Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 83, pp. 1–18.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

STABILIZATION OF THE WAVE EQUATION WITH LOCALIZED COMPENSATING FRICTIONAL AND KELVIN-VOIGT DISSIPATING MECHANISMS

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Communicated by Jerome A. Goldstein

Abstract. We consider the wave equation with two types of locally distributed damping mechanisms: a frictional damping and a Kelvin-Voigt type damping. The location of each damping is such that none of them alone is able to exponentially stabilize the system; the main obstacle being that there is a quite big undamped region. Using a combination of the multiplier techniques and the frequency domain method, we show that a convenient interaction of the two damping mechanisms is powerful enough for the exponential stability of the dynamical system, provided that the coefficient of the Kelvin-Voigt damping is smooth enough and satisfies a structural condition. When the latter coefficient is only bounded measurable, exponential stability may still hold provided there is no undamped region, else only polynomial stability is established. The main features of this contribution are: (i) the use of the Kelvin-Voigt or short memory damping as opposed to the usual long memory type damping; this makes the problem more difficult to solve due to the somewhat singular nature of the Kelvin-Voigt damping, (ii) allowing the presence of an undamped region unlike all earlier works where a combination of frictional and viscoelastic damping is used.

1. Introduction and statement of main results

The stabilization of the wave equation with localized damping has received a special attention since the seventies e.g. [3, 7, 9, 10, 11, 12, 13, 14, 19, 22, 25, 26, 27, 28, 31, 33, 34, 35, 36, 37, 40, 41, 47, 48]. The purpose of this work is to study the stabilization of a material composed of two parts: one that is elastic and the other one that is a Kelvin-Voigt type viscoelastic material. This type of material is encountered in real life when one uses patches to suppress vibrations, the modeling aspect of which may be found in [2]. This type of question was examined in the one-dimensional setting in [23] where it was shown that the longitudinal motion of an Euler-Bernoulli beam modeled by a locally damped wave equation with Kelvin-Voigt damping is not exponentially stable when the junction between the elastic part and the viscoelastic part of the beam is not smooth enough. Later on, the wave equation with Kelvin-Voigt damping in the multidimensional setting was examined

2010 *Mathematics Subject Classification.* 93D15, 35L05.
Key words and phrases. Stabilization; wave equation; frictional damping;
Kelvin-Voigt damping; viscoelastic material; localized damping.

Submitted April 26, 2016. Published March 24, 2017.

in [25]; in particular, those authors showed the exponential decay of the energy by assuming that the damping region ω is a neighborhood of the whole boundary, and the damping coefficient a satisfies [24, 25]: $a \in C^{1,1}(\bar{\Omega})$, $\Delta a \in L^\infty(\Omega)$, and $|\nabla a(x)|^2 \le M_0 a(x)$, for almost every x in Ω, for some positive constant M_0. Later on, it was shown that the exponential decay of the energy could be obtained without imposing $\Delta a \in L^\infty(\Omega)$, and for a larger class of feedback regions ω [41]. The main purpose of the present contribution is to use two damping mechanisms: one frictional damping and one viscoelastic damping of Kelvin-Voigt type, and answer the following questions: (a) under which conditions on the damping coefficients and locations do we ensure the exponential stability of the dynamical system? (b) When exponential stability fails, what type of stability do we have? For the sequel we need some notations. Let Ω be a bounded nonempty subset of $\mathbb{R}^N$, $(N \ge 2)$, with boundary Γ of class C^2. Let ν denote the unit normal vector pointing into the exterior of Ω.

Consider the damped wave system

$$\begin{gathered} y_{tt} - \Delta y + a(x)y_t - \operatorname{div}(b(x)\nabla y_t) = 0 \quad \text{in } \Omega \times (0,\infty) \\ y = 0 \quad \text{on } \Gamma \times (0,\infty) \\ y(0) = y^0, \quad y_t(0) = y^1, \end{gathered} \tag{1.1}$$

where $a, b : \Omega \to \mathbb{R}$ are nonnegative functions satisfying

$$\begin{gathered} a \in L^\infty(\Omega), \quad b \in L^\infty(\Omega), \\ a(x) > 0, \quad \text{a.e. } x \in \omega_a, \quad b(x) > 0 \quad \text{in } \omega_b, \end{gathered} \tag{1.2}$$

where ω_a and ω_b denote open subsets of Ω.

Under the above assumptions on the coefficients, if $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$, it is well-known that System (1.1) has a unique weak solution

$$y \in \mathcal{C}([0,\infty); H_0^1(\Omega)) \cap \mathcal{C}^1([0,\infty); L^2(\Omega)). \tag{1.3}$$

Similarly if $(y^0, y^1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$ then it can be shown that the unique solution of System (1.1) satisfies

$$y \in \mathcal{C}([0,\infty); H_0^1(\Omega)) \cap \mathcal{C}^1([0,\infty); H_0^1(\Omega)). \tag{1.4}$$

A close attention to (1.4) leads one to notice that there is a discrepancy on the regularity of the initial data and that of the solutions; this is due to the structure of the Kelvin-Voigt damping. This makes the stabilization problem much more difficult to solve than in the case of a frictional damping $a(x)y_t$ alone, when the presence of an undamped region is allowed. As we shall see in the proof of the various stabilization results later on, we need to introduce a new variable and a set of suitable auxiliary elliptic systems to cope with this loss of regularity. This loss of derivative seems intuitively unbelievable since strong damping would usually make the solution smoother than the initial data as the dynamical system evolves with time, but in the present framework where the strong dissipation is localized, the smoothing effect is also localized; in other words, there is no smoothing on the whole domain under consideration.

We would also like to stress that the type of stabilization problem being addressed here, that is using competing damping mechanisms to achieve polynomial and exponential decay of the energy, makes sense in space dimensions greater or equal to two. In fact, in the one-dimensional setting, one may choose the location of the damping arbitrarily small, and still get a uniform exponential decay of the energy, while in higher dimensions, a geometric constraint has to be imposed on the damping region for exponential decay of the energy to hold, [3].

We introduce the energy

$$E(t) = \frac{1}{2} \int_{\Omega} \{|y_t(x,t)|^2 + |\nabla y(x,t)|^2\}\, dx, \quad \forall t \geq 0. \tag{1.5}$$

The energy E is a nonincreasing function of the time variable t and its derivative satisfies

$$E'(t) = -\int_{\Omega} a(x)|y_t(x,t)|^2 + b(x)|\nabla(y_t(x,t)|^2\, dx, \quad \forall t \geq 0. \tag{1.6}$$

The questions that we would like to address in the rest of this work are:

(1) Does the energy $E(t)$ go to zero as the time variable t tends to infinity?
(2) If so, then how fast does $E(t)$ decay to zero, and under what conditions?

Before stating our main results we need some additional notation for the purpose of rewriting our system as an abstract evolution equation. Setting $Au = -\Delta u$, and $Z = \begin{pmatrix} y \\ y_t \end{pmatrix}$, equation (1.1) may be recast as

$$\begin{gathered} Z' - \mathcal{A}Z = 0 \quad \text{in } (0,\infty), \\ Z(0) = \begin{pmatrix} y^0 \\ y^1 \end{pmatrix}, \end{gathered} \tag{1.7}$$

where the unbounded operator $\mathcal{A}$ is given by

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ -A & -aI + \operatorname{div}(b\nabla) \end{pmatrix} \tag{1.8}$$

with $D(\mathcal{A}) = \{(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega); Au + av - \operatorname{div}(b\nabla v) \in L^2(\Omega)\}$.

