Electronic Journal of Differential Equations, Vol. 2006(2006), No. 132, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

A NOTE ON STRONG RESONANCE PROBLEMS FOR P-LAPLACIAN

CHUNHUA JIN, YUANYUAN KE, JINGXUE YIN

Abstract. In this note, we study the existence of the weak solutions for the p-Laplacian with strong resonance, which generalizes the previous results in one-dimension.

1. Introduction

In a previous paper, Bouchala [1] studied the existence of the weak solutions of the nonlinear boundary-value problem for one-dimensional case

$$\begin{gathered} -\Delta_p u = \lambda |u|^{p-2} u + g(u) - h(x), \quad x \in (0, \pi), \\ u(0) = u(\pi) = 0, \end{gathered}$$

where $p > 1$, $\lambda \in \mathbb{R}$, $h \in L^{p'}(0,\pi)$ $(p' = \frac{p}{p-1})$, and $g : \mathbb{R} \to \mathbb{R}$ is a continuous and nonlinear function of the Landesman-Lazer type. By applying the variational approach, the author translated problem into a critical points problem, and proved the existence of critical points separately for situations

$$\lambda < \lambda_1, \quad \lambda_k < \lambda < \lambda_{k+1}, \quad \lambda = \lambda_k,$$

where $\{\lambda_k\}$ is the sequence of eigenvalues and satisfies $0 < \lambda_k < \lambda_{k+1}$. The results extended a previous result by J. Bouchala and P. Drábek [5], in which, they only considered the case of $\lambda = \lambda_1$, that is, λ is the first eigenvalue.

The researches on the existence of weak solutions for the resonance problem to p-Laplacian can also be found in the other papers, such as [2, 3] and the references therein. In [2], which examined resonance problems at arbitrary eigenvalues for the analogous ODE problem. However, in [3], the author not only generalized the results in [2] into higher-dimension, but also proved the existence of weak solutions for the case of $\lambda \in \mathbb{R}$, that is λ is not only an eigenvalue.

In this short note, we would like to point a fact that the existence results that J. Bouchala has proved in [1] are also true for the higher dimensional case. In fact,

---

2000 *Mathematics Subject Classification.* 35G30, 35A15, 35B38.
Key words and phrases. p-Laplacian equations; boundary value problem; eigenvalue; strong resonance problems.

Submitted March 21, 2006. Published October 17, 2006.
Supported by the NSFC, NSFGD-06300481, China Postdoctoral Science Foundation, and the Specific Foundation for Ph.D. Specialities of Educational Department of China.

by substituting the higher dimensional domain Ω for the one-dimensional interval $(0,\pi)$, we may consider the following boundary-value problem

$$\begin{gathered} -\Delta_p u = \lambda |u|^{p-2}u + g(u) - h(x), \quad x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $\lambda \in \mathbb{R}$, $N \geq 1$, $p > 1$, $g : \mathbb{R} \to \mathbb{R}$ is a continuous function, $h \in L^{p'}(\Omega)$ ($p' = \frac{p}{p-1}$), and Δ_p is the p-Laplacian operator, that is $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. Similar to [1], we say that $\lambda \in \mathbb{R}$ is an eigenvalue of $-\Delta_p$, if there exists a nonzero function $u \in W_0^{1,p}(\Omega)$, such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\,dx = \lambda \int_\Omega |u|^{p-2}uv\,dx \quad \text{for all } v \in W_0^{1,p}(\Omega).$$

The function u is called an eigenfunction of $-\Delta_p$ corresponding to the eigenvalue λ, and we denote it by

$$u \in \ker(-\Delta_p - \lambda)\backslash\{0\}.$$

For convenience, we first introduce some notation. Consider the functional $R : W_0^{1,p}(\Omega)\backslash\{0\} \to \mathbb{R}$,

$$R(u) = \frac{\int_\Omega |\nabla u|^p\,dx}{\int_\Omega |u|^p\,dx}, \quad u \in W_0^{1,p}(\Omega)\backslash\{0\},$$

and the manifold

$$\mathcal{S} = \{u \in W_0^{1,p}(\Omega) : \|u\|_{L^p(\Omega)} = 1\}.$$

For $k \in \mathbb{N}$, let

$$\mathcal{F}_k := \{\mathcal{A} \subset \mathcal{S} : \text{there exists a continuous odd surjection } h : \mathcal{S}_{k-1} \to \mathcal{A}\},$$

where $\mathcal{S}_{k-1}$ represents the unit sphere in $\mathbb{R}^k$. Let

$$\lambda_k = \inf_{\mathcal{A}\in\mathcal{F}_k} \sup_{u\in\mathcal{A}} R(u).$$

It is known that λ_k is an eigenvalue of $-\Delta_p$, and $0 < \lambda_k < \lambda_{k+1}$ (see [3, 4, 6]). Here, we denote the norm in $W_0^{1,p}(\Omega)$ by

$$\|u\| = \Big(\int_\Omega |\nabla u|^p\,dx\Big)^{1/p} \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

By Poincaré's inequality, we see that the norm $\|\cdot\|$ parallels to the usual definition. Furthermore, we denote

$$F(u) = \begin{cases} \frac{p}{u}\int_0^u g(s)ds - g(u), & u \neq 0, \\ (p-1)g(0), & u = 0, \end{cases} \tag{1.2}$$

and set

$$\begin{gathered} \overline{F(-\infty)} = \limsup_{u\to-\infty} F(u), \quad \underline{F(-\infty)} = \liminf_{u\to-\infty} F(u), \\ \overline{F(+\infty)} = \limsup_{u\to+\infty} F(u), \quad \underline{F(+\infty)} = \liminf_{u\to+\infty} F(u). \end{gathered}$$

