      4) Let $B \ \varepsilon \ \hat{D}$; then $B^c \ \varepsilon \ \hat{D}$ because $B \subseteq S$ and $B^c \subseteq S$. Moreover, $B \cap B^c = \phi$ and $B \cup B^c = S$.

B. $\hat{P}$ is a function with probability domain $\hat{D}$ such that

1) $\hat{P}$ is additive;

2) $\hat{P}(\phi) = 0$;

3) if $B \varepsilon \hat{D}$, then $\hat{P}(B) > 0$:

$\hat{P}(B)$ is some sum of $\beta^r(1 - \beta)^{n-r}$, $0 \leq r \leq n$, r an integer, and $0 \leq \beta \leq 1$. Therefore, each $\beta^r(1 - \beta)^{n-r}$ is non-negative, and its specified sum is non-negative.

4) $\hat{P}(S) = 1$:

$$\hat{P}(S) = \hat{P}(\{A\varepsilon P_0\} \cup \{A\varepsilon P_1\} \cup . . . \cup \{A\varepsilon P_n\})$$

$$= \sum_{i=0}^{n} \hat{P}(\{A\varepsilon P_i\}) - \sum_{o\leq i<j\leq n} \hat{P}(\{A\varepsilon P_i\} \cap \{A\varepsilon P_j\})$$

$$+ \sum_{o\leq i<j<k\leq n} \hat{P}(\{A\varepsilon P_i\} \cap \{A\varepsilon P_j\} \cap \{A\varepsilon P_k\}) - . . .$$

$$- (-1)^{n+1}\hat{P}(\{A\varepsilon P_0\} \cap \{A\varepsilon P_1\} \cap . . . \cap \{A\varepsilon P_n\})$$

by Theorem 2.00

$$= \sum_{i=0}^{n} \hat{P}(\{A\varepsilon P_i\}) - 0 + 0 - . . . . - 0$$

$$= \sum_{i=0}^{n} \sum_{j=1}^{\binom{n}{i}} \overline{P}_i(Q_j)$$

$$= \sum_{i=0}^{n} \sum_{j=1}^{\binom{n}{i}} \beta^i(1 - \beta)^{n-i}$$

$$= \sum_{i=0}^{n} \binom{n}{i} \beta^i(1 - \beta)^{n-i}$$

$$= [\beta + (1 - \beta)]^n \quad \text{by Theorem 2.04}$$

$$= 1^n$$

$$= 1.$$

Therefore, by Definition 2.15, $(\hat{D},\hat{P})$ is a probability distribution.

Since $(\hat{D},\hat{P})$ is a probability distribution and if $x_i \varepsilon$ domain of $\hat{P}$, $\hat{P}(x_i) = \binom{n}{i}\beta^i(1-\beta)^{n-i}$, $i = 0,1,\ldots,n$, then by Definition 3.02, $(\hat{D},\hat{P})$ is a binomial probability distribution.

THEOREM 4.05. Suppose A is a random sample of size n from a chance variable τ. If $t_1,t_2 > 0$ are real numbers such that $P(t_1<\tau\le t_2) = \beta$ where P is the probability function for τ, and $\hat{P}$ is an additive function with domain $\hat{D}$, the collection of subsets of $\{\{A\varepsilon P_0\},\{A\varepsilon P_1\},\ldots,\{A\varepsilon P_n\}\}$, such that $\hat{P}(\phi) = 0$ and $\hat{P}(\{A\varepsilon P_i\}) = \sum_{j=1}^{\binom{n}{i}} \bar{P}_i(Q_j)$, $i = 0,1,\ldots,n$, and $k \varepsilon \{1,2,\ldots,n\}$, then $\hat{P}(\{A\varepsilon P_k\}\cup\{A\varepsilon P_{k+1}\}\cup\ldots\cup\{A\varepsilon P_n\}) = \sum_{i=k}^{n}\binom{n}{i}\beta^i(1-\beta)^{n-i}$.

Proof:

Let $t_1,t_2 > 0$ be real numbers such that $P(t_1<\tau\le t_2) = \beta$ where P is the probability function for τ, and let $\hat{P}$ be an additive function with domain $\hat{D}$, the collection of subsets of $\{\{A\varepsilon P_0\},\{A\varepsilon P_1\},\ldots,\{A\varepsilon P_n\}\}$, such that $\hat{P}(\phi) = 0$ and $\hat{P}(\{A\varepsilon P_i\}) = \sum_{j=1}^{\binom{n}{i}} \bar{P}_i(Q_j)$, $i = 0,1,\ldots,n$. Let $k \varepsilon \{1,2,\ldots,n\}$.

$$\hat{P}(\{A\varepsilon P_k\}\cup\{A\varepsilon P_{k+1}\}\cup\ldots\cup\{A\varepsilon P_n\})$$

$$= \sum_{i=k}^{n}\hat{P}(\{A\varepsilon P_i\}) - \sum_{k\le i<j\le n}\hat{P}(\{A\varepsilon P_i\}\cap\{A\varepsilon P_j\})$$

$$+ \sum_{k\le i<j<r\le n}\hat{P}(\{A\varepsilon P_i\}\cap\{A\varepsilon P_j\}\cap\{A\varepsilon P_r\}) - \ldots$$

$$- (-1)^{n-k}\hat{P}(\{A\varepsilon P_k\}\cap\{A\varepsilon P_{k+1}\}\cap\ldots\cap\{A\varepsilon P_n\})$$

by Theorem 2.00

$$= \sum_{i=k}^{n} \hat{P}(\{A \varepsilon P_i\}) - 0 + 0 \; . \; . \; . - 0 \quad \text{by Theorem 4.02}$$

$$\{A\varepsilon P_i\} \cap \{A\varepsilon P_j\} = \phi$$

$$= \sum_{i=k}^{n} \binom{n}{i} \beta^i (1 - \beta)^{n-i} \quad \text{by Theorem 4.04}$$

THEOREM 4.06. If $\tau_1, \tau_2, \ldots, \tau_n$ is a sequence of n independently and normally distributed chance variables each with mean μ and variance σ^2, then $\tau = \dfrac{\tau_1 + \tau_2 + \ldots + \tau_n}{n}$ has moment generating function $\Omega_{\bar{t}} = e^{-\mu z} e^{\frac{\sigma^2 z^2}{2n}}$ where $\frac{z}{n}$ is in the domain of Ω_t, the moment generating function for each of $\frac{\tau_1}{n}, \frac{\tau_2}{n}, \ldots, \frac{\tau_n}{n}$.

Proof:

Let $\tau_1, \tau_2, \ldots, \tau_n$ be a sequence of n independently and normally distributed chance variables each with mean μ and variance σ^2.

Since $\tau_1, \tau_2, \ldots, \tau_n$ is a sequence of n independently distributed chance variables, then $\frac{\tau_1}{n}, \frac{\tau_2}{n}, \ldots \frac{\tau_n}{n}$ is a sequence of n independently distributed chance variables.

Let $t_1, t_2, \ldots, t_n$ be a sequence of real numbers; then

$$P\left[\left(\frac{\tau_1}{n}<t_1\right) \cap \left(\frac{\tau_2}{n}<t_2\right) \cap \cdots \cap \left(\frac{\tau_n}{n}<t_n\right)\right]$$

$$= P[(\tau_1<t_1 n) \cap (\tau_2<t_2 n) \cap \ldots \cap (\tau_n<t_n n)]$$

$$= P(\tau_1<t_1 n)\, P(\tau_2<t_2 n) \ldots P(\tau_n<t_n n) \quad \text{by Definition 3.01}$$

$$= P\left(\frac{\tau_1}{n}<t_1\right) P\left(\frac{\tau_2}{n}<t_2\right) \cdots P\left(\frac{\tau_n}{n}<t_n\right).$$

Therefore, by Definition 3.01, $\frac{\tau_1}{n}, \frac{\tau_2}{n}, \ldots, \frac{\tau_n}{n}$ is an n-term sequence of independently distributed chance variables.