We introduce the Hilbert space $\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega)$ over the field of complex numbers $\mathbb{C}$, equipped with the norm (a norm indeed, thanks to the Poincaré inequality)

$$\|Z\|_{\mathcal{H}}^2 = \int_{\Omega} \{|v|^2 + |\nabla u|^2\}\, dx, \quad \forall Z = (u,v) \in \mathcal{H}. \tag{1.9}$$

We now introduce a geometric constraint (GC) on the subset ω where the dissipation is effective; we proceed as in [22], (see also [16, 21]).

(GC) There exist open sets $\Omega_j \subset \Omega$ with piecewise smooth boundary $\partial\Omega_j$, and points $x_0^j \in \mathbb{R}^N$, $j = 1, 2, \dots, J$, such that $\Omega_i \cap \Omega_j = \emptyset$, for any $1 \leq i < j \leq J$, and

$$\Omega \cap \mathcal{N}_\delta\big[\big(\cup_{j=1}^J \Gamma_j\big) \cup \big(\Omega \setminus \cup_{j=1}^J \Omega_j\big)\big] \subset \omega_a \cup \tilde{\omega}_b,$$

for some $\delta > 0$, where $\tilde{\omega}_b = \{x \in \Omega; b(x) > 0\}$, and

$$\begin{gathered} \mathcal{N}_\delta(S) = \cup_{x \in S}\{y \in \mathbb{R}^N; |x-y| < \delta\}, \quad \text{for } S \subset \mathbb{R}^N, \\ \Gamma_j = \big\{x \in \partial\Omega_j; (x - x_0^j) \cdot \nu^j(x) > 0\big\}, \end{gathered}$$

where ν^j is the unit normal vector pointing into the exterior of Ω_j.

In the sequel, $|u|_q$ denotes the $L^q(\Omega)$-norm of u when $q \geq 1$. We are now in a position to state our main results:

Theorem 1.1 (Well-posedness and strong stability)**.** *Suppose that either ω_a or ω_b is nonempty. Let the damping coefficients a and b be bounded measurable, and positive in ω_a (respectively ω_b). The operator $\mathcal{A}$ generates a C_0 semigroup of contractions $(S(t))_{t\geq 0}$ on $\mathcal{H}$, which is strongly stable:*

$$\lim_{t\to\infty}\|S(t)Z^0\|_{\mathcal{H}}=0, \quad \forall Z^0\in\mathcal{H}. \tag{1.10}$$

Theorem 1.2 (Polynomial stability)**.** *Let a and b be bounded measurable functions. Suppose that both ω_a and ω_b are nonempty with meas$(\partial\omega_b\cap\partial\Omega)>0$, and $\omega_a\cup\omega_b$ satisfies the geometric constraint* (GC) *above. Furthermore, assume that*

$$\exists a_0>0: a(x)\geq a_0 \text{ a.e. in } \omega_a, \quad \exists b_0>0: b(x)\geq b_0 \text{ a.e. in } \omega_b. \tag{1.11}$$

Then we have the polynomial decay estimate

$$\exists C_0>0: \|S(t)Z^0\|_{\mathcal{H}}\leq \frac{C_0\|Z^0\|_{D(\mathcal{A})}}{(1+t)^{1/2}}, \quad \forall t\geq 0, \quad \forall Z^0\in D(\mathcal{A}). \tag{1.12}$$

Theorem 1.3 (Exponential stability)**.** *Let a and b be bounded measurable functions. Suppose that both ω_a and ω_b are nonempty with meas$(\partial\omega_b\cap\partial\Omega)>0$, with $\omega_a\cup\omega_b$ satisfying the geometric constraint (GC) above, and that* (1.11) *holds. Furthermore, assume that either $\overline{\omega_a\cup\omega_b}=\Omega$, (closure relative to Ω), or else the viscoelastic damping coefficient b satisfies*

$$b\in W^{1,\infty}(\Omega) \quad \text{with } |\nabla b(x)|^2\leq M_0 b(x), \text{for almost every } x \text{ in } \Omega, \tag{1.13}$$

for some positive constant M_0. The semigroup $(S(t))_{t\geq 0}$ is exponentially stable, viz., there exist positive constants M and λ with

$$\|S(t)Z^0\|_{\mathcal{H}}\leq M\exp(-\lambda t)\|Z^0\|_{\mathcal{H}}, \quad \forall Z^0\in\mathcal{H}. \tag{1.14}$$

Remark 1.4. We emphasize that, though the set ω_a stands for the support of the frictional damping coefficient a in all three theorems, the set ω_b represents the support of the viscoelastic damping in the first two theorems and Theorem 1.3, Case 2 only. In Theorem 1.3, Case 1, the support of the function b is much larger than ω_b; this is due to the fact that the function b is now continuous, and so, it cannot vanish on the boundary of ω_b, as b satisfies (1.11).

Remark 1.5. Theorem 1.1 shows that for the strong stability of the semigroup, one only needs one of of the damping regions ω_a or ω_b to be nonempty; in other words, it is not necessary for both regions to be nonempty for the energy to decay to zero. However, to establish decay estimates, we need both damping mechanisms to be active and conveniently located; we do not allow any of ω_a or ω_b to exponentially stabilize the system by itself. This means that we select those two feedback control regions in such a way that there is a trapping region outside ω_a covered by ω_b, and a trapping region outside ω_b covered by ω_a. As it will be graphically shown latter, the geometric restrictions on the feedback control regions are more severe in the case of exponential decay than they are for the polynomial decay.

The rest of the article is organized as follows: Section 2 is devoted to the proofs of Theorems 1.1-1.3. Section 3 deals with some further comments and open problems.

2. Proofs of main results

The energy decay estimates will be derived from resolvent estimates. For that derivation, we will rely on the characterization of the polynomial stability of semigroups, given in [5], for Theorem 1.2, and the characterization of the exponential stability of semigroups, given in [15, 30], for Theorem 1.3.

2.1. **Proof of Theorem 1.1.**

The proof of the well-posedness is quite standard and is based on the Lumer-Philips theorem found in e.g. [29]. As for the proof of the strong stability, it relies on the strong stability criterium established in [1], and on classical unique continuation results for the wave equation. The details of the proof of Theorem 1.1 being very similar to the proof provided at the beginning of [43, Section 3], we refer the interested reader to that reference. □

2.2. **Proof of Theorem 1.2.**

We would like to quantify the strong stability property of Theorem 1.1 by establishing a polynomial decay estimate. Thanks to a recent result [5, Theorem 2.4], the polynomial decay estimate will follow from the resolvent estimate $\|(i\lambda\mathcal{I}-\mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} = O(|\lambda|^2)$ as $|\lambda| \nearrow +\infty$. To this end, let $U \in \mathcal{H}$, and let λ be a real number with $|\lambda| \geq 1$. Since the range of $i\lambda\mathcal{I}-\mathcal{A}$ is $\mathcal{H}$, there exists $Z \in D(\mathcal{A})$ such that

$$i\lambda Z - \mathcal{A}Z = U. \tag{2.1}$$

We shall prove

$$\|Z\|_{\mathcal{H}} \leq K_0|\lambda|^2\|U\|_{\mathcal{H}}, \tag{2.2}$$

where here and in the sequel, K_0 is a generic positive constant that may eventually depend on Ω, ω, a and b, but not on λ.