Throughout this paper, we assume: (i)

$$\lim_{|t|\to\infty} \frac{g(t)}{|t|^{p-1}} = 0. \tag{1.3}$$

(ii) For any $v \in \ker(-\Delta_p - \lambda)\backslash\{0\}$,

$$(p-1)\int_\Omega h(x)v(x)\,dx < \underline{F(+\infty)}\int_\Omega v^+(x)\,dx + \overline{F(-\infty)}\int_\Omega v^-(x)\,dx, \tag{1.4}$$

or for every $v \in \ker(-\Delta_p - \lambda)\backslash\{0\}$,

$$(p-1)\int_\Omega h(x)v(x)\,dx > \overline{F(+\infty)}\int_\Omega v^+(x)\,dx + \underline{F(-\infty)}\int_\Omega v^-(x)\,dx, \tag{1.5}$$

where $v^+ = \max\{0, v\}$, $v^- = \min\{0, v\}$.

The following theorem is the main result of this note.

Theorem 1.1. *If* (1.3), (1.4) *(or* (1.5)*) hold, then problem* (1.1) *admits at least one weak solution.*

Remark 1.2. *If λ is not an eigenvalue of $-\Delta_p$, then* (1.4), (1.5) *are vacuously true.*

2. Proof of Main Result

To employ the variational approach, we introduce the functional

$$J_\lambda(u) := \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \frac{\lambda}{p}\int_\Omega |u|^p\,dx - \int_\Omega G(u)\,dx + \int_\Omega h(x)u(x)\,dx,$$

where $G(t) = \int_0^t g(s)ds$. Clearly, $J_\lambda \in C_1(W_0^{1,p}(\Omega);\mathbb{R})$, and for every $v \in W_0^{1,p}(\Omega)$,

$$\langle J'_\lambda(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\,dx - \lambda\int_\Omega |u|^{p-2}uv\,dx - \int_\Omega g(u)v\,dx + \int_\Omega hv\,dx.$$

Note that the weak solutions of (1.1) correspond to the critical points of J_λ.

To show that J_λ has critical points of saddle point type, we need a fundamental lemma as follows. (see [3] or [7])

Lemma 2.1 (Deformation Lemma)**.** *Suppose that J_λ satisfies the Palais-Smale condition, i.e. if $\{u_n\}$ is a sequence of functions in $W_0^{1,p}(\Omega)$ such that $\{J_\lambda(u_n)\}$ is bounded in $\mathbb{R}$, and $J'_\lambda(u_n) \to 0$ in $(W_0^{1,p}(\Omega))^*$, then $\{u_n\}$ has a subsequence that is strongly convergent in $W_0^{1,p}(\Omega)$. Let $c \in \mathbb{R}$ be a regular value of J_λ and let $\bar\varepsilon > 0$. Then there exists $\varepsilon \in (0, \bar\varepsilon)$ and a continuous one-parameter family of homeomorphisms, $\phi : W_0^{1,p}(\Omega) \times [0,1] \to W_0^{1,p}(\Omega)$ with the properties:*

(i) *If $t = 0$ or if $|J_\lambda(u) - c| \ge \bar\varepsilon$, then $\phi(u,t) = u$;*
(ii) *if $J_\lambda(u) \le c + \varepsilon$, then $J_\lambda(\phi(u,1)) \le c - \varepsilon$.*

The following lemma is a crucial step of our argument.

Lemma 2.2. *Assume* (1.3) *and* (1.4) *(or* (1.5)*) hold. Then the functional J_λ satisfies the Palais-Smale condition.*

Proof. Assume that $\{u_n\}$ is a sequence of functions in $W_0^{1,p}(\Omega)$, and there exists an positive constant M such that

$$|J_\lambda(u_n)| \le M, \tag{2.1}$$

$$J'_\lambda(u_n) \to 0 \qquad \text{in } (W_0^{1,p}(\Omega))^*. \tag{2.2}$$

In the following, we shall show that the Palais-Smale sequence $\{u_n\}$ is bounded. Suppose to the contrary (passing to the subsequence if necessary), namely

$$\|u_n\| \to +\infty.$$

Let $v_n := \frac{u_n}{\|u_n\|}$. Due to the reflexivity of $W_0^{1,p}(\Omega)$ and the compact embedding

$$W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega),$$

there exists $v \in W_0^{1,p}(\Omega)$ such that (passing to subsequences)

$$v_n \rightharpoonup v \quad \text{in } W_0^{1,p}(\Omega), \tag{2.3}$$

$$v_n \to v \quad \text{in } L^p(\Omega). \tag{2.4}$$

From (2.2) and (2.3), we have

$$\begin{aligned}
0 \leftarrow & \frac{\langle J'_\lambda(u_n), v_n - v\rangle}{\|u_n\|^{p-1}} \\
= & \int_\Omega |\nabla v_n|^{p-2}\nabla v_n(\nabla v_n - \nabla v)\,dx - \lambda\int_\Omega |v_n|^{p-2}v_n(v_n - v)\,dx \\
& - \int_\Omega \frac{g(u_n)}{\|u_n\|^{p-1}}(v_n - v)\,dx + \int_\Omega \frac{h}{\|u_n\|^{p-1}}(v_n - v)\,dx.
\end{aligned} \tag{2.5}$$

