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C^k INVARIANT MANIFOLDS FOR INFINITE DELAY

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Abstract. For a non-autonomous delay difference equation with infinite delay, we construct smooth stable and unstable invariant manifolds for any sufficiently small perturbation of an exponential dichotomy. We consider a general class of norms on the phase space satisfying an axiom considered by Matsunaga and Murakami that goes back to earlier work by Hale and Kato for continuous time. In addition, we show that the invariant manifolds are as regular as the perturbation. Finally, we consider briefly the case of center manifolds and we formulate corresponding results.

1. Introduction

We consider the non-autonomous delay difference equation

$$x(m+1) = L_m x_m + f_m(x_m), \quad \text{for } m \in \mathbb{N}, \tag{1.1}$$

with infinite delay on a Banach space. For the corresponding dynamics, we construct stable and unstable invariant manifolds for any sufficiently small perturbation (either Lipschitz or C^k with $k \geq 1$) of an exponential dichotomy. We also show that the invariant manifolds are as regular as the perturbation. We note that in a certain sense the only delay difference equations are those with infinite delay, since otherwise one can always bring them to a standard recurrence form in some higher-dimensional space. An important aspect is that in the case of infinite delay it is crucial to choose from the beginning an appropriate norm on the infinite-dimensional phase space (on a finite-dimensional space all norms are equivalent). We consider a class of norms satisfying an axiom considered by Matsunaga and Murakami [7] that is analogous to the axiom proposed by Hale and Kato [5] in the case of continuous time; that is, for delay differential equations with infinite delay.

We consider both Lipschitz and C^k perturbations. We start by constructing Lipschitz stable and unstable invariant manifolds for any sufficiently small Lipschitz perturbation of an exponential dichotomy (see Theorems 3.5 and 3.6). More precisely, we show that:

(1) the set of initial conditions leading to a bounded forward global solution is a graph of a Lipschitz function over the stable bundle, which is precisely the stable manifold;
(2) a similar statement holds for the bounded backward global solutions, leading to the construction of the unstable manifold.

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This part of our paper can be considered to be a non-autonomous version of work in [7]. We then construct smooth stable and unstable invariant manifolds for any sufficiently small smooth perturbation of an exponential dichotomy (see Theorems 4.1 and 4.3 in the case of C^1 perturbations and see Section 4.2 in the case of higher smoothness). In view of the uniqueness of the Lipschitz invariant manifolds constructed earlier it remains to show that the function of which the invariant manifold is a graph has the required regularity properties.

We also consider briefly the case of center manifolds, for any sufficiently small perturbation of an exponential trichotomy. The arguments are simple modifications of those for the stable and unstable manifolds and so we omit them. We note that center manifold theorems are powerful tools in the analysis of the behavior of a dynamical system. We refer the reader to [3] for details and references. A detailed exposition in the case of autonomous equations is given in [9], adapting results in [11]. See also [8, 10] for the case of differential equations on infinite-dimensional spaces.

In a certain sense, the arguments in the proofs can be considered classical, although the consideration of delay difference equations with infinite delay and at the same time in a class of spaces determined by a certain axiom (see Section 2), require many modifications. We also made a strong effort to reduce all the work to first principles. In particular, we showed that the stable set, that is, the set of all initial conditions (n, ϕ) leading to bounded solutions of equation (1.1) (see Section 3 for details) is an invariant manifold, by first showing that it leads naturally to a fixed point problem (while in many other works the fixed point problem is considered from the beginning without further explanations).

2. Preliminaries

For simplicity, we denote by $(-\infty, \ell]$ and $[\ell, +\infty)$, respectively, the sets $(-\infty, \ell] \cap \mathbb{Z}$ and $[\ell, +\infty) \cap \mathbb{Z}$. Let $X = (X, |\cdot|)$ be a Banach space. Given a sequence $x\colon (-\infty, m] \to X$ and an integer $\ell \le m$, we define a new sequence $x_\ell\colon \mathbb{Z}_0^- \to X$ by $x_\ell(j) = x(\ell + j)$ for $j \in \mathbb{Z}_0^-$. Following the approach in [7], we consider a Banach space $\mathcal{B} = (\mathcal{B}, \|\cdot\|)$ of sequences $\phi\colon \mathbb{Z}_0^- \to X$ satisfying the axiom:

(A1) there exist a constant $N_0 > 0$ and sequences $K, M\colon \mathbb{Z}_0^+ \to \mathbb{R}_0^+$ with the property that if $x\colon \mathbb{Z} \to X$ is a sequence with $x_0 \in \mathcal{B}$, then for all $n \in \mathbb{Z}_0^+$ we have $x_n \in \mathcal{B}$ and
$$N_0|x(n)| \le \|x_n\| \le K(n) \sup_{0 \le l \le n} |x(l)| + M(n)\|x_0\|.$$

An example of such a space $\mathcal{B}$ is the following. Given $\gamma > 0$, let X_γ be the set of all sequences $\phi\colon \mathbb{Z}_0^- \to X$ such that
$$\|\phi\|_\gamma := \sup_{j \in \mathbb{N}} \big(|\phi(j)| e^{\gamma j}\big) < +\infty.$$

Given linear operators $L_m\colon \mathcal{B} \to X$, for $m \in \mathbb{Z}$, we consider the linear delay equation
$$x(m+1) = L_m x_m. \tag{2.1}$$
We observe that given $n \in \mathbb{N}$ and $\phi \in \mathcal{B}$, there exists a unique sequence $x = x(\cdot, n, \phi)\colon \mathbb{Z} \to X$ with $x_n = \phi$ satisfying (2.1) for all $m \ge n$. We define linear operators $T(m, \ell)\colon \mathcal{B} \to \mathcal{B}$, for $m \ge n$, by
$$T(m, n)\phi = x_m(\cdot, n, \phi), \quad \phi \in \mathcal{B}.$$

Clearly, $T(m,m)=\mathrm{Id}$ and $T(m,l)T(l,n)=T(m,n)$ for $m\ge l\ge n$.