To establish (2.2), first, we note that if $Z = (u, v)$, and $U = (f, g)$, then (2.1) may be recast as

$$\begin{gathered} i\lambda u - v = f \\ i\lambda v - \Delta u + av - \operatorname{div}(b\nabla v) = g. \end{gathered} \tag{2.3}$$

Taking the inner product with Z on both sides of (2.1), then taking the real parts, we immediately obtain

$$\int_\Omega \{a|v|^2 + b|\nabla v|^2\}\,dx \leq \|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}}. \tag{2.4}$$

It now follows from the first equation in (2.3), and (2.4):

$$\begin{aligned} \lambda^2 \int_\Omega \{a|u|^2 + b|\nabla u|^2\}\,dx &\leq 2\int_\Omega \{a|v|^2 + b|\nabla v|^2\}\,dx + 2\int_\Omega \{a|f|^2 + b|\nabla f|^2\}\,dx \\ &\leq 2\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + K_0\|U\|^2_{\mathcal{H}}. \end{aligned} \tag{2.5}$$

In the remaining portion of the proof, we will be using a first order multiplier. Now, the function u in (2.3) lies in $H^1_0(\Omega)$ only, thereby not suited for the ensuing operations as it is not smooth enough. Consequently, we are going to introduce a change of variable in order to increase smoothness; set $u_1 = u + w$, where $\Delta w = \operatorname{div}(b\nabla v)$, with $w \in H^1_0(\Omega)$. Since (u, v) lies in $D(\mathcal{A})$, elliptic regularity shows that $u_1 \in H^2(\Omega) \cap H^1_0(\Omega)$. Thanks to (2.4) and Poincaré inequality, we note that

$$\|w\|^2_{H^1_0(\Omega)} \leq K_0\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}}, \quad \|u_1\|_{H^1_0(\Omega)} \leq \|Z\|_{\mathcal{H}} + K_0\sqrt{\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}}}. \tag{2.6}$$

On the other hand, the second equation in (2.3) becomes

$$i\lambda v - \Delta u_1 + av = g. \tag{2.7}$$

It immediately follows from (2.7) that

$$\begin{aligned} |\lambda|\|v\|_{H^{-1}(\Omega)} &\le K_0\|u_1\|_{H_0^1(\Omega)} + \|av\|_{H^{-1}(\Omega)} + K_0|g|_2 \\ &\le K_0(\|Z\|_{\mathcal{H}} + \sqrt{\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}}} + \|U\|_{\mathcal{H}}). \end{aligned} \tag{2.8}$$

Let $\alpha > 0$ and β be real constants with $\alpha(N-2) < \beta < \alpha N$. Multiply (2.7) by $\beta\bar{u}_1$, integrate on Ω, and take real parts to find that

$$\begin{aligned} \beta\Re\int_\Omega g\bar{u}_1\,dx &= \beta\Re\int_\Omega (i\lambda v - \Delta u_1 + av)\bar{u}_1\,dx \\ &= \beta\|u_1\|^2_{H_0^1(\Omega)} + \beta\Re\int_\Omega v(i\lambda\bar{u} + i\lambda\bar{w} + a\bar{u}_1)\,dx. \end{aligned} \tag{2.9}$$

Using (2.3), it follows that

$$\beta\Re\int_\Omega v(i\lambda\bar{u} + i\lambda\bar{w})\,dx = \beta\Re\int_\Omega v(-\bar{v} - \bar{f} + i\lambda\bar{w})\,dx. \tag{2.10}$$

Hence

$$\beta\Re\int_\Omega g\bar{u}_1\,dx = \beta\|u_1\|^2_{H_0^1(\Omega)} - \beta|v|_2^2 - \beta\Re\int_\Omega v(\bar{f} - i\lambda\bar{w} - a\bar{u}_1)\,dx. \tag{2.11}$$

It follows from (2.6) and (2.8) that

$$\begin{aligned} &\Big|\beta\Re\int_\Omega \{g\bar{u}_1 + v(\bar{f} - i\lambda\bar{w} - a\bar{u}_1)\}\,dx\Big| \\ &\le K_0\Big(\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{1/2}\|Z\|_{\mathcal{H}}^{3/2} + \|U\|_{\mathcal{H}}^{3/2}\|Z\|_{\mathcal{H}}^{1/2}\Big). \end{aligned} \tag{2.12}$$

Whence

$$K_0\Big(\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{1/2}\|Z\|_{\mathcal{H}}^{3/2} + \|U\|_{\mathcal{H}}^{3/2}\|Z\|_{\mathcal{H}}^{1/2}\Big) \ge \beta\|u_1\|^2_{[H_0^1(\Omega)]^N} - \beta|v|_2^2. \tag{2.13}$$

For the sequel, we need some additional notations. For each $j = 1,\dots,J$, where J appears in the geometric constraint (GC) stated above, set $m^j(x) = x - x_0^j$ and $R_j = \sup\{|m^j(x)|, x \in \Omega\}$. Let $0 < \delta_0 < \delta_1 < \delta$, where δ is the one given in (GC). Set

$$S = \big(\cup_{j=1}^J \Gamma_j\big) \cup \big(\Omega \setminus \cup_{j=1}^J \Omega_j\big), \quad Q_0 = \mathcal{N}_{\delta_0}(S), \quad Q_1 = \mathcal{N}_{\delta_1}(S), \quad \omega_a \cup \omega_b = \Omega \cap Q_1,$$

and for each j, let φ_j be a function satisfying

$$\varphi_j \in W^{1,\infty}(\Omega), \quad 0 \le \varphi_j \le 1, \quad \varphi_j = 1 \quad \text{in } \bar{\Omega}_j \setminus Q_1, \quad \varphi_j = 0 \quad \text{in } \Omega \cap Q_0.$$

See Figures 1–3.

Before going on, we note that for each j, the function φ_j is built in such a way that $\varphi_j \equiv 0$ in ω_a and the support of the gradient of φ_j is contained in ω_b.

Now, multiply (2.7) by $2\alpha\varphi_j m^j \cdot \nabla\bar{u}_1$, integrate on Ω_j, and take real parts to obtain

$$\begin{aligned} &2\alpha\Re\int_{\Omega_j} (g - av)(\varphi_j m^j \cdot \nabla\bar{u}_1)\,dx \\ &= 2\alpha\Re\int_{\Omega_j} v\varphi_j m^j \cdot \nabla(-\bar{v} - \bar{f} - i\lambda\bar{w})\,dx - 2\alpha\Re\int_{\Omega_j} \Delta u_1(\varphi_j m^j \cdot \nabla\bar{u}_1)\,dx. \end{aligned} \tag{2.14}$$

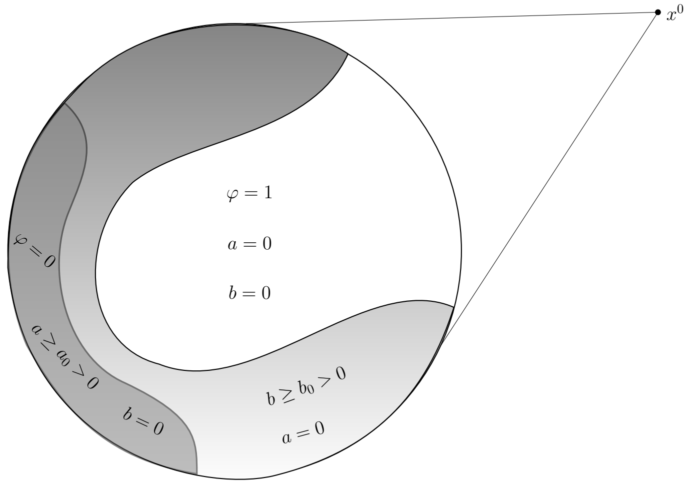

FIGURE 1. Geometric constraint in Theorem 1.2: $J = 1$, $\varphi = \varphi_1$, $N = 2$. Given that b is not continuous across the interface, only polynomial decay is expected in the presence of an undamped area.