Let $\tau = \frac{\tau_1}{n} + \frac{\tau_2}{n} + \ldots + \frac{\tau_n}{n}$

$$= \frac{\tau_1 + \tau_2 + \ldots + \tau_n}{n}$$

Let F be the distribution function and Ω_t the moment generating function for each of $\tau_1, \tau_2, \ldots, \tau_n$. Let Λ be the moment generating function for $\frac{\tau_1}{n}$. Let z ε domain of Λ and let t be a real number.

$$\Lambda(z) = \int_{-\infty}^{\infty} e^{-zt} \, d\, P\left(\frac{\tau_1}{n} \leq t\right)$$

$$= \int_{-\infty}^{\infty} e^{-zt} \, d\, P(\tau_1 \leq tn)$$

$$= \int_{-\infty}^{\infty} e^{-zt} \, d\, F(tn) \quad \text{since } \tau_1 \text{ has distribution function } F$$

$$= \int_{-\infty}^{\infty} e^{-\frac{zx}{n}} \, d\, F(x) \quad \text{where } x = tn$$

$$= \Omega_t\left(\frac{z}{n}\right) \quad \text{for } \frac{z}{n} \text{ in the domain of } \Omega_t.$$

Since $\Lambda(z) = \Omega_t\left(\frac{z}{n}\right)$, then by Theorem 3.00 $\frac{\tau_1}{n}$ is a normally distributed chance variable. The above procedure may be used to show that $\frac{\tau_2}{n}, \frac{\tau_3}{n}, \ldots, \frac{\tau_n}{n}$ are normally distributed chance variables with moment generating function $\Omega_t\left(\frac{z}{n}\right)$.

Let $\Omega_{\bar{t}}$ be the moment generating function for $\tau = \frac{\tau_1 + \tau_2 + \ldots + \tau_n}{n}$; then by Theorem 3.01 $\Omega_{\bar{t}}\left(\frac{z}{n}\right) = \prod_{i=1}^{n} \Omega_{t_i}\left(\frac{z}{n}\right)$.

$$\Omega_{\bar{t}}\left(\frac{z}{n}\right) = \left(\Omega_t\left(\frac{z}{n}\right)\right)^n$$

$$= \left(e^{-\mu\frac{z}{n}} \; e^{\frac{\sigma^2 z^2}{2n^2}} \right)^n \quad \text{by Theorem 3.05}$$

$$= e^{-\mu z} \; e^{\frac{\sigma^2 z^2}{2n}}$$

Now by Theorem 3.05, τ is normally distributed with mean μ and variance $\frac{\sigma^2}{n}$.

NOTATION 4.04. If $\tau_1, \tau_2, \ldots, \tau_n$ is a sequence of n independently and normally distributed chance variables each with mean μ and standard deviation σ, then $\bar{x} = \frac{\tau_1 + \tau_2 + \ldots + \tau_n}{n}$ has standard deviation $\frac{\sigma}{n}$, denoted $\sigma_{\bar{x}}$.

DEFINITION 4.04. If A is a random sample of size m from a chance variable τ, then the statement that $\bar{t}$ is the sample mean implies that $\bar{t} = \frac{t_1 + t_2 + \ldots + t_m}{m}$ where $t_i = \tau(s_i)$ for $(s_i, \tau(s_i)) \in A$, $i = 1, 2, \ldots, m$.

In order to acquire a more accurate concept of the purpose of this paper, an example is given illustrating a method often used in accepting or rejecting a hypothesis and the proposed method.

EXAMPLE. The average unit cost of producing a product is μ = \$5.00 with a standard deviation of \$.10. A sample of 25 units is randomly selected, and the sample is obtained from a universe which is normally distributed. The hypothesis states that the sample has been selected from the universe specified. One approach for making a judgment of acceptance or rejection concerning the hypothesis is to set up a 95% confidence interval for the sample mean. In order to do this, it must be assumed that the sample mean is the sum of independently distributed chance variables.

The 95% confidence interval for the sample mean is derived as follows:

$$P(|\bar{\bar{x}}-\mu| \leq k) = .95$$

$$P(\mu-k \leq \bar{\bar{x}} \leq \mu+k) = .95$$

NOTE. Since $\bar{\bar{x}}$ has the normal distribution, then P is a continuous function and $P(\mu-k \leq \bar{\bar{x}} \leq \mu+k) = P(\mu-k < \bar{\bar{x}} \leq \mu+k)$.

Let $z = \dfrac{\bar{\bar{x}} - \mu}{\sigma_{\bar{\bar{x}}}}$ where μ is the mean and $\sigma_{\bar{\bar{x}}}$ is the standard deviation of the chance variable $\bar{\bar{x}}$.

$$P(\mu - k \leq z\sigma_{\bar{\bar{x}}} + \mu \leq \mu + k) = .95$$

$$P(-k \leq z\sigma_{\bar{\bar{x}}} \leq k) = .95$$

$$P\left(\frac{-k}{\sigma_{\bar{\bar{x}}}} \leq z \leq \frac{k}{\sigma_{\bar{\bar{x}}}}\right) = .95$$

$$P\left(z \leq \frac{k}{\sigma_{\bar{\bar{x}}}}\right) - P\left(z < \frac{-k}{\sigma_{\bar{\bar{x}}}}\right) = .95$$

$$P\left(z \leq \frac{k}{\sigma_{\bar{\bar{x}}}}\right) - \left\{1 - P\left(z \leq \frac{k}{\sigma_{\bar{\bar{x}}}}\right)\right\} = .95$$

$$2P\left(z \leq \frac{k}{\sigma_{\bar{\bar{x}}}}\right) = 1.95$$

$$P\left(z \leq \frac{k}{\sigma_{\bar{\bar{x}}}}\right) = .975$$

A 95% confidence interval for a sample point is derived as follows:

$P(|\tau-\mu| \leq J) = .95$

$P(\mu-J \leq \tau \leq \mu+J) = .95$

NOTE. Since τ is normally distributed, then τ is a continuous chance variable and $P(\mu-J \leq \tau \leq \mu+J) = P(\mu-J < \tau \leq \mu+J)$.

Let $z = \dfrac{\tau - \mu}{\sigma}$ where μ is the mean and σ is the standard deviation of τ.

$P(\mu - J \leq z\sigma + \mu \leq \mu + J) = .95$

$P(-J \leq z\sigma \leq J) = .95$

$p\left(-\frac{J}{\sigma} \leq z \leq \frac{J}{\sigma}\right) = .95$

$P\left(z \leq \frac{J}{\sigma}\right) - P\left(z < -\frac{J}{\sigma}\right) = .95$

$P\left(z \leq \frac{J}{\sigma}\right) - \left\{1 - P\left(z \leq \frac{J}{\sigma}\right)\right\} = .95$

$2P\left(z \leq \frac{J}{\sigma}\right) = 1.95$

$P\left(z \leq \frac{J}{\sigma}\right) = .975$

g Using Table H in the appendix of STATISTICAL METHODS FOR BUSINESS DECISIONS[1], it is found that $\dfrac{k}{\sigma_{\bar{x}}}$ has a value of 1.96.

Table H is a table for area under the normal curve.

$\sigma_{\bar{x}} = \dfrac{\sigma}{\sqrt{n}} = \dfrac{.1}{\sqrt{25}} = \dfrac{.1}{5} = .02$

$\dfrac{k}{\sigma_{\bar{x}}} = \dfrac{k}{.02}$, $\dfrac{k}{.02} = 1.96$, $k = (1.96)(.02)$, $k = .0392 \doteq .04$.

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[1]Charles T. Clark and Lawrence L. Schkade, Statistical Methods for Business Decisions, Appendix.

Now

$P(\mu - k \leq \bar{\bar{x}} \leq \mu + k) = .95$

$P(5.00 - .04 \leq \bar{\bar{x}} \leq 5.00 + .04) = .95$

$P(\$4.96 \leq \bar{\bar{x}} \leq \$5.04) = .95$

A 95% confidence interval for $\bar{\bar{x}}$ where $\bar{\bar{x}}$ is the sample is thus [$4.96,$5.04], and if the sample mean falls in this interval, then the hypothesis is accepted. If the sample mean falls outside this interval, then the hypothisis is rejected.

Using Table H in the appendix of STATISTICAL METHODS FOR BUSINESS DECISIONS[2], it is found that $\frac{J}{\sigma}$ has a value of 1.96.

NOTE. The above-mentioned table will be used in the remainder of this chapter to obtain z values.

$$\frac{J}{\sigma} = \frac{J}{.1} \;,\; \frac{J}{.1} = 1.96 \;,\; J = (.1)(1.96) \;,\; J = .196 \doteq .20.$$

Now

$P(\mu - J \leq \tau \leq \mu + J) = .95$

$P(5.00 - .20 \leq \tau \leq 5.00 + .20) = .95$

$P(\$4.80 \leq \tau \leq \$5.20) = .95$

A 95% confidence interval for a sample point is thus [$4.80,$5.20].

