Electronic Journal of Differential Equations, Vol. 2004(2004), No. 78, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

GENERIC SOLVABILITY FOR THE 3-D NAVIER-STOKES EQUATIONS WITH NONREGULAR FORCE

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Abstract. We show that the existence of global strong solutions for the Navier-Stokes equations with nonregular force is generically true. Similar results for equations without the nonregular force have been obtained by Fursikov [5]. Our main tools are the Galerkin method and estimates on its solutions.

1. Introduction

We are interested in the generic solvability for the 3-dimensional Navier-Stokes equations with nonregular force on the periodic domain $\mathbb{T}^3 \times [0, \infty)$.

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \nu \Delta u + f + \frac{\partial g}{\partial t}, \tag{1.1}$$

$$\operatorname{div} u = 0, \tag{1.2}$$

$$u(x, 0) = u_0(x), \tag{1.3}$$

where u is the fluid velocity vector field, p is the scalar pressure, ν is the positive viscosity constant and $f + \frac{\partial g}{\partial t}$ s the external force. u_0 is a given initial data. The nonregular part is denoted by $\frac{\partial g}{\partial t}$. We assume $g \in C([0, \infty); V^2)$ which means $g(t)$ is a continuous function in V^2, where the space V^2 is defined below. Since we only consider the periodic domain $\mathbb{T}^3 = [0, 2\pi]^3$, every function can be regarded as a periodic vector field with period 2π, i.e., $u(x_1 + 2\pi, x_2, x_3) = u(x_1, x_2, x_3)$, etc. For this above Navier-Stokes equations with nonregular force, the existence of the weak solution was shown in [4].

Recently, Flandoli and Romito[3] proved the paths of a martingale suitable weak solution for the Navier-Stokes equations with nonregular force have a set of singular points of one-dimensional Hausdorff measure zero. Also the stochastic Navier-Stokes equations have been intensively studied by many authors (see [1], [2], [6] and references therein). One of the most important problems in nonlinear partial differential equations is to show existence of global strong solution for three-dimensional Navier-Stokes equations or to construct an example of the finite blow-up of the solution for the three-dimensional Navier-Stokes equations. Although, it is still far

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2000 *Mathematics Subject Classification.* 35A05, 76N10.
Key words and phrases. Stochastic Navier-Stokes equations, generic solvability.

Submitted February 24, 2004. Published June 4, 2004.
Supported by the post-doctoral Fellowship Program of KOSEF.

from being proved the global existence, it is known to be generically true for (1.1)–(1.3) without the nonregular part $\frac{\partial g}{\partial t}$ (see [5] and [7]). In this paper, we show that generic solvability is still true even with the nonregular force.

We assume f and g are divergence free vector fields for simplicity. In the following, we consider the Banach spaces $L^p(0,T;B)$ for any Banch spaces B, i.e. we say $f \in L^p(0,T;B)$ if and only if $|f|_{L^p(0,T;B)} < \infty$ (we use the same notation for the Banach space of 3-dimensional vector fields with the Banach space for scalar valued function for simplicity). We denote $\cup_{0<T<\infty} L^p(0,T;B)$ is denoted by $L^p_{\rm loc}(0,\infty;B)$. Let H^m be the usual Sobolev space(see [7]). Following the notation in [7], we denote the space of L^2 divergence-free vector fields by H, the space of H^m divergence free vector fields by V^m (V^1 will be denoted by V for simplicity and convention). We define the projection operator $P_{\rm div}$ as the projection to the divergence free vector fields.

Note that $\{\vec{e}_i e^{ik\cdot x} \,|\, i=1,2,3, k\in\mathbb{Z}^3\}$, where $\vec{e}_i$ is an i-th standard unit vector, is a complete orthonormal basis for L^2. Hence projection operator, $P_{\rm div}$, is defined as

$$P_{\rm div}(\vec{a}e^{ik\cdot x}) = \Big(\vec{a} - \frac{k\otimes k}{|k|}\cdot\vec{a}\Big)e^{ik\cdot x}.$$