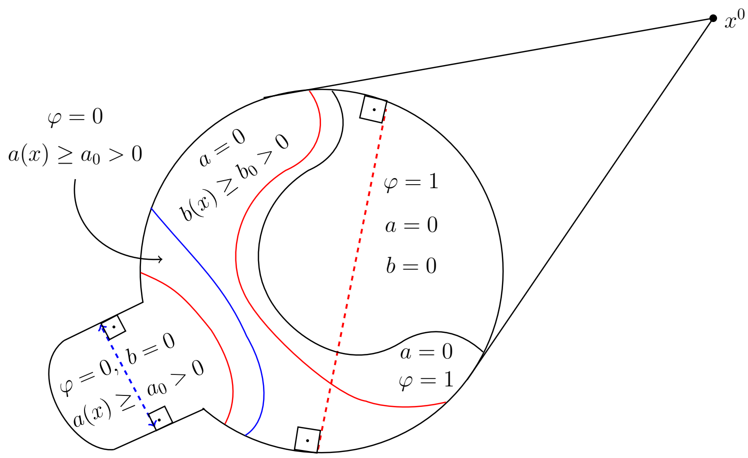

FIGURE 2. Geometric constraint in Theorem 1.3, case 1: $J = 1$, $\varphi = \varphi_1$, $N = 2$. Note that the blue ray is trapped and won't escape when the frictional damping is inactive. The red ray is trapped and cannot escape unless the viscoelastic damping is active. Thus, none of either the frictional or viscoelastic damping is enough to exponentially stabilize the system on its own; this justifies the use of both damping mechanisms to achieve the exponential stability of the system.

An application of Green's formula shows

$$\begin{aligned} &- 2\alpha\Re \int_{\Omega_j} v\varphi_j m^j \cdot \nabla \bar{v}\, dx \\ &= \alpha N \int_{\Omega_j} \varphi_j |v|^2\, dx + \alpha \int_{\Omega_j} (m^j \cdot \nabla\varphi_j)|v|^2 - \alpha \int_{\partial\Omega_j} \varphi_j (m^j \cdot \nu^j)|v|^2\, d\Gamma, \end{aligned} \tag{2.15}$$

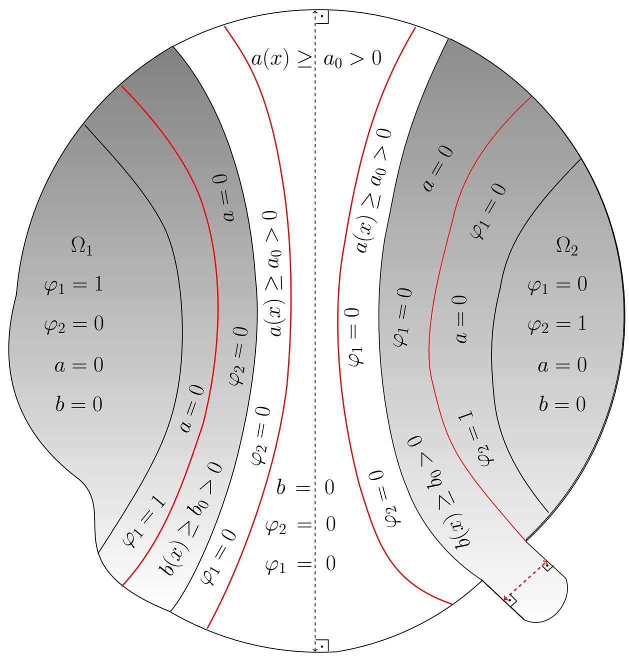

FIGURE 3. Geometric constraint in Theorem 1.3, case 1: $J = 2$, $N = 2$. Notice the trapped ray in the region where the frictional damping is active $\{a(x) \geq a_0 > 0\}$ and the one where the Kelvin-Voigt damping is active $\{b(x) \geq b_0 > 0\}$; consequently, neither of the two damping mechanisms is able by itself to exponentially stabilize the system. Ω_1 and Ω_2 are the dark regions.

and

$$
\begin{aligned}
&-2\alpha\Re\int_{\Omega_j} \Delta u_1(\varphi_j m^j \cdot \nabla \bar{u}_1)\,dx \\
&= 2\alpha\Re\int_{\Omega_j} (\nabla u_1 \cdot \nabla\varphi_j) m^j \cdot \nabla\bar{u}_1\,dx + 2\alpha\int_{\Omega_j} \varphi_j |\nabla u_1|^2\,dx \\
&\quad + 2\alpha\Re\int_{\Omega_j} \varphi_j(\partial_q u_1) m_n^j \partial_{nq}^2 \bar{u}_1\,dx - 2\alpha\Re\int_{\partial\Omega_j} (\partial_{\nu^j} u_1)\varphi_j m^j \cdot \nabla\bar{u}_1\,d\Gamma.
\end{aligned}
\tag{2.16}
$$

Now, we have

$$
2\alpha\Re\int_{\Omega_j} \varphi_j \partial_q u_1 m_n^j \partial_{nq}^2 \bar{u}_1\,dx = \alpha\int_{\Omega_j} \varphi_j m^j \cdot \nabla(|\nabla u_1|^2)\,dx, \tag{2.17}
$$

so that applying Green's formula once more, we have

$$
\begin{aligned}
&2\alpha\Re\int_{\Omega_j}\varphi_j\partial_q u_1 m_n^j\partial_{nq}^2\bar u_1\,dx\\
&= -\alpha N\int_{\Omega_j}\varphi_j|\nabla u_1|^2\,dx-\alpha\int_{\Omega_j}(m^j\cdot\nabla\varphi_j)|\nabla u_1|^2\,dx\\
&\quad+\alpha\int_{\partial\Omega_j}|\nabla u_1|^2\varphi_j m^j\cdot\nu^j\,d\Gamma.
\end{aligned}\tag{2.18}
$$

If as in [22], we set for each j, $S_j=\Gamma_j\cup(\partial\Omega_j\cap\Omega)$, then one checks that $\varphi_j=0$ on S_j. On the other hand, $\partial\Omega_j\setminus S_j\subset\Gamma_j^c\cap\partial\Omega$, ($A^c$ denotes the complement of A); consequently, for each j, one has

$$
\begin{gathered}
\int_{\partial\Omega_j}\varphi_j(m^j\cdot\nu^j)|v|^2\,d\Gamma=0\\
-2\alpha\Re\int_{\partial\Omega_j}(\partial_{\nu^j}u_1)\varphi_j m^j\cdot\nabla\bar u_1\,d\Gamma+\alpha\int_{\partial\Omega_j}|\nabla u_1|^2\varphi_j m^j\cdot\nu^j\,d\Gamma\geq 0.
\end{gathered}\tag{2.19}
$$

The last inequality follows from the fact that

$$
-2\alpha\Re\int_{\partial\Omega_j}(\partial_{\nu^j}u_1)\varphi_j m^j\cdot\nabla\bar u_1\,d\Gamma=-2\alpha\int_{\partial\Omega_j\setminus S_j}|\nabla u_1|^2\varphi_j m^j\cdot\nu^j\,d\Gamma.
$$

