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BIFURCATION OF POSITIVE SOLUTIONS FOR THE ONE-DIMENSIONAL (p, q)-LAPLACE EQUATION

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Abstract. In this article, we study the bifurcation of positive solutions for the one-dimensional (p, q)-Laplace equation under Dirichlet boundary conditions. We investigate the shape of the bifurcation diagram and prove that there exist five different types of bifurcation diagrams. As a consequence, we prove the existence of multiple positive solutions and show the uniqueness of positive solutions for a bifurcation parameter in a certain range.

1. Introduction

In this article, we study the bifurcation problem of positive solutions for the one-dimensional (p, q)-Laplace equation in the interval $(-L, L)$,

$$\begin{gathered}(|u'|^{p-2}u')' + (|u'|^{q-2}u')' + \lambda(|u|^{p-2}u + |u|^{q-2}u) = 0, \\ u(-L) = u(L) = 0,\end{gathered} \tag{1.1}$$

where $L > 0$, $1 < q < p < \infty$ and $\lambda > 0$ is a parameter. We assume that $1 < q < p$ throughout the paper. We can deal with any interval (a, b) instead of $(-L, L)$. Indeed, putting $L := (b - a)/2$ and using a translation, we can reduce (a, b) to $(-L, L)$. We call (λ, u) a solution of (1.1) if

$$u \in C^1[-L, L], \quad |u'|^{p-2}u' + |u'|^{q-2}u' \in C^1[-L, L],$$

and (λ, u) satisfies (1.1).

There has been much interest in studying autonomous equations with one-dimensional p-Laplacian or one-dimensional Laplacian. It is impossible to quote all of them. Here, we only refer to [1, 14, 17, 19, 21]. For the (p, q)-Laplace equation, the existence of positive solutions was studied in [7, 24, 25, 26]. The problems in $\mathbb{R}^N$ were investigated in [2, 3, 23]. The existence of nodal solutions was proved in [15]. The one-dimensional Φ-Laplacian problem

$$(\Phi(u'))' + \lambda f(u) = 0, \quad u(-L) = u(L) = 0,$$

which is a more general problem, is studied in [8, 9, 10, 11, 12, 18, 20], where Φ is an increasing odd homeomorphism of $\mathbb{R}$ and $f \in C(\mathbb{R})$. However, it seems difficult

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to obtain precise results for Φ-Laplacian problems. In this paper, we concentrate (1.1) and investigate the structure of solutions as precise as possible.

Several authors studied a more general problem than (1.1),

$$-\Delta_p u - \Delta_q u = \lambda(m_p(x)|u|^{p-2}u + m_q(x)|u|^{q-2}u) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$. Motreanu and Tanaka [16] dealt with sign-indefinite weights $m_p(x)$, $m_q(x)$ and proved the existence and nonexistence of positive solutions for some ranges of λ. Applying the results in [16] to our problem, we obtain the following:

(i) if $0 < \lambda < \min\{\mu(p, L), \mu(q, L)\}$, then (1.1) has no positive solutions;
(ii) if $\min\{\mu(p, L), \mu(q, L)\} < \lambda < \max\{\mu(p, L), \mu(q, L)\}$, then (1.1) has at least one positive solution,

where $\mu(p, L)$ is the first eigenvalue of the p-Laplacian, that is, the following equation has a *positive* solution if and only if $\mu = \mu(p, L)$:

$$(|\phi'|^{p-2}\phi')' + \mu|\phi|^{p-2}\phi = 0 \quad \text{in } (-L, L), \quad \phi(-L) = \phi(L) = 0. \tag{1.2}$$

However, when $\lambda \geq \max\{\mu(p, L), \mu(q, L)\}$, there is no information about the existence or nonexistence of positive solutions. Moreover, it is unknown whether a positive solution is unique or not in the case (ii). One of our purposes is to solve such problems.

The references mentioned above used mainly the variational method, a minimizing method or the mountain pass lemma. In this paper, we employ the bifurcation approach. It seems to the authors that little is known about the bifurcation diagram of (1.1). Our problem (1.1) seems simple, however the structure of the solution space is very complicated.

The p-Laplace operator Δ_p is homogeneous with degree $p-1$, that is, $\Delta_p(\lambda u) = \lambda^{p-1}\Delta_p u$ for $\lambda > 0$. However, the (p, q)-Laplace operator has no such homogeneity. This is one of difficulties to our problem.

To explain another difficulty, we consider a large positive solution of (1.1). Then it approaches the first eigenfunction of the p-Laplacian (1.2). Indeed, we rewrite (1.1) as

$$[(p-1)|u'|^{p-2} + (q-1)|u'|^{q-2}]u'' + \lambda(u^{p-2} + u^{q-2})u = 0.$$

(This expression is not rigorous because a solution does not have the C^2-regularity. However we shall prove the convergence of solutions to the eigenfunction in the strict method.) If u and $|u'|$ are large enough, the equation above is nearly equal to the following equation

$$((p-1)|u'|^{p-2})u'' + \lambda u^{p-1} = 0,$$

which is exactly (1.2). Thus a large positive solution converges to the first eigenfunction of (1.2) as the L^∞ norm of u diverges. This assertion will be proved in Section 4. On the other hand, a small positive solution is close to the first eigenfunction of the q-Laplacian because (1.1) approaches the q-Laplace equation as the L^∞ norm of u tends to zero. It is well-known that the first eigenvalue $\mu(p, L)$ of (1.2) is represented as (see [6, pp.4-5])

$$\mu(p, L) := (p-1)\left(\frac{\pi_p}{2L}\right)^p, \tag{1.3}$$

where

$$\pi_p := 2\int_0^1 (1-s^p)^{-1/p}\,ds = \frac{2\pi}{p\sin(\pi/p)}. \tag{1.4}$$

For π_p, we also refer the reader to [5] and [22]. Therefore a small positive solution of (1.1) is governed by $\mu(q,L)$ and a large one by $\mu(p,L)$. Hence the behavior of the first eigenvalue plays the key role in the bifurcation problem. If $L \le 1$, then $\mu(p,L)$ is increasing on p. However if $L > 1$, then it is not increasing. Indeed, we have the next theorem.

Theorem 1.1. (i) *If $0 < L \le 1$, then $\mu(p,L)$ is strictly increasing with respect to p, that is, $\mu_p(p,L) > 0$ for $1 < p < \infty$, where $\mu_p(p,L)$ denotes the partial derivative with respect to p.*

(ii) *If $L > 1$, then there exists a unique $p_*(L) > 0$ such that $\mu_p(p,L) > 0$ for $p \in (1, p_*(L))$ and $\mu_p(p,L) < 0$ for $p \in (p_*(L), \infty)$.*

The theorem above may be known, however we can not find a proof of it. Therefore we give a proof in Section 2. The behavior of $\mu(p,L)$ as above makes the bifurcation problem complicated. This is another difficulty in our problem.

We denote the $L^\infty(-L,L)$ norm of u by $\|u\|_\infty$. We shall prove that for any $\alpha > 0$, there exists a unique positive solution (λ, u) which satisfies $\|u\|_\infty = \alpha$. We write it as $(\lambda, u) = (\lambda(\alpha), u(x,\alpha))$. Then $(\lambda(\alpha), u(x,\alpha))$ with $\alpha > 0$ represents all positive solutions. Since $u(x,\alpha)$ is uniquely determined by α, we can identify a curve $(\lambda(\alpha), u(x,\alpha))$ with $(\lambda(\alpha), \alpha)$. It will be shown that any positive solution u of (1.1) is even and achieves its maximum at $x = 0$ only. See Lemma 3.2. Therefore $\alpha = \|u(\cdot,\alpha)\|_\infty = u(0,\alpha)$. As mentioned before, if (λ_n, u_n) is a sequence of positive solutions and if $\|u_n\|_\infty \to \infty$, then $\lambda_n \to \mu(p,L)$. This fact will be proved in Lemma 4.8. If $\|u_n\|_\infty \to 0$, then $\lambda_n \to \mu(q,L)$. See Lemma 4.11 for the proof. Since $(\lambda(\alpha), u(x,\alpha))$ is a positive solution satisfying $\|u(\cdot,\alpha)\|_\infty = \alpha$, it holds that $\lambda(\alpha) \to \mu(q,L)$ as $\alpha \to +0$ and $\lambda(\alpha) \to \mu(p,L)$ as $\alpha \to \infty$. Then the bifurcation curve $(\lambda(\alpha), \alpha)$ starts from the initial point $(\mu(q,L), 0)$ and reaches the final point $(\mu(p,L), \infty)$.

