GOOD PAIRS IN EXTRASPECIAL p-GROUPS

THESIS

Presented to the Graduate Council
of Texas State University
in Partial Fulfillment
of the Requirements

for the Degree

Master of SCIENCE

by

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San Marcos, TX
May 2006

ACKNOWLEDGEMENTS

I would like to express my great appreciation for the members of my thesis committee. To Dr. Acosta and Dr. Morey for their careful attention to detail and their valuable suggestions.

In particular, I must thank Dr. Keller. In fact the word thank is not nearly enough. Without his unending patience and support this thesis would not have been possible. My respect and admiration for him, both as a person and as a mathematician, have infinitely multiplied throughout this process, and I am so grateful for the opportunity I had to work with him and to be exposed to just a portion of the knowledge and insight he has to share.

Upon my husband Matt, I impart my utmost appreciation. For first agreeing to let me return to school, and then for his understanding and encouragement throughout the process, and most of all for his extreme patience with my, often, frazzled state. He is my pillar of sanity in a world filled with craziness.

I would also like to thank my courier service, Brian Doring, for his immeasurable kindness in delivering correspondence between San Marcos and Austin.

Last, but certainly not least, I would like to express my gratitude for all of my math professors at Texas State University (you know who you are), for the knowledge they shared with me, the effort they made to help me understand, and for taking the time to learn my name (this in particular means more than I could tell you).

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CHAPTER 1

INTRODUCTION

In the article "A Group Theoretic Characterization of M-Groups[1]", Alan Parks [6] uses character theory to prove that a certain relation on what he calls "good pairs," which we will introduce in Chapter 3, is an equivalence relation. In the paper he also proposes that it would be interesting to find a group theoretic proof that the relation is an equivalence relation. It is fairly straightforward to prove that symmetry and reflexivity hold in any group G. The trouble arises in proving that transitivity holds. There is a simple proof of transitivity for abelian groups, but we encounter more difficulty in non-abelian groups.

While the inspiration for this paper came from trying to prove that the relation mentioned above is an equivalence relation, the real focus of the paper is an exploration of how good pairs really fit into a group's structure and what it means for two good pairs to be related.

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[1]Although we do not further discuss M-groups in this thesis, for the sake of the reader we include the following definition. An M-group is a finite group G all of whose irreducible characters are induced from a linear character of some subgroup of G. For further information regarding M-groups and characters, we suggest Victor E. Hill, *Groups and Characters.* (Boca Raton: Chapman & Hall/CRC, 2000).

Since the real interest is in studying good pairs within non-abelian groups, we have chosen to study a very specific type of non-abelian group: extraspecial p-groups of exponent p. We have chosen these groups specifically because their unique properties and structure allows for detailed analysis, which offers considerable insight into the workings of good pairs.

The reader should note that extraspecial p-groups with exponent p^2 have also been included in the form of the specific examples D_8 and Q_8. However, because of the extreme similarity in structure between extraspecial p-groups of exponent p and extraspecial p-groups of exponent p^2, we feel that any in depth study of extraspecial p-groups of exponent p^2 would lend little further insight into understanding good pairs.

The paper begins with a discussion of background information on p-groups and extra-special p-groups, including some examples (Chapter 2). The second part of this thesis consists of definitions and original lemmas regarding the equivalence relation defined by Parks, as well as an in depth exploration of good pairs and related good pairs within extraspecial p-groups (Chapter 3). The final part is the suggested group theoretic proof that the relation is an equivalence relation for the specific cases of abelian groups and extraspecial p-groups of exponent p (Chapter 4).

The main findings of this paper are located in Chapter 3, and are summarized on the following page. (All necessary notation and definitions are provided in Chapters 2 and 3.)

RESULT. If G is an extraspecial p-group of exponent p then G contains exactly two types of good pairs (H,M).

Type I: $H = G$, where $M = G$ or M is a maximal subgroup of G containing $Z(G)$.

Type II: H is a maximal abelian subgroup of G, and M is a maximal subgroup of H that does not contain $Z(G)$.

Under the equivalence relation $\sim$, each good pair of Type I is in an equivalence class of its own, and there is exactly one equivalence class of good pairs of Type II.

Of the reader we assume a basic knowledge of group theory, including an understanding of group structures (including subgroup lattices), properties of normal subgroups, commutators, and the commutator subgroup. We will cite background theorems from outside sources without proof, but we prove all original lemmas and theorems presented in the paper.

Notation

All groups considered in this paper are finite. For a group G we often simply write Z to indicate the center of G, $Z(G)$, and we often refer to the trivial subgroup $\langle 1 \rangle$ simply as 1. Also, for x and g in a group G, we will use the notation g^x to denote the conjugate of g by x, or $x^{-1}gx$. By G^x we denote the conjugate group $x^{-1}Gx$.

To indicate that a group H is a subgroup of another group G we write $H \leq G$, and $H < G$ means H is a proper subgroup of G. Similarly, $\subseteq$ is used to indicate a subset and $\subset$ is used to denote a proper subset. The symbol $\trianglelefteq$ is used to denote a normal subgroup.

By $[x, y]$ we denote the commutator $x^{-1}y^{-1}xy$, and for H a group we write $[x, H]$ to denote the set of commutators $\{x^{-1}h^{-1}xh \mid h \in H\}$. We will also use the acronym WLOG to represent the phrase "without loss of generality." All other notation is standard as found in Isaacs [4], or is defined as it is used.

CHAPTER 2

p-GROUPS

2.1 Background on p-groups

We will begin by considering some background information on p-groups and extraspecial p-groups.

Definition. A group of order p^a for some prime p and some whole number a is called a *p-group*.

The following is the Fundamental Theorem for Finite Abelian Groups (see for example Aschbacher [1, p.5]) and describes an interesting aspect of the structure of p-groups.

(2.1) THEOREM. Let $P \neq 1$ be an abelian p-group. Then P is the direct product of cyclic subgroups $P_i \cong \mathbb{Z}_{p^{e_i}}$, $1 \leq i \leq n$, $e_1 \geq e_2 \geq \cdots \geq e_n \geq 1$. Moreover the integers n and $(e_i : 1 \leq i \leq n)$ are uniquely determined by P.

Several more key properties about p-groups are outlined in the following theorem from Dummit and Foote [2, p.190].

(2.2) THEOREM. Let p be a prime and let P be a group of order p^a, $a \geq 1$. Then

(1) The center of P is nontrivial: $Z(P) \neq 1$.

(2) If H is a nontrivial normal subgroup of P then H intersects the center non-trivially: $H \cap Z(P) \neq 1$. In particular, every normal subgroup of order p is contained in the center.

(3) If H is a normal subgroup of P then H contains a subgroup of order p^b that is normal in P for each divisor p^b of $|H|$. In particular, P has a normal subgroup of order p^b for every $b \in \{0,1,\ldots,a\}$.

(4) If $H < P$ then $H < N_P(H)$ (i.e., every proper subgroup of P is a proper subgroup of its normalizer in P).

(5) Every maximal subgroup of P is of index p and is normal in P.

Although p-groups as a whole have many interesting and useful properties, it is often beneficial to consider more specific types of p-groups, such as extraspecial p-groups, whose properties are even more noteworthy. First we define the Frattini subgroup.

Definition. The *Frattini subgroup* of a group G is the intersection of all maximal subgroups of G, and is denoted by $\Phi(G)$.

Definition. An *extraspecial p-group* is a finite p-group P such that $\Phi(P) = Z(P) = P'$, where P' is the commutator subgroup of P, is of order p.

Definition. If G is any group, the *exponent* of G is the smallest positive integer n such that $x^n = 1$ for all $x \in G$ (if no such integer exists the exponent of G is ∞).

Definition. An *elementary abelian p-group* is an abelian p-group of exponent p.

Combining Theorem 2.1 and the above definition, it is clear that an elementary abelian p-group P of order p^n is the direct product of n copies of $\mathbb{Z}_p$.

Now we include some results from Aschbacher [1, p.111] that reveal more information about the nature of extraspecial p-groups.

(2.3) THEOREM. Let p be an odd prime and m a positive integer. Then up to isomorphism there is a unique extraspecial p-group E of order p^{2m+1} and exponent p. E is the central product of m copies of the extraspecial p-group of exponent p and order p^3.

