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A GEOMETRIC ANALYSIS OF A SINGULAR ODE RELATED TO THE STUDY OF A QUASILINEAR PDE

JUKKA TUOMELA

Abstract. In this note we complement the analysis of a singular ODE given in [2]. Using geometric arguments we are able to settle the structure of the existence and non-existence regions in the parameter space and improve the nonexistence bound.

1. Introduction

In [2] existence and uniqueness questions of radial solutions of a class of quasilinear elliptic PDEs in a ball, having strong dependence in the gradient, are analysed in terms of the related singular ODE, see also [3]. The authors transform the ODE to an integral equation and look for the fixed points of this operator equation. We complement their analysis using only elementary geometric methods. We show that in addition to the regular solution analysed in [2] there exist also infinitely many singular solutions. It is not quite clear if these 'new' solutions can be used to construct new solutions to the original PDE, because at least they are too singular to yield strong solutions. All solutions blow up in finite time, and we give a nonexistence result which for all parameter values is sharper than the one given in [2]. Finally we prove that the regions of existence and nonexistence in the parameter space have a very simple structure, answering rather completely the question raised in [2].

2. Results

In [2] the following ODE is considered

$$t^{\varepsilon}\omega' - g_0\gamma t^{\gamma+\varepsilon-1} - f_0\omega^{\delta} = 0 \tag{1}$$

where f_0 and g_0 are assumed to be positive constants and other parameters will be specified below. We are only interested in nonnegative solutions which satisfy this equation for $t > 0$ and which can be continuously extended to $t = 0$.

To simplify the study we first observe

Lemma 1. *Let u be a solution of*

$$t^{\varepsilon}u' - \gamma t^{\gamma+\varepsilon-1} - u^{\delta} = 0 \tag{2}$$

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and let us define $\omega(t) = c_1 u(c_2 t)$*. Then* ω *is a solution of* (1) *if*

$$c_2 = \left(f_0 g_0^{\delta-1}\right)^{1/\beta}$$
$$c_1 = g_0\left(f_0 g_0^{\delta-1}\right)^{-\gamma/\beta} = g_0 c_2^{-\gamma}$$

where $\beta = \gamma\delta - \gamma - \varepsilon + 1$.

Proof. This is easy to check. □

Hence it is really sufficient to study (2) because all solutions of (1) are obtained from the solutions of (2) by simple scaling. It is useful to make a further reduction.

Lemma 2. *Let* $x = t^{\varepsilon-1}$ *and* $y = u^{\delta-1}$ $(\varepsilon \neq 1,\ \delta \neq 1)$*. The equation* (2) *written in coordinates* (x, y) *is*

$$ax^2 y' - y^2 - \gamma x^r y^s = 0 \tag{3}$$

where $a = (\varepsilon-1)/(\delta-1)$*,* $r = 1 + \gamma/(\varepsilon-1)$ *and* $s = (\delta-2)/(\delta-1)$*.*

Proof. This is also straightforward. □

Note that identically zero function is a solution of (3) (if $s > 0$), but not of (1). When in the following we say 'any solution' or 'all solutions', it will always mean any or all solutions, except identically zero solution. The notation $y \sim x^m$ will mean that there exist positive constants d_1 and d_2 such that $d_1 x^m \leq y(x) \leq d_2 x^m$ for small enough x.

Now let us list here for convenience the conditions the parameters are supposed to satisfy in the rest of the paper.

$$\delta > 1,\ \varepsilon > 1,\ \gamma > (\varepsilon-1)/(\delta-1)$$
$$a > 0,\ \gamma > 0,\ r > 1,\ s < 1,\ r + s > 2,\ \beta > 0$$

Note that inequalities in the second line are consequences of the first line. Let us also consider the following equations

$$ax^2 y' - y^2 = 0 \tag{4}$$
$$ax^2 y' - \gamma x^r y^s = 0 \tag{5}$$

By elementary integration they have solutions

$$y(x) = \frac{ax}{1 + cax} \tag{6}$$

$$y(x) = \left(x^{\gamma/(\varepsilon-1)} + c\right)^{\delta-1} \tag{7}$$

Hence (4) has an infinite number of (analytic) solutions with $y(0) = 0$ and $y'(0) = a$, but a unique solution can be specified by giving the second derivative at origin (note that $y''(0) = -2a^2 c$). This phenomenon can be understood nicely in a geometric way with jet spaces, see [4] and references therein for more information.