García-Huidobro, Manásevich and Schmitt in [10, 11] considered N-dimensional Φ-Laplacian problem

$$(r^{N-1}\Phi(u'))' + \lambda r^{N-1} f(u) = 0, \quad u'(0) = u(L) = 0, \tag{1.5}$$

and they proved the following (i)–(iii): (i) for each $\alpha > 0$, there exists a positive solution (λ, u) with $\|u\|_\infty = \alpha$; (ii) there exists a constant $\lambda_0 > 0$ such that (1.5) has no nontrivial solution if $0 < \lambda < \lambda_0$; (iii) there exists a connected component of positive solutions connecting $(\mu(q,L), 0)$ and $(\mu(p,L), \infty)$. Moreover, they also showed the existence of sign-changing solutions. See [10, Theorem 1.1] and [11, Theorem 5.1].

The purpose of the present paper is to investigate the shape of the bifurcation diagram. Moreover, observing the shape of the diagram, we study the number of positive solutions. We draw the curve $(\lambda(\alpha), \alpha)$ in the (λ, α) plain, where λ-axis and α-axis are chosen as axes of abscissa and ordinate, respectively. For simplicity, we write $\mu(p,L)$ as $\mu(p)$ if there is no confusion. In Section 4, we shall show that the curve $(\lambda(\alpha), \alpha)$ always stays in the right side of the line $\lambda = \min\{\mu(p), \mu(q)\}$. Using this fact, we classify all the bifurcation diagrams according to whether the bifurcation curve always lies in the left side of the line $\lambda = \max\{\mu(q), \mu(p)\}$ or protrudes from the line (see Figure 1):

(A) $\mu(q) < \mu(p)$ and $\mu(q) < \lambda(\alpha) \leq \mu(p)$ for all $\alpha > 0$.
(B) $\mu(q) < \mu(p)$ and $\mu(p) < \lambda(\alpha)$ at some $\alpha > 0$.
(C) $\mu(p) < \mu(q)$ and $\mu(p) < \lambda(\alpha) \leq \mu(q)$ for all $\alpha > 0$.
(D) $\mu(p) < \mu(q)$ and $\mu(q) < \lambda(\alpha)$ at some $\alpha > 0$.
(E) $\mu(p) = \mu(q)$ and $\mu(p) < \lambda(\alpha)$ for all $\alpha > 0$.

The definitions above can be rewritten as below in terms of the number of positive solutions.

(A) $\mu(q) < \mu(p)$ and (1.1) has at least one positive solution for $\lambda \in (\mu(q), \mu(p))$ and no positive solutions for $\lambda \in (0, \mu(q)] \cup (\mu(p), \infty)$.
(B) $\mu(q) < \mu(p)$ and there exists $\lambda^* > \mu(p)$ such that (1.1) has at least one positive solution for $\lambda \in (\mu(q), \mu(p)] \cup \{\lambda^*\}$, at least two positive solutions for $\lambda \in (\mu(p), \lambda^*)$ and no positive solutions for $\lambda \in (0, \mu(q)] \cup (\lambda^*, \infty)$.
(C) $\mu(p) < \mu(q)$ and (1.1) has at least one positive solution for $\lambda \in (\mu(p), \mu(q))$ and no positive solutions for $\lambda \in (0, \mu(p)] \cup (\mu(q), \infty)$.
(D) $\mu(p) < \mu(q)$ and there exists $\lambda^* > \mu(q)$ such that (1.1) has at least one positive solution for $\lambda \in (\mu(p), \mu(q)] \cup \{\lambda^*\}$, at least two positive solutions for $\lambda \in (\mu(q), \lambda^*)$ and no positive solutions for $\lambda \in (0, \mu(p)] \cup (\lambda^*, \infty)$.
(E) $\mu(p) = \mu(q)$ and there exists $\lambda^* > \mu(p)$ such that (1.1) has at least one positive solution for $\lambda = \lambda^*$, at least two positive solutions for $\lambda \in (\mu(p), \lambda^*)$ and no positive solutions for $\lambda \in (0, \mu(p)] \cup (\lambda^*, \infty)$.

We shall prove that the five types (A)–(E) of behaviors actually occur for some (p, q, L). We note that if $\mu(p, L) = \mu(q, L)$, then (E) always occurs, by recalling $(\lambda(\alpha), \alpha)$ always stays in the right side of the line $\lambda = \min\{\mu(p, L), \mu(q, L)\}$. Now we fix p and q and define

$$L_* = L_*(p, q) := \frac{1}{2}\Big(\frac{(p-1)\pi_p^p}{(q-1)\pi_q^q}\Big)^{1/(p-q)}. \tag{1.6}$$

Since $\mu(p, 1)$ is increasing with respect to p by Theorem 1.1, it holds that $\mu(q, 1) < \mu(p, 1)$ by $1 < q < p$. This proves $L_* > 1$. From (1.3) it follows that $\mu(p, L) = \mu(q, L)$ if and only if $L = L_*$. If $L > L_*$, then $\mu(p, L) < \mu(q, L)$ by (1.3). In Case (E), the bifurcation curve stays in the right side of the line $\lambda = \mu(p, L)$. Consider a small perturbation from (E). Then, if L is slightly greater than L_*, (D) occurs. Conversely, (B) occurs when $L < L_*$ and L is close to L_*. In Section 6, we shall show that if L is small enough or large enough, then $\lambda'(\alpha) > 0$ or $\lambda'(\alpha) < 0$ for all $\alpha > 0$, respectively. Therefore, if L is small enough or large enough, then (A) or (C) occurs respectively, and a positive solution is unique. Consequently, we shall prove the existence of an $\varepsilon > 0$ (see Theorem 7.15) such that (A), (B), (E), (D), (C) occur when $0 < L < \varepsilon$, $L_* - \varepsilon < L < L_*$, $L = L_*$, $L_* < L < L_* + \varepsilon$, $1/\varepsilon < L < \infty$, respectively. Furthermore, the types (A) with $0 < L < \varepsilon$ and (C) with $1/\varepsilon < L < \infty$ have no turning points and they are monotone. Therefore, the structure of solutions to (1.1) changes depending on L. Such phenomena have been reported in [4] and [13] for mean curvature equations.

Contents of the article: In Section 2, we give a proof of Theorem 1.1 and study some properties of the first eigenvalue, which play an important role for our proofs.

In Section 3, we prove the uniqueness of solutions for the initial value problem corresponding to our problem (1.1). After that, we introduce a time map which denotes the first zero in $(0, \infty)$ of a solution for the initial value problem.