If $z_{\frac{\alpha}{2}}$ is the value of the normal distribution leaving an area of $\frac{\alpha}{2}$ to the right of μ and σ_τ is the standard deviation of the normally distributed chance variable τ for which an interval is being calculated, then an α% confidence

[2] *Ibid.*

interval for τ with universe mean μ and standard deviation σ_τ is $\mu - z_{\frac{\alpha}{2}}\sigma_\tau \leq \tau \leq \mu + z_{\frac{\alpha}{2}}\sigma_\tau$.

In the proposed approach for making a decision of acceptance or rejection of the hypothesis an α% confidence interval is set up for a sample point. In addition to this, the number of sample points which should fall in this interval with a probability of β% is determined.

Suppose a 99% confidence interval is specified.

$$P(\mu - z_{\frac{\alpha}{2}}\sigma \leq \tau \leq \mu + z_{\frac{\alpha}{2}}\sigma) = .99$$

$$P(5.00 - 2.58(.1) \leq \tau \leq 5.00 + 2.58(.1)) = .99$$

The $z_{\frac{\alpha}{2}}$ value was obtained from Table H as mentioned before.

$$P(5.00 - .258 \leq \tau \leq 5.00 + .258) = .99$$

$$P(4.742 \leq \tau \leq 5.258) = .99$$

$$P(\$4.74 \leq \tau \leq \$5.26) = .99$$

Therefore, the 99% confidence interval for τ is [\$4.74,\$5.26].

Likewise, if α = 90%,

$$P(\mu - z_{\frac{\alpha}{2}}\sigma \leq \tau \leq \mu + z_{\frac{\alpha}{2}}\sigma) = .9$$

$$P(5.00 - 1.64(.1) \leq \tau \leq 5.00 + 1.64(.1)) = .9$$

$$P(5.00 - .164 \leq \tau \leq 5.00 + .164) = .9$$

$$P(4.836 \leq \tau \leq 5.164) = .9$$

$$P(\$4.84 \leq \tau \leq \$5.16) = .9$$

Therefore, the 90% confidence interval for τ is [\$4.84,\$5.16].

Using Table D on cumulative binomial probability distributions from the appendix of STATISTICAL METHODS FOR BUSINESS DECISIONS[3], the following probabilities are calculated.

NOTATION. Let $r \geq i$, $i = 0,1,\ldots,25$, denote the set such that $r = \bigcup_{k=i}^{25} \{s \; \varepsilon \; P_k\}$.

A. For a 95 % confidence interval; that is,

$P(\$4.80 \leq \tau \leq \$5.20) = .95$:

1) $P(r \geq 25 | 25, .95) = 1 = P(r \geq 1 | 25, .05)$ by Theorem 3.04

$= 1 - .7266$

$= .2774.$

2) $P(r \geq 24 | 25, .95) = 1 - P(r \geq 2 | 25, .05)$

$= 1 - .3567$

$= .6433.$

3) $P(r \geq 23 | 25, .95) = 1 - P(r \geq 3 | 25, .05)$

$= 1 - .1271$

$= .8729.$

4) $P(r \geq 22 | 25, .95) = 1 - P(r \geq 4 | 25, .05)$

$= 1 - .0258$

$= .9742.$

B. For a 99% confidence interval; that is,

$P(\$4.74 \leq \tau \leq \$5.26) = .99$:

1) $)(r \geq 25 | 25, .99) = 1 - P(r \geq 1 | 25, .01)$

$= 1 - .2222$

$= .7778$

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[3] Ibid.

2) $P(r \geq 24 | 25, .99) = 1 - P(r \geq 2 | 25, .01)$

$= 1 - .0258$

$= .9742.$

3) $P(r \geq 23 | 25, .99) = 1 - P(r \geq 3 | 25, .01)$

$= 1 - .0020$

$= .9980$

C. For a 90% confidence interval; that is,

$P(\$4.84 \leq \tau \leq \$5.16) = .90$:

1) $P(r \geq 25 | 25, .9) = 1 - P(r \geq 1 | 25, .1)$

$= 1 - .9282$

$= .0718.$

2) $P(r \geq 24 | 25, .9) = 1 - P(r \geq 2 | 25, .1)$

$= 1 - .7288$

$= .2712.$

3) $P(r \geq 23 | 25, .9) = 1 - P(r \geq 3 | 25, .1)$

$= 1 - .4629$

$= .5371$

4) $P(r \geq 22 | 25, .9) = 1 - P(r \geq 4 | 25, .1)$

$= 1 - .2364$

$= .7636.$

5) $P(r \geq 21 | 25, .9) = 1 - P(r \geq 5 | 25, .1)$

$= 1 - .0980$

$= .9020.$

6) $P(r \geq 20 | 25, .9) = 1 - P(r \geq 6 | 25, .1)$

$= 1 - .0334$

$= .9666.$

7) $P(r \geq 19 | 25, .9) = 1 - P(r \geq 7 | 25, .1)$

$= 1 - .0095$

$= .9905$

With the proposed example in mind, several different samples will be taken in order to illustrate the advantage and disadvantage of each method. Each sample point will be designated by cost per item.

Sample A:

x_1 = \$4.90	x_{10} = \$5.02	x_{18} = \$4.93
x_2 = \$4.99	x_{11} = \$5.04	x_{19} = \$5.02
x_3 = \$4.98	x_{12} = \$5.00	x_{20} = \$5.04
x_4 = \$5.01	x_{13} = \$5.00	x_{21} = \$4.75
x_5 = \$5.10	x_{14} = \$4.95	x_{22} = \$4.90
x_6 = \$5.06	x_{15} = \$5.04	x_{23} = \$5.20
x_7 = \$5.03	x_{16} = \$5.05	x_{24} = \$5.10
x_8 = \$4.99	x_{17} = \$4.98	x_{25} = \$4.98
x_9 = \$4.96		

$$\bar{x} = \frac{\sum_{i=1}^{25} x_i}{25} = \frac{125.01}{25} \doteq \$5.00$$

Therefore, $\bar{x}$ falls within the confidence interval [\$4.96,\$5.04], and the hypothesis is accepted.

As shown by previous calculations, one can be 96.59% confident that at least 22 of the sample points will fall in the interval [\$4.80,\$5.20]. By going back and checking each sample point, it is found that 24 of them lie in the designated confidence interval, and the hypothesis is accepted.

Also, one can be 97.42% confident that at least 24 of the sample points will fall in the interval [\$4.74,\$5.26]. By checking each sample point it is found that all 25 of them lie in the designated confidence interval, and the hypothesis is accepted.

One can be 90% confident that at least 21 of the sample points will fall in the interval [\$4.84,\$5.16]. By checking each sample point, it is found that 23 of them lie in the designated confidence interval, and the hypothesis is accepted.

Consider another sample.

Sample B:

x_1 = \$4.50	x_{10} = \$5.00	x_{18} = \$4.99
x_2 = \$4.55	x_{11} = \$5.25	x_{19} = \$5.05
x_3 = \$4.60	x_{12} = \$5.40	x_{20} = \$5.31
x_4 = \$4.58	x_{13} = \$5.48	x_{21} = \$4.80
x_5 = \$4.52	x_{14} = \$5.30	x_{22} = \$4.70
x_6 = \$5.00	x_{15} = \$5.35	x_{23} = \$4.91
x_7 = \$4.75	x_{16} = \$5.45	x_{24} = \$5.11
x_8 = \$4.70	x_{17} = \$4.98	x_{25} = \$5.00
x_9 = \$5.49		

$$\bar{x} = \frac{\sum_{i=1}^{25} x_i}{25} = \frac{124.77}{25} \doteq \$4.99$$

Therefore, $\bar{x}$ falls within the confidence interval [\$4.96,\$5.04], and the hypothesis is accepted.

As shown by previous calculations, one can be 96.59% confident that at least 22 of the sample points will fall in the interval [\$4.80,\$5.20]. By going back and checking each sample point, it is found that only 9 of them lie in the designated confidence interval, and the hypothesis is rejected.

In this particular case, the calculation of $\bar{x}$ gives a misleading conclusion of the values of the sample points. Actually the alternate approach of finding the number of sample points within the specified confidence interval reveals the wide variation of unit costs. This variation may not be detected if only the sample mean is calculated.