These solutions can be used to study the solutions of (3). In the following we will often specify some point (x_0, y_0). It is always assumed that x_0 and y_0 are positive.

Lemma 3. *The equation* (3) *does not have any solutions with* $\lim_{x\searrow 0} y(x) > 0$*.*

Proof. Let y (resp. z) be a solution of (3) (resp. (4)), going through the point (x_0, y_0). Comparing the derivatives we see that $y(x) \leq z(x)$ for all $0 \leq x \leq x_0$ from which the claim follows. □

We are interested in positive solutions of (3) which can be continuously extended to origin. We need the following simple result.

Lemma 4. *Let y be a solution of* (3) *such that $y(x) > 0$ for $0 < x \le x^*$. Then the (right hand) limit $\lim_{x\searrow 0} y(x)$ exists.*

Proof. Writing (3) as
$$y' = y^2/(ax^2) + \gamma x^{r-2} y^s/a$$
we see that y' is positive (in the relevant region), and hence y is monotonically increasing. The result follows because monotonic functions have right (and left) hand limits. □

Corollary 1. *Let y be a solution of* (3) *such that $y(x) > 0$ for $0 < x \le x^*$. Let us set $y(0) = 0$. Then $y : [0, x^*] \to \mathbb{R}$ is continuous.*

Proof. By Lemma 4 the limit $\lim_{x\searrow 0} y(x)$ exists. By Lemma 3 this limit must be zero. □

Hence positive solutions can be continuously extended to zero, if they exist on $(0, x^*]$. Our next task is then to prove existence.

Lemma 5. *For x^* small enough, equation* (3) *has an infinite number of solutions with $y(x) > 0$ for $0 < x \le x^*$ and $\lim_{x\searrow 0} y(x) = 0$.*

Proof. Let us define
$$\Omega_d^c = \{(x, y) \in \mathbb{R}^2 \,|\, 0 \le x \le d,\ y \ge cx\}$$
$$\Gamma_d^c = \{(x, y) \in \mathbb{R}^2 \,|\, 0 < x \le d,\ y = cx\} \cup \{(x, y) \in \mathbb{R}^2 \,|\, x = d,\ y \ge cd\}$$
Consider the line $y = cx$ with some $c > 0$. If y is a solution of (3), then on this line
$$y'(x) = c^2/a + \gamma c^s x^{r+s-2}/a$$
Because $r + s > 2$ this implies that there exist c and d such that $y'(x) < c$ for all $0 < x \le d$. Hence no solution of (3) enters Ω_d^c through Γ_d^c. Hence by Corollary 1 we can continuously extend the solutions of (3) which meet Γ_d^c backwards in 'time' to origin. □

Lemma 6. *Let y be a solution of* (3) *with $\lim_{x\searrow 0} y(x) = 0$ and $y(x) > 0$ for $x > 0$. Then for any $b > 0$ we have*
$$x^\alpha \le y(x) \le a(1 + b)x$$
where $\alpha = \gamma(\delta - 1)/(\varepsilon - 1) > 1$. The first inequality is valid for all x and the second for sufficiently small x.

Proof. The second inequality follows from the proof of Lemma 3. To prove the other inequality let (x_0, y_0) be a point on the curve $y = x^\alpha$ or below it. Further let y (resp. z) be a solution of (3) (resp. (5)) going through (x_0, y_0). We must show that there exists $\tilde{x} > 0$ such that $y(\tilde{x}) = 0$. Now if $y_0 < x_0^\alpha$, $y(x) \le z(x) = (x^{\gamma/(\varepsilon-1)} + c)^{\delta-1}$ for some $c < 0$ which shows that $\tilde{x} \ge (-c)^{(\varepsilon-1)/\gamma} > 0$. If $y_0 = x_0^\alpha$, essentially the same argument applies because the solutions of (3) intersect the curve $y = x^\alpha$ transversely. □

It is a straightforward task to check that no solution can approach the origin as $y \sim x^m$ with $1 < m < \alpha$.

Theorem 1. *There is precisely one solution of* (3) *with* $y \sim x^\alpha$ *for small* x.