Leedham-Green and McKay [5, p.28] describes the unique extraspecial p-group of exponent p and order p^3 as having the following presentation:

$$E = \langle x, y, z \mid x^p = y^p = z^p = 1, y^x = yz, z^x = z, z^y = z \rangle.$$

So, clearly a minimal set of generators for E is $\{x, y\}$. Further we can extract the following description of an extraspecial p-group of order p^{2m+1} and exponent p from the same source [5, p.33].

(2.4) LEMMA. Any subgroup of an extraspecial p-group E that does not contain the center is abelian.

Proof. Let H be a subgroup of E that does not contain $Z(E) = Z$. Then since Z is cyclic, we know $H \cap Z = 1$. Let $x, y \in H$. We know $[x, y] \in E' = Z$ and clearly $[x, y] \in H$. Thus $[x, y] = 1$, and this completes the proof. ■

(2.5) LEMMA. Let E be an extraspecial p-group. Let H be a subgroup of E with $Z(E) \leq H$. Then $H \trianglelefteq E$. *(In particular, if H is maximal abelian in E, then* $H \trianglelefteq E$.)

Proof. To show that $H \trianglelefteq E$ we must show that for all $h \in H$ and $g \in E$, $h^g \in H$. This is equivalent to showing that $h^{-1}h^g \in H$ for all $g \in E$, where $h^{-1}h^g = [h, g] \in E' = Z(E)$. But $Z(E) \leq H$. This completes our proof. ■

We now include a result from Huppert [3, p.353].

(2.6) THEOREM. Let G be a non-abelian p-group such that $G/Z(G)$ is elementary abelian where $Z(G)$ is cyclic. Then:

(1) $|G/Z(G)|$ is a square, i.e. $|G/Z(G)| = p^{2m}$ for some integer m.

(2) If $|G/Z(G)| = p^{2m}$ then all maximal abelian normal subgroups of G have order $p^m|Z(G)|$.

(3) For each maximal abelian normal subgroup A_1 of G there exists a maximal normal subgroup A_2 of G such that $A_1A_2 = G$ and $A_1 \cap A_2 = Z(G)$.

Clearly an extraspecial p-group of exponent p satisfies the hypotheses of Theorem 2.6. Thus we know that all maximal abelian (normal) subgroups of G, where $|G| = p^{2m+1}$, have order $p^m|Z(G)| = p^{m+1}$.

(2.7) LEMMA. Let $A \leq B$ where B is elementary abelian of order p^m. Suppose that $\{a_1, \ldots, a_k\}$ is a minimal set of generators of A, i.e. $|A| = p^k$ and thus $k \leq m$. Then there exists $a_{k+1}, \ldots, a_m$ such that $\{a_1, \ldots, a_m\}$ is a minimal set of generators for B.

Proof: We will write A and B additively. Let $F=\{0,1,\ldots,p-1\}$ be the field of order p. Then $\langle a_1\rangle=\{0,a_1,2a_1,\ldots,(p-1)a_1\}=Fa_1$. So $A=Fa_1\oplus Fa_2\oplus\ldots\oplus Fa_k$, i.e. A is a k-dimensional F-vector space with basis $\{a_1,\ldots,a_k\}$. Also we know A is a subspace of B which is an m-dimensional F-vector space. It is well-known from linear algebra that this means we can find $a_{k+1},\ldots,a_m$ so that $\{a_1,\ldots,a_m\}$ is a basis for B, i.e. $\{a_1,\ldots,a_m\}$ is a minimal generating set for B. ∎

(2.8) LEMMA. Let P be a p-group of order p^{n+1} and let B be a maximal subgroup of P. Suppose $U\le P$ with $|U|=p^m$. Then $|U\cap B|\ge p^{m-1}$.

Proof: Consider $p^{n+1}=|P|\ge|UB|=\dfrac{|U||B|}{|U\cap B|}=\dfrac{p^{m+n}}{|U\cap B|}$. Thus $|U\cap B|\ge\dfrac{p^{m+n}}{p^{n+1}}=p^{m-1}$. ∎

Now we can prove a fundamental result about the structure of extraspecial p-groups of exponent p.

(2.9) THEOREM. Let E be an extraspecial p-group of order p^{2m+1} and exponent p. Write $Z=Z(E)=\langle z\rangle$. Let $A=\langle a_1,\ldots,a_n,z^k\rangle$ for some $n\le m$ and $k\in\{0,1\}$ be an abelian subgroup of E of order p^{n+k}. Then we can find $a_{n+1},\ldots,a_m,b_1,\ldots,b_m\in E$, where $E=\langle a_1,\ldots,a_m,b_1,\ldots,b_m,z\rangle=\langle a_1,\ldots,a_m,b_1,\ldots,b_m\rangle$, with $[a_i,a_j]=[b_i,b_j]=1$ for all i,j, $[a_i,b_j]=1$ for all $i\neq j$ and $[a_i,b_i]=z$ for all i.

Proof: We know that $H=\langle A,z\rangle=\langle a_1,\ldots,a_n,z\rangle$ is normal in E (Lemma 2.5). We consider two cases.

Case I: If $H=\langle a_1,\ldots,a_n,z\rangle$ is maximal abelian in *E*, i.e. $n=m$, then by Theorem 2.6 we know there exists a maximal abelian subgroup B such that $HB=E$ and $H\cap B=Z$. We next prove the existence of $b_1,\ldots,b_n\in B$ by an inductive process as follows. Let $i\in\{1,\ldots,n\}$. Consider the map $\varphi_i:B\to Z$ defined by $\varphi_i(b)=[a_i,b]$. Clearly $[a_i,b]\in Z$, since $Z=E'$. First we check that φ_i is indeed a group homomorphism. Let $x_1,x_2\in B$. It is well known that $[a,bc]=[a,c][a,b]^c$ ([5, p.3]). Thus $\varphi_i(x_1x_2)=[a_i,x_1x_2]=[a_i,x_2][a_i,x_1]^{x_2}$. Since $[a_i,x_1],[a_i,x_2]\in Z$ we know $[a_i,x_2][a_i,x_1]^{x_2}=[a_i,x_2][a_i,x_1]=[a_i,x_1][a_i,x_2]=\varphi_i(x_1)\varphi_i(x_2)$. So φ_i is a group homomorphism for each *i*.

Write $B_i=Ker(\varphi_i)=\{x\in B \mid [a_i,x]=1\}=C_B(a_i)$. We know $\varphi_i(B)\le Z$. So φ_i is either surjective or trivial. If φ_i is trivial then $B_i=Ker(\varphi_i)=C_B(a_i)=B$. Thus $\langle B,a_i\rangle$ is an abelian subgroup of *E*, which contradicts the fact that *B* is maximal abelian in *E*. So φ_i is surjective for each *i*. Thus for each *i* we know $B/B_i\cong Z$. So $|B/B_i|=p$ which implies $|B_i|=p^n$ (since *B* is maximal abelian, thus $|B|=p^{n+1}$). So for each *i*, B_i is a maximal subgroup of *B*. Write $C=\bigcap_{i=1}^{n-1}B_i$. Then

$|C| = \left|\bigcap_{i=1}^{n-1} B_i\right| \geq p^2$ (This is a clear result of Lemma 2.8). So $Z < C$. Now, since H is maximal abelian and $H \cap B = Z$, we know that $C_B(H) = Z$. But $\bigcap_{i=1}^{n} B_i = C_B(H)$, and thus $\bigcap_{i=1}^{n} B_i = Z$. Consider then $\varphi_n|_C$. If $\varphi_n|_C$ is not surjective, i.e. $\varphi_n|_C = 1$, then $C \subseteq B_n$. This tells us that $\bigcap_{i=1}^{n} B_i = C \cap B_n > Z$ which is clearly a contradiction. So there is some $b_n \in C$ such that $\varphi_n(b_n) = [a_n, b_n] = z$. Also, since $b_n \in C$ we know $b_n \in B_i = Ker(\varphi_i)$ for all $i < n$. Thus $\varphi_i(b_n) = [a_i, b_n] = 1$ for all $i < n$.

Now let $H_0 = \langle a_1, \ldots a_{n-1}, z \rangle$, $E_0 = \langle a_n, b_n \rangle$ and $E_1 = H_0 B_n \leq E$. Then $E_0 \cap E_1 = Z$ and $[E_0, E_1] = 1$. It is easy to check that $E = E_0 E_1$ using a simple order argument. Thus, if $n = 1$ then $E_1 = Z$ and so $E = \langle a_n, b_n, z \rangle = \langle a_n, b_n \rangle$ and we are done. Otherwise, we can show that E_1 is extraspecial of exponent p.