Let $\alpha_i(k) = |P_{\rm div}(\vec{e}_i e^{ik\cdot x})|_{L^2}$. Hence $K = \big\{\frac{1}{\alpha_i(k)}P_{\rm div}(\vec{e}_i e^{ik\cdot x}) : k\in\mathbb{Z}^3,\ i=1,2,3\big\}$ is a complete orthonormal basis for H. Define $B(u,v) = -P_{\rm div}(u\cdot\nabla)v$, $\Lambda^2 u = -P_{\rm div}\Delta u$, where $P_{\rm div}$ is the L^2 projection operator as above. By projecting (1.1)–(1.2) to the divergence-free vector field, we obtain

$$\frac{du}{dt} + \nu\Lambda^2 u(t) - B(u,u) = f(t) + \frac{dg(t)}{dt}. \tag{1.4}$$

For the three-dimensional Navier-Stokes equations with regular force, Fursikov[5] and Temam [7] proved that for any initial data $u_0\in V$, there exists a set F which is included in $L^2(0,T;H)$ and dense in $L^q(0,T;V')$ with $1\le q<\frac{4}{3}$ (V' is the dual space of V), such that for every external force $f\in F$, the equations have a unique strong solution u. Using methods similar to those developed in [4], [5] and [7], we obtain the following generic existence and uniqueness of the Navier-Stokes equations with nonregular force.

Theorem 1.1. *Assume that the initial data is $u_0\in V$. We also assume that $f\in L^2_{\rm loc}(0,\infty;H)$. There exist $f_m\in L^2_{\rm loc}(0,\infty;H)$ satisfying $f_m\to f$ in $L^q_{\rm loc}(0,\infty;L^{6/5})$ for all q satisfying $1\le q<\frac{4}{3}$ such that (1.4) corresponding to u_0 and f_m possesses a unique strong solution in $L^\infty_{\rm loc}(0,\infty;V)\cap L^2_{\rm loc}(0,\infty;V^2)$.*

We remark that since $L^{6/5}\subset V'$, Theorem 1.1 can be regarded as a slight generalization of the results in [7].

2. Proof of main Theorem

For the proof of Theorem 1.1, we use the methods developed in [4], [5], and [7]. First, we consider the following Galerkin approximation of the system (1.1)–(1.2).

$$\begin{gathered}\frac{du_m}{dt} + \nu\Lambda^2 u_m - P_m B(u_m,u_m) = P_m f + P_m\frac{dg}{dt},\\ u_m(0) = P_m u_0.\end{gathered} \tag{2.1}$$

The projection onto the space spanned by $\{\frac{1}{\alpha_i(k)}P_{\rm div}(\vec{e}_i e^{ik\cdot x})\,|\,|k|\le m\}$ is denoted by P_m. Note that (2.1) is equivalent to the integral equation

$$\begin{aligned} &u_m(t)+\nu\int_0^t \Lambda^2 u_m(s)ds-\int_0^t P_m B(u_m(s),u_m(s))ds\\ &=u_m(0)+\int_0^t P_m f(s)ds+P_m g(t). \end{aligned}\tag{2.2}$$

Using contraction mapping argument, we can show, local in time, existence of solution u_m for (2.2). Following the argument given in [4], we consider the following auxiliary equation for given $z_0\in V$.

$$z_m(t)+\nu\int_0^t \Lambda^2 z_m(s)ds=P_m z_0+P_m g(t),\quad t\ge 0.\tag{2.3}$$

This equation has a unique solution, which is continuous with values in V and global in time. We show z_m converges in $C([0,\infty);V)\cap L^2_{\rm loc}(0,\infty;V^2)$. (2.3) is equivalent to

$$z_m(t)=e^{-\nu t\Lambda^2}P_m z_0+P_m g(t)-\nu\int_0^t \Lambda^2 e^{-\nu(t-s)\Lambda^2}P_m g(s)ds.$$

Note that for $c(\gamma)=\max_{x\ge 0}x^{2\gamma}e^{-x}$, we have

$$\|\Lambda^{2\gamma}e^{-t\Lambda^2}\|_{L(H)}\le c(\gamma)\frac{e^{-\frac{t}{2}\lambda_1}}{t^\gamma},\tag{2.4}$$

where $\|F\|_{L(H)}=\sup_{\|f\|_H\le 1}\|F(f)\|_H$ and λ_1 is the smallest eigenvalue of Λ^2. Hence

$$\Lambda z_m(t)=e^{-\nu t\Lambda^2}\Lambda P_m z_0+\Lambda P_m g-\nu\int_0^t \Lambda^{2-2\epsilon}e^{-\nu(t-s)\Lambda^2}\Lambda^{1+2\epsilon}P_m g(s)ds$$

is a continuous function in H. Since $z_m\in C([0,\infty);V)$, it follows that z_m converges in $C([0,\infty);V)$. We let z be the limit of z_m. Set

$$\rho_m(t)=e^{-\nu t\Lambda^2}P_m z_0-\int_0^t \nu\Lambda^2 e^{-\nu(t-s)\Lambda^2}P_m g(s)ds.$$