Since (1.3) and (2.4), it follows that the last three terms approach to 0 as $n \to \infty$. Then we have

$$\int_\Omega |\nabla v_n|^{p-2}\nabla v_n(\nabla v_n - \nabla v)\,dx \to 0.$$

Furthermore, we have

$$\begin{aligned}
0 &\leftarrow \int_\Omega |\nabla v_n|^{p-2}\nabla v_n(\nabla v_n - \nabla v)\,dx - \int_\Omega |\nabla v|^{p-2}\nabla v(\nabla v_n - \nabla v)\,dx \\
&= \int_\Omega |\nabla v_n|^p\,dx - \int_\Omega |\nabla v_n|^{p-2}\nabla v_n\nabla v\,dx - \int_\Omega |\nabla v|^{p-2}\nabla v\nabla v_n\,dx + \int_\Omega |\nabla v|^p\,dx \\
&\geq \|v_n\|^p - \|v_n\|^{p-1}\|v\| - \|v\|^{p-1}\|v_n\| + \|v\|^p \\
&= (\|v_n\|^{p-1} - \|v\|^{p-1})(\|v_n\| - \|v\|) \geq 0,
\end{aligned} \tag{2.6}$$

which implies

$$\|v_n\| \to \|v\|, \quad n \to \infty. \tag{2.7}$$

Noticing that $v_n \rightharpoonup v$ in $W_0^{1,p}(\Omega)$, and combining with the uniform convexity of $W_0^{1,p}(\Omega)$, we infer that

$$v_n \to v \quad \text{in } W_0^{1,p}(\Omega), \quad \|v\| = 1. \tag{2.8}$$

Moreover, for any $w \in W_0^{1,p}(\Omega)$, as $n \to \infty$,

$$\begin{aligned}
\frac{\langle J'_\lambda(u_n), w\rangle}{\|u_n\|^{p-1}} &= \int_\Omega |\nabla v_n|^{p-2}\nabla v_n\nabla w\,dx - \lambda\int_\Omega |v_n|^{p-2}v_n w\,dx \\
&\quad - \int_\Omega \frac{g(u_n)}{\|u_n\|^{p-1}}w\,dx + \int_\Omega \frac{h}{\|u_n\|^{p-1}}w\,dx \to 0.
\end{aligned}$$

Clearly the last two terms approach to zero. Hence for all $w \in W_0^{1,p}(\Omega)$:

$$\int_\Omega |\nabla v_n|^{p-2}\nabla v_n\nabla w\,dx - \lambda\int_\Omega |v_n|^{p-2}v_n w\,dx \to 0, \quad \text{as } n \to \infty, \tag{2.9}$$

which implies

$$\int_\Omega |\nabla v|^{p-2}\nabla v\nabla w\,dx = \lambda\int_\Omega |v|^{p-2}vw\,dx, \quad \forall\, w \in W_0^{1,p}(\Omega)$$

and $v \in \ker(-\Delta_p - \lambda)\backslash\{0\}$, $\|v\| = 1$. The boundedness of $\{J_\lambda(u_n)\}$, $J'_\lambda(u_n) \to 0$, and $\|u_n\| \to \infty$ imply

$$\begin{aligned}
0 \leftarrow & \frac{\langle J'_\lambda(u_n), u_n\rangle - pJ_\lambda(u_n)}{\|u_n\|} \\
&= \int_\Omega \frac{pG(u_n) - g(u_n)u_n}{\|u_n\|}\, dx - (p-1)\int_\Omega h\frac{u_n}{\|u_n\|}\, dx \\
&= \int_\Omega F(u_n)\frac{u_n}{\|u_n\|}\, dx - (p-1)\int_\Omega h\frac{u_n}{\|u_n\|}\, dx,
\end{aligned}$$

that is,

$$\lim_{n\to\infty}\int_\Omega F(u_n)\frac{u_n}{\|u_n\|}\, dx = (p-1)\int_\Omega hv\, dx. \tag{2.10}$$

Now we assume that (1.4) (the other case (1.5) can be treated similarly) holds. It follows that

$$\underline{F(+\infty)} > -\infty \quad \text{and} \quad \overline{F(-\infty)} < +\infty.$$

For arbitrary $\varepsilon > 0$, set

$$\begin{aligned}
c_\varepsilon &:= \begin{cases} \underline{F(+\infty)} - \varepsilon & \text{if } \underline{F(+\infty)} \in \mathbb{R}, \\ 1/\varepsilon & \text{if } \underline{F(+\infty)} = +\infty; \end{cases} \\
d_\varepsilon &:= \begin{cases} \overline{F(-\infty)} + \varepsilon & \text{if } \overline{F(-\infty)} \in \mathbb{R}, \\ -1/\varepsilon & \text{if } \overline{F(-\infty)} = -\infty. \end{cases}
\end{aligned}$$

Then for every $\varepsilon > 0$ there exists $K > 0$ such that

$$\begin{aligned}
F(t) &\geq c_\varepsilon \quad \text{for all } t > K, \\
F(t) &\leq d_\varepsilon \quad \text{for all } t < -K.
\end{aligned} \tag{2.11}$$