Clearly E_1 is of exponent p. We know $Z \leq E_1$ and thus $Z \leq Z(E_1)$. Suppose then that $a \in Z(E_1) - Z$. Then since $[E_0, E_1] = 1$, we know that a commutes with all of $E = E_0 E_1$. Thus $a \in Z$ which is a contradiction. So $Z(E_1) = Z$. Now we must show $E_1' = \Phi(E_1) = Z$. Consider first E_1'. Clearly $E_1' \leq E' = Z$. So we need only show that $E_1' \neq 1$. Assume otherwise. Then $E_1' = 1$, which implies E_1 is abelian and thus $Z = Z(E_1) = E_1$, which is a contradiction. Thus $E_1' = Z$. Now we consider

$\Phi(E_1)$. It is well known that for a group G with $N \trianglelefteq G$, $\Phi(N) \le \Phi(G)$ ([3, p.269]). Since $Z \le E_1$ we know, by Lemma 2.5, that $E_1 \trianglelefteq E$ and thus $\Phi(E_1) \le \Phi(E) = Z$. Also since E_1 is a p-group Theorem 2.2 tells us that every maximal subgroup of E_1 is normal and thus intersects nontrivially with $Z(E_1) = Z$. But since Z is cyclic of order p, this means every maximal subgroup of E_1 contains the center. Thus $Z \le \Phi(E_1)$. So $\Phi(E_1) = Z$, which means E_1 is extraspecial of exponent p.

We know that $p^{2(n-1)+1} = |E_1| < |E| = p^{2n+1}$, so by induction (applied to $A_0 = \langle a_1, \ldots, a_{n-1}, z^k \rangle \le E_1$) there exist $b_1, \ldots b_{n-1} \in E_1$ such that

$$E_1 = \langle a_1, \ldots, a_{n-1}, b_1, \ldots b_{n-1}, z \rangle = \langle a_1, \ldots, a_{n-1}, b_1, \ldots b_{n-1} \rangle,$$

with $[a_i, a_j] = [b_i, b_j] = 1$ for $1 \le i, j \le n-1$, $[a_i, b_j] = 1$ for all $i \ne j$ and $[a_i, b_i] = z$ for all i. Thus since $E = E_0 E_1$ we are done.

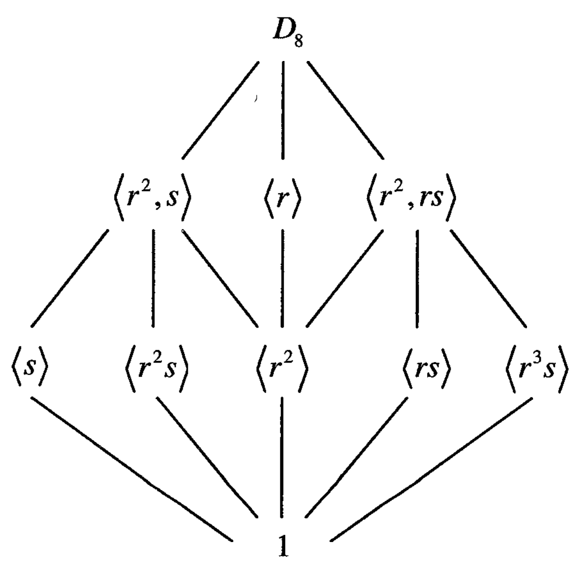
Case II: If $H = \langle a_1, \ldots, a_n, z \rangle$ is not maximal abelian in E, i.e. $n < m$, then pick a maximal abelian subgroup G of E such that $H \le G$. By Lemma 2.7, we know that we can find $a_{n+1}, \ldots, a_m$ such that $G = \langle H, a_{n+1}, \ldots, a_m \rangle = \langle a_1, \ldots, a_m, z \rangle$. Since G is maximal abelian in E, Case I applies, and our proof is complete. ∎

2.2 Examples of Extraspecial p-groups

In order to better understand the structure of extraspecial p-groups we find it useful to study some examples.

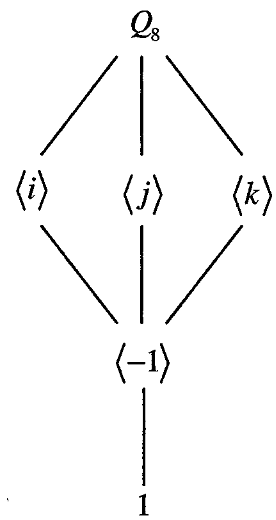
First we consider $p=2$. It is well known that up to isomorphism, there are two extraspecial p-groups of order $2^3=8$. The first is $D_8=\langle r,s\rangle$, where $r^4=s^2=1$ and $rs=sr^{-1}$. Note that D_8 has exponent 4 and that $Z(D_8)=\langle r^2\rangle$.

The subgroup lattice of D_8 is shown below.



The second extraspecial p-group of order $2^3=8$ is $Q_8=\{1,-1,i,j,k,-i,-j,-k\}$, with the following rules of multiplication: $i^2=j^2=k^2=-1$, $ij=k$, $jk=i$, and $ki=j$. This group also has exponent 4 and $Z(Q_8)=\langle -1\rangle$.

The subgroup lattice for Q_8 is included below.

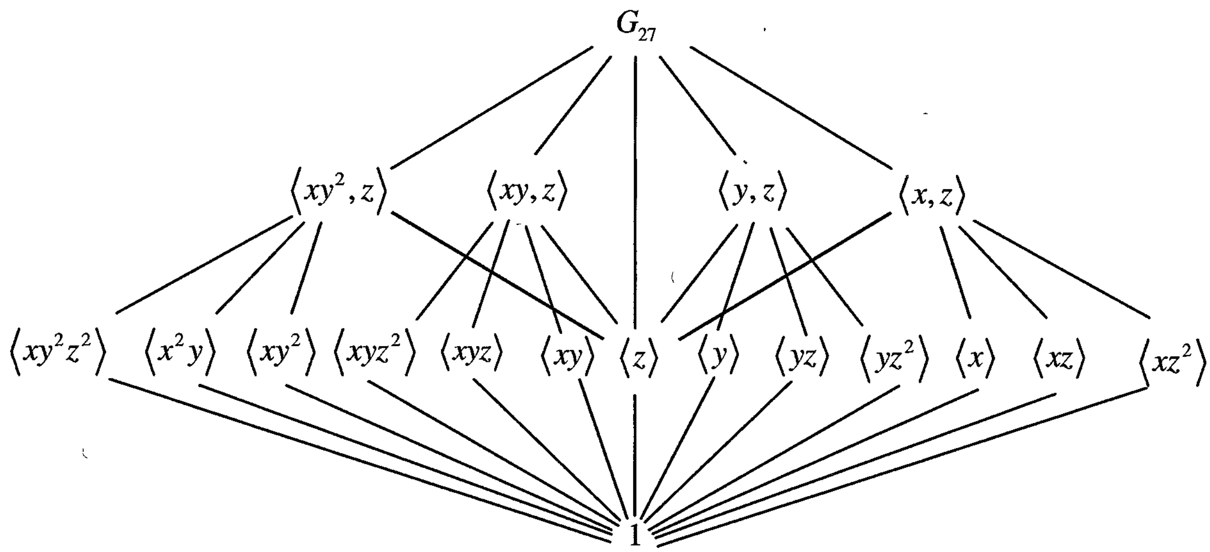


Now we consider an example where $p = 3$. It is known that up to isomorphism there are two extraspecial p-groups of order $3^3 = 27$, one has exponent 3 and the other has exponent 9 [4, p.35]. Here we will look only at the one with exponent 3. We will call it G_{27}. Listed element-wise we have:

$$G_{27} = \left\{ \begin{array}{l} 1, x, x^2, y, y^2, z, z^2, xy, x^2y, xy^2, x^2y^2, xz, x^2z, \\ xz^2, x^2z^2, yz, y^2z, yz^2, y^2z^2, xyz, x^2yz, xy^2z, \\ xyz^2, x^2y^2z, x^2yz^2, xy^2z^2, x^2y^2z^2 \end{array} \right\} = \langle x, y, z \rangle,$$

where $x^3 = y^3 = z^3 = 1$, $[x, y] = z$, $yz = zy$, and $xz = zx$. Obviously the center of G_{27} is $Z(G_{27}) = \langle z \rangle$.

The subgroup lattice of G_{27} is shown below.