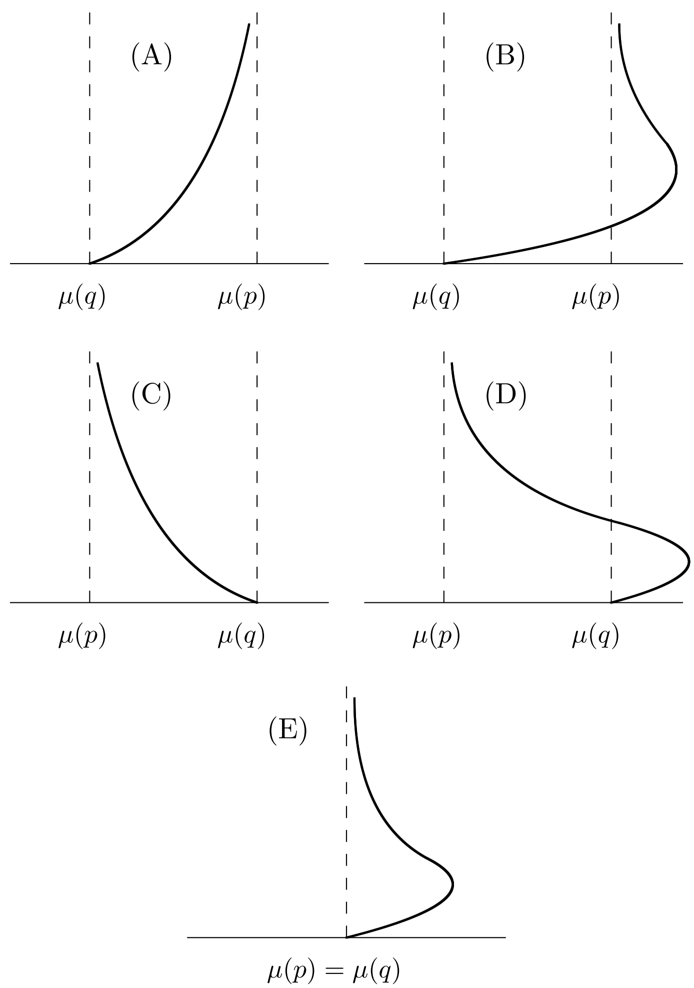

FIGURE 1. Bifurcation diagrams

In Section 4, we prove that all positive solutions are represented as a smooth curve $(\lambda(\alpha), u(x, \alpha))$ with one parameter $\alpha = u(0, \alpha) = \|u\|_\infty$ and study the properties of this bifurcation curve.

In Section 5, to consider the behavior of the bifurcation curve, we estimate a derivative of the time map, which will be used in Sections 6 and 7.

In Section 6, we give sufficient conditions for which the shape of the bifurcation curve is exactly type (A) or (C).

In Section 7, we shall study the direction in which the bifurcation curve moves near the initial point $(\mu(q, L), 0)$ and near the final point $(\mu(p, L), \infty)$. Using these results, we shall construct types (B) and (D). This is a different way from the perturbation method of type (E) as mentioned after (1.6). As a consequence, we shall obtain the existence of multiple positive solutions and moreover the uniqueness of positive solutions in some cases of p, q, and L.

2. The first eigenvalue

In this section, we shall prove Theorem 1.1 and investigate the properties of the first eigenvalue of the p-Laplacian.

Proof of Theorem 1.1. We put $g(x) := \log \mu(p, L)$ and $p := \pi/x$. Then we find that

$$\begin{aligned} g(x) = \log \mu(p, L) &= \log(p-1) + p \log[\pi/(Lp \sin(\pi/p))] \\ &= \log((\pi - x)/x) - (\pi/x) \log(\sin x/x) - (\pi/x) \log L. \end{aligned} \tag{2.1}$$

Differentiating it, we have

$$g'(x) = \frac{-\pi}{x(\pi - x)} + \pi x^{-2} \log(\sin x/x) - \pi x^{-1} \frac{\cos x}{\sin x} + \pi x^{-2} + \pi x^{-2} \log L.$$

Putting $h(x) = \pi^{-1} x^2 g'(x)$, we obtain

$$h(x) := \pi^{-1} x^2 g'(x) = \frac{-x}{\pi - x} + \log(\sin x/x) - \frac{x \cos x}{\sin x} + \log L + 1.$$

Since $p \in (1, \infty)$, x lies in $(0, \pi)$. We compute $h'(x)$ as

$$\begin{aligned} h'(x) &= \frac{-\pi}{(\pi - x)^2} + \frac{x^2 - \sin^2 x}{x \sin^2 x} \\ &= \frac{x^2(\pi - x)^2 - [(\pi - x)^2 + \pi x] \sin^2 x}{(\pi - x)^2 x \sin^2 x}. \end{aligned} \tag{2.2}$$

We shall show that $h'(x) < 0$ for $0 < x < \pi$ in Lemma 2.1 later on. Thus $h(x)$ is strictly decreasing.

We shall show the assertion (i). Let $0 < L \le 1$. Since $\lim_{x \to +0} h(x) = \log L \le 0$ and $h(x)$ is decreasing, it holds that $h(x) < 0$ for $0 < x < \pi$. Accordingly, $g'(x) < 0$. Since $g(x) = \log \mu(p, L)$ and $p = \pi/x$, we find that

$$\mu(p, L) = \exp(g(x)), \quad \mu_p(p, L) = -\pi p^{-2} g'(x) \exp(g(x)). \tag{2.3}$$

Consequently, $\mu_p(p, L) > 0$ and the assertion (i) is obtained.

Let us show (ii). Let $L > 1$. Then $\lim_{x \to +0} h(x) = \log L > 0$. Using L'Hospital's rule, we compute

$$\frac{-x}{\pi - x} - \frac{x \cos x}{\sin x} \to 0 \quad \text{as } x \to \pi - 0. \tag{2.4}$$

Hence $\lim_{x \to \pi - 0} h(x) = -\infty$. Since $h(x)$ is strictly decreasing, there exists a unique point $x_* \in (0, \pi)$ such that

$$h(x) > 0 \quad \text{for } 0 < x < x_*, \quad h(x) < 0 \quad \text{for } x_* < x < \pi. \tag{2.5}$$

Put $p_* = \pi/x_*$. When $x \in (0, x_*)$, p lies in (p_*, ∞). Since $g'(x) > 0$ for $x \in (0, x_*)$, $\mu_p(p, L) < 0$ for $p \in (p_*, \infty)$ by (2.3). In the same way, it holds that $\mu_p(p, L) > 0$ for $p \in (0, p_*)$. The proof is complete. □

We denote the numerator of $h'(x)$ in (2.2) by $k(x)$, i.e.,

$$k(x) := x^2(\pi - x)^2 - (x^2 - \pi x + \pi^2) \sin^2 x.$$

We have used the next lemma in the proof of Theorem 1.1.

Lemma 2.1. *With the above notation,*

$$k(x) < 0 \quad \text{for } x \in (0, \pi).$$

Proof. Since $k(x)$ is symmetric with respect to the line $x = \pi/2$, it is enough to show that $k(x) < 0$ for $0 < x \le \pi/2$. We use the inequality

$$\sin x > x - \frac{1}{6}x^3 > 0 \quad \text{for } 0 < x \le \pi/2,$$

to obtain

$$\begin{aligned} k(x) &\le x^2(\pi - x)^2 - (x^2 - \pi x + \pi^2)(x - x^3/6)^2 \\ &= -\frac{x^3}{36}\left[x^5 - \pi x^4 - (12 - \pi^2)x^3 + 12\pi x^2 - 12\pi^2 x + 36\pi\right]. \end{aligned}$$

Put

$$K(x) := x^5 - \pi x^4 - (12 - \pi^2)x^3 + 12\pi x^2 - 12\pi^2 x + 36\pi.$$

Then it has a derivative,

$$\begin{aligned} K'(x) &= 5x^4 - 4\pi x^3 - 3(12 - \pi^2)x^2 + 24\pi x - 12\pi^2 \\ &= -(4\pi - 5x)x^3 - 3(12 - \pi^2)x^2 - 12\pi(\pi - 2x) < 0, \end{aligned}$$

because $4\pi - 5x > 0$ and $\pi - 2x > 0$ for $0 < x < \pi/2$. Hence $K(x)$ is decreasing. Moreover,

$$K(\pi/2) = \frac{3}{32}\pi^5 - \frac{9}{2}\pi^3 + 36\pi > 0.$$

Accordingly, $K(x) > 0$ for $0 < x \le \pi/2$ and therefore $k(x) < 0$ for $0 < x \le \pi/2$. The proof is complete. □

To study the bifurcation problem, we need to investigate more properties of the first eigenvalue $\mu(p, L)$ of the p-Laplacian. As proved in Theorem 1.1, for each $L > 1$ fixed, $\mu(p, L)$ has a unique maximum point p in $(1, \infty)$. We denote it by $p_*(L)$.