Consider the following sample:

Sample C:

x_1 = \$5.10	x_{10} = \$5.15	x_{18} = \$5.11
x_2 = \$5.10	x_{11} = \$5.17	x_{19} = \$5.12
x_3 = \$5.10	x_{12} = \$5.20	x_{20} = \$5.14
x_4 = \$5.15	x_{13} = \$5.17	x_{21} = \$5.18
x_5 = \$5.15	x_{14} = \$5.25	x_{22} = \$5.13
x_6 = \$5.16	x_{15} = \$5.19	x_{23} = \$5.14
x_7 = \$5.16	x_{16} = \$5.19	x_{24} = \$5.18
x_8 = \$5.09	x_{17} = \$5.15	x_{25} = \$5.20
x_9 = \$5.20		

$$\bar{x} = \frac{\sum_{i=1}^{25} x_i}{25} = \$5.15$$

Therefore, $\bar{x}$ does not fall within the confidence interval [$4.96,$5.04], and the hypothesis is rejected.

By checking each sample point it is found that 24 of them lie in the confidence interval [$4.80,$5.20]. Since one can be 96.59% confident that at least 22 of the sample points will fall in the designated interval, the hypothesis is accepted.

This example points out a disadvantage of using the alternate approach for making a judgment concerning a sample rather than finding the value of the sample mean. In this case the production cost of each unit is more than average which indicates that total production costs are probably more than average. However, this fact may not be detected if an acceptable number of unit costs fall within the desired confidence interval.

Samples which are composed of normally distributed chance variables have been used in the chapter in order to illustrate the difference between the two methods of using statistical measures for making decisions concerning a sample or a universe. The normal distribution function was chosen because the distribution of the sample mean is known in this case. However, not all universes are normally and independently distributed and thus the distribution of the sample mean is not known. Upon this event, the proposed approach of finding the number of sample points within a designated confidence interval would be of possible use. This alternate approach may be used provided the distribution of the universe is known.

CHAPTER V

THE PROPOSED APPROACH INTRODUCED IN N SPACE

NOTATION 5.00. E_n denotes the collection of all n-term sequences of real numbers.

DEFINITION 5.00. The statement that $M_{n,m}$ is a random sample from E_n means

1) $n \geq 1$ and $m \geq 2$ are positive integers;
2) there exists an n-term sequence of chance variables, $\tau_1, \tau_2, \ldots, \tau_n$;
3) $M_{n,m}$ is an $n \times m$ matrix;
4) R_i, the i th row, is a random sample from a chance variable τ_i, $i = 1,2,\ldots,n$;
5) $\bar{\bar{f}}$ is a function with domain the collection of statements

   $\{x_{ij} \varepsilon (\tau_i \leq a_i) \mid (x_{ij}, \tau_i(x_{ij})) \varepsilon R_i$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$, and a_i, $i = 1,2,\ldots,n$, is a real number$\} \cup \{x_{ij} \not\varepsilon (\tau_i \leq a_i) \mid (x_{ij}, \tau_i(x_{ij})) \varepsilon R_i$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$, and $a_i, i = 1,2,\ldots,n$, is a real number$\}$

   such that $\bar{\bar{f}}[x_{ij} \varepsilon (\tau_i \leq a_i)] = P_i(\tau_i \leq a_i)$ and $\bar{\bar{f}}[x_{ij} \not\varepsilon (\tau_i \leq a_i)] = P_i(\tau_i > a_i)$ where (D_i, P_i) is a probability distribution, and $(\tau_i \leq a_i) \in D_i$, $i = 1,2,...,n$;

6) given a collection of number intervals $[a_i,b_i]$, $i = 1,2,...,n$, if A_i, $i = 1,2,...,n$, represents either of $[x_{ij} \in (a_i<\tau_i\leq b_i)]$ or $[x_{ij} \in (a_i<\tau_i\leq b_i)]'$, then $\bar{\bar{f}}(A_1 \wedge A_2 \wedge \; . \; . \; . \wedge A_k) = \sum_{i=1}^{k} \bar{\bar{f}}(A_i)$ for $k = 2,3,...,n$.

NOTATION 5.01 Suppose $\hat{x} = (\hat{x}_1, \hat{x}_2, \; . \; . \; . \; , \hat{x}_n) \in E_n$ and $x_j = (x_{1j}, x_{2j}, \; . \; . \; . \; , x_{nj}) \in E_n$ for each x_j, $j = 1,2,...,m$. The notation

$$\sum_{j=1}^{m} |x_j - \hat{x}| = \sum_{j=1}^{m} [(x_{1j} - \hat{x}_1)^2 + (x_{2j} - \hat{x}_2)^2 + \; . \; . \; . \; + (x_{nj} - \hat{x}_n)^2],$$

and the notation

$$|x_j - \hat{x}| = \sum_{i=1}^{n} (x_{ij} - \hat{x}_i)^2.$$

DEFINITION 5.01. The statement that B is the center of the random sample $M_{n,m}$ from E_n means that $B \in E_n$ and if $x \in E_n$, then $\sum_{j=1}^{m} |\tau(s_j) - x| \geq \sum_{j=1}^{m} |\tau(s_j) - B|$ where

$\tau(s_j) = [\tau_1(s_{1j}), \tau_2(s_{2j}), \; . \; . \; . \; , \tau_n(s_{nj})]$

is from the j th column of $M_{n,m}$, $j = 1,2,...,m$.

THEOREM 5.00. Suppose $\tau_1, \tau_2, \ldots, \tau_n$ is a sequence of chance variables with distribution functions $F_1, F_2, \ldots, F_n$ respectively. Let $\mu_1, \mu_2, \ldots, \mu_n$ denote the means of $\tau_1, \tau_2, \ldots, \tau_n$ respectively. If $(x_1, x_2, \ldots, x_n) \in E_n$ and $\int_{-\infty}^{\infty} (t - x_i)^2 \, d\, F_i(t)$ and $\int_{-\infty}^{\infty} (t - \mu_i)^2 \, d\, F_i(t)$ exist for $i = 1,2,\ldots,n$, then

$$\sum_{i=1}^{n} \left[\int_{-\infty}^{\infty} (t - x_i)^2 \, d\, F_i(t)\right] \geq \sum_{i=1}^{n} \left[\int_{-\infty}^{\infty} (t - \mu_i)^2 \, d\, F_i(t)\right].$$

Proof:

Assume that the conclusion is false; then there exists $(x_1, x_2, \ldots, x_n) \in E_n$ such that

$$\sum_{i=1}^{n} \left[\int_{-\infty}^{\infty} (t - x_i)^2 \, d\, F_i(t)\right] < \sum_{i=1}^{n} \left[\int_{-\infty}^{\infty} (t - \mu_i)^2 \, d\, F_i(t)\right].$$

$$\sum_{i=1}^{n} \left[\int_{-\infty}^{\infty} (t^2 - 2tx_i + x_i^2) \, d\, F_i(t)\right]$$

$$< \sum_{i=1}^{n} \left[\int_{-\infty}^{\infty} (t^2 - 2t\mu_i + \mu_i^2) \, d\, F_i(t)\right]$$

$$\sum_{i=1}^{n} \left[\int_{-\infty}^{\infty} t^2 \, d\, F_i(t) - \int_{-\infty}^{\infty} 2tx_i \, d\, F_i(t) + \int_{-\infty}^{\infty} x_i^2 \, d\, F_i(t)\right]$$

$$< \sum_{i=1}^{n} \left[\int_{-\infty}^{\infty} t^2 \, d\, F_i(t) - \int_{-\infty}^{\infty} 2t\mu_i \, d\, F_i(t) + \int_{-\infty}^{\infty} \mu_i^2 \, d\, F_i(t)\right]$$

$$\sum_{i=1}^{n} \int_{-\infty}^{\infty} t^2 \, d\, F_i(t) - \sum_{i=1}^{n} \int_{-\infty}^{\infty} 2tx_i \, d\, F_i(t) + \sum_{i=1}^{n} \int_{-\infty}^{\infty} x_i^2 \, d\, F_i(t)$$

$$< \sum_{i=1}^{n} \int_{-\infty}^{\infty} t^2 \, d\, F_i(t) - \sum_{i=1}^{n} \int_{-\infty}^{\infty} 2t\mu_i \, d\, F_i(t)$$

$$+ \sum_{i=1}^{n} \int_{-\infty}^{\infty} \mu_i^2 \, d\, F_i(t)$$

$$- \sum_{i=1}^{n} \int_{-\infty}^{\infty} 2tx_i \, d\, F_i(t) + \sum_{i=1}^{n} \int_{-\infty}^{\infty} x_i^2 \, d\, F_i(t)$$