Then ρ_m satisfies the linear equation

$$\frac{d\rho_m}{dt}+\nu\Lambda^2\rho_m=-\nu\Lambda^2 P_m g,\rho_m(0)=P_m z_0.$$

It follows that

$$\frac{d}{dt}|\Lambda\rho_m|^2_{L^2}+\nu|\Lambda^2\rho_m|^2_{L^2}\le\nu|\Lambda^2 P_m g|^2_{L^2}.$$

Therefore, integrating we have

$$\nu\int_0^t|\Lambda^2\rho_m(s)|^2_{L^2}ds\le|\Lambda P_m z_0|^2_{L^2}+\nu\int_0^t|\Lambda^2 P_m g(s)|^2_{L^2}ds.$$

By taking subsequence, ρ_m converges in $L^2{\rm loc}(0,\infty;V^2)$. Let ρ be the limit of ρ_m. Since we have $g\in L^2_{\rm loc}(0,\infty;V^2)$, we have $z_m=\rho_m+P_m g$ converges to $z:=\rho+g$ in $L^2{\rm loc}(0,\infty;V^2)$. Now define $\tilde{u}_m=u_m-z_m$. Let $\tilde{u}_m$ satisfy the equations

$$\frac{d\tilde{u}_m}{dt}+\nu\Lambda^2\tilde{u}_m-P_m B(\tilde{u}_m+z_m,\tilde{u}_m+z_m)=P_m f,t\in[0,\infty),\tag{2.5}$$

$$\tilde{u}_m(0)=u_m(0)-P_m z_0.\tag{2.6}$$

In [4], the boundedness of $\tilde{u}_m$ in $L^\infty(0,T;L^2)\cap L^2(0,T;V)$ is shown, i.e.,

$$\begin{aligned}&|\tilde{u}_m(t)|_{L^2}^2+\nu\int_0^t|\Lambda\tilde{u}_m(s)|_{L^2}^2ds\\&\le|\tilde{u}_m(0)|_{L^2}^2+C\int_0^t|\tilde{u}_m|_{L^2}^2|z_m|_{L^4}^8+|z_m|_{L^4}^4+|z_m|_{L^2}^2+|P_mf|_{V'}^2ds.\end{aligned}$$

Thus $\nu\Lambda^2\tilde{u}_m\in L^2_{\rm loc}(0,\infty;V')\subset L^{4/3}_{\rm loc}(0,\infty;V')$. Note that

$$\begin{gathered}\|\tilde{u}_m\cdot\nabla\tilde{u}_m\|_{V'}\le C\|\tilde{u}_m\|_{L^2}^{1/2}\|\tilde{u}_m\|_{V'}^{3/2},\\\|\tilde{u}_m\cdot\nabla z_m\|_{V'}\le C\|\tilde{u}_m\|_{L^2}\|z_m\|_V^{1/2}\|z_m\|_{V^2}^{1/2},\\\|\tilde{z}_m\cdot\nabla\tilde{u}_m\|_{V'}\le C\|z_m\|_{L^2}^{1/2}\|z_m\|_V^{1/2}\|\tilde{u}_m\|_V,\\\|z_m\cdot\nabla z_m\|_{V'}\le C\|z_m\|_{L^2}^{1/2}\|z_m\|_V^{3/2}.\end{gathered}$$

Hence, we have $P_mB(\tilde{u}_m+z_m,\tilde{u}_m+z_m)\in L^{4/3}_{\rm loc}(0,\infty;V')$. Thus we conclude that $\frac{\partial\tilde{u}_m}{\partial t}$ is bounded in $L^{4/3}_{\rm loc}(0,\infty;V')$. Since $\{\tilde{u}_m\}$ is bounded in $W^{1,\frac{4}{3}}{\rm loc}([0,\infty);V')$, we have $\tilde{u}_m$ converges strongly in $L^2{\rm loc}(0,\infty;H)$. Thus there exists u such that u_m converges to u in $L^2(0,T;V^{1-\epsilon})$ and $L^{\frac{1}{\epsilon}}(0,T;H)$ for any small $\epsilon>0$. Since u_m is a finite Galerkin approximation, we have $|u_m|_{V^m}\le C(m)|u_m|_{L^2}$. Hence u_m is in $L^\infty(0,T;V)\cap L^2(0,T;V^2)$. Thus we have showed the following lemma.