CHAPTER 3

GOOD PAIRS AND A RELATION ON GOOD PAIRS

3.1 Definitions and Lemmas

We will consider the following definitions from Parks [6].

Definition. Let G be a group. Then (H,M) is called a *pair* in G if $M \trianglelefteq H \leq G$ and H/M is cyclic.

Definition. For a subgroup H of a group G and an element $g \in G$, we write $F_H(g) = \left[g, H \cap H^{g^{-1}}\right]$ to denote the set $\left\{[g,x] \mid x \in H \cap H^{g^{-1}}\right\}$, where $[g,x] = g^{-1}x^{-1}gx$.

Definition. A pair (H,M) in a group G is called a *good pair* if $F_H(g) \not\subseteq M$ for all $g \in G - H$.

Definition. Two good pairs (H,M) and (K,L) in a group G are *related in G* if there is some $g \in G$ such that $H^g \cap L = K \cap M^g$. We write $(H,M) \sim (K,L)$.

The reader should note that if (H_0,M_0) is a good pair in G, then $(H_0{}^g,M_0{}^g)$ is a good pair in G for all $g \in G$. For any two related good pairs (H_0,M_0) and (K,L) in G, we know $H_0{}^g \cap L = K \cap M_0{}^g$ for some $g \in G$. Thus by writing $(H,M) = \left(H_0{}^g, M_0{}^g\right)$ we have $(H,M) \sim (K,L)$, with $H \cap L = K \cap M$. So for any two related good pairs (H,M) and (K,L) in G, we may assume that $H \cap L = K \cap M$ when it is convenient to do so.

We now consider some lemmas relating to goodness and $\sim$.

(3.1) LEMMA. Let G be a group and H a subgroup of G. Then $F_H(g) \subseteq H$.

Proof: Let $g \in G$ and $h \in H \cap H^{g^{-1}}$. Then $h = gkg^{-1}$ for some $k \in H$. So $[g,h] = g^{-1}h^{-1}gh = g^{-1}(gk^{-1}g^{-1})gh = k^{-1}h$. Also $h,k \in H$, so $k^{-1}h \in H$. Thus $F_H(g) \subseteq H$. ■

(3.2) LEMMA. Let G be a group and let $(H,M) \sim (K,L)$ be related good pairs in G. Then for $M_0 = \bigcap_{g \in G} M^g$ and $L_0 = \bigcap_{g \in G} L^g$ we have $M_0 = L_0$.

Proof: Since $(H,M)\sim(K,L)$ are related good pairs, we may assume (by possibly replacing (H,M) by a G-conjugate) that $H\cap L=K\cap M$. This implies $H\cap L\leq M$, and so we have $H\cap L=M\cap L$. Assume then that $M_0\neq L_0$. Then without loss of generality we can assume that there is an $x\in L_0$ such that $x\notin M_0$. Therefore there exists some $g\in G$ such that $x\notin M^g$, so $x^{g^{-1}}\notin M$ and $x^{g^{-1}}\in L$. So by replacing x by $x^{g^{-1}}$, we may assume that $g=1$. Thus we have $x\notin M$ and $x\in L$. Then $H\cap L=M\cap L$ tells us that $x\notin H\cap L$, so $x\notin H$. Therefore, because (H,M) is a good pair we know that $F_H(x)=\left[x,H\cap H^{x^{-1}}\right]\not\subseteq M$, and from Lemma 3.1 we know that $F_H(x)\subseteq H$. Let $a\in H\cap H^{x^{-1}}$. Then $[x,a]=x^{-1}a^{-1}xa\in H$. Also we know that $L_0\trianglelefteq G$, so $a^{-1}xa\in L_0$ and $x^{-1}\in L_0$. Thus $[x,a]=x^{-1}a^{-1}xa\in L_0$. So we have that $[x,a]\in L$. Therefore $[x,a]\in H\cap L=M\cap L\subseteq M$. As $a\in H\cap H^{x^{-1}}$ was arbitrary, this is a contradiction to the fact that (H,M) is a good pair. So x must be in M_0, and the proof is complete. ■

The reader should note that because of Lemma 3.2 we often can assume that $M_0=L_0=1$ when it is convenient to do so, since if $M_0\neq 1$ we may mod out by $M_0=L_0$ and consider the factor groups.

(3.3) LEMMA. Let G be a group with $H,K \leq G$, $N \trianglelefteq G$, $N \leq H$, and $N \leq K$. Then $(H/N) \cap (K/N) = (H \cap K)/N$.

Proof: First we show that $(H/N) \cap (K/N) \subseteq (H \cap K)/N$. Let $xN \in (H/N) \cap (K/N)$. Then $x \in H$ and $x \in K$. Thus $x \in H \cap K$ which implies $xN \in (H \cap K)/N$. Now suppose that $xN \in (H \cap K)/N$. Then $x \in H \cap K$, which tells us that $x \in H$ and $x \in K$. Thus $xN \in H/N$ and $xN \in K/N$, which implies $xN \in (H/N) \cap (K/N)$. ∎

(3.4) LEMMA. Suppose (H,M) is a good pair in G and that $N \trianglelefteq G$ with $N \leq M$. Then $(H/N, M/N)$ is a good pair in G/N. Moreover, if (K,L) is another good pair in G with $N \leq L$, then $(H,M) \sim (K,L)$ if and only if $(H/N, M/N) \sim (K/N, L/N)$.

Proof: Clearly $M/N \trianglelefteq H/N \leq G/N$. Also we know that $(H/N)/(M/N) \cong H/M$. Since (H,M) is a pair in G we know that H/M is cyclic, thus $(H/N)/(M/N)$ is cyclic. So $(H/N, M/N)$ is a pair in G/N. We now want to show that the goodness condition is satisfied. Let $g \in G$ such that $gN \in (G/N) - (H/N)$. Then $g \in G - H$. Since (H,M) is a good pair in G there exists some $x \in H \cap H^{g^{-1}}$ such that $[g,x] \notin M$. First we show $xN \in (H/N) \cap (H/N)^{(g^{-1}N)}$. We know that $x \in H \cap H^{g^{-1}}$ which tells us that $xN \in \left(H \cap H^{g^{-1}}\right)N/N$. But

$N \le H \cap H^{g^{-1}}$, so we have $xN \in \left(H \cap H^{g^{-1}}\right)/N$. So by Lemma 3.3 $xN \in (H/N) \cap \left(H^{g^{-1}}/N\right)$, where it is clear that $H^{g^{-1}}/N = (H/N)^{(g^{-1}N)}$. Thus $xN \in (H/N) \cap (H/N)^{(g^{-1}N)}$. Assume then that $[gN, xN] \in M/N$. Then $[g,x]N \in M/N$ which implies that $[g,x] \in M$, which is clearly a contradiction. So $[gN, xN] \notin M/N$, which tells us that $(H/N, M/N)$ is a good pair in G/N.

Finally, we want to show that $(H,M) \sim (K,L)$ if and only if $(H/N, M/N) \sim (K/N, L/N)$. We begin by showing that $(H,M) \sim (K,L)$ implies that $(H/N, M/N) \sim (K/N, L/N)$. Assume $(H,M) \sim (K,L)$, then there is some $g \in G$ such that $H^g \cap L = K \cap M^g$. Thus $\left(H^g \cap L\right)/N = \left(K \cap M^g\right)/N$. So by Lemma 3.3 we have $\left(H^g/N\right) \cap (L/N) = (K/N) \cap \left(M^g/N\right)$. Therefore $(H/N)^{gN} \cap (L/N) = (K/N) \cap (M/N)^{gN}$, since $H^g/N = (H/N)^{gN}$ and $M^g/N = (M/N)^{gN}$, which implies $(H/N, M/N) \sim (K/N, L/N)$.

Now suppose that $(H/N, M/N) \sim (K/N, L/N)$. Then for some $gN \in G/N$ we know $(H/N)^{gN} \cap (L/N) = (K/N) \cap (M/N)^{gN}$ which implies $\left(H^g/N\right) \cap (L/N) = (K/N) \cap \left(M^g/N\right)$. Thus Lemma 3.3 tells us that $\left(H^g \cap L\right)/N = \left(K \cap M^g\right)/N$. Therefore $H^g \cap L = K \cap M^g$, and this completes our proof. ∎

(3.5) LEMMA. Let G be a p-group and let (H,M) be a good pair in G. Then $Z(G) \le H$.