On the other hand, the continuity of F on $\mathbb{R}$ implies that for any $K > 0$ there exists $c(K) > 0$ such that

$$|F(t)| \leq c(K) \quad \text{for all } t \in [-K, K]. \tag{2.12}$$

Choose $\varepsilon > 0$ and consider the corresponding $K > 0$ and $c(K) > 0$ given by (2.11) and (2.12), respectively. Set

$$\int_\Omega F(u_n)\frac{u_n}{\|u_n\|}\, dx = A_{K,n} + B_{K,n} + C_{K,n} + D_{K,n} + E_{K,n}, \tag{2.13}$$

where

$$\begin{aligned}
A_{K,n} &= \int_{\{x\in\Omega: |u_n(x)| \leq K\}} F(u_n)\frac{u_n}{\|u_n\|}\, dx, \\
B_{K,n} &= \int_{\{x\in\Omega: u_n(x) > K, v(x) > 0\}} F(u_n)\frac{u_n}{\|u_n\|}\, dx, \\
C_{K,n} &= \int_{\{x\in\Omega: u_n(x) > K, v(x) \leq 0\}} F(u_n)\frac{u_n}{\|u_n\|}\, dx, \\
D_{K,n} &= \int_{\{x\in\Omega: u_n(x) < -K, v(x) < 0\}} F(u_n)\frac{u_n}{\|u_n\|}\, dx, \\
E_{K,n} &= \int_{\{x\in\Omega: u_n(x) < -K, v(x) \geq 0\}} F(u_n)\frac{u_n}{\|u_n\|}\, dx.
\end{aligned}$$

Before estimating these integrals we claim that for any $K > 0$ the following assertions are true, since that $\|u_n\| \to +\infty$ and $u_n/\|u_n\| \to v$ in $W_0^{1,p}(\Omega)$ as $n \to \infty$.

$$\lim_{n\to\infty} \int_{\{x\in\Omega: u_n(x)\le K, v(x)>0\}} v_n\,dx = 0, \tag{2.14}$$

$$\lim_{n\to\infty} \int_{\{x\in\Omega: u_n(x)> K, v(x)\le 0\}} v_n\,dx = 0, \tag{2.15}$$

$$\lim_{n\to\infty} \int_{\{x\in\Omega: u_n(x)\ge -K, v(x)<0\}} v_n\,dx = 0, \tag{2.16}$$

$$\lim_{n\to\infty} \int_{\{x\in\Omega: u_n(x)< -K, v(x)\ge 0\}} v_n\,dx = 0. \tag{2.17}$$

In fact, for the first equality (2.14), we have

$$\begin{aligned}
&\lim_{n\to\infty} \int_{\{x\in\Omega: u_n(x)\le K, v(x)>0\}} v_n\,dx \\
&= \lim_{n\to\infty} \int_{\{x\in\Omega: u_n(x)< -K, v(x)>0\}} v_n\,dx + \lim_{n\to\infty} \int_{\{x\in\Omega: -K\le u_n(x)\le K, v(x)>0\}} v_n\,dx \\
&= \lim_{n\to\infty} \int_{\{x\in\Omega: u_n(x)< -K, v(x)>0\}} v_n\,dx \le 0.
\end{aligned}$$

Moreover, since $v_n \to v$ in $L^p(\Omega)$, it follows that

$$\int_{\{x\in\Omega: u_n(x)< -K, v(x)>0\}} |v_n - v|\,dx \le |\Omega|^{1-1/p}\|v_n - v\|_{L^p} \to 0, \quad \text{as } n\to\infty,$$

which implies

$$0 \ge \lim_{n\to\infty} \int_{\{x\in\Omega: u_n(x)< -K, v(x)>0\}} v_n\,dx = \lim_{n\to\infty} \int_{\{x\in\Omega: u_n(x)< -K, v(x)>0\}} v\,dx \ge 0,$$

and so proves the limit equality (2.14). For the other three equalities (2.15)–(2.17), the proofs are similar and we omit the details. Furthermore, have

$$\begin{gathered}
|A_{K,n}| \le \frac{Kc(K)|\Omega|}{\|u_n\|} \to 0, \\
\begin{aligned}
B_{K,n} &\ge c_\varepsilon\Big(\int_{\{x\in\Omega: v(x)>0\}} v_n\,dx - \int_{\{x\in\Omega: u_n(x)\le K, v(x)>0\}} v_n\,dx\Big) \\
&\to c_\varepsilon \int_{\{x\in\Omega: v(x)>0\}} v\,dx,
\end{aligned} \\
C_{K,n} \ge c_\varepsilon \int_{\{x\in\Omega: u_n(x)>K, v(x)\le 0\}} v_n\,dx \to 0, \\
\begin{aligned}
D_{K,n} &\ge d_\varepsilon\Big(\int_{\{x\in\Omega: v(x)<0\}} v_n\,dx - \int_{\{x\in\Omega: u_n(x)\ge -K, v(x)<0\}} v_n\,dx\Big) \\
&\to d_\varepsilon \int_{\{x\in\Omega: v(x)<0\}} v\,dx,
\end{aligned} \\
E_{K,n} \ge d_\varepsilon \int_{\{x\in\Omega: u_n(x)<-K, v(x)\ge 0\}} v_n\,dx \to 0.
\end{gathered}$$