We say that equation (2.1) has an *exponential dichotomy* if:

(1) there exist projections $P_n\colon \mathcal{B}\to\mathcal{B}$, for $n\in\mathbb{Z}$, such that for $m\ge n$ we have
$$P_mT(m,n)=T(m,n)P_n;$$
(2) writing $Q_n=\mathrm{Id}-P_n$, the operator $T(m,n)Q_n$ is invertible from $\operatorname{Im}Q_n$ onto $\operatorname{Im}Q_m$ for each $m\ge n$;
(3) there exist constants $\lambda,N>0$ such that for $m\ge n$ we have
$$\|T(m,n)P_n\|\le Ne^{-\lambda(m-n)},\quad \|(T(m,n)Q_n)^{-1}\|\le Ne^{-\lambda(m-n)}. \tag{2.2}$$

For each $m\in\mathbb{Z}$ we then define
$$E_m=\operatorname{Im}P_m \quad\text{and}\quad F_m=\operatorname{Im}Q_m.$$

Now we consider perturbations given by continuous maps $f_m\colon\mathcal{B}\to X$ with $f_m(0)=0$ for $m\in\mathbb{Z}$. The delay equation
$$x(m+1)=L_mx_m+f_m(x_m) \tag{2.3}$$
determines the dynamics
$$x_m=T(m,n)\phi+\sum_{l=n}^{m-1}T(m,l+1)(\Gamma f_l(x_l)) \tag{2.4}$$
for each $m\ge n$, where $\Gamma(0)=\mathrm{Id}$ and $\Gamma(l)=0$ for $l<0$; that is,
$$(\Gamma f_l(x_l))(j)=\Gamma(j)f_l(x_l)=\begin{cases} f_l(x_l) & \text{if } j=0,\\ 0 & \text{if } j<0.\end{cases}$$

We shall write $x_m=v_m(\cdot,n,\phi)$ for $m\in\mathbb{Z}$, with v_m given by (2.4) for $m\ge n$ and by ϕ_{m-n} for $m\le n$. It follows from (A1) that $\Gamma f_l(x_l)\in\mathcal{B}$ and
$$\|\Gamma f_l(x_l)\|\le K(1)|f_l(x_l)|. \tag{2.5}$$

3. Lipschitz manifolds

In this section we construct Lipschitz stable and unstable invariant manifolds for equation (2.3).

3.1. **Stable manifolds.** We first construct Lipschitz stable invariant manifolds. The *stable set* V^s of equation (2.3) is the set of all $(n,\phi)\in\mathbb{Z}\times\mathcal{B}$ for which the map $m\mapsto v_m(\cdot,n,\phi)$ is bounded on $[n,+\infty)$.

Proposition 3.1. *If $(n,\phi)\in V^s$, then $(m,v_m(\cdot,n,\phi))\in V^s$ for all $m\ge n$.*

Proof. It suffices to show that
$$v(m,k,v_k(\cdot,n,\phi))=v(m,n,\phi) \tag{3.1}$$
for all $m\ge k\ge n$. Indeed, since $v(\cdot,n,\phi)$ is bounded on $[n,+\infty)$, this implies that $v(\cdot,k,v_k(m,n,\phi))$ is bounded on $[k,+\infty)$ and so $(k,v_k(m,n,\phi))\in V^s$ for all $k\ge n$.

To establish identity (3.1) we note that applying $T(m,n)$ to (2.4) with $m=k$, that is,
$$x_k=T(k,n)\phi+\sum_{l=n}^{k-1}T(k,l+1)(\Gamma f_l(x_l)),$$

we obtain

$$\begin{aligned} &T(m,k)x_k + \sum_{l=k}^{m-1} T(m,l+1)(\Gamma f_l(x_l)) \\ &= T(m,n)\phi + \sum_{l=n}^{m-1} T(m,l+1)(\Gamma f_l(x_l)) = x_m. \end{aligned} \tag{3.2}$$

Since (2.4) determines the solution recursively, it follows from (3.2) that (3.1) holds. □

The following result gives an alternative characterization of the stable set when we have an exponential dichotomy.

Proposition 3.2. *Assume that equation* (2.1) *has an exponential dichotomy. Then the stable set is composed of the pairs* $(n,\phi) \in \mathbb{Z}\times\mathcal{B}$ *for which there exists a sequence* $x\colon \mathbb{Z} \to X$ *bounded on* $\mathbb{N}$ *such that* $x_n = \phi$ *and*

$$\begin{aligned} x_m = T(m,n)P_n\phi &+ \sum_{l=n}^{m-1} T(m,l+1)P_{l+1}(\Gamma f_l(x_l)) \\ &- \sum_{l=m}^{+\infty} T(m,l+1)Q_{l+1}(\Gamma f_l(x_l)) \end{aligned} \tag{3.3}$$

for every $m \ge n$ *(with the first sum vanishing for* $m = n$*).*

Proof. First we assume that x is a solution of equation (2.3) bounded on $\mathbb{N}$ with $x_n = \phi$. Then

$$\begin{aligned} P_k x_k &= T(k,j)P_j x_j + \sum_{l=j}^{k-1} T(k,l+1)P_{l+1}(\Gamma f_l(x_l)), \\ Q_k x_k &= T(k,j)Q_j x_j + \sum_{l=j}^{k-1} T(k,l+1)Q_{l+1}(\Gamma f_l(x_l)) \end{aligned} \tag{3.4}$$

for $k \ge j \ge n$. Taking $j = m$, we write the second identity in the form

$$Q_m x_m = T(m,k)Q_k x_k - \sum_{l=m}^{k-1} T(k,l+1)Q_{l+1}(\Gamma f_l(x_l)),$$

where $T(m,k) = (T(k,m)Q_m)^{-1}$. By the second inequality in (2.2), since the sequence x_k is bounded on $\mathbb{N}$, we obtain

$$\|T(m,k)Q_k x_k\| \le Ne^{-\lambda(k-m)}\|x_k\| \to 0$$

when $k \to +\infty$. Hence,

$$Q_m x_m = -\sum_{l=m}^{+\infty} T(m,l+1)Q_{l+1}(\Gamma f_l(x_l)),$$

which added to the first identity in (3.4) with $k = m$ and $j = n$ yields (3.3).