Proof. Let $c > 1$ and let K_1 (resp. K_2) be the curve $y = x^\alpha$ (resp. $y = cx^\alpha$). Let

$$\Omega_d = \left\{(x, y) \in \mathbb{R}^2 \,|\, x^\alpha \le y \le cx^\alpha \ , \quad 0 \le x \le d\right\}$$

Then the boundary of Ω_d consists of the parts of the curves K_1 and K_2 and the vertical part which will be called Γ_d. Then it is easy to check that for d small enough, the solutions go out of Ω_d only through Γ_d. Now extend all solutions going through Γ_d backwards in 'time'. By continuity of the flow, there is at least one solution y such that $(x, y(x)) \in \Omega_d$ for $0 \le x \le d$.

It remains to prove the uniqueness. We will use some simple modifications of the results in [1]. Suppose that there are two solutions y_1 and y_2 such that $(x, y_i(x)) \in \Omega_d$ for $0 \le x \le d$, $i = 1, 2$. Denote $v(x) = y_2(x) - y_1(x)$; without loss of generality we may assume that $v(x) \ge 0$. The uniqueness will follow if we succeed in proving that $v(d) = 0$. Let us first write (3) as

$$y' = f(x, y) = y^2/(ax^2) + \alpha x^{r-2} y^s$$

Then we can express v using

$$v'(x) = y_2'(x) - y_1'(x) = f(x, y_2) - f(x, y_1) = \int_{y_1(x)}^{y_2(x)} \frac{\partial f}{\partial y}(x, s) ds$$

Hence we need an estimate for $\partial f/\partial y$. A simple computation shows that in Ω_d

$$\frac{\partial f}{\partial y} = \frac{2y}{ax^2} + s\alpha x^{r-2} y^{s-1} \le \begin{cases} \frac{2c}{a} x^{\alpha-2} + c^{s-1}\alpha x^{-1} & , \ 0 < s < 1 \\ \frac{2c}{a} x^{\alpha-2} & , \ s \le 0 \end{cases} \tag{8}$$

Case 1: $s \le 0$. Now $v'(x) \le c_1 x^{\alpha-2} v(x)$ where $c_1 = 2c/a$. Consider the differential equation $w'(x) = c_1 x^{\alpha-2} w(x)$ with initial condition $w(d) = v(d)$. The solution is given by

$$w(x) = v(d) \exp(-\int_x^d c_1 s^{\alpha-2} ds)$$

and because $\alpha - 2 > -1$ this implies that $w(0) = c_2 v(d)$ for some $c_2 > 0$. On the other hand $v(x) \ge w(x)$ if $x \le d$. Combined with $\lim_{x \searrow 0} v(x) = 0$, this is possible only if $v(d) = 0$. This is essentially Theorem 4.7.5 (or rather its proof) in [1, p. 188].

Case 2: $0 < s < 1$. Let us write the inequality in (8) a bit differently:

$$\frac{\partial f}{\partial y} \le \Big(\frac{2cx^{\alpha-1}}{a} + \frac{\gamma}{c^{1-s}a}\Big)x^{-1} \le \Big(\frac{2cd^{\alpha-1}}{a} + \frac{\gamma}{c^{1-s}a}\Big)x^{-1}$$

Now we can first choose c sufficiently big so that $\gamma/(c^{1-s}a) \le 1/2$ and then d sufficiently small so that $2cd^{\alpha-1}/a \le 1/2$. Hence for suitable c and d we have $\partial f/\partial y \le 1/x$ in Ω_d.

Now proceed as before: $v'(x) \le x^{-1} v(x)$ and let w be a solution of $w'(x) = x^{-1} w(x)$ with $w(d) = v(d)$. By elementary integration $w(x) = v(d)x/d$ and again $v(x) \ge w(x)$ if $x \le d$. Now $v(x) \le (c-1)x^\alpha$ which implies that

$$v(d)x/d \le (c-1)x^\alpha$$

for all $0 \le x \le d$. This is possible only if $v(d) = 0$ because $\alpha > 1$. This is essentially the extension of Theorem 4.7.5 in [1, p. 188], outlined in Exercise 4.7#3 [1, p. 196] $\square$

Hence there is one solution behaving like x^α (let us call this the regular solution) and infinitely many solutions which behave like x near the origin (called singular solutions). Translating the above results back to the original coordinates we get

Corollary 2. *If T is small enough, equation* (1) *has*

- *infinitely many solutions in V_m if $m \le (\varepsilon - 1)/(\delta - 1)$*
- *a unique solution in V_m if $(\varepsilon - 1)/(\delta - 1) < m \le \gamma$*
- *no solutions in V_m if $m > \gamma$*

where

$$V_m = \{u \in C[0,T] \mid 0 \le u(t) \le M_u t^m\}$$

and M_u may depend on u.