Recalling (2.13), for $\varepsilon > 0$, we obtain

$$\begin{aligned}
&\liminf \int_\Omega F(u_n)\frac{u_n}{\|u_n\|}\,dx \\
&= \liminf(A_{K,n} + B_{K,n} + C_{K,n} + D_{K,n} + E_{K,n}) \\
&\geq c_\varepsilon \int_{\{x\in\Omega: v(x)>0\}} v(x)\,dx + d_\varepsilon \int_{\{x\in\Omega: v(x)<0\}} v(x)\,dx.
\end{aligned}$$

By the definition of c_ε and d_ε together with (2.10) and the above inequality, we conclude that

$$(p-1)\int_\Omega h(x)v(x)\,dx \geq \underline{F(+\infty)}\int_\Omega v^+(x)\,dx + \overline{F(-\infty)}\int_\Omega v^-(x)\,dx,$$

clearly which contradicts (1.4), and so we complete the proof of the boundedness of $\{u_n\}$.

Since $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, then there exists $u \in W_0^{1,p}(\Omega)$, such that (passing to subsequences)

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega), \quad u_n \to u \quad \text{in } L^p(\Omega). \tag{2.18}$$

Taking (2.2) and (1.3) into account, it follows that

$$\begin{aligned}
0 &= \lim\langle J'_\lambda(u_n), u_n - u\rangle \\
&= \lim \int_\Omega |\nabla u_n|^{p-2}\nabla u_n(\nabla u_n - \nabla u)\,dx - \lambda\int_\Omega |u_n|^{p-2}u_n(u_n-u)\,dx \\
&\quad - \int_\Omega g(u_n)(u_n-u)\,dx + \int_\Omega h(u_n-u)\,dx.
\end{aligned}$$

Recalling (1.3) and combining with the continuity of $g(t)$, we have that for any $\varepsilon > 0$, there exists $M > 0$, such that $|g(u_n)| \leq M + \varepsilon|u_n|^{p-1}$, which together with (2.18) yield that the last three terms goes to zero, and

$$\lim \int_\Omega |\nabla u_n|^{p-2}\nabla u_n(\nabla u_n - \nabla u)\,dx = 0.$$

Similar to (2.6), we obtain $\|u_n\| \to \|u\|$. The uniform convexity of $W_0^{1,p}(\Omega)$ then yields $u_n \to u$ in $W_0^{1,p}(\Omega)$, which complete the proof. □

Next, we prove the main theorem. As in [1], we divide it into three lemmas for different cases separately:

$$\lambda < \lambda_1, \quad \lambda_k < \lambda < \lambda_{k+1}, \quad \lambda = \lambda_k.$$

Lemma 2.3. *Assume* (1.3) *holds, and* $\lambda < \lambda_1$. *Then* (1.1) *admits at least one weak solution.*

Proof. By the definition of $J_\lambda(u)$ and the assumption on $g(t)$, for any $\varepsilon > 0$ we have

$$\begin{aligned}
J_\lambda(u) &= \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \frac{\lambda}{p}\int_\Omega |u|^p\,dx - \int_\Omega G(u)\,dx + \int_\Omega h(x)u(x)\,dx \\
&\geq \frac{\lambda_1-\lambda}{p}\int_\Omega |u|^p\,dx - C\int_\Omega |u|\,dx - \frac{\varepsilon}{p}\int_\Omega |u|^p\,dx - \int_\Omega |h(x)u(x)|\,dx \\
&\geq \frac{\lambda_1-\lambda-\varepsilon}{p}\|u\|^p_{L^p(\Omega)} - C\|u\|_{L^1(\Omega)} - \|h\|_{L^{p'}}\|u\|_{L^p(\Omega)},
\end{aligned}$$

which implies that the functional J_λ is bounded from below on $W_0^{1,p}(\Omega)$. Moreover, from Lemma 2.2, we have J_λ satisfies the Palais-Smale condition. Hence J_λ attains its global minimum on $W_0^{1,p}(\Omega)$. $\square$

Lemma 2.4. *Assume* (1.3), (1.4) *(or* (1.5)*) hold, and there exists* $k \in \mathbb{N}$ *such that* $\lambda_k < \lambda < \lambda_{k+1}$. *Then* (1.1) *admits at least one weak solution.*

Proof. Let $m \in (\lambda_k, \lambda)$, and let $\mathcal{A} \in \mathcal{F}_k$, such that $\sup_{u\in\mathcal{A}} R(u) \le m$. Then for all $u \in \mathcal{A}$, $t > 0$ and all $\varepsilon > 0$, by (1.3) there exists $c > 0$, such that

$$\begin{aligned} J_\lambda(tu) &= \frac{1}{p}t^p\Big(\int_\Omega |\nabla u|^p\,dx - \lambda\int_\Omega |u|^p\,dx\Big) - \int_\Omega G(tu)\,dx + t\int_\Omega h(x)u(x)\,dx \\ &\le \frac{1}{p}t^p(m-\lambda)\|u\|^p_{L^p(\Omega)} + ct\|u\|_{L^1(\Omega)} + \frac{\varepsilon}{p}t^p\|u\|^p_{L^p(\Omega)} + t\|h\|_{L^{p'}(\Omega)}\|u\|_{L^p(\Omega)} \\ &= \frac{1}{p}t^p(m-\lambda+\varepsilon)\|u\|^p_{L^p(\Omega)} + t(c\|u\|_{L^1(\Omega)} + \|h\|_{L^{p'}(\Omega)}\|u\|_{L^p(\Omega)}). \end{aligned}$$