Thus, using (2.18) and (2.19) in (2.16), and combing (2.15) and (2.16), we find that

$$
\begin{aligned}
&-2\alpha\Re\int_{\Omega_j}v\varphi_j m^j\cdot\nabla\bar v\,dx-2\alpha\Re\int_{\Omega_j}\Delta u_1(\varphi_j m^j\cdot\nabla\bar u_1)\,dx\\
&\geq\alpha N\int_{\Omega_j}|v|^2\,dx+\alpha N\int_{\Omega_j}(\varphi_j-1)|v|^2\,dx+\alpha\int_{\Omega_j}(m^j\cdot\nabla\varphi_j)|v|^2\\
&\quad+2\alpha\Re\int_{\Omega_j}(\nabla u_1\cdot\nabla\varphi_j)m^j\cdot\nabla\bar u_1\,dx-(N-2)\alpha\int_{\Omega_j}|\nabla u_1|^2\,dx\\
&\quad-(N-2)\alpha\int_{\Omega_j}(\varphi_j-1)|\nabla u_1|^2\,dx-\alpha\int_{\Omega_j}|\nabla u_1|^2(m^j\cdot\nabla\varphi_j)\,dx.
\end{aligned}\tag{2.20}
$$

Adding the utmost right term in the first line of (2.14) in (2.20), then taking the sums over j, we obtain

$$
\begin{aligned}
&-2\alpha\Re\sum_{j=1}^J\int_{\Omega_j}\{v\varphi_j m^j\cdot\nabla\bar v+\Delta u_1(\varphi_j m^j\cdot\nabla\bar u_1)+ibv\varphi_j m^j\cdot\nabla\bar w\}\,dx\\
&\geq\alpha N\sum_{j=1}^J\int_{\Omega_j}|v|^2\,dx+\alpha N\sum_{j=1}^J\int_{\Omega_j}(\varphi_j-1)|v|^2\,dx+\alpha\int_{\Omega_j}(m^j\cdot\nabla\varphi_j)|v|^2\,dx\\
&\quad+2\alpha\Re\sum_{j=1}^J\int_{\Omega_j}(\nabla u_1\cdot\nabla\varphi_j)m^j\cdot\nabla\bar u_1\,dx-(N-2)\alpha\sum_{j=1}^J\int_{\Omega_j}|\nabla u_1|^2\,dx\\
&\quad-(N-2)\alpha\sum_{j=1}^J\int_{\Omega_j}(\varphi_j-1)|\nabla u_1|^2\,dx-\alpha\sum_{j=1}^J\int_{\Omega_j}|\nabla u_1|^2(m^j\cdot\nabla\varphi_j)\,dx\\
&\quad-2\alpha\Re i\lambda\sum_{j=1}^J\int_{\Omega_j}v\varphi_j m^j\cdot\nabla\bar w\,dx,
\end{aligned}\tag{2.21}
$$

which is equivalent to

$$
\begin{aligned}
&2\alpha\Re\sum_{j=1}^{J}\int_{\Omega_j}\{(g-av)(\varphi_j m^j\cdot\nabla\bar{u}_1)+v\varphi_j m^j\cdot\nabla\bar{f}\}\,dx\\
&\geq \alpha N\sum_{j=1}^{J}\int_{\Omega_j}|v|^2\,dx+\alpha N\sum_{j=1}^{J}\int_{\Omega_j}(\varphi_j-1)|v|^2\,dx+\alpha\int_{\Omega_j}(m^j\cdot\nabla\varphi_j)|v|^2\,dx\\
&\quad+2\alpha\Re\sum_{j=1}^{J}\int_{\Omega_j}(\nabla u_1\cdot\nabla\varphi_j)m^j\cdot\nabla\bar{u}_1\,dx-(N-2)\alpha\sum_{j=1}^{J}\int_{\Omega_j}|\nabla u_1|^2\,dx\\
&\quad-(N-2)\alpha\sum_{j=1}^{J}\int_{\Omega_j}(\varphi_j-1)|\nabla u_1|^2\,dx-\alpha\sum_{j=1}^{J}\int_{\Omega_j}|\nabla u_1|^2(m^j\cdot\nabla\varphi_j)\,dx\\
&\quad-2\alpha\Re i\lambda\sum_{j=1}^{J}\int_{\Omega_j}v\varphi_j m^j\cdot\nabla\bar{w}\,dx.
\end{aligned}
\tag{2.22}
$$

Applying Hölder inequality to the terms in the left hand side of (2.22), and using (2.6), one immediately gets

$$
\begin{aligned}
&2\alpha\Re\sum_{j=1}^{J}\int_{\Omega_j}\{(g-av)(\varphi_j m^j\cdot\nabla\bar{u}_1)+v\varphi_j m^j\cdot\nabla\bar{f}\}\,dx\\
&\leq K_0(\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}}+\|U\|_{\mathcal{H}}^{1/2}\|Z\|_{\mathcal{H}}^{3/2}+\|U\|_{\mathcal{H}}^{3/2}\|Z\|_{\mathcal{H}}^{1/2}).
\end{aligned}
\tag{2.23}
$$

Now we are going to estimate the terms in the right hand side of (2.22). The use of Poincaré inequality and (2.4) lead to (adding the second term in the right hand side of (2.13), and keeping in mind that the support of the gradient of φ_j lies in ω_b)

$$
\begin{aligned}
&(\alpha N-\beta)\sum_{j=1}^{J}\int_{\Omega_j}|v|^2\,dx+\alpha N\sum_{j=1}^{J}\int_{\Omega_j}(\varphi_j-1)|v|^2\,dx\\
&+\alpha\int_{\Omega_j}(m^j\cdot\nabla\varphi_j)|v|^2\,dx\\
&\geq(\alpha N-\beta)|v|_2^2-K_0\int_{\omega_1}|v|^2\,dx-K_0\int_{\omega_b}|v|^2\,dx\\
&\geq(\alpha N-\beta)|v|_2^2-K_0\int_{\omega_a}|v|^2\,dx-K_0\int_{\omega_b}|v|^2\,dx\\
&\geq(\alpha N-\beta)|v|_2^2-K_0\int_{\omega_a}|v|^2\,dx-K_0\int_{\omega_b}|\nabla v|^2\,dx\\
&\geq(\alpha N-\beta)|v|_2^2-K_0\int_{\Omega}a|v|^2-K_0\int_{\Omega}b|\nabla v|^2\,dx\\
&\geq(\alpha N-\beta)|v|_2^2-K_0\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}}.
\end{aligned}
\tag{2.24}
$$

Thanks to Hölder inequality, Poincaré inequality, and (2.6), it easily follows that

$$
\Big|2\alpha\Re i\lambda\sum_{j=1}^{J}\int_{\Omega_j}v\varphi_j m^j\cdot\nabla\bar{w}\,dx\Big|\leq K_0|\lambda|\|U\|_{\mathcal{H}}^{1/2}\|Z\|_{\mathcal{H}}^{3/2}. \tag{2.25}
$$

On the other hand, given that $N \geq 2$ and $\beta > (N-2)\alpha$, adding the first term in the right hand side of (2.13), one arrives to

$$\begin{aligned}
&\beta \int_\Omega |\nabla u_1|^2\,dx - (N-2)\alpha \sum_{j=1}^J \int_{\Omega_j} |\nabla u_1|^2\,dx \\
&\quad - \alpha \sum_{j=1}^J \int_{\Omega_j} |\nabla u_1|^2 (m^j \cdot \nabla \varphi_j)\,dx - (N-2)\alpha \sum_{j=1}^J \int_{\Omega_j} (\varphi_j - 1)|\nabla u_1|^2\,dx \\
&= \beta \int_\Omega |\nabla u_1|^2\,dx - (N-2)\alpha \int_\Omega |\nabla u_1|^2\,dx + (N-2)\alpha \int_{\omega_a \cup \omega_b} |\nabla u_1|^2\,dx \\
&- \alpha \sum_{j=1}^J \int_{\Omega_j} |\nabla u_1|^2 (m^j \cdot \nabla \varphi_j)\,dx - (N-2)\alpha \sum_{j=1}^J \int_{\Omega_j} (\varphi_j - 1)|\nabla u_1|^2\,dx \\
&\geq K_0 \int_\Omega |\nabla u_1|^2\,dx - K_0 \int_{\omega_b} |\nabla u_1|^2\,dx.
\end{aligned} \tag{2.26}$$

We note that there is no integral over ω_a in the last line of (2.26); this is so because it has a nonnegative factor, and so, it is dropped.