Lemma 2.1. *If $u_0\in H$, then u_m converges in $L^2(0,T;V^{1-\epsilon})$ and $L^{\frac{1}{\epsilon}}(0,T;H)$ for any small $\epsilon>0$ as $m\to\infty$. The sequence u_m is bounded in $L^\infty(0,T;H)\cap L^2(0,T;V)$. Furthermore, u_m is in $L^\infty_{\rm loc}(0,\infty;V)\cap L^2_{\rm loc}(0,\infty;V^2)$.*

To proceed further, we consider the linear equations

$$\begin{gathered}\frac{\partial v_m}{\partial t}+\nu\Lambda^2v_m=(I-P_m)f+(I-P_m)\frac{\partial g}{\partial t},\\v_m(0)=(I-P_m)u_0,\end{gathered}\tag{2.7}$$

where $I-P_m$ is the projection onto the space spanned by $\{\frac{1}{\alpha_i(k)}P_{\rm div}(\vec{e}_ie^{ik\cdot x})\,|\,|k|>m\}$.

Lemma 2.2. *If $u_0\in H$, then there exist a unique solution v_m of* (2.7) *in the space $L^2(0,T;V)\cap L^\infty(0,T;H)$ and $v_m\to0$ in $L^2(0,T;V)\cap L^\infty(0,T;H)$ as $m\to\infty$. Furthermore, if $u_0\in V$, then $v_m\in L^2(0,T;V^2)\cap L^\infty(0,T;V)$ and $v_m\to0$ in $L^2(0,T;V^2)\cap L^\infty(0,T;V)$ as $m\to\infty$.*

Proof. Since (2.7) is a simple linear dissipative system, the existence and uniqueness are immediate consequence of the standard results. Similarly to the proof of Lemma 2.1, we can prove $v_m\in L^2(0,T;V)\cap L^\infty(0,T;H)$ and $v_m\to0$ in $L^2(0,T;V)\cap L^\infty(0,T;H)$. We only provide the proof of the second claim of this lemma. Equation (2.7) is equivalent to the integral equation

$$\begin{aligned}v_m(t)=&e^{-\nu t\Lambda^2}(I-P_m)u_0+(I-P_m)g(t)+\int_0^te^{-\nu(t-s)\Lambda^2}(I-P_m)f(s)\,ds\\&-\int_0^t\nu\Lambda^2e^{-\nu(t-s)\Lambda^2}(I-P_m)g(s)\,ds.\end{aligned}$$

Since

$$\begin{aligned}\Lambda v_m(t) =& e^{-\nu t\Lambda^2}\Lambda(I-P_m)u_0 + \Lambda(I-P_m)g(t) + \int_0^t \Lambda e^{-\nu(t-s)\Lambda^2}(I-P_m)f(s)\,ds\\ &- \int_0^t \nu\Lambda^{2(1-\epsilon)}e^{-\nu(t-s)\Lambda^2}\Lambda^{1+2\epsilon}(I-P_m)g(s)\,ds\end{aligned}$$

is a continuous function in H (see (2.4)), we have $v_m \in C([0,T);V)$. Set

$$\begin{aligned}h_m(t) =& e^{-\nu t\Lambda^2}(I-P_m)u_0 - \int_0^t \nu\Lambda^2 e^{-\nu(t-s)\Lambda^2}(I-P_m)g(s)\,ds\\ &+ \int_0^t e^{-\nu(t-s)\Lambda^2}(I-P_m)f(s)\,ds,\end{aligned}$$

i.e., $v_m(t) = (I-P_m)g(t) + h_m(t)$. It follows that h_m satisfies the equation

$$\frac{dh_m}{dt} + \nu\Lambda^2 h_m = -\nu\Lambda^2(I-P_m)g + (I-P_m)f.$$

Taking inner product with $\Lambda^2 h_m$ in L^2 produces

$$\frac{1}{2}\frac{d}{dt}|\Lambda h_m|_{L^2}^2 + \nu|\Lambda^2 h_m(t)|_{L^2}^2 \le \frac{\nu}{2}|\Lambda^2 h_m|_{L^2}^2 + C|\Lambda^2(I-P_m)g|_{L^2}^2 + C|(I-P_m)f|_{L^2}^2.$$