Clearly,

$$\lim_{t\to+\infty} J_\lambda(tu) = -\infty \quad \text{uniformly for any } \ u \in \mathcal{A}. \tag{2.19}$$

Now let

$$\varepsilon_{k+1} := \{u \in W_0^{1,p}(\Omega);\ \int_\Omega |\nabla u|^p\,dx \ge \lambda_{k+1}\int_\Omega |u|^p\,dx\}.$$

By noting that for all $u \in \varepsilon_{k+1}$, and all $\varepsilon > 0$, there exists $c > 0$, such that

$$J_\lambda(u) \ge \frac{1}{p}(\lambda_{k+1} - \lambda - \varepsilon)\|u\|^p_{L^p(\Omega)} - c\|u\|_{L^1(\Omega)} - \|h\|_{L^{p'}(\Omega)}\|u\|_{L^p(\Omega)}.$$

Hence $J_\lambda(u)$ is bounded from below in ε_{k+1}. Let

$$\alpha = \inf_{u\in\varepsilon_{k+1}} J_\lambda(u). \tag{2.20}$$

From (2.19) and (2.20), we see that there exists $T > 0$ such that

$$\gamma := \max\{J_\lambda(tu);\ u \in \mathcal{A},\ t \ge T\} < \alpha.$$

Define

$$\begin{gathered} T\mathcal{A} := \{tu \in W_0^{1,p}(\Omega);\ u \in \mathcal{A},\ t \ge T\}, \\ \Gamma := \{h \in C^0(B_k, W_0^{1,p}(\Omega));\ h|_{\mathcal{S}_{k-1}} \to T\mathcal{A} \text{ is an odd map}\}, \end{gathered}$$

where B_k is a unit ball centered at the origin in $\mathbb{R}^k$. Then we see that Γ is nonempty. In fact, recalling the definition of $\mathcal{F}_k$, we see that there exists a continuous odd surjection $h : \mathcal{S}_{k-1} \to \mathcal{A}$. Define

$$\begin{gathered} \overline{h} : B_k \to W_0^{1,p}(\Omega), \\ \overline{h}(tx) = tTh(x) \quad \text{for } x \in \mathcal{S}_{k-1},\ t \in [0,1]. \end{gathered}$$

Obviously, $\overline{h} \in \Gamma$. Furthermore, if $h \in \Gamma$, then

$$h(B_k) \cap \varepsilon_{k+1} \ne \phi. \tag{2.21}$$

In fact, if $0 \in h(B_k)$, then (2.21) holds clearly. Otherwise, considering the mapping $\widetilde{h} : \mathcal{S}_k \to \mathcal{S}$,

$$\widetilde{h}(x_1,\dots,x_{k+1}) = \begin{cases} \pi\cdot h(x_1,\dots,x_k), & x_{k+1} \ge 0, \\ -\pi\cdot h(-x_1,\dots,-x_k), & x_{k+1} < 0, \end{cases}$$

where π represents radial projection onto $\mathcal{S}$ in $W_0^{1,p}(\Omega)\backslash\{0\}$, clearly, we have $\widetilde{h}(\mathcal{S}_k) \in \mathcal{F}_{k+1}$. From the definition of λ_{k+1}, we see that

$$\sup_{u\in\widetilde{h}(\mathcal{S}_k)} R(u) \geq \lambda_{k+1},$$

which implies that there exists $u = \pi \cdot h(x) \in \widetilde{h}(\mathcal{S}_k)$ such that $R(u) \geq \lambda_{k+1}$. That is $u = \pi \cdot h(x) \in \varepsilon_{k+1}$, which also implies that $h(\bar{x}) \in \varepsilon_{k+1}$, where $\bar{x} = x/\|x\|$. Thus $h(B_k) \cap \varepsilon_{k+1} \neq \phi$.

Moreover, recalling the Deformation Lemma, we see that

$$C = \inf_{h\in\Gamma} \sup_{x\in B_k} J_\lambda(h(x))$$

is a critical value of J_λ. In fact, we assume by contradiction that C is a regular value of J_λ, from $h(B_k) \cap \varepsilon_{k+1} \neq \phi$, it is easy to see that $C \geq \alpha > \gamma$. Let $\overline{\varepsilon}$ be an arbitrary given constant in $(0, C-\gamma)$. By the definition of C, for any $\varepsilon \in (0, \overline{\varepsilon})$, there exists a corresponding $h \in \Gamma$, such that

$$\sup_{x\in B_k} J_\lambda(h(x)) < C + \varepsilon.$$

Then by the Deformation Lemma, there exists ε and a corresponding $\varphi : W_0^{1,p}(\Omega) \times [0,1] \to W_0^{1,p}(\Omega)$ such that

$$J_\lambda(\varphi(h,1)) \leq C - \varepsilon.$$

For any $x \in \mathcal{S}_{k-1}$, $h(x) \in T\mathcal{A}$,

$$J_\lambda(h(x)) < \gamma < C - \overline{\varepsilon}.$$

Hence, $\varphi(h,1) = h \in \Gamma$, which contradicts the definition of C. $\square$

Lemma 2.5. *Let us assume* (1.3), (1.4) *or (*(1.5)*), and there exists $k \in \mathbb{N}$ such that $\lambda = \lambda_k$. Then* (1.1) *admits at least one weak solution.*

Proof. We split the proof into several steps, in the first step, we show the case of (1.4), then the second step is devoted to the case of (1.5).