Now we assume that property (3.3) holds for some sequence x bounded on $\mathbb{N}$ with $x_n = \phi$. Taking $m = n$ we obtain

$$x_n = P_n x_n - \sum_{l=n}^{+\infty} T(n,l+1)Q_{l+1}(\Gamma f_l(x_l))$$

and hence,

$$Q_n x_n = -\sum_{l=n}^{+\infty} T(n,l+1)Q_{l+1}(\Gamma f_l(x_l)).$$

Therefore,

$$\begin{aligned}
&T(m,n)x_n + \sum_{l=n}^{m-1} T(m,l+1)(\Gamma f_l(x_l)) \\
&= T(m,n)P_n x_n + \sum_{l=n}^{m-1} T(m,l+1)P_{l+1}(\Gamma f_l(x_l)) \\
&\quad + \sum_{l=n}^{m-1} T(m,l+1)Q_{l+1}(\Gamma f_l(x_l)) + T(m,n)Q_n x_n \\
&= x_m + \sum_{l=m}^{+\infty} T(m,l+1)Q_{l+1}(\Gamma f_l(x_l)) + \sum_{l=n}^{m-1} T(m,l+1)Q_{l+1}(\Gamma f_l(x_l)) \\
&\quad - T(m,n)\sum_{l=n}^{+\infty} T(n,l+1)Q_{l+1}(\Gamma f_l(x_l)) = x_m
\end{aligned}$$

and so x is a solution of equation (2.3). □

We use Proposition 3.2 to show that when equation (2.1) has an exponential dichotomy and the perturbations are Lipschitz, all bounded solutions of equation (2.3) decay exponentially.

Proposition 3.3. *Assume that equation* (2.1) *has an exponential dichotomy and that there exists $\delta > 0$ such that*

$$|f_m(u) - f_m(v)| \le \delta \|u - v\| \tag{3.5}$$

for every $m \in \mathbb{Z}$ and $u, v \in \mathcal{B}$. Then each bounded solution x of equation (2.3) *bounded on $\mathbb{N}$ with $v_n = \phi$ satisfies*

$$\|v_m\| \le 2Ne^{-(\lambda - 2\delta NK(1))(m-n)}\|P_n\phi\| \tag{3.6}$$

for every $m \ge n$.

Proof. By Proposition 3.2, each solution x of equation (2.3) bounded on $\mathbb{N}$ with $v_n = \phi$ satisfies (3.3). Using (3.5) and the fact that $f_n(0) = 0$, it follows from (2.5) and (3.3) that

$$\begin{aligned}
\|x_m\| \le Ne^{-\lambda(m-n)}\|P_n\phi\| &+ \delta NK(1)\sum_{l=n}^{m-1} e^{-\lambda(m-l-1)}\|x_l\| \\
&+ \delta NK(1)\sum_{l=m}^{+\infty} e^{\lambda(m-l-1)}\|x_l\|.
\end{aligned} \tag{3.7}$$

Before continuing, we formulate an auxiliary result.

Lemma 3.4 (see [1, Lemma 2.5])**.** *Let $(v(m))_{m\ge n}$ be a bounded sequence in $\mathbb{R}_0^+$ such that*

$$v(m) \le \beta e^{-\gamma(m-n)} + \delta D\sum_{l=n}^{m-1} e^{-\gamma(m-l-1)}v(l) + \delta D\sum_{l=m}^{+\infty} e^{-\gamma(l+1-m)}v(l)$$

for $m \geq n$. Then, for any sufficiently small δ, we have

$$v(m) \leq 2\beta e^{-(\gamma-2\delta D)(m-n)} \quad \text{for } m \geq n.$$

Proof. Consider a sequence $u(m)$ such that for $m \geq n$

$$u(m) = \beta e^{-\gamma(m-n)} + \delta D \sum_{l=n}^{m-1} e^{-\gamma(m-l-1)} u(l) + \delta D \sum_{l=m}^{+\infty} e^{-\gamma(l+1-m)} u(l) \tag{3.8}$$

for $m \geq n$. One can verify in a straightforward manner that it satisfies the recurrence

$$u(m+1) = (e^{\gamma} + e^{-\gamma})u(m) - (1 + \delta(e^{\gamma} + e^{-\gamma}))u(m-1). \tag{3.9}$$

In order that the solution $u(m)$ of recurrence (3.9) is bounded we must have $u(m) = u(n)e^{-\tilde{\gamma}(m-n)}$, where

$$\tilde{\gamma} = -\log\Big(\cosh\gamma - \sqrt{\cosh^2\gamma - 2(1 + \delta D \sinh\gamma)}\Big) \geq \gamma - 2\delta D.$$

Note that for δ sufficiently small we have $e^{-\tilde{\gamma}} \leq e^{-(\gamma-2\delta D)} < 1$.

Taking $m = n$ in (3.8) we obtain

$$\begin{aligned} u(n) &= \beta + \delta D e^{-\gamma} u(n) \sum_{l=n}^{+\infty} e^{-(\gamma+\tilde{\gamma})(l-n)} \\ &= \beta + u(n)\frac{\delta D e^{-\gamma}}{1 - e^{-(\gamma+\tilde{\gamma})}}, \end{aligned}$$

which yields

$$u(n) = \frac{\beta(e^{\gamma+\tilde{\gamma}} - 1)}{e^{\gamma+\tilde{\gamma}} - 1 - \delta D e^{\tilde{\gamma}}} \leq 2\beta$$

for δ sufficiently small. Hence,

$$u(m) \leq 2\beta e^{-\tilde{\gamma}(m-n)} \leq 2\beta e^{-(\gamma-2\delta D)(m-n)}.$$

Now let $w(n) = v(m) - u(m)$ for $m \geq n$. Then

$$w(n) \leq \delta D \sum_{l=n}^{m-1} e^{-\gamma(m-l-1)} w(l) + \delta D \sum_{l=m}^{+\infty} e^{-\gamma(l+1-m)} w(l)$$