Proof: Since G is a p-group, we know that $Z(G) \ne 1$. Suppose then that $g \in Z(G)$ with $g \ne 1$, and $g \in G-H$. Then for all $h \in H$ we have $[g,h] = g^{-1}h^{-1}gh = g^{-1}gh^{-1}h = 1$. Clearly $H^{g^{-1}} = gHg^{-1} = H$. So we have $F_H(g) = [g,H] = \{[g,h] \mid h \in H\} = 1$. But this implies $F_H(g) \subseteq M$, which is a contradiction to (H,M) being a good pair. So $Z(G) \le H$. ■

We conclude this section with a result that gives us further insight into the nature of good pairs in p-groups, but which will not be used at any other point in this thesis.

(3.6) LEMMA. Let G be a p-group. Let (H,M) be a good pair in G with $H \ne G$ and $\bigcap_{g\in G} M^g = 1$. Then $Z(G)$ is cyclic and if $Z(G) = \langle z \rangle$ then $z \in H - M$.

Proof: Let $z_0 \in Z(G)$ with $z_0 \ne 1$. From Lemma 3.5, we know that $z_0 \in H$. Suppose then that $z_0 \in M$. Then $\langle z_0 \rangle \le M$. This implies $\bigcap_{g\in G} \langle z_0 \rangle^g \le \bigcap_{g\in G} M^g = 1$,

where $\bigcap_{g\in G} \langle z_0 \rangle^g = \bigcap_{g\in G} \{g^{-1}xg \mid x \in \langle z_0 \rangle \le Z(G)\}$

$$= \bigcap_{g\in G} \{x \mid x \in \langle z_0 \rangle \le Z(G)\}$$

$$= \bigcap_{g \in G} \langle z_0 \rangle = \langle z_0 \rangle \neq 1.$$

Thus we have a contradiction. So $z_0 \in H - M$, and since z_0 was an arbitrary non-identity element, we know that $Z(G) \cap M = 1$. Now, since $M \trianglelefteq H$ and $Z(G) \leq H$, we have $Z(G)M/M \cong Z(G)/Z(G) \cap M = Z(G)$, where $Z(G)M/M \leq H/M$. Thus $Z(G)$ is isomorphic to a subgroup of H/M which implies that $Z(G)$ is cyclic. So, $Z(G) = \langle z \rangle$, with $z \in H - M$. ■

Here it is important to note that $L_0 = \bigcap_{g \in G} L^g$ is a normal subgroup of *G*, *K*, and *L*. Thus, if $L_0 \neq 1$ we may consider the factor group G/L_0 in which $L_0 = 1$ and $(K/L_0, L/L_0)$ is a good pair by Lemma 3.4, and so all of the conditions of Lemma 3.6 hold.

3.2 Classifying Good Pairs in Extraspecial *p*-groups

In order to gain further insight into good pairs and related good pairs, it is useful to study good pairs within specific types of groups. We begin by considering abelian groups.

(3.7) LEMMA. Let G be an abelian group and let (H, M) be a good pair in *G*. Then $H = G$.

Proof: Since G is an abelian group, $G = Z(G)$. So for all $H < G$, we have $F_H(g) = 1$ for all $g \in G$. Thus if (H,M) is a good pair in G, it must be that $H = G$. ∎

The reader should note that every subgroup of an abelian group is normal, thus for any $M \le G$ such that G/M is cyclic, (G,M) is a good pair.

Now we consider related pairs in an abelian group.

(3.8) LEMMA. All good pairs in an abelian group are related only to themselves.

Proof: Let G be an abelian group and suppose that (G,M) and (G,L) are related good pairs in G. Then $G^g \cap L = G \cap M^g$ for some $g \in G$. But this implies that $G \cap L = G \cap M$. Thus $L = M$. So each good pair in an abelian group is related only to itself. ∎

Now we turn to classifying good pairs and related good pairs in some non-abelian groups. We will consider a very specific type of non-abelian group: extraspecial p-groups of exponent p.

Suppose that G is an extraspecial p-group of exponent p with order p^{2m+1} and that (H,M) is a good pair in G. Write $Z(G) = Z$. We know from Lemma 3.5 that $Z \le H$. So we need to consider two possibilities: (1) $Z \le M$ and (2) $Z \not\le M$.

(3.9) LEMMA. Let G be an extraspecial p-group of exponent p with order p^{2m+1} and let (H,M) be a good pair in G such that $Z \leq M$. Then $H = G$ and $M = G$ or M is a maximal subgroup of G with $Z \leq M$.

Proof: If $Z \leq M$, then by Lemma 3.4 $(H/Z, M/Z)$ is a good pair in G/Z. We know G/Z has exponent p and since $Z = G'$ implies G/Z is abelian, we know G/Z is elementary abelian. So from Lemma 3.7 we know that $H/Z = G/Z$. Thus $H = G$. Also $(G/Z)/(M/Z)$ is cyclic, thus of order 1 or p. Therefore $G/Z = M/Z$ or M/Z is a maximal subgroup of G/Z. Thus $M = G$ or M is a maximal subgroup of G with $Z \leq M$. ■

(3.10) LEMMA. Let G be an extraspecial p-group with exponent p, and $|G| = p^{2m+1}$. Let (H,M) be a good pair in G such that $H \neq G$. Then H is a maximal abelian subgroup of G.

Proof: First we show that H is abelian. We know that H/M is cyclic. Since G has exponent p it must be that $|H/M| = p$, since $|H/M| = 1$ implies $F_H(g) \subseteq M = H$ for all $g \in G$. By Lemma 3.9 we also know that $H \neq G$ implies $Z \not\leq M$, thus M is abelian by Lemma 2.4. So $|M| \leq p^m$, since M cannot be a maximal abelian subgroup because it does not contain the center. Also Lemma 3.5 tells us that

$Z(G) \le H$. Since $|H/M| = p$ it must be that $H = \langle M, z \rangle$, where $Z(G) = \langle z \rangle$ has order p. Therefore H is abelian and $|H| \le p^{m+1}$.

Now we want to show that H is maximal abelian. Assume otherwise. Then $H \subset C_G(H)$. So there is some g in $G - H$ such that $g \in C_G(H)$. But this tells us that for all $h \in H$, $[g,h] = g^{-1}h^{-1}gh = g^{-1}gh^{-1}h = 1$. Thus $F_H(g) = \{[g,h] \mid h \in H \cap H^{g^{-1}}\} = 1 \subseteq M$, which is a contradiction to the goodness of (H, M) in G. So $H = C_G(H)$ which implies H is a maximal abelian subgroup of G. ■

(3.11) LEMMA. Let G be an extraspecial p-group of exponent p with order p^{2m+1} and let (H, M) be a good pair in G such that $Z \not\le M$. Then H is a maximal abelian subgroup of G and M is a maximal subgroup of H with $Z \not\le M$.

Proof: If $Z \not\le M$, then because Z is cyclic, $M \cap Z = 1$. Since every nontrivial normal subgroup of a p-group intersects nontrivially with the center of G (part 2 of Theorem 2.2) we know $M \not\trianglelefteq G$. Thus $H \ne G$. So by Lemma 3.10 we know that H is a maximal abelian subgroup of G with $|H| = p^{m+1}$ and $H \trianglelefteq G$. Also H/M is cyclic and thus of order p, therefore M is a maximal subgroup of H with $|M| = p^m$. ■

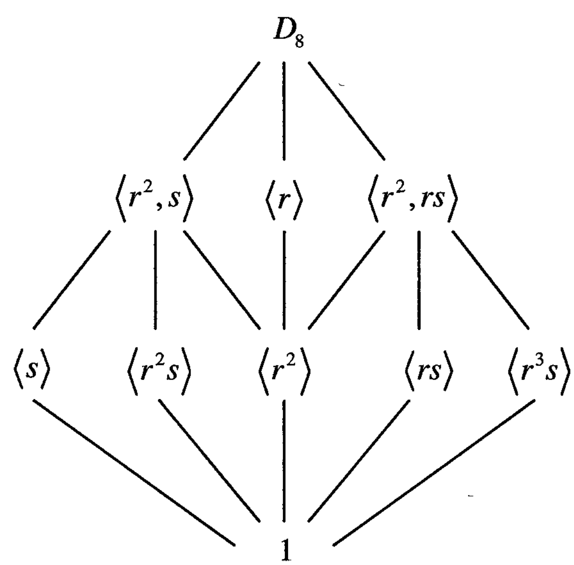
(3.12) LEMMA. Let G be an extraspecial p-group with exponent p and $|G| = p^{2m+1}$. Let H be a maximal abelian subgroup of G and M a maximal subgroup of H with $Z \not\leq M$. Then (H,M) is a good pair in G.