Proposition 2.2. (i) *$\mu(p, L)$ has the following properties.*

$$\lim_{p\to 1+0} \mu(p, L) = 1/L \quad \textit{for all } L > 0, \tag{2.6}$$

$$\lim_{p\to\infty} \mu(p, L) = \infty, \quad \textit{when } L \le 1, \tag{2.7}$$

$$\lim_{p\to\infty} \mu(p, L) = 0, \quad \textit{when } L > 1. \tag{2.8}$$

(ii) *The unique maximum point $p_*(L)$ of $\mu(p, L)$ is strictly decreasing with respect to $L \in (1, \infty)$ and satisfies*

$$\lim_{L\to 1+0} p_*(L) = \infty, \quad \lim_{L\to\infty} p_*(L) = 1, \tag{2.9}$$

$$\lim_{L\to 1+0} \mu(p_*(L), L) = \infty, \quad \lim_{L\to\infty} \mu(p_*(L), L) = 0. \tag{2.10}$$

Proof. We shall prove (i). Since $\mu(p, L) = \mu(p, 1)L^{-p}$ by (1.3), we have only to prove that $\lim_{p\to 1+0} \mu(p, 1) = 1$, which ensures (2.6). Let us show $\mu(p, 1)^{1/p} \to 1$. Put $x = 1/p$. Then

$$\begin{aligned} \mu(p, 1)^{1/p} &= (p - 1)^{1/p}(\pi_p/2) = \Big(\frac{1 - x}{x}\Big)^x \frac{\pi x}{\sin \pi x} \\ &= x^{1-x}(1 - x)^{x-1}\frac{\pi(1 - x)}{\sin \pi x}. \end{aligned}$$

As $p \to 1+0$, x converges to $1-0$. We see readily that

$$\lim_{x\to 1-0}(1-x)^{x-1}=1, \quad \lim_{x\to 1-0}\frac{\pi(1-x)}{\sin \pi x}=1,$$

which imply

$$\lim_{x\to 1-0} x^{1-x}(1-x)^{x-1}\frac{\pi(1-x)}{\sin \pi x}=1.$$

Therefore $\mu(p,1)^{1/p} \to 1$ and hence (2.6) holds.

We shall show (2.7) with $L=1$. Put

$$y := \mu(p,1)=(p-1)\Big(\frac{\pi_p}{2}\Big)^p, \quad x := 1/p.$$

Then we see that

$$\log y = \log\Big(\frac{1-x}{x}\Big)+\frac{1}{x}\log\Big(\frac{\pi x}{\sin \pi x}\Big). \tag{2.11}$$

Since $p>1$, x is in $(0,1)$. Since $(\pi x)/\sin \pi x > 1$, the second term on the right hand side of (2.11) is positive. Accordingly, we have

$$\log y \ge \log\Big(\frac{1-x}{x}\Big) \to \infty \quad \text{as } x \to +0.$$

Therefore $\lim_{p\to\infty}\mu(p,1)=\infty$. When $L \le 1$, it follows that

$$\mu(p,L)=\mu(p,1)L^{-p} \ge \mu(p,1) \to \infty \quad \text{as } p \to \infty.$$

Consequently, (2.7) holds.

Putting $x=1/p$, we have

$$\pi_p/2 = \frac{\pi x}{\sin \pi x} \to 1 \quad \text{as } x \to +0.$$

Therefore, when $L>1$, we obtain

$$\mu(p,L)=(p-1)\Big(\frac{\pi_p}{2L}\Big)^p \to 0 \quad \text{as } p \to \infty,$$

which ensures (2.8).

We shall show that $p_* = p_*(L)$ is decreasing. We define

$$\begin{aligned} H(x) :=& \frac{x}{\pi - x} - \log(\sin x/x) + \frac{x\cos x}{\sin x} \\ =& -h(x)+1+\log L, \end{aligned}$$

where $h(x)$ is the function introduced in the proof of Theorem 1.1. Since $h(x)$ is decreasing, $H(x)$ is increasing. By recalling the proof of Theorem 1.1, there exists a unique $x_* \in (0,\pi)$ such that (2.5) holds and $p_*(L)=\pi/x_*$. Since $h(x_*)=0$ (see (2.5)), x_* satisfies $H(x_*)=\log L+1$, or equivalently $x_* = H^{-1}(\log L+1)$. Therefore x_* is an increasing function of L and hence $p_*(L)$ is decreasing. Since $h(x) \to \log L$ as $x \to +0$ and $h(x) \to -\infty$ as $x \to \pi - 0$, we have

$$\lim_{x\to 1+0} H^{-1}(x) = +0, \quad \lim_{x\to\infty} H^{-1}(x)=\pi,$$

which show that $p_*(L)=\pi/H^{-1}(\log L+1) \to \infty$ as $L \to 1+0$ and that $p_*(L) = \pi/H^{-1}(\log L+1) \to 1$ as $L \to \infty$. Consequently we obtain (2.9).

We shall show (2.10). Since $h(x_*)=0$, we have

$$\log(\sin x_*/x_*)+\log L = \frac{x_*}{\pi - x_*}+\frac{x_*\cos x_*}{\sin x_*}-1.$$

Let $g(x)$ be as in the proof of Theorem 1.1. Substituting the equation above into $g(x_*)$ (see (2.1)), we find that

$$g(x_*) = \log((\pi - x_*)/x_*) - \frac{\pi}{\pi - x_*} - (\pi/x_*)\Big(\frac{x_* \cos x_*}{\sin x_*} - 1\Big).$$

As $L \to 1+0$, $x_* = H^{-1}(\log L + 1) \to 0$. An easy computation shows that

$$(\pi/x_*)\Big(\frac{x_* \cos x_*}{\sin x_*} - 1\Big) \to 0 \quad \text{as } x_* \to 0.$$

Therefore $g(x_*) \to \infty$ as $x_* \to +0$, and hence $\mu(p_*(L), L) = \exp(g(x_*)) \to \infty$ as $L \to 1+0$.

As $L \to \infty$, $x_* = H^{-1}(\log L + 1) \to \pi - 0$. We rewrite $g(x_*)$ as

$$g(x_*) = \log((\pi - x_*)/x_*) + \frac{\pi}{x_*}\Big(\frac{-x_*}{\pi - x_*} - \frac{x_* \cos x_*}{\sin x_*} + 1\Big).$$

Applying (2.4), we compute $g(x_*) \to -\infty$ as $L \to \infty$ and therefore $\mu(p_*(L), L)$ converges to 0. The proof is complete. □

3. Initial value problem and time map

First we shall show the uniqueness of solutions for the initial value problem

$$\begin{gathered}(|u'|^{p-2}u')' + (|u'|^{q-2}u')' + \lambda(|u|^{p-2}u + |u|^{q-2}u) = 0, \\ u(x_0) = \alpha, \quad u'(x_0) = \beta,\end{gathered} \tag{3.1}$$

because we shall use it several times. Define

$$f(t) := |t|^{p-2}t + |t|^{q-2}t. \tag{3.2}$$

Then the first equation of (3.1) becomes

$$f(u')' + \lambda f(u) = 0. \tag{3.3}$$

Now, we define the energy $E(u)$ by

$$E(u)(x) := \frac{p-1}{p}|u'(x)|^p + \frac{q-1}{q}|u'(x)|^q + \frac{\lambda}{p}|u(x)|^p + \frac{\lambda}{q}|u(x)|^q.$$

Multiplying (3.3) by u', we see that $E(u)$ is constant on x if u is a solution of (3.3). We put

$$\Phi(t) := \frac{p-1}{p}t^p + \frac{q-1}{q}t^q, \quad F(t) := \frac{1}{p}t^p + \frac{1}{q}t^q. \tag{3.4}$$

Then $E(u)$ is rewritten as

$$E(u)(x) = \Phi(|u'(x)|) + \lambda F(|u(x)|).$$

Lemma 3.1. *The initial value problem* (3.1) *has a unique solution.*

Proof. Since (3.1) is autonomous, we may assume that $x_0 = 0$, that is, we consider the initial value $u(0) = \alpha$ and $u'(0) = \beta$. By putting $v = f(u')$, problem (3.1) is rewritten as

$$\frac{d}{dx}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f^{-1}(v) \\ -\lambda f(u) \end{pmatrix}, \quad \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} \alpha \\ f(\beta) \end{pmatrix}.$$

Then the existence of a local solution of (3.1) is guaranteed by the Peano existence theorem.