Now, the definition of u_1, and (2.5)-(2.6) show (keeping in mind that $|\lambda| \geq 1$)

$$\begin{aligned}
\int_{\omega_b} |\nabla u_1|^2\,dx &= \int_{\omega_b} |\nabla u + \nabla w|^2\,dx \\
&\leq K_0 \int_\Omega b|\nabla u|^2\,dx + 2\int_\Omega |\nabla w|^2\,dx \\
&\leq K_0(\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^2),
\end{aligned} \tag{2.27}$$

by Cauchy-Schwarz inequality. Gathering (2.22)-(2.27), we find that

$$\begin{aligned}
&|v|_2^2 + \int_\Omega |\nabla u_1|^2\,dx \\
&\leq K_0\Big(|\lambda|\|U\|_{\mathcal{H}}^{1/2}\|Z\|_{\mathcal{H}}^{3/2} + \|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{3/2}\|Z\|_{\mathcal{H}}^{1/2} + \|U\|_{\mathcal{H}}^2\Big).
\end{aligned} \tag{2.28}$$

The definition of u_1 and (2.6), as in (2.27), yields

$$\begin{aligned}
&|v|_2^2 + \int_\Omega |\nabla u|^2\,dx \\
&\leq K_0\Big(|\lambda|\|U\|_{\mathcal{H}}^{1/2}\|Z\|_{\mathcal{H}}^{3/2} + \|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{3/2}\|Z\|_{\mathcal{H}}^{1/2} + \|U\|_{\mathcal{H}}^2\Big),
\end{aligned} \tag{2.29}$$

or

$$\|Z\|_{\mathcal{H}}^2 \leq K_0(|\lambda|\|U\|_{\mathcal{H}}^{1/2}\|Z\|_{\mathcal{H}}^{3/2} + \|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{3/2}\|Z\|_{\mathcal{H}}^{1/2} + \|U\|_{\mathcal{H}}^2). \tag{2.30}$$

The use of Young inequality in (2.30) leads at once to (2.2). Applying [5, Theorem 2.4], one gets the claimed polynomial decay estimate, thereby completing the proof of Theorem 1.2. □

2.3. **Proof of Theorem 1.3. Case 1: $\overline{\omega_a \cup \omega_b} \neq \Omega$.** In this setting, the proof of Theorem 1.3 is very similar to that of Theorem 1.2; only estimating the last term in the right-hand side of (2.22) is distinct in the present proof. Instead of the rough estimate (2.25), we must now get an estimate that is independent of λ. So, thanks to the proof of Theorem 1.2, we already have

$$\begin{aligned} \|Z\|_{\mathcal{H}}^2 &\le K_0(\|U\|_{\mathcal{H}}^{1/2}\|Z\|_{\mathcal{H}}^{3/2} + \|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{3/2}\|Z\|_{\mathcal{H}}^{1/2} + \|U\|_{\mathcal{H}}^2) \\ &\quad + K_0\big|\Re i\lambda\sum_{j=1}^J\int_{\Omega_j} v\varphi_j m^j\cdot\nabla\bar{w}\,dx\big|. \end{aligned} \tag{2.31}$$

We shall now estimate the last term in (2.31) independently of λ. To this end, introduce for each $j\in\{1,\dots,J\}$, the function $z^j\in H_0^1(\Omega)$, solution of the system

$$\Delta z^j = \operatorname{div}(1_{\Omega_j}v\varphi_j m^j) \quad \text{in } \Omega, \quad p=1,\dots,N \tag{2.32}$$

where 1_{Ω_j} stands for the characteristic function of Ω_j. Multiplying that system by $\bar{w}$, and applying Green's formula over Ω, we obtain

$$\int_{\Omega_j} v\varphi_j m^j\cdot\nabla\bar{w}\,dx = \int_\Omega \nabla z^j\cdot\nabla\bar{w}\,dx = \int_\Omega b\nabla\bar{v}\cdot\nabla z^j\,dx, \tag{2.33}$$

where the last equality comes from the equation satisfied by $\bar{w}$, and the variational method.

Now, if we multiply the system (2.32) by $b\bar{v}$, and apply Green's formula once more, we find that

$$\begin{aligned} &\int_\Omega (\nabla z^j\cdot\nabla b)\bar{v}\,dx + \int_\Omega b(\nabla z^j\cdot\nabla\bar{v})\,dx \\ &= \int_{\Omega_j} \varphi_j(m^j\cdot\nabla b)|v|^2\,dx + \int_{\Omega_j} bv\varphi_j m^j\cdot\nabla\bar{v}\,dx, \end{aligned} \tag{2.34}$$

Adding (2.33) and (2.34), it follows that

$$\begin{aligned} &\int_{\Omega_j} v\varphi_j m^j\cdot\nabla\bar{w}\,dx \\ &= -\int_\Omega (\nabla z^j\cdot\nabla b)\bar{v}\,dx + \int_{\Omega_j}\varphi_j(m^j\cdot\nabla b)|v|^2\,dx + \int_{\Omega_j} bv\varphi_j m^j\cdot\nabla\bar{v}\,dx, \end{aligned} \tag{2.35}$$

Consequently, by (2.35), one has

$$\Re i\lambda\int_{\Omega_j} v\varphi_j m^j\cdot\nabla\bar{w}\,dx = -\Re i\lambda\int_\Omega(\nabla z^j\cdot\nabla b)\bar{v}\,dx + \Re i\lambda\int_{\Omega_j} bv\varphi_j m^j\cdot\nabla\bar{v}\,dx, \tag{2.36}$$

We shall now estimate the two terms in the right hand side of (2.36). Thanks to Cauchy-Schwarz inequality and the inequality constraint on the gradient of the damping coefficient b, estimating the left term yields

$$\big|\Re i\lambda\int_\Omega(\nabla z^j\cdot\nabla b)\bar{v}\,dx\big| \le K_0|\lambda|\|\sqrt{b}v|_2\|Z\|_{\mathcal{H}}, \tag{2.37}$$

where we used the estimate $\|z^j\|_{H_0^1(\Omega)}\le K_0|v|_2$, for all j. As for the other term, applying the Cauchy-Schwarz inequality, we have

$$\big|\Re i\lambda\int_{\Omega_j} bv\varphi_j m^j\cdot\nabla\bar{v}\,dx\big| \le K_0|\lambda|\|\sqrt{b}v|_2\Big(\int_\Omega b|\nabla v|^2\,dx\Big)^{1/2}. \tag{2.38}$$

Then from (2.36)–(2.38) we obtain

$$\big|\Re i\lambda\sum_{j=1}^J\int_{\Omega_j} v\varphi_j m^j\cdot\nabla\bar{w}\,dx\big| \le K_0|\lambda|\|\sqrt{b}v|_2(\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + \|Z\|_{\mathcal{H}}^2)^{1/2}. \tag{2.39}$$