Finally, taking $w = \sup_{m \geq n} w(m)$ we obtain

$$w \leq \delta D w \sup_{m \geq n} \sum_{l=n}^{m-1} e^{-\gamma(m-l-1)} + \delta D w \sup_{m \geq n} \sum_{l=m}^{+\infty} e^{-\gamma(l+1-m)} \leq \delta D w \frac{1 + e^{-\gamma}}{1 - e^{-\gamma}}$$

and so $w \leq 0$ for δ sufficiently small. Thus, $v(m) \leq u(m)$ for $m \geq n$, which yields the desired result. □

In view of (3.7), applying Lemma 3.4 with

$$v(m) = \|x_m\|, \quad \beta = N\|P_n\phi\|, \quad \gamma = \lambda \quad \text{and} \quad D = NK(1)$$

we obtain inequality (3.6). □

Now we formulate the Lipschitz stable manifold theorem. Assume that equation (2.1) has an exponential dichotomy. Let $\mathcal{L}$ be the set of all maps

$$z\colon \{(n, a) \in \mathbb{Z} \times \mathcal{B} : a \in E_n\} \to \mathcal{B} \tag{3.10}$$

such that for each $n \in \mathbb{Z}$:

(1) $z(n,0)=0$ and $z(n,E_n)\subset F_n$;
(2) for $a,\bar a\in E_n$ we have

$$\|z(n,a)-z(n,\bar a)\|\le\|a-\bar a\|.$$

For each function $z\in\mathcal{L}$ we consider its graph

$$\operatorname{graph} z=\big\{(n,a+z(n,a)):(n,a)\in\mathbb{Z}\times E_n\big\}\subset\mathbb{Z}\times\mathcal{B}.$$

Theorem 3.5. *Assume that* (2.1) *has an exponential dichotomy and that there exists $\delta>0$ such that* (3.5) *holds for every $n\in\mathbb{Z}$ and $u,v\in\mathcal{B}$. Then, for any sufficiently small δ, there exists a function $z\in\mathcal{L}$ such that $V^s=\operatorname{graph} z$. Moreover,*

$$\|v_m(\cdot,n,a+z(n,a))-v_m(\cdot,n,\bar a+z(n,\bar a))\|\le 2Ne^{-(\lambda-2\delta NK(1))(m-n)}\|a-\bar a\| \tag{3.11}$$

for every $m,n\in\mathbb{Z}$ with $m\ge n$ and $a,\bar a\in E_n$.

Proof. Take $n\in\mathbb{Z}$ and $a\in E_n$. We consider the set $\mathcal{L}_a$ of all sequences $x\colon\mathbb{Z}\to X$ with $P_nx_n=a$ such that

$$\|x_m\|\le 3Ne^{-(\lambda-2\delta NK(1))(m-n)}\|a\|$$

for all $m\ge n$. One can easily verify that $\mathcal{L}_a$ is a complete metric space when equipped with the norm

$$|x|_a:=\sup_{m\ge n}\big(\|x_m\|e^{(\lambda-2\delta NK(1))(m-n)}\big).$$

We define an operator J on $\mathcal{L}_a$ by

$$\begin{aligned}(Jx)_m&=T(m,n)a+\sum_{l=n}^{m-1}T(m,l+1)P_{l+1}(\Gamma f_l(x_l))\\&\quad-\sum_{l=m}^{+\infty}T(m,l+1)Q_{l+1}(\Gamma f_l(x_l))\end{aligned}$$

for $x\in\mathcal{L}_a$ and $m\ge n$. Note that

$$(Jx)_n=a-\sum_{l=n}^{+\infty}T(n,l+1)Q_{l+1}(\Gamma f_l(x_l))$$

and so $P_n(Jx)_n=a$.

We want to show that $J(\mathcal{L}_a)\subset\mathcal{L}_a$ and that J is a contraction. For each $m\ge n$ and $x,y\in\mathcal{L}_a$, by (3.5) we have

$$\begin{aligned}\|(Jx)_m-(Jy)_m\|&\le\delta NK(1)\sum_{l=n}^{m-1}e^{-\lambda(m-l-1)}\|x_l-y_l\|\\&\quad+\delta NK(1)\sum_{l=m}^{+\infty}e^{\lambda(m-l-1)}\|x_l-w_l\|\\&\le\delta NK(1)\sum_{l=n}^{m-1}e^{-\lambda(m-l-1)}e^{-(\lambda-2\delta NK(1))(l-n)}|x-y|_a\\&\quad+\delta NK(1)\sum_{l=m}^{+\infty}e^{\lambda(m-l-1)}e^{-(\lambda-2\delta NK(1))(l-n)}|x-y|_a\end{aligned}$$

$$
\begin{aligned}
&\le \delta NK(1)e^{-\lambda(m-1-n)}\sum_{l=n}^{m-1} e^{2\delta NK(1)(l-n)}|x-y|_a\\
&\quad + \delta NK(1)e^{\lambda(m-n)}\sum_{l=m}^{+\infty} e^{-2(\lambda-\delta NK(1))(l-n)}|x-y|_a\\
&\le \frac{\delta NK(1)e^{\lambda}}{e^{2\delta NK(1)}-1}e^{-(\lambda-2\delta NK(1))(m-n)}|x-y|_a\\
&\quad + \frac{\delta NK(1)}{1-e^{-2(\lambda-\delta NK(1))}}e^{-(\lambda-2\delta NK(1))(m-n)}|x-y|_a\\
&\le \Big(\frac{1}{2}+\frac{\delta NK(1)}{1-e^{-2(\lambda-\delta NK(1))}}\Big)e^{-(\lambda-2\delta NK(1))(m-n)}|x-y|_a
\end{aligned}
$$

(since $x/(e^{2x}-1)<1/2$), and so

$$
|Jx-Jy|_a \le \theta|x-y|_a, \quad \text{where } \theta = \frac{1}{2}+\frac{\delta NK(1)}{1-e^{-2(\lambda-\delta NK(1))}}.
$$

Taking δ sufficiently small so that $\theta<2/3$, the operator becomes a contraction. Moreover, we have $|J0|_a \le N\|a\|$ and hence,

$$
\begin{aligned}
|Jx|_a &\le |J0|_a + |Jx-J0|_a \le N\|a\| + \theta|x|_a\\
&\le N\|a\| + \frac{2}{3}\cdot 3N\|a\| = 3N\|a\|.
\end{aligned}
$$

This shows that $J(\mathcal{L}_a)\subset\mathcal{L}_a$ and so there exists a unique $x=x^{n,a}\in\mathcal{L}_a$ such that $Jx=x$.