Next we shall show the uniqueness of local solutions. Let u be a local solution of (3.1). Since the value $E(u)$ is constant on x, we conclude that

$$E(u)(x) = E(u)(0) = \Phi(|\beta|) + \lambda F(|\alpha|). \tag{3.5}$$

If $\alpha = \beta = 0$, then $E(u)(x) \equiv 0$, which means that $u(x) \equiv 0$ is a unique solution. From (3.5) it follows that

$$|u'(x)| = \Phi^{-1}\big(\Phi(|\beta|) + \lambda F(|\alpha|) - \lambda F(|u(x)|)\big),$$

where Φ^{-1} is the inverse function of Φ. Now we assume that $\beta > 0$. Then $u'(x) > 0$ near $x = 0$ and hence

$$u'(x) = \Phi^{-1}\big(\Phi(|\beta|) + \lambda F(|\alpha|) - \lambda F(|u(x)|)\big), \tag{3.6}$$

near $x = 0$. Since $\Phi^{-1}\big(\Phi(|\beta|) + \lambda F(|\alpha|) - \lambda F(|t|)\big)$ is continuously differentiable near $t = \alpha$, Picard's existence and uniqueness theorem implies that a solution of (3.6) with the initial condition $u(0) = \alpha$ is unique. In the same way, we can prove the uniqueness when $\beta < 0$. Finally, we assume that $\alpha \neq 0$ and $\beta = 0$. We consider the case $\alpha > 0$ only, since the case $\alpha < 0$ can be treated similarly. Then $u(x) > 0$ for $x \in [-\varepsilon, \varepsilon]$ with a small $\varepsilon > 0$. Since $f(u')' = -\lambda f(u) < 0$ in $[-\varepsilon, \varepsilon]$, $u'(x)$ is decreasing in this interval. Since $u'(0) = \beta = 0$, $u'(x) < 0$ in $(0, \varepsilon]$. Thus $u(x) > 0$ and $u'(x) < 0$ for $x \in (0, \varepsilon]$. Using these facts with $\beta = 0$ and computing as in (3.6), we have

$$u'(t) = -\Phi^{-1}\big(\lambda F(\alpha) - \lambda F(u(t))\big),$$

or equivalently

$$\frac{-u'(t)}{\Phi^{-1}(\lambda[F(\alpha) - F(u(t))])} = 1 \quad \text{for } t \in (0, \varepsilon].$$

Integrating both sides over $(0, x)$, changing the variables $u(t) = \alpha s$ and using $u(0) = \alpha$, we obtain

$$\mathcal{T}(u(x)) = x \tag{3.7}$$

for $x \in (0, \varepsilon]$, where

$$\mathcal{T}(t) = \int_{t/\alpha}^{1} \frac{\alpha}{\Phi^{-1}(\lambda[F(\alpha) - F(\alpha s)])}\, ds. \tag{3.8}$$

We note that $\mathcal{T}(t)$ is strictly decreasing in $t \in (u(\varepsilon), \alpha)$ and it has an inverse function. Consequently, any solution of (3.1) is uniquely represented as $u(x) = \mathcal{T}^{-1}(x)$, which guarantees the uniqueness of solutions.

We have shown that (3.1) has a unique local solution u. The energy identity (3.5) implies that both u and u' are bounded as far as u exists. Thus, by a standard argument, we conclude that u exists on $\mathbb{R}$ and is unique. The proof is complete. □

We summarize the properties of positive solutions for (1.1) in the next lemma.

Lemma 3.2. *Any positive solution $u(x)$ of (1.1) is concave, even and $u'(x) < 0$ in $(0, L]$.*

Proof. Let u be any positive solution of (1.1). Since $u > 0$, the first equation of (1.1) ensures that $f(u')' < 0$ in $(-L, L)$, where f is the function defined by (3.2). Therefore u' is decreasing and hence u is concave. Since u is concave, it has a unique critical point x_0, i.e., $u'(x_0) = 0$. Put $v(x) := u(2x_0 - x)$, which satisfies the first equation of (1.1). Since $v(x_0) = u(x_0)$ and $v'(x_0) = u'(x_0) = 0$, v is identically equal to u because of Lemma 3.1. Accordingly, $u(x)$ is symmetric with respect to the line $x = x_0$. Since $u(L) = u(-L) = 0$, it holds that $x_0 = 0$. Thus $u(x)$ is even. Since u is even and concave, $u'(x) < 0$ in $(0, L]$. □

Lemma 3.2 implies that, to find all positive solutions, it is sufficient to consider the initial value problem

$$\begin{gathered}(|u'|^{p-2}u')' + (|u'|^{q-2}u')' + \lambda(|u|^{p-2}u + |u|^{q-2}u) = 0,\\ u'(0) = 0, \quad u(0) = \alpha > 0.\end{gathered} \tag{3.9}$$

Then Lemma 3.1 implies that (3.9) has a unique solution, which is periodic and has zeros by a standard argument. Since the energy $E(u)$ is constant on x, the solution of (3.9) satisfies

$$E(u) = \Phi(|u'(x)|) + \lambda F(|u(x)|) = \lambda F(\alpha) \quad \text{for all } x \in \mathbb{R}.$$

Denote the smallest zero in $(0, \infty)$ of the solution of (3.9) by $T(\lambda, \alpha)$. Then we have the following lemma.

Lemma 3.3.

$$T(\lambda, \alpha) = \int_0^1 \frac{\alpha}{\Phi^{-1}(\xi)}\, ds, \tag{3.10}$$

where Φ^{-1} is the inverse function of Φ given by (3.4) *and ξ is defined by*

$$\xi := \frac{\lambda}{p}(1 - s^p)\alpha^p + \frac{\lambda}{q}(1 - s^q)\alpha^q. \tag{3.11}$$

Proof. By the same argument as in the proof of Lemma 3.1, we have (3.7) for $x \in (0, T(\lambda, \alpha)]$, where $\mathcal{T}$ is given by (3.8). Letting $x = T(\lambda, \alpha)$ in (3.7), we have

$$\int_0^1 \frac{\alpha}{\Phi^{-1}(\lambda[F(\alpha) - F(\alpha s)])}\, ds = T(\lambda, \alpha).$$

We put $\xi := \lambda(F(\alpha) - F(\alpha s))$, which is reduced to (3.11). The proof is complete. □

4. Existence and nonexistence of positive solutions

In this section, we investigate the range of λ for which (1.1) has a positive solution by using the time map and the energy. Moreover, we prove that all positive solutions are represented as a smooth curve $(\lambda(\alpha), u(x, \alpha))$ with one parameter $\alpha = u(0, \alpha) = \|u\|_\infty$ and study the properties of this bifurcation curve. Recall that $1 < q < p$ is assumed throughout the paper. By Theorem 1.1, if $L \le 1$, then $\mu(q, L) < \mu(p, L)$. However, if $L > 1$, then all cases $\mu(q, L) < \mu(p, L)$, $\mu(q, L) > \mu(p, L)$ and $\mu(q, L) = \mu(p, L)$ can occur. We state one of the main results.