Proof: Clearly $|H:M| = p$ tells us that (H,M) is a pair. Also we know $F_H(g) \subseteq G' = Z$ for all $g \in G-H$. Since H is normal in G and $Z \cap M = 1$ we need only show that for each $g \in G-H$ there is some $h \in H$ such that $[g,h] \neq 1$. Let $g \in G-H$. If $[g,h]=1$ for all $h \in H$ then the group $\langle H,g \rangle$ (the subgroup of G generated by all elements of H and by g) would be abelian contradicting the fact that H is a maximal abelian subgroup of G. So there must be some $h \in H$ such that $[g,h] \neq 1$, and this completes our proof. ∎

3.3 Examples of Good Pairs in Extraspecial p-groups

Having considered what good pairs look like in a general extraspecial p-group, we find it valuable to look at some specific examples. We will consider the same extraspecial p-groups that we looked at in Chapter 2, beginning with D_8.

Recall the subgroup lattice for D_8.



Write (H,M) for a general good pair. By Lemma 3.5, we know that $Z(D_8)=\langle r^2\rangle \leq H$. Also we know that if $H=Z(D_8)$ then $F_H(g)=1$ for all $g\in G$, thus goodness would fail. So the possible candidates for *H* are: D_8, $\langle r^2,s\rangle$, $\langle r\rangle$, and $\langle r^2,rs\rangle$.

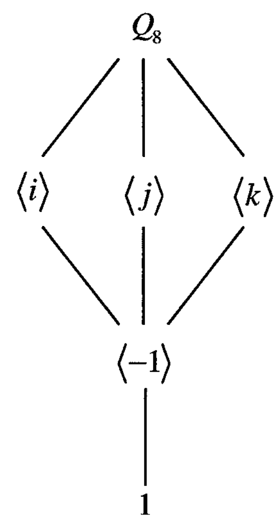
Now we note that although D_8 is of exponent p^2 and not *p*, it is still easily verified that if $Z\leq M$ then $H=G$, as shown below.

> If $Z\leq M$, then $(H/Z, M/Z)$ is a good pair in G/Z by Lemma 3.4. We know $Z=G'$ and thus G/Z is abelian. So by Lemma 3.7 we know that $H/Z=G/Z$. Thus $H=G$.

So we know that for $H \neq G$ we have $Z(D_8) = \langle r^2 \rangle \not\leq M$. Also, since H/M must be cyclic, we can eliminate all choices where $M = 1$, except $(\langle r \rangle, 1)$. Thus we find the following good pairs in D_8.

Good pairs in D_8: (D_8, D_8), $(D_8, \langle r^2, s \rangle)$, $(D_8, \langle r \rangle)$, $(D_8, \langle r^2, rs \rangle)$, $(\langle r^2, s \rangle, \langle s \rangle)$, $(\langle r^2, s \rangle, \langle r^2 s \rangle)$, $(\langle r^2, rs \rangle, \langle rs \rangle)$, $(\langle r^2, rs \rangle, \langle r^3 s \rangle)$, $(\langle r \rangle, 1)$

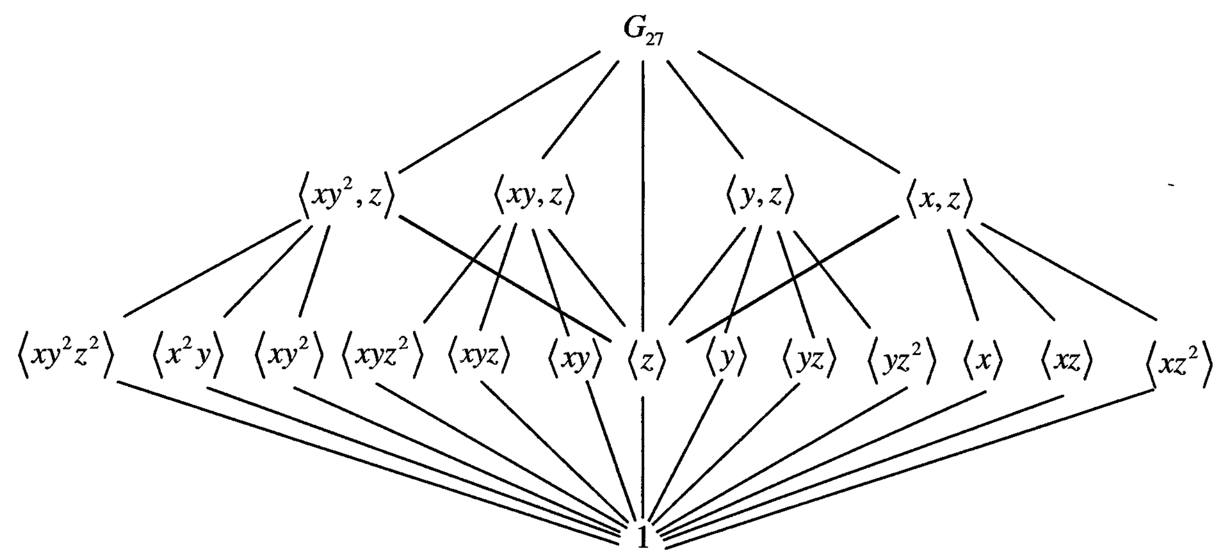
Now we consider good pairs in Q_8.



Good pairs in Q_8: (Q_8, Q_8), $(Q_8, \langle i \rangle)$, $(Q_8, \langle j \rangle)$, $(Q_8, \langle k \rangle)$, $(\langle i \rangle, 1)$, $(\langle j \rangle, 1)$, $(\langle k \rangle, 1)$

The reader should note that it turns out that in D_8 and Q_8, the good pairs (H, M) have the same properties as the good pairs in an extraspecial p-group of exponent p, i.e. they can be classified into two distinct types: (1) those where $H = G$ and $Z \leq M$, and (2) those where H is a maximal abelian subgroup of G and M is maximal subgroup of H that does not contain Z.

Finally, we consider good pairs in G_{27}. First we recall the subgroup lattice for G_{27}.



Write (H,M) for a general good pair. We note that by Lemma 3.9, if $H \neq G_{27}$ we know that $Z(G_{27}) = \langle z\rangle \not\leq M$. So, we can apply Lemma 3.10 and we have that H is a maximal abelian subgroup of G_{27}. So the possible candidates for H are: G_{27}, $\langle xy^2, z\rangle$, $\langle xy, z\rangle$, $\langle y, z\rangle$, and $\langle x, z\rangle$. Additionally, Lemmas 3.9 and 3.10 tells us that if $H \neq G_{27}$

then M is a maximal subgroup of H which does not contain the center, so $M \neq \langle z \rangle$ for any H.

Thus we find the following good pairs in G_{27}.

Good Pairs in G_{27}: (G_{27}, G_{27}), $(G_{27}, \langle xy^2, z \rangle)$, $(G_{27}, \langle xy, z \rangle)$, $(G_{27}, \langle y, z \rangle)$, $(G_{27}, \langle x, z \rangle)$, $(\langle xy^2, z \rangle, \langle xy^2 z^2 \rangle)$, $(\langle xy^2, z \rangle, \langle x^2 y \rangle)$, $(\langle xy^2, z \rangle, \langle xy^2 \rangle)$, $(\langle xy, z \rangle, \langle xyz^2 \rangle)$, $(\langle xy, z \rangle, \langle xyz \rangle)$, $(\langle xy, z \rangle, \langle xy \rangle)$, $(\langle y, z \rangle, \langle y \rangle)$, $(\langle y, z \rangle, \langle yz \rangle)$, $(\langle y, z \rangle, \langle yz^2 \rangle)$, $(\langle x, z \rangle, \langle x \rangle)$, $(\langle x, z \rangle, \langle xz \rangle)$, $(\langle x, z \rangle, \langle xz^2 \rangle)$

3.4 Classifying Related Good Pairs in Extraspecial p-groups

We found in Section 3.2 that we have two types of good pairs (H, M) within an extraspecial p-group G of exponent p, those where $H = G$ and $Z \leq M$, and those where H is a maximal abelian subgroup of G and M is maximal subgroup of H that does not contain Z. We now want to determine which of these good pairs are related.