Step 1. Assume (1.4). Take sequence $\{\mu_n\}$ with $\lambda_k < \mu_n < \lambda_{k+1}$ and $\mu_n \searrow \lambda_k$. By means of Lemma 2.4, there exists a sequence $\{u_n\}$ of critical points associated with the functional $\{J_{\mu_n}\}$ such that

$$C_n = J_{\mu_n}(u_n) \geq \alpha_n := \inf\{J_{\mu_n}(u) : u \in \varepsilon_{k+1}\}.$$

For all $u \in \varepsilon_{k+1}$,

$$\begin{aligned} J_{\mu_n}(u) &= \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \frac{\mu_n}{p}\int_\Omega |u|^p\,dx - \int_\Omega G(u)\,dx + \int_\Omega h(x)u(x)\,dx \\ &\geq \frac{1}{p}(\lambda_{k+1} - \mu_n - \varepsilon)\|u\|^p_{L^p}(\Omega) - C\|u\|_{L^1(\Omega)} - \|h\|_{L^{p'}}\|u\|_{L^p}, \end{aligned}$$

which implies that C_n is bounded from below uniformly.

In the following, we pay our attention to the boundedness of the corresponding sequence of critical points $\{u_n\}$. Suppose to the contrary, there exists a subsequence of $\{u_n\}$, for simplify, we might as well assume to be itself, such that $\|u_n\| \to \infty$.

Similar to Lemma 2.2, we can show that there exists $v \in \ker(-\Delta_p - \lambda_k)\backslash\{0\}$, such that (up to subsequence) $\frac{u_n}{\|u_n\|} \to v$. Since C_n is bounded from below, then we have

$$\begin{aligned}
0 &\le \liminf \frac{pC_n}{\|u_n\|} \le \limsup \frac{pC_n}{\|u_n\|} \\
&= \limsup \frac{pJ_{\mu_n}(u_n) - \langle J'_{\mu_n}(u_n), u_n\rangle}{\|u_n\|} \\
&= \limsup \Big(- \frac{p\int_\Omega G(u_n)\,dx - \int_\Omega g(u_n)u_n\,dx}{\|u_n\|} + (p-1)\int_\Omega hv_n\,dx\Big) \\
&= -\liminf \Big(\int_\Omega F(u_n)\frac{u_n}{\|u_n\|}\,dx\Big) + (p-1)\int_\Omega hv\,dx.
\end{aligned}$$

Similar to Lemma 2.2, we obtain

$$\underline{F(+\infty)}\int_\Omega v^+(x)\,dx + \overline{F(-\infty)}\int_\Omega v^-(x)\,dx \le (p-1)\int_\Omega h(x)v(x)\,dx,$$

which contradicts to the assumption (1.4), that is $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Thus, there exists $u \in W_0^{1,p}(\Omega)$, such that (passing to subsequence)

$$u_n \rightharpoonup u \text{ in } W_0^{1,p}(\Omega), \quad u_n \to u \quad \text{in } L^p(\Omega).$$

Therefore,

$$\begin{aligned}
0 &= \lim_{n\to\infty} \langle J'_{\mu_n}(u_n), u_n - u\rangle \\
&= \lim_{n\to\infty} \int_\Omega |\nabla u_n|^{p-2}\nabla u_n(\nabla u_n - \nabla u)\,dx - \mu_n\int_\Omega |u_n|^{p-2}u_n(u_n-u)\,dx \\
&\quad - \int_\Omega g(u_n)(u_n-u)\,dx + \int_\Omega h(u_n-u)\,dx \\
&= \lim_{n\to\infty} \int_\Omega |\nabla u_n|^{p-2}\nabla u_n(\nabla u_n - \nabla u)\,dx.
\end{aligned}$$

Recalling Hölder's inequality, we conclude that

$$\begin{aligned}
0 &\leftarrow \int_\Omega |\nabla u_n|^{p-2}\nabla u_n(\nabla u_n - \nabla u)\,dx - \int_\Omega |\nabla u|^{p-2}\nabla u(\nabla u_n - \nabla u)\,dx \\
&= \int_\Omega |\nabla u_n|^p\,dx - \int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla u\,dx - \int_\Omega |\nabla u|^{p-2}\nabla u\nabla u_n\,dx + \int_\Omega |\nabla u|^p\,dx \\
&\ge \|u_n\|^p - \|u_n\|^{p-1}\|u\| - \|u\|^{p-1}\|u_n\| + \|u\|^p \\
&= (\|u_n\|^{p-1} - \|u\|^{p-1})(\|u_n\| - \|u\|) \ge 0,
\end{aligned}$$

which implies that $\|u_n\| \to \|u\|$. The uniform convexity of $W_0^{1,p}(\Omega)$ yields

$$u_n \to u \quad \text{in } \ W_0^{1,p}(\Omega).$$

Considering the sequence $\{J_{\mu_n}(u_n)\}$ (passing to a subsequence if necessary), letting $n \to \infty$, and combining with the Lebesgue dominated convergence theorem, we finally arrive at

$$J_{\mu_n}(u_n) \to J_{\lambda_k}(u) = C \text{ and } \ J'_{\lambda_k}(u) = 0,$$

which implies that u is a critical point of J_{λ_k}.