Theorem 4.1. *Let $1 < q < p$ and $L > 0$. Then the following assertions hold.*

(i) *For any $\alpha > 0$, there exists a unique positive solution (λ, u) of* (1.1) *which satisfies $u(0) = \alpha$. Denote it by $(\lambda(\alpha), u(x, \alpha))$.*
(ii) *The set of all positive solutions consists only of $(\lambda(\alpha), u(x, \alpha))$ with $\alpha > 0$.*
(iii) $\lim_{\alpha \to +0} \lambda(\alpha) = \mu(q, L)$ *and* $\lim_{\alpha \to \infty} \lambda(\alpha) = \mu(p, L)$.
(iv) *Let $\phi(x)$ and $\psi(x)$ be the first eigenfunctions of the p and q Laplacian, respectively, which satisfy $\phi(0) = \psi(0) = 1$. Then it holds that*

$$\lim_{\alpha \to +0} u(x, \alpha)/\|u(\cdot, \alpha)\|_\infty = \psi(x), \quad \lim_{\alpha \to \infty} u(x, \alpha)/\|u(\cdot, \alpha)\|_\infty = \phi(x),$$

where the convergence is the strong topology in $C^1[-L, L]$.

(v) $\lambda(\alpha)$ *is a* $C^1(0,\infty)$ *function which satisfies, for* $\alpha > 0$

$$\begin{aligned} &\min\{\mu(p,L),\mu(q,L)\} \\ &< \lambda(\alpha) < \max\{(p-1)(\pi_q/(2L))^p,(q-1)(\pi_q/(2L))^q\}. \end{aligned} \tag{4.1}$$

García-Huidobro, Manásevich and Schmitt [11, Theorem 5.1] proved the existence part of (i) and (iii) of Theorem 4.1 for more general problem (1.5). By using the theorem above, we obtain the next corollary.

Corollary 4.2. *Let* $1 < q < p$ *and* $L > 0$.
(i) Suppose that $\mu(p,L) \neq \mu(q,L)$. *Then* (1.1) *has a positive solution when*

$$\min\{\mu(p,L),\mu(q,L)\} < \lambda < \max\{\mu(p,L),\mu(q,L)\}.$$

(ii) Suppose that $L > 1$ *and* $\mu(p,L) = \mu(q,L)$. *Then there exists a* $\lambda^* > \mu(p,L)$ *such that* (1.1) *has no positive solutions when* $\lambda \le \mu(p,L)$, *at least two positive solutions when* $\mu(p,L) < \lambda < \lambda^*$, *at least one positive solution when* $\lambda = \lambda^*$, *and no positive solutions when* $\lambda > \lambda^*$.

Corollary 4.2 (ii) gives the bifurcation type (E) stated in Section 1. The assertion (i) has been proved in Motreanu and Tanaka [16] also by using the variational method. We shall prove the theorem by applying the bifurcation method.

We shall show the nonexistence of a positive solution when λ does not satisfy (4.1). To this end, we shall show the next lemma.

Lemma 4.3. *Suppose that* $L > 1$, $1 < q < p$ *and* $\mu(p,L) = \mu(q,L)$. *Then no first eigenfunction of the* p*-Laplacian is equal to that of the* q*-Laplacian.*

Proof. Suppose on the contrary that u is a first eigenfunction of both p-Laplacian and q-Laplacian. By the scalar multiplication, we may assume that $u(0) = 1$. Then u is positive, even, concave and $0 < u(x) < u(0) = 1$ for $x \in (0,L)$. Put $\lambda := \mu(p,L) = \mu(q,L)$. Since u is a first eigenfunction of the p-Laplacian and $u'(0) = 0$ and $u(0) = 1$, we have the energy identity

$$\frac{p-1}{p}|u'(x)|^p + \frac{\lambda}{p}u(x)^p = \frac{\lambda}{p},$$

or equivalently,

$$|u'(x)| = (\lambda/(p-1))^{1/p}(1-u(x)^p)^{1/p} \quad \text{for all } x \in [0,L].$$

Similarly, we have

$$|u'(x)| = (\lambda/(q-1))^{1/q}(1-u(x)^q)^{1/q}.$$

Therefore it follows that for $x \in [0,L]$,

$$(\lambda/(p-1))^{1/p}(1-u(x)^p)^{1/p} = (\lambda/(q-1))^{1/q}(1-u(x)^q)^{1/q}.$$

As x varies on $[0,L]$, $u(x)$ takes all values on $[0,1]$. Accordingly, we see that

$$(\lambda/(p-1))^{1/p}(1-t^p)^{1/p} = (\lambda/(q-1))^{1/q}(1-t^q)^{1/q} \quad \text{for all } t \in [0,1].$$

This causes a contradiction. Indeed, differentiating the equation above with respect to t and dividing it by t^{q-1}, we obtain

$$(\lambda/(p-1))^{1/p}(1-t^p)^{-(p-1)/p}t^{p-q} = (\lambda/(q-1))^{1/q}(1-t^q)^{-(q-1)/q}.$$

As $t \to +0$, we find a contradiction. The proof is complete. $\square$

For simplicity, we write $\mu(p,L)$ as $\mu(p)$ if there is no confusion.

Lemma 4.4. *If $\lambda \le \min\{\mu(p), \mu(q)\}$, then* (1.1) *has no positive solutions.*

Proof. Hereafter $W_0^{1,p}(-L, L)$ denotes the Sobolev space and $\|u\|_p$ denotes the $L^p(-L, L)$ norm of u. Since the first eigenvalue is the minimum of the Rayleigh quotient, we have

$$\mu(p)\|u\|_p^p \le \|u'\|_p^p \quad \text{for } u \in W_0^{1,p}(-L, L). \tag{4.2}$$

Here the equality occurs if and only if $u \equiv 0$ or it is a first eigenfunction of the p-Laplacian. Let $\lambda \le \min\{\mu(p), \mu(q)\}$ but assume that (1.1) has a positive solution u. We divide the proof into three cases.

Case 1. Assume that $\lambda < \min\{\mu(p), \mu(q)\}$. Multiplying (1.1) by u, integrating it on $(-L, L)$ and using (4.2), we have

$$\begin{aligned} \|u'\|_p^p + \|u'\|_q^q &= \lambda(\|u\|_p^p + \|u\|_q^q) \\ &< \mu(p)\|u\|_p^p + \mu(q)\|u\|_q^q \\ &\le \|u'\|_p^p + \|u'\|_q^q. \end{aligned} \tag{4.3}$$

This is impossible.

Case 2. Assume that $\lambda = \min\{\mu(p), \mu(q)\}$ and $\mu(p) \ne \mu(q)$. Then (4.3) is still true. This is a contradiction.

Case 3. Assume that $\lambda = \mu(p) = \mu(q)$. Then (4.3) remains valid. Indeed, if the equality holds, then we have

$$\|u'\|_p^p + \|u'\|_q^q = \lambda(\|u\|_p^p + \|u\|_q^q) = \mu(p)\|u\|_p^p + \mu(q)\|u\|_q^q,$$

which with (4.2) shows that

$$\|u'\|_p^p = \mu(p)\|u\|_p^p \quad \text{and} \quad \|u'\|_q^q = \mu(q)\|u\|_q^q.$$

Hence u is a first eigenfunction of both p-Laplacian and q-Laplacian. This contradicts Lemma 4.3. The proof is complete. ☐

By using the unique solution of the initial value problem (3.9) introduced in Section 3, we shall solve our problem (1.1). Thus, we denote the unique solution of (3.9) by $U(x, \lambda, \alpha)$. Let $T(\lambda, \alpha)$ be as in (3.10). Consider the case where $U(x, \lambda, \alpha)$ becomes a positive solution of (1.1), that is,

$$U(x, \lambda, \alpha) > 0 \quad \text{in } [0, L), \quad U(L, \lambda, \alpha) = 0.$$

Then the next lemma readily follows.