To complete the proof of Theorem 1.3, we shall now estimate the term $|\lambda|\|\sqrt{b}v|_2$. To this end, multiplying the second equation in (2.3) by $-i\lambda b\bar{v}$ and applying Green's formula, one finds

$$\begin{aligned}
&\lambda^2 \int_\Omega b|v|^2\,dx \\
&= \Re i\lambda \int_\Omega \{b(\nabla u\cdot\nabla\bar{v}) + (\nabla u\cdot\nabla b)\bar{v}\}\,dx + \Re i\lambda\int_\Omega \{ab|v|^2 + b^2|\nabla v|^2\}\,dx \\
&\quad + \Re i\lambda\int_\Omega b\bar{v}\nabla v\cdot\nabla b\,dx - \Re i\lambda\int_\Omega bg\cdot\bar{v}\,dx \\
&= \Re i\lambda \int_\Omega \{b(\nabla u\cdot\nabla\bar{v}) + (\nabla u\cdot\nabla b)\bar{v}\}\,dx + \Re i\lambda\int_\Omega b\bar{v}\nabla v\cdot\nabla b\,dx - \Re i\lambda\int_\Omega bg\cdot\bar{v}\,dx \\
&= \Re\int_\Omega (\nabla v+\nabla f)\cdot(b\nabla\bar{v}+\bar{v}\nabla b)\,dx + \Re i\lambda\int_\Omega b\bar{v}\nabla v\cdot\nabla b\,dx - \Re i\lambda\int_\Omega bg\cdot\bar{v}\,dx,
\end{aligned} \tag{2.40}$$

where in the last line we use the equation: $i\lambda u = v+f$. Thanks to Cauchy-Schwarz inequality and (2.4), one gets the estimate

$$\begin{aligned}
&\Big|\Re\int_\Omega (\nabla v+\nabla f)\cdot(b\nabla\bar{v}+\bar{v}\nabla b)\,dx\Big| \\
&\le K_0\Big[\Big(\int_\Omega b|\nabla v|^2\,dx\Big)^{1/2} + \|f\|_{H_0^1(\Omega)}\Big]\Big(|v|_2^2 + \int_\Omega b|\nabla v|^2\,dx\Big)^{1/2} \\
&\le K_0(\|U\|_{\mathcal{H}}^{1/2}\|Z\|_{\mathcal{H}}^{1/2} + \|U\|_{\mathcal{H}})(\|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{1/2}\|Z\|_{\mathcal{H}}^{1/2}) \\
&\le K_0(\|U\|_{\mathcal{H}}^{1/2}\|Z\|_{\mathcal{H}}^{3/2} + \|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{3/2}\|Z\|_{\mathcal{H}}^{1/2}).
\end{aligned} \tag{2.41}$$

Now, using Young inequality and (2.4) once more, one obtains

$$\begin{aligned}
&\Big|\Re i\lambda\int_\Omega b\bar{v}\nabla v\cdot\nabla b\,dx - \Re i\lambda\int_\Omega bg\cdot\bar{v}\,dx\Big| \\
&\le \frac{\lambda^2}{4}\int_\Omega b|v|^2\,dx + K_0\int_\Omega b|\nabla v|^2\,dx + \frac{\lambda^2}{4}\int_\Omega b|v|^2\,dx + K_0|g|_2^2 \\
&\le \frac{\lambda^2}{2}\int_\Omega b|v|^2\,dx + K_0(\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^2).
\end{aligned} \tag{2.42}$$

Using (2.41) and (2.42) in (2.40), we have

$$\lambda^2\int_\Omega b|v|^2\,dx \le K_0(\|U\|_{\mathcal{H}}^{1/2}\|Z\|_{\mathcal{H}}^{3/2} + \|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{3/2}\|Z\|_{\mathcal{H}}^{1/2} + \|U\|_{\mathcal{H}}^2). \tag{2.43}$$

Then combining (2.39) and (2.43) yields

$$\begin{aligned}
&\Big|\Re i\lambda\sum_{j=1}^J\int_{\Omega_j} v\varphi_j m^j\cdot\nabla\bar{w}\,dx\Big| \\
&\le K_0(\|U\|_{\mathcal{H}}^{1/2}\|Z\|_{\mathcal{H}}^{3/2} + \|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{3/2}\|Z\|_{\mathcal{H}}^{1/2} \\
&\quad + \|U\|_{\mathcal{H}}^2)^{1/2}(\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + \|Z\|_{\mathcal{H}}^2)^{1/2} \\
&\le K_0(\|U\|_{\mathcal{H}}^{3/4}\|Z\|_{\mathcal{H}}^{5/4} + \|U\|_{\mathcal{H}}^{\frac14}\|Z\|_{\mathcal{H}}^{\frac74} + \|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{1/2}\|Z\|_{\mathcal{H}}^{3/2} \\
&\quad + \|U\|_{\mathcal{H}}^{5/4}\|Z\|_{\mathcal{H}}^{3/4} + \|U\|_{\mathcal{H}}^{3/2}\|Z\|_{\mathcal{H}}^{1/2}).
\end{aligned} \tag{2.44}$$

Using (2.44) in (2.31), we obtain

$$\begin{aligned} \|Z\|_{\mathcal{H}}^2 \le K_0(&\|U\|_{\mathcal{H}}^{3/4}\|Z\|_{\mathcal{H}}^{5/4} + \|U\|_{\mathcal{H}}^{\frac{1}{4}}\|Z\|_{\mathcal{H}}^{\frac{7}{4}} + \|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{1/2}\|Z\|_{\mathcal{H}}^{3/2} \\ &+ \|U\|_{\mathcal{H}}^{5/4}\|Z\|_{\mathcal{H}}^{3/4} + \|U\|_{\mathcal{H}}^{3/2}\|Z\|_{\mathcal{H}}^{1/2} + \|U\|_{\mathcal{H}}^2). \end{aligned} \tag{2.45}$$

Using Young's inequality, one derives the desired estimate from (2.45) for large enough $|\lambda|$. By the continuity of the resolvent, one obtains the desired estimate for the remaining values of λ, thereby completing the proof of Theorem 1.3 in this case.

Case 2: $\overline{\omega_a \cup \omega_b} = \Omega$. This case is much easier to handle since we now have dissipation everywhere in Ω albeit of different types. Using the weak formulation of (2.3), we obtain the identity

$$\int_\Omega \{|v|^2 + |\nabla u|^2\}dx = 2\int_\Omega |v|^2\,dx + \Re\int_\Omega \{(g-av)\bar{u} - b\nabla v\cdot\nabla\bar{u} + v\bar{f}\}\,dx \tag{2.46}$$

Now, thanks to the coerciveness of the damping coefficients a and b, and the Poincaré inequality, one has

$$\begin{aligned} \int_\Omega |v|^2dx &= \int_{\omega_a} |v|^2\,dx + \int_{\omega_b} |v|^2\,dx \\ &\le K_0\int_\Omega \{a|v|^2 + b|\nabla v|^2\}\,dx \\ &\le K_0\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}}. \end{aligned} \tag{2.47}$$

On the other hand, the combination of the Cauchy-Schwarz inequality and Poincaré inequality yields

$$\Big|\int_\Omega \{(g-av)\bar{u} - b\nabla v\cdot\nabla\bar{u} + v\bar{f}\}\,dx\Big| \le K_0(\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{1/2}\|Z\|_{\mathcal{H}}^{3/2}). \tag{2.48}$$

Using (2.47)-(2.48) in (2.46), we obtain

$$\int_\Omega \{|v|^2 + |\nabla u|^2\}dx \le K_0(\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{1/2}\|Z\|_{\mathcal{H}}^{3/2}), \tag{2.49}$$

from which one derives, by the Young's inequality,

$$\|Z\|_{\mathcal{H}} \le K_0\|U\|_{\mathcal{H}}. \tag{2.50}$$

Thanks to the exponential stability of semigroups criterion given in [15, 30], one gets the claimed exponential decay of the energy, which completes the proof of Theorem 1.3. $\square$

3. Further results and open problems

The purpose of this section is to discuss some extensions of our results, and some open problems. First, we point out that the proof of the Case 2 in Theorem 1.3 shows that one may choose the fractional damping region ω_a as small as one wishes. Given that the Kelvin-Voigt damping coefficient b is not continuous in that case, it is known, at least in the one dimensional setting, that the exponential stability of the semigroup fails if the viscoelastic damping only is active; so we note that this failure can be compensated by introducing a small frictional damping.