$$< - \sum_{i=1}^{n} \int_{-\infty}^{\infty} 2t\mu_i \, d\, F_i(t) + \sum_{i=1}^{n} \int_{-\infty}^{\infty} \mu_i^2 \, d\, F_i(t)$$

$$- \sum_{i=1}^{n} 2x_i \int_{-\infty}^{\infty} t \, d\, F_i(t) + \sum_{i=1}^{n} x_i^2 \int_{-\infty}^{\infty} 1 \, d\, F_i(t)$$

$$< - \sum_{i=1}^{n} 2\mu_i \int_{-\infty}^{\infty} t \, d\, F_i(t) + \sum_{i=1}^{n} \mu_i^2 \int_{-\infty}^{\infty} 1 \, d\, F_i(t)$$

$$- \sum_{i=1}^{n} 2x_i \, (\mu_i) + \sum_{i=1}^{n} x_i^2 \, (1) < - \sum_{i=1}^{n} 2\mu_i \, (\mu_i) + \sum_{i=1}^{n} \mu_i^2 \, (1)$$

by Theorem 2.03 and Definition 2.25

$$- \sum_{i=1}^{n} 2x_i\mu_i + \sum_{i=1}^{n} x_i^2 < - \sum_{i=1}^{n} 2\mu_i^2 + \sum_{i=1}^{n} \mu_i^2$$

$$- \sum_{i=1}^{n} 2x_i\mu_i + \sum_{i=1}^{n} x_i^2 < - \sum_{i=1}^{n} (2\mu_i^2 - \mu_i^2)$$

$$- \sum_{i=1}^{n} 2x_i\mu_i + \sum_{i=1}^{n} x_i^2 < - \sum_{i=1}^{n} \mu_i^2$$

$$\sum_{i=1}^{n} \mu_i^2 - \sum_{i=1}^{n} 2x_i\mu_i + \sum_{i=1}^{n} x_i^2 < 0$$

$$\sum_{i=1}^{n} (\mu_i^2 - 2x_i\mu_i + x_i^2) < 0$$

$$\sum_{i=1}^{n} (\mu_i - x_i)^2 < 0.$$

A contradiction has been reached, for if A is a real number, then $A^2 \geq 0$. Now for each μ_i, x_i, $i = 1,2,\ldots,n$, $\mu_i - x_i$ is a real number. If $a,b \geq 0$ are real, then $a + b \geq 0$. Consequently, it follows that

$$\sum_{i=1}^{n} [\int_{-\infty}^{\infty} (t - x_i)^2 \, d\, F_i(t)] \geq \sum_{i=1}^{n} [\int_{-\infty}^{\infty} (t - \mu_i)^2 \, d\, F_i(t)].$$

THEOREM 5.01. Suppose $(y_1, y_2, \ldots, y_n) \in E_n$ such that $\int_{-\infty}^{\infty} (t - y_i)^2 \, d\, F_i(t)$ and $\int_{-\infty}^{\infty} (t - x_i)^2 \, d\, F_i(t)$ exist where $(x_1, x_2, \ldots, x_n) \in E_n$ and F_i is the distribution function for τ_i, $i = 1,2,\ldots,n$, and

$$\sum_{i=1}^{n} \int_{-\infty}^{\infty} (t - y_i)^2 \, d\, F_i(t) \leq \sum_{i=1}^{n} \int_{-\infty}^{\infty} (t - x_i)^2 \, d\, F_i(t),$$

then $y_i = \mu_i$, $i = 1,2,\ldots,n$, where μ_i denotes the mean of τ_i.

Proof:

Assume that the conclusion is false; then there is at least one y_i, μ_i such that $y_i \neq \mu_i$, $i = 1,2,\ldots,n$. Let $\mu_1, \mu_2, \ldots, \mu_n$ denote the means of $\tau_1, \tau_2, \ldots, \tau_n$ respectively such that $\int_{-\infty}^{\infty} (t - \mu_i)^2 \, d\, F_i(t)$ exists where F_i is the distribution function for τ_i, $i = 1,2,\ldots,n$. From the hypothesis of this theorem and by Theorem 5.00, it is concluded that

$$\sum_{i=1}^{n} \int_{-\infty}^{\infty} (t - y_i)^2 \, d\, F_i(t) = \sum_{i=1}^{n} \int_{-\infty}^{\infty} (t - \mu_i)^2 \, d\, F_i(t).$$

$$\sum_{i=1}^{n} \int_{-\infty}^{\infty} (t^2 - 2ty_i + y_i^{\,2}) \, d\, F_i(t)$$

$$= \sum_{i=1}^{n} \int_{-\infty}^{\infty} (t^2 - 2t\mu_i + \mu_i^{\,2}) \, d\, F_i(t)$$

$$\sum_{i=1}^{n} [\int_{-\infty}^{\infty} t^2 \, d\, F_i(t) - \int_{-\infty}^{\infty} 2ty_i \, d\, F_i(t) + \int_{-\infty}^{\infty} y_i^{\,2} \, d\, F_i(t)]$$

$$= \sum_{i=1}^{n} [\int_{-\infty}^{\infty} t^2 \, d\, F_i(t) - \int_{-\infty}^{\infty} 2t\mu_i \, d\, F_i(t) + \int_{-\infty}^{\infty} \mu_i^{\,2} \, d\, F_i(t)]$$

$$\sum_{i=1}^{n} \int_{-\infty}^{\infty} t^2 \, d\, F_i(t) - \sum_{i=1}^{n} \int_{-\infty}^{\infty} 2ty_i \, d\, F_i(t) + \sum_{i=1}^{n} \int_{-\infty}^{\infty} y_i^{\,2} \, d\, F_i(t)$$

$$= \sum_{i=1}^{n} \int_{-\infty}^{\infty} t^2 \, d\, F_i(t) - \sum_{i=1}^{n} \int_{-\infty}^{\infty} 2t\mu_i \, d\, F_i(t)$$

$$+ \sum_{i=1}^{n} \int_{-\infty}^{\infty} \mu_i^{\,2} \, d\, F_i(t)$$

$$- \sum_{i=1}^{n} \int_{-\infty}^{\infty} 2ty_i \, d\, F_i(t) + \sum_{i=1}^{n} \int_{-\infty}^{\infty} y_i^{\,2} \, d\, F_i(t)$$

$$= - \sum_{i=1}^{n} \int_{-\infty}^{\infty} 2t\mu_i \, d\, F_i(t) + \sum_{i=1}^{n} \int_{-\infty}^{\infty} \mu_i^{\,2} \, d\, F_i(t)$$

$$- \sum_{i=1}^{n} 2y_i \int_{-\infty}^{\infty} t \, d\, F_i(t) + \sum_{i=1}^{n} y_i^{\,2} \int_{-\infty}^{\infty} 1 \, d\, F_i(t)$$

$$= - \sum_{i=1}^{n} 2\mu_i \int_{-\infty}^{\infty} t \, d\, F_i(t) + \sum_{i=1}^{n} \mu_i^{\,2} \int_{-\infty}^{\infty} 1 \, d\, F_i(t)$$

$$- \sum_{i=1}^{n} 2y_i \, (\mu_i) + \sum_{i=1}^{n} y_i^{\,2} \, (1) = - \sum_{i=1}^{n} 2\mu_i \, (\mu_i) + \sum_{i=1}^{n} \mu_i^{\,2} \, (1)$$

by Theorem 2.03 and Definition 2.25

$$-\sum_{i=1}^{n} 2y_i\mu_i + \sum_{i=1}^{n} y_i^2 = -\sum_{i=1}^{n} 2\mu_i^2 + \sum_{i-1}^{n} \mu_i^2$$

$$-\sum_{i=1}^{n} 2y_i\mu_i + \sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} (-2\mu_i^2 + \mu_i^2)$$

$$-\sum_{i=1}^{n} 2y_i\mu_i + \sum_{i=1}^{n} y_i^2 = -\sum_{i=1}^{n} \mu_i^2$$

$$\sum_{i=1}^{n} \mu_i^2 - \sum_{i=1}^{n} 2y_i\mu_i + \sum_{i=1}^{n} y_i^2 = 0$$

$$\sum_{i=1}^{n} (\mu_i^2 - 2y_i\mu_i + y_i^2) = 0$$

$$\sum_{i=1}^{n} (\mu_i - y_i)^2 = 0.$$

Therefore, $\mu_i - y_i = 0$ for each μ_i, y_i, $i = 1,2,\ldots,n$, and $\mu_i = y_i$, $i = 1,2,\ldots,n$.