Lemma 4.5. *Problem* (1.1) *has a positive solution at λ if and only if $T(\lambda, \alpha) = L$ at some $\alpha > 0$. In this case, $U(x, \lambda, \alpha)$ is a positive solution.*

Lemma 4.6. *Problem* (1.1) *has no positive solutions if*

$$\lambda \ge \max\{(p-1)(\pi_q/(2L))^p, (q-1)(\pi_q/(2L))^q\}. \tag{4.4}$$

Proof. Let λ satisfy the inequality above. We shall show that $T(\lambda, \alpha) < L$ for all $\alpha > 0$. Then the conclusion follows from Lemma 4.5. We shall estimate the time map $T(\lambda, \alpha)$. Let ξ be as in (3.11). For $s \in [0, 1]$, we have

$$\begin{aligned} \xi &= \frac{\lambda}{p}(1 - s^p)\alpha^p + \frac{\lambda}{q}(1 - s^q)\alpha^q \\ &= \frac{p-1}{p}\Big[\lambda^{1/p}\frac{(1 - s^p)^{1/p}}{(p-1)^{1/p}}\alpha\Big]^p + \frac{q-1}{q}\Big[\lambda^{1/q}\frac{(1 - s^q)^{1/q}}{(q-1)^{1/q}}\alpha\Big]^q. \end{aligned}$$

Define

$$m(\lambda) := \min\Big\{\frac{\lambda^{1/p}}{(p-1)^{1/p}}, \frac{\lambda^{1/q}}{(q-1)^{1/q}}\Big\}.$$

Since $(1-s^p)^{1/p} > (1-s^p)^{1/q} > (1-s^q)^{1/q}$ for $s \in (0,1)$, we have

$$\begin{aligned}\xi &> \frac{p-1}{p}\big[m(\lambda)(1-s^q)^{1/q}\alpha\big]^p + \frac{q-1}{q}\big[m(\lambda)(1-s^q)^{1/q}\alpha\big]^q \\ &= \Phi(m(\lambda)(1-s^q)^{1/q}\alpha),\end{aligned}$$

which is rewritten as $\Phi^{-1}(\xi) > m(\lambda)(1-s^q)^{1/q}\alpha$. Therefore we see that

$$T(\lambda,\alpha) = \int_0^1 \frac{\alpha}{\Phi^{-1}(\xi)}ds < \frac{1}{m(\lambda)}\int_0^1 (1-s^q)^{-1/q}ds = \frac{\pi_q}{2m(\lambda)}.$$

Condition (4.4) is equivalent to the inequality $\pi_q/(2m(\lambda)) \le L$. Consequently, $T(\lambda,\alpha) < L$ for all $\alpha > 0$. The proof is complete. $\square$

Combining Lemmas 4.4 and 4.6, we have

Lemma 4.7. *If λ satisfies either $\lambda \le \min\{\mu(p), \mu(q)\}$ or* (4.4)*, then* (1.1) *has no positive solutions.*

We shall investigate the behavior of positive solution (λ, u) when $\|u\|_\infty$ diverges to infinity.

Lemma 4.8. *Let (λ_n, u_n) be a sequence of positive solutions to* (1.1) *which satisfies $\|u_n\|_\infty \to \infty$. Put $v_n := u_n/\|u_n\|_\infty$. Then (λ_n, v_n) converges to $(\mu(p,L), \phi)$ in $\mathbb{R} \times C^1[-L,L]$, where $\phi(x)$ is the first eigenfunction of the p-Laplacian which satisfies $\phi(0) = 1$.*

To show the lemma above, we need an a priori estimate of positive solutions.

Lemma 4.9. *Let (λ, u) be a positive solution of* (1.1)*. Then there exists a constant $C > 0$ independent of u, λ and L such that*

$$|u'(x)| \le C\lambda^{1/(p-1)}L^{1/(p-1)}(\|u\|_\infty + 1) \quad \textit{for } x \in [-L,L].$$

Proof. Since u is even, it is enough to show the inequality above for $x \in [0,L]$. Since $1 < q < p$, there exists a constant $C > 0$ such that

$$|f(t)| \le C(|t|^{p-1} + 1) \quad \text{for } t \in \mathbb{R}, \tag{4.5}$$

where $f(t)$ is given by (3.2). Since $f(t) \ge t^{p-1}$ for $t \ge 0$, the inverse function f^{-1} satisfies

$$f^{-1}(t) \le t^{1/(p-1)} \quad \text{for } t \ge 0. \tag{4.6}$$

We rewrite (1.1) as

$$f(u')' + \lambda f(u) = 0.$$

Integrating this equation on $[0,x]$, using $u'(0) = 0$ and operating f^{-1}, we obtain

$$u'(x) = -f^{-1}\Big(\lambda\int_0^x f(u(t))dt\Big). \tag{4.7}$$

Using (4.5) and (4.6), we obtain, for $x \in [0,L]$

$$\begin{aligned}|u'(x)| &\le \Big(\lambda\int_0^x f(u(t))dt\Big)^{1/(p-1)} \\ &\le \big(\lambda LC(\|u\|_\infty^{p-1} + 1)\big)^{1/(p-1)}\end{aligned}$$

$$\le \lambda^{1/(p-1)} L^{1/(p-1)} C'(\|u\|_\infty + 1).$$

□

Proof of Lemma 4.8. Let (λ_n, u_n) be as in Lemma 4.8. Since λ_n is bounded and bounded away from zero by Lemma 4.7, a subsequence of λ_n (again denoted by λ_n) converges to a limit $\lambda > 0$. Integrating (1.1) on $(0, L)$ and using $u'(0) = 0$, we have

$$|u_n'(x)|^{p-2}u_n'(x) + |u_n'(x)|^{q-2}u_n'(x) = -\lambda_n \int_0^x (u_n(t)^{p-1} + u_n(t)^{q-1})dt. \quad (4.8)$$

Put $v_n := u_n/\|u_n\|_\infty$. Dividing the both sides by $\|u_n\|_\infty^{p-1}$, we have

$$|v_n'|^{p-2}v_n' = -\lambda_n \int_0^x (v_n(t)^{p-1} + u_n(t)^{q-1}/\|u_n\|_\infty^{p-1})dt - |u_n'|^{q-2}u_n'/\|u_n\|_\infty^{p-1}. \quad (4.9)$$

The second term $u_n(t)^{q-1}/\|u_n\|_\infty^{p-1}$ of the integrand uniformly converges to zero. By Lemma 4.9, we estimate the last term as

$$\|u_n'\|_\infty^{q-1}/\|u_n\|_\infty^{p-1} \le C^{q-1}(\|u_n\|_\infty + 1)^{q-1}/\|u_n\|_\infty^{p-1} \to 0.$$

Since $\|v_n\|_\infty = 1$ by the definition of v_n, the right-hand side of (4.9) is uniformly bounded. Thus $\|v_n'\|_\infty$ is bounded. By the Ascoli-Arzela theorem, a subsequence of v_n uniformly converges to a limit v. Denote the right hand side of (4.9) by $w_n(x)$. Then $w_n(x)$ uniformly converges to

$$w(x) := -\lambda \int_0^x v(t)^{p-1}dt. \quad (4.10)$$

Since $|v_n'|^{p-2}v_n' = w_n$ by (4.9), we rewrite this equation as

$$v_n(x) = -\int_x^L \operatorname{sgn}(w_n)|w_n(t)|^{1/(p-1)}dt.$$

Since v_n and w_n uniformly converge to v and w, respectively, we find that

$$v(x) = -\int_x^L \operatorname{sgn}(w)|w(t)|^{1/(p-1)}dt.$$

Differentiating it and using (4.10), we obtain

$$\begin{gathered}(|v'|^{p-2}v')' + \lambda v^{p-1} = 0, \quad v \ge 0, \quad \text{in } (-L, L),\\ v(-L) = v(L) = 0.\end{gathered}$$

Since $v_n(0) = 1$, it holds that $v(0) = 1$. Hence $v(x) > 0$ for $x \in (-L, L)$. Therefore v is a positive eigenfunction and so λ must be the first eigenvalue. Consequently, $\lambda = \mu(p, L)$ and $v = \phi$. From the uniqueness of the limit, (λ_n, v_n) itself (without extracting a subsequence) converges. □

We next consider the case where a sequence of positive solutions converges to zero. To this end, we prepare another a priori estimate of positive solutions.