Now we use the function $x^{n,a}$ to construct the stable manifold. Taking $m=n$ in (3.3) we obtain

$$
Q_n x_n = -\sum_{l=n}^{+\infty} T(n,l+1)Q_{l+1}(\Gamma f_l(x_l)).
$$

We define a function z as in (3.10) by

$$
z(n,a) := Q_n x_n^{n,a} = -\sum_{l=n}^{+\infty} T(n,l+1)Q_{l+1}(\Gamma f_l(x_l)). \tag{3.12}
$$

Note that by construction

$$
x^{n,a} = v(\cdot,n,a+z(n,a)). \tag{3.13}
$$

Now we show that $z\in\mathcal{L}$. Taking $\phi\in\mathcal{B}$ with $a=P_n\phi=0$, the function $x=0$ satisfies (3.3) (recall that $f_m(0)=0$ for $m\in\mathbb{Z}$) and so $z(n,0)=0$. Moreover, it follows from Lemma 3.4 with

$$
v(m)=\|x_m^{n,a}-x_m^{n,\bar a}\|, \quad \beta=N\|a-\bar a\|, \quad \gamma=\lambda, \quad D=NK(1)
$$

that

$$
\|x_m^{n,a}-x_m^{n,\bar a}\| \le 2Ne^{-(\lambda-2\delta NK(1))(m-n)}\|a-\bar a\|.
$$

This establishes (3.11). On the other hand, it follows from (3.12) that

$$
\|z(n,a)-z(n,\bar a)\| \le \delta NK(1)\sum_{l=n}^{+\infty} e^{\lambda(n-l-1)}\|x_l^{n,a}-x_l^{n,\bar a}\|
$$

$$
\begin{aligned}
&\le 2\delta N^2 K(1) \sum_{l=n}^{+\infty} e^{-2(\lambda-\delta NK(1))(l-n)} \|a-\bar{a}\| \\
&\le \frac{2\delta N^2 K(1)}{1-e^{-2(\lambda-\delta NK(1))}} \|a-\bar{a}\|.
\end{aligned}
$$

Taking δ sufficiently small we have $z \in \mathcal{L}$.

Finally, we show that $\operatorname{graph} z = V^s$. First observe that given $(n,\phi) \in \operatorname{graph} z$, we have $\phi = a + z(n,a)$, where $a = P_n\phi$. Hence, by (3.12) we obtain $x^{n,a} = x(\cdot,n,\phi)$ and since $(x_m^{n,a})_{m\ge 0}$ is bounded, it follows from (A1) that $(n,\phi) \in V^s$. Conversely, if $(n,\phi) \in V^s$, then by Propositions 3.1 and 3.2 there exists a sequence $x\colon \mathbb{Z} \to X$ bounded on $\mathbb{N}$ with $x_n = \phi$ satisfying (3.3). Moreover, in view of the uniqueness of solutions we have $x = x^{n,a}$, where $a = P_n\phi$. By (3.12) and (3.13) we obtain

$$
z(n,a) = Q_n x_n = Q_n\phi
$$

and so

$$
(n,\phi) = (n, P_n\phi + Q_n\phi) = (n, a + z(n,a)) \in \operatorname{graph} z.
$$

This completes the proof of the theorem. $\square$

3.2. **Unstable manifolds.** The *unstable set* V^u of equation (2.3) is the set of all $(n,\phi) \in \mathbb{Z} \times \mathcal{B}$ for which:

(1) $v(\cdot,n,\phi)$ satisfies (2.3) for all $m \in \mathbb{Z}$;
(2) the map $m \mapsto v_m(\cdot,n,\phi)$ is bounded on $(-\infty,n]$.

Note that the first condition corresponds to require a certain compatibility from the entries of ϕ since $v_m = \phi_{m-n}$ for all $m \le n$. It can be shown as in the proof of Proposition 3.1 that if $(n,\phi) \in V^u$, then $(m, v_m(\cdot,n,\phi)) \in V^k$ for all $m \le n$.

Now we formulate the Lipschitz unstable manifold theorem. Assume that equation (2.1) has an exponential dichotomy. Let $\mathcal{M}$ be the set of all maps

$$
w\colon \{(n,b) \in \mathbb{Z} \times \mathcal{B} : b \in F_n\} \to \mathcal{B}
$$

such that for each $n \in \mathbb{Z}$:

(1) $w(n,0) = 0$ and $w(n,F_n) \subset E_n$;
(2) for $b, \bar{b} \in F_n$ we have

$$
\|w(n,b) - w(n,\bar{b})\| \le \|b - \bar{b}\|.
$$

For each function $w \in \mathcal{M}$ we consider its graph

$$
\operatorname{graph} w = \{(n, w(n,b) + b) : (n,b) \in \mathbb{Z} \times F_n\} \subset \mathbb{Z} \times \mathcal{B}.
$$

Theorem 3.6. *Assume that the equation* (2.1) *has an exponential dichotomy and that there exists $\delta > 0$ such that* (3.5) *holds for every $m \in \mathbb{Z}$ and $u, v \in \mathcal{B}$. Then, for any sufficiently small δ, there exists a function $w \in \mathcal{M}$ such that $V^u = \operatorname{graph} w$. Moreover,*

$$
\|v_m(\cdot,n,w(n,b)+b) - v_m(\cdot,n,w(n,\bar{b})+\bar{b})\| \le 2Ne^{-(\lambda-2\delta NK(1))(n-m)}\|b-\bar{b}\|
$$

for every $m, n \in \mathbb{Z}$ with $m \le n$ and $b, \bar{b} \in F_n$.