(3.13) LEMMA. Let G be any group and let $Z \leq Z(G)$ with $Z \neq 1$. If (G, M) is a good pair in G, with $Z \leq M$, and (K, L) is a good pair in G such that $K < G$ and $Z \not\leq L$, then $(K, L) \nsim (G, M)$.

(3.13) LEMMA. Let G be any group and let $Z \leq Z(G)$ with $Z \neq 1$. If (G,M) is a good pair in G, with $Z \leq M$, and (K,L) is a good pair in G such that $K < G$ and $Z \not\leq L$, then $(K,L) \nsim (G,M)$.

Proof: Assume that $(K,L) \sim (G,M)$. Then by Lemma 3.2 we know that $\bigcap_{g \in G} M^g = \bigcap_{g \in G} L^g$. But $Z \leq \bigcap_{g \in G} M^g$ and $Z \not\leq L$ thus $Z \not\leq \bigcap_{g \in G} L^g$, and so we have a contradiction. ■

(3.14) LEMMA. Let G be any group. All good pairs (H,M) with $H = G$ are related only to themselves.

Proof: Suppose that (G,M) and (G,L) are related good pairs in G. Then $G^g \cap L = G \cap M^g$ for some $g \in G$. But this implies that $G \cap L = G \cap M$. Thus $L = M$. So each good pair is related only to itself. ■

The reader should note that while Lemmas 3.13 and 3.14 apply to any groups that contain the specific types of good pairs described in the hypotheses of the lemmas, they are especially relevant here because we have shown that these are the only types of good pairs in an extraspecial p-group of exponent p. Since we have not explored good pairs in a general non-abelian group, we do not know whether, or how, these lemmas might apply outside of the context of extraspecial p-groups.

(3.15) THEOREM. Let G be an extraspecial p-group with exponent p and $|G| = p^{2m+1}$. Let (H, M_1) and (H, M_2) be good pairs in G with $H \neq G$ (so H is a maximal abelian subgroup of G). Then $(H, M_1) \sim (H, M_2)$.

Proof: Let $Z = Z(G) = \langle z \rangle$. By Lemma 2.5 we know $H \trianglelefteq G$, so we need to show that $M_1^g = M_2$ for some $g \in G$. If $M_1 = M_2$ we are done, so we will assume that $M_1 \neq M_2$. Recall from Lemma 3.9 that $Z \not\leq M_1$ and $Z \not\leq M_2$, but from Lemma 3.5 we know that $z \in H$. We begin by considering the cardinality of $M_1 \cap M_2$. We know that $|H| = p^{m+1}$, so we have $p^{m+1} = |H| \geq |M_1 M_2| = \dfrac{p^m p^m}{|M_1 \cap M_2|} = \dfrac{p^{2m}}{|M_1 \cap M_2|}$. Thus $|M_1 \cap M_2| \geq \dfrac{p^{2m}}{p^{m+1}} = p^{m-1}$. But we know that $p^m > |M_1 \cap M_2|$, so we have $p^m > |M_1 \cap M_2| \geq p^{m-1}$, and thus $|M_1 \cap M_2| = p^{m-1}$. Since $M_1 \cap M_2$ is elementary abelian, we can write $M_1 \cap M_2 = \langle x_1, \ldots, x_{m-1} \rangle$, for suitable $x_i \in G$, $i = 1, \ldots, m-1$.

So $M_1 = \langle x_1, \ldots, x_{m-1}, b \rangle$ and $M_2 = \langle x_1, \ldots, x_{m-1}, x_m \rangle$ for appropriate $x_m, b \in G$ with $x_m \neq b$, which implies $H = M_1 M_2 = \langle x_1, \ldots, x_{m-1}, x_m, b \rangle$. Also because G is an extraspecial p-group of exponent p we can find appropriate $y_1, \ldots, y_m$ and write $G = \langle x_1, \ldots, x_{m-1}, x_m, y_1, \ldots, y_m \rangle$, where $z = [x_m, y_m]$ and $[x_i, y_m] = 1$ for $i = 1, \ldots, m-1$ (by Theorem 2.9). We also know that $z \in H$, thus we can write $z = b^r x_m{}^s w$ for some $w \in M_1 \cap M_2$ and integers r, s. As $z \notin M_1$ and $z \notin M_2$, we see that we may assume that $1 \leq r \leq p-1$ and $1 \leq s \leq p-1$. This implies

$b^r = x_m^{-s} w^{-1} z$. We can find $l \in \{1, \ldots, p-1\}$ such that $rl \equiv 1 \bmod p$, thus $(b^r)^l = b$ and so $b = (x_m^{-s} w^{-1} z)^l = x_m^{\ k} v z^l$ for some $v \in M_1 \cap M_2$ and an integer *k*. If *k* is a multiple of *p* then $b = vz^l$, which implies $1 \neq z^l = bv^{-1} \in M_1$, contradicting the fact that $Z \cap M_1 = 1$. So WLOG we may assume that $1 \leq k \leq p-1$.

Now, we know that $(x_a)^{y_m^n} = x_a$ for all $1 \leq a \leq m-1$ and for all integers *n*. So if we can show that $b^{y_m^n} \in M_2$ for some *n* then we will have $M_1^{\ y_m^n} = M_2$. Consider then

$$b^{y_m} = (x_m^{\ k} v z^l)^{y_m} = (y_m^{-1} x_m^{\ k} y_m) v z^l = (y_m^{-1} x_m y_m)^k v z^l = (x_m z)^k v z^l = x_m^{\ k} v z^{k+l}$$

(since $z = [x_m, y_m]$). If $k + l \equiv 0 \bmod p$, then $b^{y_m} = x_m^{\ k} v \in M_2$ and so we are done. If not, then consider $b^{y_m^2} = (b^{y_m})^{y_m} = (x_m^{\ k} v z^{k+l})^{y_m} = x_m^{\ k} v z^{2k+l}$. If $2k + l \equiv 0 \bmod p$ we are done, if not then continue on in this manner until we find *n* such that $nk + l \equiv 0 \bmod p$. (We know it is possible to find such an *n* since $1 \leq k, l \leq p-1$ and $\mathbb{Z}/p\mathbb{Z}$ is a cyclic group of order *p* under addition.) Then we know $b^{y_m^n} = x_m^{\ k} v \in M_2$. So $M_1^{\ y_m^n} = M_2$. As $H \trianglelefteq G$, we conclude that $H^{y_m^n} \cap M_2 = H \cap M_1^{\ y_m^n}$. Thus $(H, M_1) \sim (H, M_2)$. ∎

(3.16) THEOREM. Let *G* be an extraspecial *p*-group with exponent *p* and $|G| = p^{2m+1}$. Let (H, M) and (K, L) be good pairs in *G* with $H \neq G$ and $K \neq G$. Then $(H, M) \sim (K, L)$.

Proof: By Lemma 3.10, H and K are maximal abelian subgroups of G. Let $H \cap K = S$ and let T be a maximal subgroup of S with $Z \not\leq T$. (Note that we know $Z \leq S = H \cap K$ so $S \neq T$, and, in fact, $|S:T| = p$.) Then $T \leq H$ and $T \leq K$. Now choose $M_0 \leq H$ such that $|H:M_0| = p$, $T \leq M_0$, and $Z \not\leq M_0$. Similarly choose $L_0 \leq K$ such that $|K:L_0| = p$, $T \leq L_0$, and $Z \not\leq L_0$. So (H,M_0) and (K,L_0) are good pairs in G (by Lemma 3.12). We know

$$T \leq H \cap L_0 \leq H \cap K = S \qquad (1)$$

And since $Z \not\leq L_0$ we know $Z \not\leq H \cap L_0$ and thus $H \cap L_0 \neq H \cap K$. Thus as $|S:T| = p$, (1) implies that $H \cap L_0 = T$. Similarly $K \cap M_0 = T$. Therefore $H \cap L_0 = K \cap M_0$, which implies $(H,M_0) \sim (K,L_0)$.