3.1. **Unbounded frictional damping.** So far in this work, we have assumed that the coefficient a of the frictional damping belongs to $L^\infty(\Omega)$. A natural question then arises: what can be said about the stability of the system at hand, involving competing viscous and viscoelastic damping mechanisms, when the coefficient a is in $L^r(\Omega)$ for some $r > N$? The restriction on r is helpful for well-posedness. It is known that if the frictional damping only is active, then we have a polynomial decay of the energy; the decay rate depends on r and the decay is exponential when $r \nearrow \infty$ [36, 39]. We will restrict our attention to the situation in Theorem 1.3 where the semigroup is exponentially stable. It can be asserted that the exponential decay property is kept when the damping coefficient a lies in some $L^r(\Omega)$; indeed the restriction on a matters only when estimating the term $\int_\Omega av\bar{u}\,dx$ in (2.12) or (2.46), and the term $\int_{\Omega_j} av\varphi_j(m^j\cdot\nabla\bar{u}_1)\,dx$ in (2.23). The latter term is zero thanks to the fact that the function a vanishes on the support of each φ_j. As for the former term, it can be estimated either by using a combination of the Cauchy-Schwarz inequality, Poincaré inequality and a Sobolev embedding theorem, or else, by using the Cauchy-Schwarz inequality and estimate (2.5), provided $|\lambda|$ is large enough.

3.2. **Wave equation with a potential.** Our results extend to the system

$$\begin{gathered} y_{tt} - \Delta y + py + a(x)y_t - \operatorname{div}(b(x)\nabla y_t) = 0 \quad \text{in } \Omega\times(0,\infty) \\ y = 0 \quad \text{on } \Gamma\times(0,\infty) \\ y(0) = y^0, \quad y_t(0) = y^1, \end{gathered} \tag{3.1}$$

where $p \in L^r(\Omega)$ is a nonnegative function with $r > N$, and the other parameters of the system are given as before.

The well-posedness of this new system is established following the same pattern as before. Concerning stability issues, we note that the frequency domain analogue of (3.1) is the counterpart of (2.3), and is given by

$$\begin{gathered} i\lambda u - v = f \\ i\lambda v - \Delta u + pu + av - \operatorname{div}(b\nabla v) = g. \end{gathered} \tag{3.2}$$

All of the estimates are the same as before except that now we need to estimate the terms $\int_\Omega p|u|^2\,dx$ and $\sum_{j=1}^J \int_{\Omega_j} pu\varphi_j(2\alpha m^j\cdot\nabla\bar{u}_1 + \beta\bar{u}_1)\,dx$. To appropriately estimate either of those two terms, we need the following Gagliardo-Nirenberg inequality.

Lemma 3.1. *Let $1 \le q \le s \le \infty$, $1 \le r \le s$, $0 \le k < m < \infty$, where k and m are nonnegative integers, and $\theta \in [0,1]$. Let $v \in W^{m,q}(\Omega)\cap L^r(\Omega)$. Suppose that*

$$k - \frac{N}{s} \le \theta\big(m - \frac{N}{q}\big) - \frac{N(1-\theta)}{r}. \tag{3.3}$$

Then $v \in W^{k,s}(\Omega)$, and there exists a positive constant C such that

$$\|v\|_{W^{k,s}(\Omega)} \le C\|v\|^\theta_{W^{m,q}(\Omega)}|v|_r^{1-\theta}. \tag{3.4}$$

Using Hölder's inequality, Lemma 3.1, (with $\theta = N/2r$), and Young's inequality, we find

$$\int_\Omega p|u|^2\,dx \le |p|_r|u|^2_{\frac{2r}{r-1}} \le K_0|u|_2^{\frac{2r-N}{r}}|\nabla u|_2^{\frac{N}{r}} \le \varepsilon\|Z\|^2_{\mathcal{H}} + \frac{K_0}{\varepsilon}|u|_2^2, \quad \forall\varepsilon > 0. \tag{3.5}$$

Now, using the generalized Hölder inequality, Poincaré inequality, Lemma 3.1 and Young inequality once more, we obtain

$$\begin{aligned}
&\Big|\sum_{j=1}^{J}\int_{\Omega_j} pu\varphi_j(2\alpha m^j\cdot\nabla\bar{u}_1+\beta\bar{u}_1)\,dx\Big| \\
&\le K_0|p|_r|u|_{\frac{2r}{r-2}}|\nabla u_1|_2 \\
&\le K_0|u|_2^{\frac{r-N}{r}}|\nabla u|_2^{\frac{N}{r}}|\nabla u_1|_2 \\
&\le K_0|u|_2^{\frac{r-N}{r}}\|Z\|_{\mathcal{H}}^{\frac{N}{r}}\Big(\|Z\|_{\mathcal{H}}+\|U\|_{\mathcal{H}}^{1/2}\|Z\|_{\mathcal{H}}^{1/2}\Big), \quad \text{by (2.6)} \\
&\le K_0|u|_2^{\frac{r-N}{r}}\|Z\|_{\mathcal{H}}^{\frac{N+r}{r}}+K_0\|U\|_{\mathcal{H}}^{1/2}\|Z\|_{\mathcal{H}}^{3/2} \\
&\le \varepsilon\|Z\|_{\mathcal{H}}^2+\frac{K_0}{\varepsilon}|u|_2^2+K_0\|U\|_{\mathcal{H}}^{1/2}\|Z\|_{\mathcal{H}}^{3/2}, \quad \forall\varepsilon>0.
\end{aligned} \tag{3.6}$$

Once (3.5) and (3.6) are established, one chooses ε appropriately in order to get rid of the term involving $\|Z\|_{\mathcal{H}}$ from the right hand side. Then, noticing that

$$\|Z\|_{\mathcal{H}}^2 \ge \frac{\|Z\|_{\mathcal{H}}^2}{2}+\frac{|v|_2^2}{2} \ge \frac{\|Z\|_{\mathcal{H}}^2}{2}+\frac{\lambda^2|u|_2^2}{4}-\frac{|f|_2^2}{2}, \tag{3.7}$$

one absorb the term involving $|u|_2$ by choosing $|\lambda|$ large enough.

3.3. **Some open problems.** It is worth noting that when using the Kelving-Voigt damping, one critically relies on the Poincaré inequality to estimate the norm of the localized velocity by the norm of its gradient in the region where that damping is active. This leads us to wonder what would happen if we were to replace the Dirichlet boundary conditions by either Neumann or Robin boundary conditions; this is by now an open problem worth exploring. To the best of our knowledge, all earlier works used the Dirichlet boundary conditions. A very challenging problem would be to investigate stability issues for the wave equation when only the localized Kelvin-Voigt damping is active and the control region is arbitrarily small; in the case of fractional damping, we know, thanks to [20] and some related subsequent works that the stability is logarithmic. The stabilization of the Euler-Bernoulli plate equation with localized Kelvin-Voigt damping is also an open problem worth investigating; the corresponding beam equation with localized Kelvin-Voigt damping is exponentially stable with no smoothness condition on the damping coefficient, [23].

Acknowledgements. The authors are indebted to Dr Wellington Corréa for his help with the drawings. The authors also thank the anonymous reviewer for helping with the presentation of this article.

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