The proof of Theorem 3.6 is entirely analogous to the proof of Theorem 3.5 and so we omit it.

4. C^1 MANIFOLDS

In this section we turn to the construction of smooth stable and unstable invariant manifolds for equation (2.3).

4.1. **Stable manifolds.** Now we assume that f is of class C^1, with $f_m(0)=0$ and $d_0 f_m = 0$ for all $m \in \mathbb{Z}$. The stable manifold is obtained as a graph of a function of class C^1 in the second variable. In Theorem 3.5 we already showed that the stable set is a graph of a Lipschitz function z in the second variable. Thus, it remains to verify (with the current assumptions) that this function is of class C^1 in the second variable.

Let $\mathcal{L}^1$ be the set of all functions $z \in \mathcal{L}$ of class C^1 in the second variable such that $\partial z(n,0)=0$ for $n \in \mathbb{Z}$.

Theorem 4.1. *Assume that equation* (2.1) *has an exponential dichotomy and that there exists $\delta > 0$ such that*
$$\|d_v f_m\| \le \delta \tag{4.1}$$
for every $m \in \mathbb{Z}$ and $v \in \mathcal{B}$. Then, for any sufficiently small δ, the function $z \in \mathcal{L}$ given by Theorem 3.5 is in $\mathcal{L}^1$.

Proof. We first recall a result from [6] (see [4]).

Lemma 4.2. *Given Banach spaces Y and Z, let $g\colon A \to Z$ be a Lipschitz function on some open ball $A \subset Y$. Then g is of class C^1 if and only if for each $y \in A$ we have*
$$|g(v+h)-g(v)-g(u+h)+g(u)| = o(|h|),$$
when $(v,h) \to (u,0)$.

Let $z \in \mathcal{L}$ be the function given by Theorem 3.5. The map $E_n \ni a \mapsto z(n,a)$ is of class C^1 if and only if the same happens with
$$a \mapsto a + z(n,a) = v_n(\cdot, n, a + z(n,a)).$$
Hence, in view of Lemma 4.2 and (A1), writing
$$y_m^a = v_m(\cdot, n, a+z(n,a))$$
it suffices to show that
$$\|y_m^{b+h} - y_m^b - y_m^{a+h} + y_m^a\| = o(\|h\|)$$
when $(b,h) \to (a,0)$ for each $m \in \mathbb{Z}$ and $a \in E_n$. We define
$$w_l(a,b,h) = y_l^{b+h} - y_l^b - y_l^{a+h} + y_l^a.$$
By Taylor's formula we have
$$\begin{aligned} f_l(y_l^{a+h}) &= f_l(y_l^a + (y_l^{a+h} - y_l^a)) \\ &= f_l(y_l^a) + d_{y_l^a} f_l(y_l^{a+h} - y_l^a) + \Delta(l,a,h), \end{aligned}$$
where
$$\begin{aligned} \Delta(l,a,h) &= \int_0^1 d_{y_l^a + t(y_l^{a+h} - y_l^a)} f_l(y_l^{a+h} - y_l^a)\,dt - d_{y_l^a} f_l(y_l^{a+h} - y_l^a) \\ &= \int_0^1 \big[d_{y_l^a + t(y_l^{a+h} - y_l^a)} f_l - d_{y_l^a} f_l\big](y_l^{a+h} - y_l^a)\,dt. \end{aligned}$$

Therefore,

$$
\begin{aligned}
G(l) :&= f_l(y_l^{b+h}) - f_l(y_l^b) - f_l(y_l^{a+h}) + f_l(y_l^a) \\
&= f_l(y_l^{b+h}) - f_l(y_l^b) - d_{y_l^a} f_l(y_l^{a+h} - y_l^a) - \Delta(l,a,h) \\
&= f_l(y_l^{b+h}) - f_l(y_l^b) + d_{y_l^a} f_l(w_l(a,b,h) - y_l^{b+h} + y_l^b) - \Delta(l,a,h) \\
&= d_{y_l^b} f_l(y_l^{b+h} - y_l^b) + \Delta(l,b,h) + d_{y_l^a} f_l w_l(a,b,h) \\
&\quad - d_{y_l^a} f_l(y_l^{b+h} + y_l^b) - \Delta(l,a,h) \\
&= d_{y_l^b} f_l w_l(a,b,h) + (d_{y_l^b} f_l - d_{y_l^a} f_l)(y_l^{b+h} - y_l^b) + \Delta(l,b,h) - \Delta(l,a,h).
\end{aligned}
$$

On the other hand, since

$$
y_m^a = T(m,n)a + \sum_{l=n}^{m-1} T(m,l+1)P_{l+1}(\Gamma f_l(x_l)) - \sum_{l=m}^{+\infty} T(m,l+1)Q_{l+1}(\Gamma f_l(x_l))
$$

for $m \geq n$, we obtain

$$
\begin{aligned}
w_m(a,b,h) &= \sum_{l=n}^{m-1} T(m,l+1)P_{l+1}(\Gamma G(l)) - \sum_{l=m}^{+\infty} T(m,l+1)Q_{l+1}(\Gamma G(l)) \\
&= \sum_{l=n}^{m-1} T(m,l+1)P_{l+1}(\Gamma d_{y_l^a} f_l w_l(a,b,h)) \\
&\quad + \sum_{l=n}^{m-1} T(m,l+1)P_{l+1}(\Gamma(d_{y_l^b} f_l - d_{y_l^a} f_l)(y_l^{b+h} - y_l^b)) \\
&\quad + \sum_{l=n}^{m-1} T(m,l+1)P_{l+1}(\Gamma(\Delta(l,b,h) - \Delta(l,a,h))) \qquad (4.2) \\
&\quad - \sum_{l=m}^{+\infty} T(m,l+1)Q_{l+1}(\Gamma d_{y_l^a} f_l w_l(a,b,h)) \\
&\quad - \sum_{l=m}^{+\infty} T(m,l+1)Q_{l+1}(\Gamma(d_{y_l^b} f_l - d_{y_l^a} f_l)(y_l^{b+h} - y_l^b)) \\
&\quad - \sum_{l=m}^{+\infty} T(m,l+1)Q_{l+1}(\Gamma(\Delta(l,b,h) - \Delta(l,a,h))).
\end{aligned}
$$