Integrating over $[0,T)$, we obtain

$$\begin{aligned}&|\Lambda^2 h_m(t)|_{L^2}^2 + \nu\int_0^T |\Lambda^2 h_m(t)|_{L^2}^2 dt\\ &\le |\Lambda(I-P_m)u_0|_{L^2}^2 + C\int_0^T |\Lambda^2(I-P_m)g(t)|_{L^2}^2 dt + C\int_0^T |(I-P_m)f(t)|_{L^2}^2 dt.\end{aligned}$$

Thus $h_m \in L^2(0,T;V^2)$. Since $g \in L^2(0,T;V^2)$, we obtain $v_m \in L^2(0,T;V^2)$. Lebesgue's dominated convergence Theorem produces $v_m \to 0$ in $L^2(0,T;V^2) \cap L^\infty(0,T;V)$ as $m \to \infty$. For the uniqueness, if there exists two solutions v_m^1 and v_m^2 of (2.7), then we denote by $\rho(t) = v_m^1(t) - v_m^2(t)$. We have the following deterministic equations.

$$\frac{d\rho}{dt} + \nu\Lambda^2\rho = 0, \quad \rho(0) = 0.$$

Thus we have $\rho(t) = 0$, which completes the proof. $\square$

From Lemma 2.1, we have u_m is in $L^\infty(0,T;V) \cap L^2(0,T;V^2)$. Now for every m, we consider also the solution v_m of the linearized problems (2.7) We then set $w_m = u_m + v_m$ and observe w_m satisfies $w_m \in L^2(0,T;V^2) \cap L^\infty(0,T;V)$ if $u_0 \in V$. By adding two equations, we have the following Navier-Stokes equations with nonregular force

$$\frac{dw_m}{dt} + \nu\Lambda^2 w_m - B(w_m,w_m) = f_m + \frac{dg}{dt}, \tag{2.8}$$

where $f_m = f - B(v_m,v_m) - B(v_m,u_m) - B(u_m,v_m) - (I-P_m)B(u_m,u_m)$. Let $\tilde w_m$ be another solution of (2.8). Then by letting $\tilde\rho_m = w_m - \tilde w_m$, we have

$$\frac{d\tilde\rho_m}{dt} + \nu\Lambda^2\tilde\rho_m = B(\tilde\rho_m,w_m) + B(\tilde w_m,\tilde\rho_m).$$

Hence we have

$$\begin{aligned}\frac{d}{dt}|\tilde\rho_m|_{L^2}^2 &\le -2\nu|\Lambda\tilde\rho_m|_{L^2}^2 + C|\tilde\rho_m|_{L^6}|\nabla w_m|_{L^3}|\tilde\rho_m|_{L^2}\\ &\le -\nu|\Lambda\tilde\rho_m|_{L^2}^2 + C|w_m|_{V^2}|w_m|_V|\tilde\rho_m|_{L^2}^2.\end{aligned}$$

Using Gronwall's inequality, we have

$$|\tilde\rho_m(t)|_{L^2}^2 \le |\tilde\rho_m(0)|_{L^2}^2 \exp\Big(C\int_0^t |w_m|_{V^2}|w_m|_V\,ds\Big).$$

Since $w_m \in L^2(0,T;V^2)\cap L^\infty(0,T;V)$, it is clear that w_m is the unique solution in $L^2_{\rm loc}(0,\infty;V^2)\cap L^\infty_{\rm loc}(0,\infty;V)$. Hence for the proof of Theorem 1.1, it is sufficient to show that f_m converges to f in $L^q(0,T;L^{6/5})$ with $1\le q<\frac{4}{3}$.