By Theorem 3.15, we know $(H,M) \sim (H,M_0)$. So, for some $g \in G$ $H^g \cap M_0 = H \cap M^g$, which implies $H \cap M_0 = H \cap M^g$. Thus $M_0 = M^g$. Likewise there is some $h \in G$ such that $L_0 = L^h$. So we have $H \cap L_0 = K \cap M_0$ which implies $H \cap L^h = K \cap M^g$. Conjugating by h^{-1} gives us $H^{h^{-1}} \cap (L^h)^{h^{-1}} = K^{h^{-1}} \cap (M^g)^{h^{-1}}$, which results in $H \cap L = K \cap M^{gh^{-1}}$. Then because $H \trianglelefteq G$, we have $H^{gh^{-1}} \cap L = K \cap M^{gh^{-1}}$ which implies $(H,M) \sim (K,L)$. ■

It is now clear that under $\sim$ the equivalence classes of good pairs in G_{27} are:

(1) $\{(G_{27}, G_{27})\}$

(2) $\{(G_{27},\langle xy^2,z\rangle)\}$

(3) $\{(G_{27},\langle xy,z\rangle)\}$

(4) $\{(G_{27},\langle y,z\rangle)\}$

(5) $\{(G_{27},\langle x,z\rangle)\}$

(6) $$\left\{\begin{array}{l}(\langle xy^2,z\rangle,\langle xy^2z^2\rangle),(\langle xy^2,z\rangle,\langle x^2y\rangle),(\langle xy^2,z\rangle,\langle xy^2\rangle),(\langle xy,z\rangle,\langle xyz^2\rangle),\\(\langle xy,z\rangle,\langle xyz\rangle),(\langle xy,z\rangle,\langle xy\rangle),(\langle y,z\rangle,\langle y\rangle),(\langle y,z\rangle,\langle yz\rangle),(\langle y,z\rangle,\langle yz^2\rangle),\\(\langle x,z\rangle,\langle x\rangle),(\langle x,z\rangle,\langle xz\rangle),(\langle x,z\rangle,\langle xz^2\rangle)\end{array}\right\}$$

Additionally, as we saw in Section 3.3, the good pairs in D_8 and Q_8 can be classified into the same two types that good pairs in extraspecial *p*-groups of exponent *p* can be classified. Thus we can use Lemmas 3.13 and 3.14 as a starting point for considering the equivalence classes of good pairs in these groups. We begin by looking at Q_8.

Recall that the good pairs in Q_8 are (Q_8,Q_8), $(Q_8,\langle i\rangle)$, $(Q_8,\langle j\rangle)$, $(Q_8,\langle k\rangle)$, $(\langle i\rangle,1)$, $(\langle j\rangle,1)$, and $(\langle k\rangle,1)$. So for a general good pair (H,M), Lemma 3.13 tells us that none of the pairs where $H=Q_8$ are related to the pairs where $H\neq Q_8$. Then, Lemma 3.14 tells us that each pair where $H=Q_8$ is related only to itself. So we need only consider relatedness among the good pairs where $H\neq Q_8$.

In this case, we have $\langle i\rangle\cap 1=\langle j\rangle\cap 1=\langle k\rangle\cap 1=1$. Thus we can clearly see that all good pairs in Q_8 with $H\neq Q_8$ are related. So our equivalence classes in Q_8 are:

(1) $\{(Q_8,Q_8)\}$

(2) $\{(Q_8,\langle i\rangle)\}$

(3) $\{(Q_8,\langle j\rangle)\}$

(4) $\{(Q_8,\langle k\rangle)\}$

(5) $\{(\langle i\rangle,1),(\langle j\rangle,1),(\langle k\rangle,1)\}$.

The example of D_8 is a bit more complicated than Q_8, but still quite reasonable to consider. Recall that the good pairs in D_8 are: (D_8,D_8), $(D_8,\langle r^2,s\rangle)$, $(D_8,\langle r\rangle)$, $(D_8,\langle r^2,rs\rangle)$, $(\langle r^2,s\rangle,\langle s\rangle)$, $(\langle r^2,s\rangle,\langle r^2s\rangle)$, $(\langle r^2,rs\rangle,\langle rs\rangle)$, $(\langle r^2,rs\rangle,\langle r^3s\rangle)$, and $(\langle r\rangle,1)$.

Again for a general good pair (H,M), Lemma 3.13 tells us that none of the pairs where $H=D_8$ are related to the pairs where $H\neq D_8$. Also Lemma 3.14 tells us that each pair where $H=D_8$ is related only to itself.

Since we know that for all $H\neq D_8$,

$$H\cap 1=\langle r\rangle\cap\langle s\rangle=\langle r\rangle\cap\langle rs\rangle=\langle r\rangle\cap\langle r^2s\rangle=\langle r\rangle\cap\langle r^3s\rangle=1$$

we can easily see that $(\langle r\rangle,1)$ is related to all good pairs (H,M) in D_8 where $H\neq D_8$.

Similarly, since

$$\langle r^2,s\rangle\cap\langle rs\rangle=\langle r^2,s\rangle\cap\langle r^3s\rangle=\langle r^2,rs\rangle\cap\langle s\rangle=\langle r^2,rs\rangle\cap\langle r^2s\rangle=1$$

it is clear that the good pairs in D_8 with $H=\langle r^2,s\rangle$ are related to the good pairs with $H=\langle r^2,rs\rangle$.

Finally, since $\langle s\rangle^r=\langle r^2s\rangle$ and $\langle rs\rangle^r=\langle r^3s\rangle$, i.e. for a good pair (H,M) in D_8 any two subgroups M of the same group $H\neq D_8$ are conjugates, we can see that any two good pairs with the same $H \neq D_8$ are related. Thus the equivalence classes of related good pairs in D_8 are:

(1) $\{(D_8, D_8)\}$

(2) $\{(D_8, \langle r^2, s\rangle)\}$

(3) $\{(D_8, \langle r\rangle)\}$

(4) $\{(D_8, \langle r^2, rs\rangle)\}$

(5) $\{(\langle r^2, s\rangle, \langle s\rangle), (\langle r^2, s\rangle, \langle r^2 s\rangle), (\langle r^2, rs\rangle, \langle rs\rangle), (\langle r^2, rs\rangle, \langle r^3 s\rangle), (\langle r\rangle, 1)\}$.

Note that in D_8 and Q_8 all good pairs in which H is the whole group are in their own equivalence class and there is exactly one equivalence class of good pairs (H, M) with H a proper subgroup. So the examples of D_8 and Q_8 seem to indicate that the results for related good pairs in extraspecial p-groups of exponent p^2 are the same as the results for related good pairs in extraspecial p-groups of exponent p. (And, in fact, from the character theory we know this is true.) Although these two examples by no means provide a proof of this fact.

CHAPTER 4

A GROUP THEORETIC PROOF

We begin this section by recalling that two good pairs (H,M) and (K,L) in a group G are *related in* G if there is some $g \in G$ such that $H^g \cap L = K \cap M^g$. In this case, we write $(H,M) \sim (K,L)$.

Now we consider the problem proposed by Parks [6] to find a group theoretic proof that $\sim$ is an equivalence relation. We begin by proving reflexivity and symmetry for all groups G.

(4.1) PROPOSITION. Let G be any group. Then the relation $\sim$ on good pairs is reflexive and symmetric.

Proof: Let (H,M) and (K,L) be good pairs in G. Let $g=1$. Then we have $H^g \cap M = H \cap M = H \cap M^g$ which implies $(H,M) \sim (H,M)$. So $\sim$ is reflexive. Now assume $(H,M) \sim (K,L)$. Then there is some $g \in G$ such that $H^g \cap L = K \cap M^g$. So we have $(H^g)^{g^{-1}} \cap L^{g^{-1}} = K^{g^{-1}} \cap (M^g)^{g^{-1}}$ and thus $K^{g^{-1}} \cap M = H \cap L^{g^{-1}}$. Therefore $(K,L) \sim (H,M)$. ■

(4.2) PROPOSITION. Let G be an abelian group. Then the relation $\sim$ on good pairs is transitive.

Proof: By Lemma 3.8 all good pairs in G are related only to themselves thus transitivity trivially holds. ■

So for any abelian group G, the relation $\sim$ on good pairs is an equivalence relation. We now shift our focus to non-abelian groups and prove that transitivity holds for extraspecial p-groups with exponent p.

(4.3) PROPOSITION. Let G be an extraspecial p-group with exponent p. Then the relation $\sim$ on good pairs is transitive.

Proof: As we found in Section 3.2 of this paper, we have two types of good pairs (H,M) to consider: (1) those where $H=G$, and (2) those where $H \neq G$. By Lemma 3.13 we know that no two good pairs are related unless they are of the same type. By Lemma 3.14 we know that any good pair of the form (G,M) is related only to itself, thus transitivity trivially holds. Finally, by Theorem 3.16 we know that any two good pairs of the form (H,M) and (K,L) where $H,K \neq G$ are related. Thus transitivity holds. ■

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Vita

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