A contradiction has been reached, and it follows that $\mu_i = y_i$ for each μ_i, y_i, $i = 1,2,\ldots,n$.

DEFINITION 5.02. If f is a function which maps a number or a set of numbers into the real numbers and if a is an element of the domain of f such that for each x, $x \in$ domain of f, $f(a) \le f(x)$, then f(a) is a minimum for f.

THEOREM 5.02. Suppose that p, q, r is a real-number sequence and that $f(x) = px^2 - 2qx + r$ for each real number x.

A) In order that $f(x) \ge 0$ for each real number x, it is necessary and sufficient that $p \ge 0$, $r \ge 0$, $pr \ge q^2$.

B) If $f(x) \geq 0$ for each real number x and $p > 0$, then

$f(x) \geq f\left(\frac{q}{p}\right)$ for each real number x, and $f\left(\frac{q}{p}\right) = \frac{pr - q^2}{p}$.

Proof:

1. Let p, q, r be a real number sequence and let

   $f(x) = px^2 - 2qx + r$ for each real number x.

2. Let x be a real number.

Part A:

1. Let $p \geq 0$, $r \geq 0$, $pr \geq q$ and show that $f(x) \geq 0$.

2. If $p \neq 0$, then

$$f(x) = px^2 - 2qx + r$$

$$\frac{f(x)}{p} = x^2 - \frac{2qx}{p} + \frac{r}{p}$$

$$= x^2 - \frac{2qx}{p} + \frac{q^2}{p^2} - \frac{q^2}{p^2} + \frac{r}{p}$$

$$= \left(x - \frac{q}{p}\right)^2 + \left(\frac{r}{p} - \frac{q^2}{p^2}\right)$$

$$= \left(x - \frac{q}{p}\right)^2 + \left(\frac{rp - q^2}{p^2}\right)$$

$\frac{f(x)}{p} \geq 0$ because $\left(x - \frac{q}{p}\right)^2 \geq 0$ and since $p > 0$, then

$p^2 > 0$. By the hypothesis $pr \geq q^2$; therefore,

$pr - q^2 \geq 0$, and $\frac{pr - q^2}{p^2} \geq 0$.

Consequently, $x - \frac{q}{p}^{\,2} + \frac{pr - q^2}{p^2} \geq 0$.

$f(x) \geq 0$,

3. Suppose $p = 0$; then

$$f(x) = px^2 - 2qx + r$$

$$= -2qx + r$$

$= -2(0)x + r$ because $pr \geq q^2$

$0r \geq q^2$

$0 \geq q^2$

Therefore, $q = 0$.

$= 0 + r$

$= r$

≥ 0 by the hypothesis.

Therefore, $f(x) > 0$.

4. Conversely, let $f(x) \geq 0$ and show that $p \geq 0$, $r \geq 0$, $pr \geq q^2$.

5. For $x = 0$

$0 \leq f(0)$

$= p(0)^2 - 2q(0) + r$

$= 0 + 0 + r$

$= r$.

Therefore, $0 \leq r$.

6. For $x = \frac{q}{p}$

$$0 \leq f\left(\frac{q}{p}\right) = p\left(\frac{q}{p}\right)^2 - 2q\left(\frac{q}{p}\right) + r$$

$$= \frac{q^2}{p} - 2\frac{q^2}{p} + r$$

$$= -\frac{q^2}{p} + r$$

$$= \frac{pr - q^2}{p}$$

Therefore, $0 \leq \frac{pr - q^2}{p}$

$0 \leq pr - q^2$

$q^2 \leq pr$.

7. $(px - q)^2 \geq 0.$

$$p^2x^2 - 2pqx + q^2 \geq 0$$

$$p^2x^2 - 2pqx + q^2 + pr \geq pr$$

$$p^2x^2 - 2pqx + pr \geq pr - q^2$$

$p(px^2 - 2qx + r) \geq 0$ since $pr - q^2 \geq 0$

$p \geq 0$ since $px^2 - 2qx + r \geq 0.$

Part B:

1. Let $f(x) \geq 0.$

2. Let $p > 0.$

3. For $x = \frac{q}{p}$

$$f\left(\frac{q}{p}\right) = p\left(\frac{q}{p}\right)^2 - 2q\left(\frac{q}{p}\right) + r$$

$$= \frac{q^2}{p} - 2\frac{q^2}{p} + r$$

$$= r - \frac{q^2}{p}$$

$$= \frac{pr - q^2}{p}$$

Therefore, $f\left(\frac{q}{p}\right) = \frac{pr - q^2}{p}$

4. $(px - q)^2 \geq 0$

$$p^2x^2 - 2pqx + q^2 \geq 0$$

$$p^2x^2 - 2pqx \geq -q^2$$

$$p^2x^2 - 2pqx + pr \geq pr - q^2$$

$$p(px^2 - 2pqx + pr) \geq pr - q^2$$

$px^2 - 2pqx + pr \geq \frac{pr - q^2}{p}$ since $p > 0$

Consequently, $f(x) \geq f\left(\frac{q}{p}\right).$

THEOREM 5.03. If $M_{n,m}$ is a random sample from E_n, then the center of $M_{n,m}$ is $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ where

$$\bar{x}_i = \frac{1}{m} \sum_{j=1}^{m} \tau_i(s_{ij}), \; i = 1,2,\ldots,n.$$

Proof:

Let $M_{n,m}$ be a random sample from E_n. Let f be a function from E_n into the reals such that if $(x_1, x_2, \ldots, x_n) \in E_n$, and

$(\tau_1(s_{1j}), \tau_2(s_{2j}), \ldots, \tau_n(s_{nj})) = (x_{1j}, x_{2j}, \ldots, x_{nj})$

is from the j th column in $M_{n,m}$, $j = 1,2,\ldots,m$, then

$$f(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{m} [(x_{1j} - x_1)^2 + (x_{2j} - x_2)^2 + \ldots + (x_{nj} - x_n)^2].$$

Let $g_i(x_i) = \sum_{j=1}^{m} (x_{ij} - x_i)^2$ for $i = 1,2,\ldots,n$.

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) &= \sum_{j=1}^{m} [(x_{1j} - x_1)^2 + (x_{2j} - x_2)^2 + \ldots + (x_{nj} - x_n)^2] \\ &= \sum_{j=1}^{m} (x_{1j} - x_1)^2 + \sum_{j=1}^{m} (x_{2j} - x_2)^2 \\ &\quad + \ldots + \sum_{j=1}^{m} (x_{nj} - x_n)^2 \\ &= g_1(x_1) + g_2(x_2) + \ldots + g_n(x_n). \end{aligned}$$

Therefore, $f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} g_i(x_i)$.

Let $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) \in E_n$ such that

$$\bar{x}_i = \frac{1}{m}\sum_{j=1}^{m} x_{ij},\ i = 1,2,\ldots,n,$$

$$f(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) = g_1(\bar{x}_1) + g_2(\bar{x}_2) + \ldots + g_n(\bar{x}_n)$$

$$= \sum_{j=1}^{m} (x_{1j} - \bar{x}_1)^2 + \sum_{j=1}^{m} (x_{2j} - \bar{x}_2)^2$$

$$+ \ldots + \sum_{j=1}^{m} (x_{nj} - \bar{x}_n)^2.$$

Now $g_1(\bar{x}_1) = \sum_{j=1}^{m} (x_{1j} - \bar{x}_1)^2 \geq 0$.

$$\sum_{j=1}^{m} (x_{ij} - \bar{x}_1)^2 = \sum_{j=1}^{m} (x_{1j}{}^2 - 2x_{1j}\bar{x}_1 + \bar{x}_1{}^2)$$

$$= \sum_{j=1}^{m} x_{1j}{}^2 - 2\bar{x}_1 \sum_{j=1}^{m} x_{1j} + m\bar{x}_1{}^2$$

$$= m\bar{x}_1{}^2 - 2\sum_{j=1}^{m} x_{1j}\bar{x}_1 + \sum_{j=1}^{m} x_{1j}{}^2$$

Therefore, $g_1(\bar{x}_1) = m\bar{x}_1{}^2 - 2\sum_{j=1}^{m} x_{1j}\bar{x}_1 + \sum_{j=1}^{m} x_{1j}{}^2 \geq 0$, and $g_1(\bar{x}_1)$ has the same form as the trinomial in Theorem 5.02 where $p = m$, $q = \sum_{j=1}^{m} x_{1j}$, and $r = \sum_{j=1}^{m} x_{1j}{}^2$.