Proof of Theorem 1.1. For the remaining of this proof, we use only the weaker assumption $u_0\in H$ instead of $u_0\in V$. Since we have $v_m\to 0$ in $L^2(0,T;V)\cap L^\infty(0,T;H)$ as $m\to\infty$, it is clear that $B(v_m,v_m)\to 0$ in $L^q(0,T;L^{6/5})$ by the inequalities

$$\begin{aligned}\int_0^T |B(v_m,v_m)|_{L^{6/5}}^q dt &\le C\int_0^T |v_m|_{L^3}^q\,|v_m|_V^q dt\\ &\le C\int_0^T |v_m|_{L^2}^{q/2}|v_m|_V^{\frac32 q}dt\\ &\le C\Big(\int_0^T |v_m|_{L^2}^{\frac{2q}{4-3q}}dt\Big)^{\frac{4-3q}{4}}\Big(\int_0^T |v_m|_V^2 dt\Big)^{3q/4}\\ &\le CT^{\frac{4-3q}{4}}|v_m|_{L^\infty(0,T;H)}^{\frac{4q}{4-3q}}|v_m|_{L^2(0,T;V)}^{3q/2}.\end{aligned}$$

It is well known from the interpolation inequality that

$$|B(u_m,v_m)|_{L^{6/5}} \le C|u_m|_{L^6}|\nabla v_m|_{L^{3/2}} \le C|\nabla u_m|_{L^2}|v_m|_{L^2}^{1/2}|v_m|_V^{1/2}.$$

Then

$$\begin{aligned}&\int_0^T |B(u_m,v_m)|_{L^{6/5}}^q dt\\ &\le C\int_0^T |\nabla u_m|_{L^2}^q|v_m|_{L^2}^{q/2}|v_m|_V^{q/2}dt\\ &\le C\Big(\int_0^T|\nabla u_m|_{L^2}^2dt\Big)^{q/2}\Big(\int_0^T |v_m|_{L^2}^{\frac{2q}{4-3q}}dt\Big)^{\frac{4-3q}{4}}\Big(\int_0^T |v_m|_V^2 dt\Big)^{q/4}\\ &\le CT^{\frac{4-3q}{4}}|u_m|_{L^2(0,T;V)}^q|v_m|_{L^\infty(0,T;H)}^{q/2}v_m|_{L^2(0,T;V)}^{q/2}\to 0\,.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\int_0^T |B(v_m,u_m)|_{L^{6/5}}^q dt &\le \int_0^T |v_m|_{L^3}^q|\nabla u_m|_{L^2}^q dt\\ &\le CT^{\frac{4-3q}{4}}|u_m|_{L^2(0,T;V)}^q|v_m|_{L^\infty(0,T;H)}^{q/2}|v_m|_{L^2(0,T;V)}^{q/2}\to 0\,.\end{aligned}$$

To complete the proof, it is sufficient to show that

$$(I-P_m)B(u_m,u_m)\to 0 \quad \text{in } L^q(0,T;L^{6/5}) \text{ as } m\to\infty.$$

First we recall that u_m converges to its limit u from Lemma 2.1. We rewrite u as an expansion by the complete orthonormal basis K, i.e.,

$$u = \sum_{k\in\mathbb{Z}^3\ i=1,2,3} u_k^i \frac{1}{\alpha_i(k)} P_{\rm div}(\vec{e}_i e^{ik\cdot x}) =: \sum_{k\in\mathbb{Z}^3} u_k e_k(x),$$

where u_k^i is the corresponding coefficient, and for simplicity of notation we introduced the right-hand-side. Then we have

$$\begin{aligned}
(I-P_m)B(u,u) &= (I-P_m)P_{\rm div}((u\cdot\nabla)u)\\
&= (I-P_m)P_{\rm div}\sum_{k'\in\mathbb{Z}^3}\big(\sum_{k\in\mathbb{Z}^3} u_k e_k(x)\cdot k'\big)u_{k'}e_{k'}(x)\\
&= P_{\rm div}\Big(\big(\sum_{|k|\geq[\frac{m}{2}]} u_k e_k(x)\cdot\nabla\big)\sum_{k'}^{*} u_{k'}e_{k'}(x)\Big)\\
&\quad + P_{\rm div}\Big(\big(\sum_{k}^{*} u_k e_k(x)\cdot\nabla\big)\sum_{|k'|\geq[\frac{m}{2}]} u_{k'}e_{k'}(x)\Big),
\end{aligned} \tag{2.9}$$

where $[a]$ denotes the largest integer less than or equal to a, and $\sum_h^*$ denotes the summation over all $h\in\mathbb{Z}^3$ satisfying $|h+j|>m$ when $|j|\geq[\frac{m}{2}]$. Using the identity (2.9), we obtain

$$\begin{aligned}
|(I-P_m)B(u,u)| &\leq C|\sum_{|k|\geq[\frac{m}{2}]} u_k e_k(x)|_{L^3}|\nabla u|_{L^2} + C|u|_{L^6}|\nabla\sum_{|k'|\geq\frac{m}{2}} u_{k'}e_{k'}(x)|_{L^{3/2}}\\
&\leq C|\sum_{|k|\geq[\frac{m}{2}]} u_k e_k(x)|_{L^2}^{1/2}|\nabla u|_{L^2}^{3/2}.
\end{aligned}$$