Step 2. Next, we transfer our attention to the case of (1.5). First of all, we consider the case of $k = 1$. Take sequence $\{\mu_n\}$ with $0 < \mu_n < \lambda_1$ and $\mu_n \nearrow \lambda_1$. We can find a sequence $\{u_n\}$ of critical points associated with the functional $\{J_{\mu_n}\}$

such that $C_n = J_{\mu_n}(u_n)$ is decreasing. Now we are going to show $\{u_n\}$ is bounded. Suppose, by contradiction, $\|u_n\| \to \infty$, then there exists $v \in \ker(-\Delta_p - \lambda_1)\backslash\{0\}$ such that (up to a subsequence) $u_n/\|u_n\| \to v$, and

$$\begin{aligned}
0 &\geq \limsup \frac{pC_n}{\|u_n\|} \\
&\geq \liminf \frac{pC_n}{\|u_n\|} \\
&= \liminf \frac{pJ_{\mu_n}(u_n) - \langle J'_{\mu_n}(u_n), u_n\rangle}{\|u_n\|} \\
&= \liminf \Big(- \frac{p\int_\Omega G(u_n)\,dx - \int_\Omega g(u_n)u_n\,dx}{\|u_n\|} + (p-1)\int_\Omega h\frac{u_n}{\|u_n\|}\,dx\Big) \\
&= -\limsup \Big(\frac{p\int_\Omega G(u_n)\,dx - \int_\Omega g(u_n)u_n\,dx}{\|u_n\|}\Big) + (p-1)\int_\Omega hv\,dx \\
&= -\limsup \Big(\int_\Omega F(u_n)\frac{u_n}{\|u_n\|}\,dx\Big) + (p-1)\int_\Omega hv\,dx > 0,
\end{aligned}$$

which is a contradiction. The following argument is completely parallel to Step 1, so we omit it.

In the following, we focus on the case of $k > 1$. Let $\{\mu_n\}$ be a sequence in $(\lambda_{k-1}, \lambda_k)$ with $\mu_n \nearrow \lambda_k$. We can find a sequence $\{u_n\}$ of critical points associated with the functional $\{J_{\mu_n}\}$ such that $C_n = J_{\mu_n}(u_n)$ is decreasing. Then we obtain that $\{u_n\}$ is bounded. Suppose, by contradiction, $\|u_n\| \to \infty$, then there exists $v \in \ker(-\Delta_p - \lambda_k)\backslash\{0\}$ such that (up to subsequence) $\frac{u_n}{\|u_n\|} \to v$, and

$$\begin{aligned}
0 &\geq \limsup \frac{pC_n}{\|u_n\|} \\
&\geq \liminf \frac{pC_n}{\|u_n\|} \\
&= \liminf \frac{pJ_{\mu_n}(u_n) - \langle J'_{\mu_n}(u_n), u_n\rangle}{\|u_n\|} \\
&= \liminf \Big(- \frac{p\int_\Omega G(u_n)\,dx - \int_\Omega g(u_n)u_n\,dx}{\|u_n\|} + (p-1)\int_\Omega h\frac{u_n}{\|u_n\|}\,dx\Big) \\
&= -\limsup \Big(\frac{p\int_\Omega G(u_n)\,dx - \int_\Omega g(u_n)u_n\,dx}{\|u_n\|}\Big) + (p-1)\int_\Omega hv\,dx \\
&= -\limsup \Big(\int_\Omega F(u_n)\frac{u_n}{\|u_n\|}\,dx\Big) + (p-1)\int_\Omega hv\,dx > 0,
\end{aligned}$$

which is a contradiction. The remaining argument is quite simple, similar to the above discussion, and so we omit it here. □

Proof of Theorem 1.1. Combining Lemma 2.3 – Lemma 2.5, Theorem 1.1 holds clearly. The proof is complete. □

References

[1] J. Bouchala; *Strong resonance problems for the one-dimensional p-Laplacian*, Elec. J. Diff. Equ. **2005** (2005), 1–10.

[2] P. Drábek, S. B. Robinson; *Resonance problems for the one-dimensional p-Laplacian*, Pro. Amer. Math. Soc. **128** (1999), 755–765.

[3] P. Drábek, S. B. Robinson; *Resonance problems for the p-Laplacian*, J. Func. Anal. **169** (1999), 189–200.
[4] W. Walter; *Sturm-Liouville theory for the radial Δ_p-operator*, Math. Z. **227** (1998), 175–185.
[5] J. Bouchala, P. Drábek; *Strong Resonance for Some Quasilinear Elliptic Equations*, J. Math. Anal. Appl. **245** (2000), 7–19.
[6] B. Xuan; *The eigenvalue problem for a singular quasilinear elliptic equation*, Elec. J. Diff. Equ. **2004** (2004), 1–11.
[7] M. Struwe; *"Variational Methods; Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems"*, Springer-Verlag, New York, 1990.

Chunhua Jin
Department of Applied Mathematics, Jilin University, Changchun 130012, China
E-mail address: diffusion@jlu.edu.cn

Yuanyuan Ke
Department of Mathematics, Sun Yat-Sen University, Guangzhou 510275, China.
Department of Applied Mathematics, Jilin University, Changchun 130012, China
E-mail address: keyy@jlu.edu.cn (corresponding author)

Jingxue Yin
Department of Applied Mathematics, Jilin University, Changchun 130012, China
E-mail address: yjx@jlu.edu.cn