$$g_1(\bar{x}_1) = m\bar{x}_1{}^2 - 2\sum_{j=1}^{m} x_{ij}\bar{x}_1 + \sum_{j=1}^{m} x_{1j}{}^2$$

$$= m\bar{x}_1{}^2 - 2\frac{m}{m}\sum_{j=1}^{m} x_{1j}\bar{x}_1 + \sum_{j=1}^{m} x_{1j}{}^2$$

$$= m\bar{x}_1{}^2 - 2m\bar{x}_1 \frac{\sum_{j=1}^{m} x_{1j}}{m} + \sum_{j=1}^{m} x_{1j}{}^2$$

$$= m\bar{x}_1^{\,2} - 2m\bar{x}_1^{\,2} + \sum_{j=1}^{m} x_{1j}^{\,2}$$

$$= \sum_{j=1}^{m} x_{1j}^{\,2} - m\bar{x}_1^{\,2}$$

$$= \sum_{j=1}^{m} x_{1j}^{\,2} - m\left(\frac{\sum_{j=1}^{m} x_{1j}}{m}\right)^2$$

$$= \frac{m \sum_{j=1}^{m} x_{1j}^{\,2} - m^2 \dfrac{\left(\sum_{j=1}^{m} x_{1j}\right)^2}{m^2}}{m}$$

$$= \frac{m \sum_{j=1}^{m} x_{1j}^{\,2} - \left(\sum_{j=1}^{m} x_{1j}\right)^2}{m}$$

Therefore, by Theorem 5.02, $g_1(\bar{x}_1)$ is a minimum for g_1, and by the same procedure as above it can be shown that $g_2(\bar{x}_2)$, $g_3(\bar{x}_3)$, . . . , $g_n(\bar{x}_n)$ are minimums for g_2, g_3, . . . , g_n respectively.

Consequently, for $(\tau_1(s_{1j}), \tau_2(s_{2j}), \ldots, \tau_n(s_{nj})) = (x_{1j}, x_{2j}, \ldots, x_{nj})$ from the j th column in $M_{n,m}$, $j = 1,2,\ldots,m$, and $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n) \in E_n$,

$$f(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) = g_1(\bar{x}_1) + g_2(\bar{x}_2) + \ldots + g_n(\bar{x}_n)$$

$$\leq g_1(\hat{x}_1) + g_2(\hat{x}_2) + \ldots + g_n(\hat{x}_n)$$

since $g_i(\bar{x}_i)$ is a minimum for g_i, $i = 1,2,\ldots,n$.

$$= \sum_{j=1}^{m} (x_{1j} - \hat{x}_1)^2 + \sum_{j=1}^{m} (x_{2j} - \hat{x}_2)^2$$

$$+ \ . \ . \ . + \sum_{j=1}^{m} (x_{nj} - \hat{x}_n)^2$$

$$= \sum_{j=1}^{m} [(x_{1j} - \hat{x}_1)^2 + (x_{2j} - \hat{x}_2)^2$$

$$+ \ . \ . \ . + (x_{nj} - \hat{x}_n)^2]$$

$$= f(\hat{x}_1, \hat{x}_2, \ . \ . \ . \ , \hat{x}_n).$$

Therefore, $f(x_1, x_2, \ . \ . \ . \ , x_n) \leq f(\hat{x}_1, \hat{x}_2, \ . \ . \ . \ , \hat{x}_n)$.

If follows then that for every $x \in E_n$ $f(B) \leq f(x)$ where $B = (\bar{x}_1, \bar{x}_2, \ . \ . \ . \ , \bar{x}_n)$. Since for $(x_1, x_2, \ . \ . \ . \ , x_n) \in E_n$,

$$\sum_{j=1}^{m} [(x_{1j} - \bar{x}_1)^2 + (x_{2j} - \bar{x}_2)^2 + \ . \ . \ . + (x_{nj} - \bar{x}_n)^2]$$

$$\leq \sum_{j=1}^{m} [(x_{1j} - x_1)^2 + (x_{2j} - x_2)^2 + \ . \ . \ . + (x_{nj} - x_n)^2],$$

then

$$\sum_{j=1}^{m} \left|(x_{1j}, x_{2j}, \ . \ . \ . \ , x_{nj}) - (\bar{x}_1, \bar{x}_2, \ . \ . \ . \ , \bar{x}_n)\right|$$

$$\leq \sum_{j=1}^{m} \left|(x_{1j}, x_{2j}, \ . \ . \ . \ , x_{nj}) - (x_1, x_2, \ . \ . \ . \ , x_n)\right|.$$

Consequently, by Definition 5.01 $B = (\bar{x}_1, \bar{x}_2, \ . \ . \ . \ , \bar{x}_n)$ where $\bar{x}_i = \frac{1}{m} \sum_{j=1}^{m} \tau_i(s_{ij})$, $i = 1,2,\ldots,n$, and $M_{n,m}$ is a random sample from E_n.

NOTE. The property just pointed out for B, the center of a random sample from E_n, is an interesting one. This information might be useful in determining the location of places such as an airport, a space station, a medical center, an entertainment center, a recovery station in time of war, etc.

NOTATION 5.02. Given an n-term sequence of chance variables $\tau_1, \tau_2, \ldots, \tau_n$, and an integer $m \geq 2$, $Y^{\tau}_{n,m}$ denotes the set such that $x \in Y^{\tau}_{n,m}$ if and only if x is an $n \times m$ matrix, and R_i, the i th row, is a random sample from a chance variable τ_i, $i = 1,2,\ldots,n$.

NOTE. Let the rows of x be arranged such that if $[a_i,b_i]$, $i = 1,2,\ldots,n$, is a given collection of n intervals and $[x,\tau_i(x)] \in R_i$, $i = 1,2,\ldots,n$, then each $[x,\tau_i(x)]$ such that $a_i < \tau_i(x) \leq b_i$ is placed in order before those $[x,\tau_i(x)]$'s $\in R_i$ such that $\tau_i(x) \not\in (a_i,b_i]$, $i = 1,2,\ldots,n$.

DEFINITION 5.03. The statement that $(x_1, \tau_1(x_1)), (x_2, \tau_2(x_2)), \ldots, (x_n, \tau_n(x_n))$ has property Z with respect to $[a_i, b_i]$, $i = 1,2,\ldots,n$, means that $[a_i, b_i]$ is a collection of n number intervals and $a_1 < \tau_1(x_1) \leq b_1$, $a_2 < \tau_2(x_2) \leq b_2$, $\ldots$, $a_n < \tau_n(x_n) \leq b_n$.

NOTATION. 5.03. Let ρ be a function with domain $Y^{\tau}_{n,m}$ such that if $x \in Y^{\tau}_{n,m}$, then $\rho(x)$ is the number of columns in x that have property Z with respect to $[a_i, b_i]$, $i = 1,2,\ldots,n$.

THEOREM 5.04. For each $k \varepsilon \{0,1,\ldots,m\}$, let $P_k \subseteq Y^{\tau}_{n,m}$ such that $s \varepsilon P_k$ if and only if $\rho(s) = k$, then $\bigcup_{i=0}^{m} P_k = Y^{\tau}_{n,m}$. Moreover, if $k \neq r$ and $s \varepsilon P_k$, then $s \notin P_r$.

Proof:

Let $k,r = 0, 1, \ldots, m$, and let $s \varepsilon P_k$; then $\rho(s) = k$. If $k \neq r$, then $\rho(s) \neq r$ and $s \notin P_r$.

The proof showing that $\bigcup_{i=0}^{m} P_k = Y^{\tau}_{n,m}$ is similar to the proof of Theorem 4.02.

NOTE. In the remainder of this chapter P_k, $k = 0,1,\ldots,m$, will denote the set described in the above theorem.

NOTATION 5.04. Let m be a positive integer, and let $k \varepsilon \{0,1,\ldots,m\}$. Let $\{r_1, r_2, \ldots, r_{\binom{m}{k}}\}$ be the collection of k-term increasing sequences from $\{1,2,\ldots,m\}$. For each $j \varepsilon \{1,2,\ldots,\binom{m}{k}\}$ let $T_j \subseteq P_k$ such that $x \varepsilon T_j$ if and only if the columns in x specified by r_j have property Z with respect to $[a_i, b_i]$, $i = 1,2,\ldots,n$.

THEOREM 5.05. If $k \varepsilon \{0,1,\ldots,m\}$, then $\{T_1, T_2, \ldots, T_{\binom{m}{k}}\}$ is a partition of P_k.