Using the inequalities in (2.2) and (4.1) we obtain

$$
\begin{aligned}
&\sum_{l=n}^{m-1} \|T(m,l+1)P_{l+1}(\Gamma d_{y_l^a} f_l w_l(a,b,h))\| \\
&\leq \delta N K(1) \sum_{l=n}^{m-1} e^{-\lambda(m-l-1)} \sup_{l \geq n} \|w_l(a,b,h)\| \\
&\leq \frac{\delta N K(1)}{1 - e^{-\lambda}} \sup_{m \geq n} \|w_m(a,b,h)\|
\end{aligned}
$$

and

$$\sum_{l=m}^{+\infty} \|T(m,l+1)Q_{l+1}(\Gamma d_{y_l^a} f_l w_l(a,b,h))\| \le \frac{\delta NK(1)}{1-e^{-\lambda}} e^{\lambda} \sup_{m\ge n} \|w_m(a,b,h)\|.$$

Moreover, by Theorem 3.5 we have

$$\|y_l^{b+h} - y_l^b\| \le 2N\|h\| e^{-(\lambda-2\delta NK(1))(m-n)} \tag{4.3}$$

for $m \ge n$, and so

$$\begin{aligned}
&\sum_{l=n}^{m-1} \|T(m,l+1)P_{l+1}(\Gamma(d_{y_l^b} f_l - d_{y_l^a} f_l)(y_l^{b+h} - y_l^b))\| \\
&\le 2N^2K(1)\|h\| \sum_{l=n}^{m-1} e^{-\lambda(m-l-1)} e^{-(\lambda-2\delta NK(1))(l-n)} \sup_{l\ge n} \|d_{y_l^b} f_l - d_{y_l^a} f_l\| \\
&\le 2N^2K(1)\|h\| e^{\lambda} e^{-(\lambda-2\delta NK(1))(m-n)} \sup_{l\ge n} \|d_{y_l^b} f_l - d_{y_l^a} f_l\| \sum_{l=n}^{m-1} e^{2\delta NK(1)(l-m)} \\
&\le N_1\|h\| \sup_{l\ge n} \|d_{y_l^b} f_l - d_{y_l^a} f_l\|,
\end{aligned}$$

where N_1 is some positive constant. Similarly,

$$\begin{aligned}
&\sum_{l=m}^{+\infty} \|T(m,l+1)Q_{l+1}(\Gamma(d_{y_l^b} f_l - d_{y_l^a} f_l)(y_l^{b+h} - y_l^b))\| \\
&\le N_2\|h\| \sup_{l\ge n} \|d_{y_l^b} f_l - d_{y_l^a} f_l\|,
\end{aligned}$$

where N_2 is some positive constant. Finally, letting

$$\begin{aligned}
S(a,b,h) &= \sup_{m\ge n} \Big\| \int_0^1 \big(d_{y_m^a + t(y_m^{a+h} - y_m^a)} f_m - d_{y_m^a} f_m\big)\,dt\Big\| \\
&\quad + \sup_{m\ge n} \Big\| \int_0^1 \big(d_{y_m^b + t(y_m^{b+h} - y_m^b)} f_m - d_{y_m^b} f_m\big)\,dt\Big\|,
\end{aligned}$$

it follows from (4.3) that

$$\begin{aligned}
&\sum_{l=n}^{m-1} \|T(m,l+1)P_{l+1}(\Gamma(\Delta(l,b,h) - \Delta(l,a,h)))\| \\
&\le 2N^2K(1)\|h\| S(a,b,h) \sum_{l=n}^{m-1} e^{-\lambda(m-l-1)} e^{-(\lambda-2\delta NK(1))(l-n)} \\
&\le N_3\|h\| S(a,b,h) e^{-(\lambda-2\delta NK(1))(m-n)} \\
&\le N_3\|h\| S(a,b,h)
\end{aligned}$$

and

$$\sum_{l=m}^{+\infty} \|T(m,l+1)Q_{l+1}(\Gamma(\Delta(l,b,h) - \Delta(l,a,h)))\| \le N_4\|h\| S(a,b,h)$$

for some positive constants N_3 and N_4. Therefore, by (4.2), for δ sufficiently small we obtain

$$\sup_{m\geq n}\|w_m(a,b,h)\|\leq N_5\|h\|\Big(\sup_{m\geq n}\|d_{y_m^b}f_m-d_{y_m^a}f_m\|+S(a,b,h)\Big), \tag{4.4}$$

where N_5 is some positive constant. It follows from the continuity of the maps f_m, (4.3) and (4.4) that

$$\lim_{(b,a)\to(a,0)}\frac{1}{\|h\|}\sup_{m\geq n}\|w_m(a,b,h)\|=0.$$

Hence, by Lemma 3.4, each map $E_n\ni a\mapsto z(n,a)$ is of class C^1. Moreover, it follows from (3.12) that $\partial z(n,0)=0$ and so $z\in\mathcal{L}^1$. $\square$

We have a corresponding result for the unstable manifold. Let $\mathcal{M}^1$ be the set of all functions $w\in\mathcal{M}$ of class C^1 in the second variable such that $\partial w(n,0)=0$ for $n\in\mathbb{Z}$.

Theorem 4.3. *Assume that equation* (2.1) *has an exponential dichotomy and that there exists $\delta>0$ such that* (4.1) *holds for every $m\in\mathbb{Z}$ and $v\in\mathcal{B}$. Then, for any sufficiently small δ, the function $w\in\mathcal{M}$ given by Theorem 3.6 is in $\mathcal{M}^1$.*

The proof of Theorem 4.3 is entirely analogous to the proof of Theorem 4.1.

4.2. **Higher smoothness.** In this section we formulate C^k stable and unstable manifold theorems, for $k\in\mathbb{N}$, following closely the approaches in the former sections. The results can be obtained in a more or less straightforward manner using induction on k together with the Faà di Bruno formula and so we omit the proofs.