We obtain that for any $q<4/3$,

$$\begin{aligned}
&\int_0^T |(I-P_m)B(u,u)|_{L^{6/5}}^q dt\\
&\leq C\int_0^T |\nabla u|_{L^2}^{\frac{3q}{2}}|\sum_{|k|\geq[\frac{m}{2}]} u_k e_k(x)|_{L^2}^{q/2} dt\\
&\leq C\Big(\int_0^T |\nabla u|_{L^2}^2 dt\Big)^{3q/4}\Big(\int_0^T |\sum_{|k|\geq[\frac{m}{2}]} u_k e_k(x)|_{L^2}^{\frac{2q}{4-3q}} dt\Big)^{\frac{4-3q}{4}}.
\end{aligned}$$

Since K is a complete orthonormal basis for H, we have

$$(I-P_m)B(u,u)\to 0 \quad \text{in } L^q(0,T;L^{6/5}) \text{ as } m\to\infty.$$

Thus it only remains to prove that $B(u_m,u_m)-B(u,u)\to 0$ in $L^q(0,T;L^{6/5})$. From Lemma 2.1, we have

$$u_m\to u \quad \text{in } L^2(0,T;V^{1-\epsilon})\cap L^{\frac{1}{\epsilon}}(0,T;H) \text{ for any } \epsilon>0.$$

We complete the proof by showing that

$$B(u_m-u,u_m), B(u,u_m-u)\to 0 \quad \text{in } L^q(0,T;L^{6/5}) \text{ for all } q<4/3.$$

By the interpolation inequality, we have for $\epsilon < 1/2$,

$$|B(u_m - u, u_m)|_{L^{6/5}} \le C|u_m - u|_{L^2}^{\frac{1-2\epsilon}{2(1-\epsilon)}} |u_m - u|_{V^{1-\epsilon}}^{\frac{1}{2(1-\epsilon)}} |\nabla u_m|_{L^2},$$
$$|B(u, u_m - u)|_{L^{6/5}} \le C|u|_{L^6}|u_m - u|_{L^2}^{1/2}|u_m - u|_V^{1/2}.$$

Setting $r = \frac{2q(1-2\epsilon)}{4-3q-2\epsilon(2-q)}$, we have

$$\frac{q}{2} + \frac{q}{4(1-\epsilon)} + \frac{q(1-2\epsilon)}{2r(1-\epsilon)} = 1.$$

By Hölder's inequality and Lemma 2.1, we obtain

$$\begin{aligned}&\int_0^T |B(u_m - u, u_m)|_{L^{6/5}}^q dt \\ &\le C\Big(\int_0^T |u_m - u|_{L^2}^r dt\Big)^{\frac{q(1-2\epsilon)}{2r(1-\epsilon)}} \Big(\int_0^T |u_m - u|_{V^{1-\epsilon}}^2 dt\Big)^{\frac{q}{4(1-\epsilon)}} \Big(\int_0^T |\nabla u_m|_{L^2}^2 dt\Big)^{q/2}\end{aligned}$$

which approaches zero as m approaches ∞. Again using Hölder's inequality(note that $\frac{q}{2} + \frac{q}{4} + \frac{4-3q}{4} = 1$), we have

$$\begin{aligned}&\int_0^T |B(u, u_m - u)|_{L^{6/5}}^q dt \\ &\le C\Big(\int_0^T |u|_V^2 dt\Big)^{q/2} \Big(\int_0^T |u_m - u|_{L^2}^{\frac{2q}{4-3q}} dt\Big)^{\frac{4-3q}{4}} \Big(\int_0^T |u_m - u|_V^2 dt\Big)^{q/4} \to 0\,.\end{aligned}$$

This completes the proof of Theorem 1.1. □

Acknowledgements. The author would like to thank Professor D. Chae for the many helpful suggestions on the generic solvability, and Professor Ya Sinai for his helpful discussions. The author also wants to thank the anonymous referee for his/her suggestions and corrections.

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