Let us then take a look at the 'lifespan' of these solutions.

Theorem 2. *All solutions of* (3) *blow up in a finite time.*

Proof. If $p = (x_0, y_0)$ is above the line $y = ax$, then the solution of (3) through p blows up by comparing it to the corresponding solution of (4). Further, solutions of (3) intersect $y = ax$ transversely. It remains to show that any solution starting below this line will eventually meet it. Let $p = (x_0, y_0)$ be below the line $y = ax$; let y be a solution of (3) through p and z a solution of (5) through p. Then $y(x) \ge z(x)$ for $x \ge x_0$. On the other hand z will meet $y = ax$ because it grows superlinearly. □

Finally let us consider the problem raised in [2]. What is the structure of the set of parameter values (f_0, g_0) for which the problem (1) has (resp. does not have) a solution on a given interval $[0,T]$? Using Lemma 1 this question has a very simple answer.

Theorem 3. *There exists $c > 0$ (depending on T, γ, ε and δ) such that the problem* (1)

- *has a solution if $f_0 g_0^{\delta-1} < c$*
- *does not have a solution if $f_0 g_0^{\delta-1} \ge c$*

Proof. Let u be the regular solution of (2). Then there exists a well defined number T_b such that $\lim_{t \nearrow T_b} u(t) = \infty$ and no $\tilde{T} < T_b$ has this property. Now the regular solution is the last one to blow up, because initially the regular solution ($u \sim t^\gamma$) must be 'below' the singular solutions ($u_s \sim t^{(\varepsilon-1)/(\delta-1)}$, $\gamma > (\varepsilon - 1)/(\delta - 1)$), and later the solutions cannot cross by the uniqueness of the solutions.

Hence it follows that (2) has (infinitely many) solutions on the interval $[0,T]$ if $T < T_b$ and does not have any solutions on $[0,T]$ if $T \ge T_b$. To get the general result we scale the 'time' variable by $c_2 = \left(f_0 g_0^{\delta-1}\right)^{1/\beta}$ like in Lemma 1. Then it is easily seen that $c = (T_b/T)^\beta$. □

So the whole problem is essentially characterized by one parameter, namely T_b. It remains to estimate T_b. The results in [2] give

$$\left(\frac{(\gamma\delta - \varepsilon + 1)(\delta - 1)^{\delta-1}}{\delta^\delta}\right)^{1/\beta} < T_b \le \left(\gamma\delta^{\delta/(\delta-1)}\right)^{1/\beta} \tag{9}$$

We can improve the upper bound.

Lemma 7.

$$T_b \leq \left(\gamma \left(\frac{\alpha}{\alpha - 1} \right)^{\alpha - 1} \right)^{1/\beta}$$

Proof. Consider the problem (3). All solutions remain above the curve $y = x^\alpha$. This curve meets the line $y = dax$ at the point (x_0, dax_0) where

$$x_0 = (da)^{(\varepsilon - 1)/\beta}$$

Let $p = (x_0, dax_0)$ where $d > 1$ and let y (resp. z) be a solution of (3) (resp. (4)) through p. Then y blows up before

$$x = \frac{d}{d-1} (da)^{(\varepsilon - 1)/\beta}$$

because z blows up there. Getting back to (t, u) coordinates this means that

$$T_b \leq \left(\frac{ad^\alpha}{(d-1)^{\alpha - 1}} \right)^{1/\beta}$$

The right hand side is minimised if $d = \alpha$ which gives the result. □

Note that our estimate is better than (9) for all parameter values because

$$\left(\frac{\alpha}{\alpha - 1} \right)^{\alpha - 1} < e < \delta^{\delta/(\delta - 1)}$$

For example if one takes $\gamma = 1$, $\delta = 8$ and $\varepsilon = 7$, then our result gives

$$T_b \leq 7^{1/6} \approx 1.38$$

while (9) gives

$$T_b \leq 8^{8/7} \approx 10.8$$

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Jukka Tuomela
Department of Mathematics, University of Joensuu, PL 111, 80101 Joensuu, Finland
E-mail address: jukka.tuomela@joensuu.fi