We assume that f is of class C^k, with $f_m(0)=0$ and $d_0f_m=0$ for $m\in\mathbb{Z}$. The following results are C^k versions of Theorems 4.1 and 4.3. Let $\mathcal{L}^k$ be the set of all functions $z\in\mathcal{L}$ of class C^k in a such that $\partial z(n,0)=0$ and

$$\|\partial^j z(n,a)\|\leq 1$$

for every $n\in\mathbb{Z}$, $a\in E_n$ and $j=1,\dots,k$.

Theorem 4.4. *Assume that equation* (2.1) *has an exponential dichotomy and that there exists $\delta>0$ such that $\|d_v^jf_m\|\leq\delta$ for every $m\in\mathbb{Z}$, $v\in\mathcal{B}$ and $j=1,\dots,k$. Then, for any sufficiently small δ, the function $z\in\mathcal{L}$ given by Theorem 3.5 is in $\mathcal{L}^k$.*

Similarly, let $\mathcal{M}^k$ be the set of all functions $w\in\mathcal{M}$ of class C^k in b such that $\partial w(n,0)=0$ and

$$\|\partial^j w(n,b)\|\leq 1$$

for every $n\in\mathbb{Z}$, $b\in F_n$ and $j=1,\dots,k$.

Theorem 4.5. *Assume that equation* (2.1) *has an exponential dichotomy and that there exists $\delta>0$ such that $\|d_v^jf_m\|\leq\delta$ for every $n\in\mathbb{Z}$, $v\in\mathcal{B}$ and $j=1,\dots,k$. Then, for any sufficiently small δ, the function $w\in\mathcal{M}$ given by Theorem 3.6 is in $\mathcal{M}^k$.*

5. Center manifolds

In this section we formulate corresponding center manifold theorems. We first introduce the notion of an exponential trichotomy.

We say that equation (2.1) has a *exponential trichotomy* if:

(1) there exist projections $P_n, Q_n, R_n\colon \mathcal{B} \to \mathcal{B}$, for $n \in \mathbb{Z}$, satisfying $P_n + Q_n + R_n = \mathrm{Id}$ such that for $m \ge n$ we have
$$P_m T(m,n) = T(m,n) P_n, \quad Q_m T(m,n) = T(m,n) Q_n,$$
$$R_m T(m,n) = T(m,n) R_n;$$
(2) $T(m,n)$ is invertible from $\ker P_n$ onto $\ker P_m$ for each $m \ge n$;
(3) there exist constants $\lambda, N, \mu > 0$ with $\mu < \lambda$ such that for $m \ge n$ we have
$$\|T(m,n)R_n\| \le N e^{\mu(m-n)}, \quad \|T(m,n)P_n\| \le N e^{-\lambda(m-n)},$$
$$\|(T(m,n)R_n)^{-1}\| \le N e^{\mu(m-n)}, \quad \|(T(m,n)Q_n)^{-1}\| \le N e^{-\lambda(m-n)}.$$

For each $m \in \mathbb{Z}$ we then define
$$E_m = \operatorname{Im} P_m, \quad F_m = \operatorname{Im} Q_m, \quad G_m = \operatorname{Im} R_m.$$

Now we introduce the notion of the center set of equation (2.3). Let $\mathcal{K}_\kappa$ be the set of all sequences $v\colon \mathbb{Z} \to \mathcal{B}$ such that
$$\sup_{m\in\mathbb{Z}} \left(\|v_m\| e^{-\kappa|m|}\right) < +\infty$$
for any sufficiently large $\kappa < \lambda$. The *center set* V^c of equation (2.3) is the set of all $(n,\phi) \in \mathbb{Z} \times \mathcal{B}$ for which there exists a solution $x(\cdot, n, \phi) \in \mathcal{K}$ with $x_n = \phi$. The center set has the invariance property in Proposition 3.1. More precisely, if $(n,\phi) \in V^c$, then $(m, v_m(\cdot, n, \phi)) \in V^c$ for all $m \in \mathbb{Z}$.

Assume that equation (2.1) has an exponential trichotomy. Proceeding as in Section 3.1 one can establish a Lipschitz center manifold theorem. Let $\mathcal{N}$ be the set of all maps
$$u\colon \{(n,c) \in \mathbb{Z} \times \mathcal{B} : c \in G_n\} \to \mathcal{B}$$
such that for each $n \in \mathbb{Z}$:

(1) $u(n,0) = 0$ and $u(n, G_n) \subset E_n \oplus F_n$;
(2) for $c, \bar{c} \in G_n$ we have
$$\|u(n,c) - u(n,\bar{c})\| \le \|c - \bar{c}\|.$$

For each function $w \in \mathcal{N}$ we consider its graph
$$\operatorname{graph} u = \{(n, c + u(n,c)) : (n,c) \in \mathbb{Z} \times G_n\} \subset \mathbb{Z} \times \mathcal{B}.$$

Theorem 5.1. *Assume that equation* (2.1) *has an exponential trichotomy and that there exists $\delta > 0$ such that* (3.5) *holds for every $m \in \mathbb{Z}$ and $u, v \in \mathcal{B}$. Then, for any sufficiently small δ, there exists a function $u \in \mathcal{N}$ such that $V^c = \operatorname{graph} u$.*

Now let $\mathcal{N}^k$ be the set of all functions $u \in \mathcal{N}$ of class C^k in c such that $\partial u(n,0) = 0$, $\|\partial^j u(n,c)\| \le 1$ for $j = 1, \dots, k$. Repeating arguments in [2] we obtain the following smooth center manifold theorem.

Theorem 5.2. *Assume that equation* (2.1) *has an exponential trichotomy and that there exists $\delta > 0$ such that $\|d_v^j f_m\| \le \delta$ for every $m \in \mathbb{Z}$, $v \in \mathcal{B}$ and $j = 1, \dots, k$. If $\lambda > (k+1)\mu$ and δ is sufficiently small, then the function $u \in \mathcal{N}$ given by Theorem 5.1 is in $\mathcal{N}^k$.*

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