Proof:

Let $k \varepsilon \{0,1,\ldots,m\}$. Since for $T_j \subseteq P_k$ and $x \varepsilon T_j$, $j = 1,2,\ldots,\binom{m}{k}$, x is specified by a k-term increasing sequence, then $x \notin T_w$ if $j \neq w$ and $j,w = 1,2,\ldots,\binom{m}{k}$.

Therefore $T_j \cap T_w = \phi$ if $j \neq w$.

Now, it must be shown that $\bigcup_{j=1}^{\binom{m}{k}} T_j = P_k$, $k = 0,1,\ldots,m$.

Let $s \in \bigcup_{j=1}^{\binom{m}{k}} T_j$, $k = 0,1,\ldots,m$; then there is an integer q, $0 \leq q \leq m$, such that $s \in T_q$. Since $T_q \subsetneq P_k$, then $s \in P_k$ and $\bigcup_{j=1}^{\binom{m}{k}} T_j \subseteq P_k$, $k = 0,1,\ldots,m$.

Conversely, let $s \in P_k$, $k = 0,1,\ldots,m$; then there is an integer v, $0 \leq v \leq \binom{m}{k}$, $k = 0,1,\ldots,m$, such that $s \in T_v$. Since $s \in T_v$ and v, $0 \leq v \leq \binom{m}{k}$, $k = 0,1,\ldots,m$, is an integer, then $s \in \bigcup_{j=1}^{\binom{m}{k}} T_j$ and $P_k \subsetneq \bigcup_{j=1}^{\binom{m}{k}} T_j$.

Since $P_k \subsetneq \bigcup_{j=1}^{\binom{m}{k}} T_j$ and $\bigcup_{j=1}^{\binom{m}{k}} T_j \subsetneq P_k$, $k = 0,1,\ldots,m$, then $\bigcup_{j=1}^{\binom{m}{k}} T_j = P_k$. Therefore, by Definition 2.18, it follows that $\{T_1, T_2, \ldots, T_{\binom{m}{k}}\}$ is a partition of P_k, $k = 0,1,\ldots,m$.

DEFINITION 5.04. Let $0 \leq k \leq m$ where k is an integer. Define $\bar{\bar{P}}_k$ as the additive function with domain the partition $\{T_1, T_2, \ldots, T_{\binom{m}{k}}\}$ of P_k, $k = 0,1,\ldots,m$, and $\bar{\bar{P}}_k(T_j) = \beta^k(1 - \beta)^{m-k}$, $k = 0,1,\ldots,m$, $j = 1,2,\ldots,\binom{m}{k}$, where

$$\beta = \bar{\bar{f}}[(x_1 \in (a_1<\tau_1\leq b_1)) \wedge (x_2 \in (a_2<\tau_2\leq b_2)) \wedge \ldots \wedge (x_n \in (a_n<\tau_n\leq b_n))]$$

and $(x_1, x_2, \ldots, x_n) \in E_n$.

THEOREM 5.06. Suppose $M_{\hat{n},m}$ is a random sample from E_n. If $a_i, b_i > 0$, $i = 1,2,\ldots,n$, are real numbers such that $P_i(a_i<\tau_i\le b_i) = \sqrt[n]{\beta}$ for each a_i, b_i, τ_i, P_i, $i = 1,2,\ldots,n$, where P_i is the probability function for τ_i and $\overset{\Delta}{P}$ is an additive function with domain $\overset{\Delta}{D}$, the collection of subsets of $\{\{M_{n,m} \in P_0\}, \{M_{n,m} \in P_1\}, \ldots, \{M_{n,m} \in P_m\}\}$ such that $\overset{\Delta}{P}(\phi) = 0$ and $\overset{\Delta}{P}(\{M_{n,m} \in P_k\}) = \sum_{j=1}^{\binom{m}{k}} \bar{\bar{P}}_k(T_j)$, $k = 0,1,\ldots,m$, then $\overset{\Delta}{P}(\{M_{n,m} \in P_k\}) = \binom{m}{k}\beta^k(1-\beta)^{m-k}$, $k = 0,1,\ldots,m$. Moreover, $(\overset{\Delta}{D},\overset{\Delta}{P})$ is a binomial probability distribution.

Proof:

Let $a_i, b_i > 0$, $i = 1,2,\ldots,n$, be real numbers such that $P_i(a_i<\tau_i\le b_i) = \sqrt[n]{\beta}$ for each a_i, b_i, τ_i, P_i, $i = 1,2,\ldots,n$, such that P_i is the probability function for τ_i. Let $\overset{\Delta}{P}$ be an additive function with domain $\overset{\Delta}{D}$, the collection of subsets of $\{\{M_{n,m} \in P_0\}, \{M_{n,m} \in P_1\}, \ldots, \{M_{n,m} \in P_m\}\}$, such that $\overset{\Delta}{P}(\phi) = 0$ and $\overset{\Delta}{P}(\{M_{n,m} \in P_k\}) = \sum_{j=1}^{\binom{m}{k}} \bar{\bar{P}}_k(T_j)$, $k = 0,1,\ldots,m$.

1. Let $(x_{i_1}, x_{i_2}, \ldots, x_{i_m}) \in R_i$, $i = 1,2,\ldots,n$.

$$\bar{\bar{f}}[(x_{i_j} \in (a_1<\tau_1\le b_1)) \wedge (x_{2_j} \in (a_2<\tau_2\le b_2)) \wedge \ldots$$

$$\wedge (x_{n_j} \in (a_n<\tau_n\le b_n))] \quad j = 1,2,\ldots,m$$

$$= \prod_{i=1}^{n} \bar{\bar{f}}[x_{i_j} \in (a_i<\tau_i\le b_i)] \quad \text{by Definition 5.00}$$

$$= \prod_{i=1}^{n} P_i(a_i<\tau_i\le b_i) \quad \text{by Definition 5.00}$$

$$= \prod_{i=1}^{n} \sqrt[n]{\beta}$$

$$= \left(\sqrt[n]{\beta}\right)^n$$

$$= \beta.$$

2. $$\overset{\Delta}{P}(\{M_{n,m} \ \varepsilon \ P_k\}) = \sum_{j=1}^{\binom{m}{k}} \overset{=}{P}_k(T_j)$$

$$= \sum_{j=1}^{\binom{m}{k}} \beta^k(1 - \beta)^{m-k} \quad \text{by Definition 5.04}$$

$$= \binom{m}{k} \beta^k(1 - \beta)^{m-k}$$

3. It can be shown that $(\overset{\Delta}{D},\overset{\Delta}{P})$ is a binomial probability distribution in a manner similar to that used in part 3 of Theorem 4.04 to show that $(\hat{D},\hat{P})$ is a binomial probability distribution.

THEOREM 5.07. Suppose $M_{n,m}$ is a random sample from E_n. If $a_i,b_i > 0$, $i = 1,2,\ldots,n$, are real numbers such that $P_i(a_i<\tau_i\le b_i) = \sqrt[n]{\beta}$ for each a_i, b_i, τ_i, P_i, $i = 1,2,\ldots,n$, where P_i is the probability function for τ_i, and $\overset{\Delta}{P}$ is an additive function with domain $\overset{\Delta}{D}$, the collection of subsets of $\{\{M_{n,m} \ \varepsilon \ P_0\}, \{M_{n,m} \ \varepsilon \ P_1\}, \ . \ . \ . \ , \{M_{n,m} \ \varepsilon \ P_m\}\}$ such that $\overset{\Delta}{P}(\phi) = 0$ and $\overset{\Delta}{P}(\{M_{n,m} \ \varepsilon \ P_k\}) = \sum_{j=1}^{\binom{m}{k}} \overset{=}{P}_k(T_j)$, $k = 0,1,\ldots,m$, and $g \ \varepsilon \ \{1,2,\ldots,m\}$, then

$$\overset{\Delta}{P}\{\{M_{n,m} \ \varepsilon \ P_g\} \cup \{M_{n,m} \ \varepsilon \ P_{g+1}\} \cup \ . \ . \ . \ \cup \{M_{n,m} \ \varepsilon \ P_m\}\}$$

$$= \sum_{k=g}^{m} \binom{m}{k} \beta^k(1 - \beta)^{m-k}.$$

Proof:

The proof of this theorem is similar to that of Theorem 4.05.

It is pointed out that Theorems 5.06 and 5.07 are useful in special situations only. For instance, β must be very large in order for $\hat{P}$ to be of any significance.

B I B L I O G R A P H Y

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