Lemma 4.10. *For any positive solution (λ, u) satisfying $\|u\|_\infty \le 1$, it holds*

$$|u'(x)| \le (2\lambda L)^{1/(q-1)}\|u\|_\infty \quad \text{for } x \in [-L, L].$$

Proof. It is sufficient to show the inequality above for $x \in [0, L]$, because u is even. Let $f(t)$ be as in (3.2). Then we have

$$|f(t)| \le 2|t|^{q-1} \text{ for } |t| \le 1, \quad f^{-1}(t) \le t^{1/(q-1)} \text{ for } t \ge 0.$$

Using these inequalities, we estimate (4.7) for $x \in [0, L]$ as

$$|u'(x)| \le \Big(\lambda \int_0^x f(u(t))dt\Big)^{1/(q-1)} \le (2\lambda L)^{1/(q-1)} \|u\|_\infty.$$

□

We shall show that a sequence of positive solutions converging to zero approaches the first eigenfunction of the q-Laplacian.

Lemma 4.11. *Let (λ_n, u_n) be a sequence of positive solutions of (1.1) which satisfies $\|u_n\|_\infty \to 0$. Put $v_n := u_n/\|u_n\|_\infty$. Then (λ_n, v_n) converges to $(\mu(q, L), \psi)$ in $\mathbb{R} \times C^1[-L, L]$, where $\psi(x)$ denotes the first eigenfunction of the q-Laplacian satisfying $\psi(0) = 1$.*

Proof. We use the same method as in the proof of Lemma 4.8. By Lemma 4.7, λ_n is bounded and bounded away from zero. A subsequence of λ_n converges to a limit $\lambda > 0$. Dividing (4.8) by $\|u_n\|_\infty^{q-1}$, we have

$$\begin{aligned} &|v_n'|^{q-2}v_n' \\ &= -\lambda_n \int_0^x (v_n(t)^{q-1} + u_n(t)^{p-1}/\|u_n\|_\infty^{q-1})dt - |u_n'|^{p-2}u_n'/\|u_n\|_\infty^{q-1}. \end{aligned} \tag{4.11}$$

Since $1 < q < p$ and $\|u_n\|_\infty \to 0$, we use Lemma 4.10 to get

$$\|u_n\|_\infty^{p-1}/\|u_n\|_\infty^{q-1} \to 0, \quad \|u_n'\|_\infty^{p-1}/\|u_n\|_\infty^{q-1} \to 0.$$

Therefore the right hand side of (4.11) is uniformly bounded and hence $\|v_n'\|_\infty$ is bounded. By the Ascoli-Arzela theorem, a subsequence of v_n uniformly converges to a limit v. The rest of proof is the same as in the proof of Lemma 4.8. □

We are now in a position to prove Theorem 4.1.

Proof of Theorem 4.1. It is easy to verify that for any $\alpha > 0$ fixed, $T(\lambda, \alpha)$ is strictly decreasing on λ and satisfies

$$\lim_{\lambda \to +0} T(\lambda, \alpha) = \infty, \quad \lim_{\lambda \to \infty} T(\lambda, \alpha) = 0.$$

Therefore for each $\alpha > 0$, the equation $T(\lambda, \alpha) = L$ has a unique solution λ. We denote it by $\lambda(\alpha)$, that is,

$$T(\lambda(\alpha), \alpha) = L \quad \text{for } \alpha > 0. \tag{4.12}$$

Recall that $U(x, \lambda, \alpha)$ denotes a solution of (3.9). We define

$$u(x, \alpha) := U(x, \lambda(\alpha), \alpha).$$

Then $(\lambda(\alpha), u(x, \alpha))$ is a positive solution of (1.1) with λ replaced by $\lambda(\alpha)$, which satisfies $u(0, \alpha) = \alpha$. We shall show the uniqueness of such a solution. Let $\alpha > 0$. If $(\mu, v(x))$ is a positive solution satisfying $v(0) = \alpha$, then $T(\mu, \alpha) = L$ by the definition of T. From the uniqueness of $\lambda(\alpha)$ satisfying (4.12), it follows that $\lambda(\alpha) = \mu$. Since $u(0, \alpha) = \alpha = v(0)$ and $u'(0, \alpha) = 0 = v'(0)$, it holds that $u(x, \alpha) = v(x)$ from Lemma 3.1. Therefore $(\mu, v(x)) = (\lambda(\alpha), u(x, \alpha))$ and we obtain the assertion (i). By definition, $(\lambda(\alpha), u(x,\alpha))$ with $\alpha > 0$ represents all positive solutions. Thus (ii) is valid. The integrand

$$\alpha/\Phi^{-1}(\xi) = \alpha/\Phi^{-1}((\lambda/p)(1-s^p)\alpha^p + (\lambda/q)(1-s^q)\alpha^q)$$

in (3.10) is differentiable with respect to λ and α and, it holds:

$$\begin{gathered}
\frac{\partial}{\partial\lambda}(\alpha/\Phi^{-1}(\xi)) = -\frac{\alpha d\xi/d\lambda}{(\Phi^{-1}(\xi))^2\Phi'(\Phi^{-1}(\xi))},\\
\frac{\partial}{\partial\alpha}(\alpha/\Phi^{-1}(\xi)) = \frac{\Phi^{-1}(\xi)\Phi'(\Phi^{-1}(\xi)) - \alpha d\xi/d\alpha}{(\Phi^{-1}(\xi))^2\Phi'(\Phi^{-1}(\xi))},
\end{gathered}$$

where

$$\begin{gathered}
\Phi'(\Phi^{-1}(\xi)) = (p-1)\Phi^{-1}(\xi)^{p-1} + (q-1)\Phi^{-1}(\xi)^{q-1},\\
\frac{d\xi}{d\lambda} = (1-s^p)\alpha^p/p + (1-s^q)\alpha^q/q,\\
\frac{d\xi}{d\alpha} = \lambda(1-s^p)\alpha^{p-1} + \lambda(1-s^q)\alpha^{q-1}.
\end{gathered}$$

Hence, for each compact set D in $(0,\infty)\times(0,\infty)$, there exists a constant $C > 0$ such that

$$\Big|\frac{\partial}{\partial\lambda}(\alpha/\Phi^{-1}(\xi))\Big| \le C(1-s^q)^{-1/q} \quad \text{and} \quad \Big|\frac{\partial}{\partial\alpha}(\alpha/\Phi^{-1}(\xi))\Big| \le C(1-s^q)^{-1/q}$$

for $s \in (0,1)$ and $(\alpha,\lambda) \in D$. See also the proof of Proposition 5.1. Therefore, since $(1-s^q)^{-1/q} \in L^1(0,1)$, $T(\lambda,\alpha)$ has partial derivatives. Denote them by T_λ and T_α. It is easy to verify that $T_\lambda < 0$. Applying the implicit function theorem to (4.12), we find that $\lambda(\alpha)$ is a C^1 function which satisfies

$$\lambda'(\alpha) = -\frac{T_\alpha(\lambda(\alpha),\alpha)}{T_\lambda(\lambda(\alpha),\alpha)}. \tag{4.13}$$

Combining Lemmas 4.8 and 4.11, we have (iii) and (iv). Lemma 4.7 ensures (v). The proof is complete. □

Let $u(x,\alpha)$ be as in Theorem 4.1. Since $u(x,\alpha)$ is uniquely determined by α, we can identify the bifurcation curve $(\lambda(\alpha), u(x,\alpha))$ with $(\lambda(\alpha),\alpha)$. Then Theorem 4.1 (iii) means that the bifurcation curve $(\lambda(\alpha),\alpha)$ starts from the initial point $(\mu(q),0)$ and goes to the final point $(\mu(p),\infty)$. Using these results, we shall prove Corollary 4.2.

Proof of Corollary 4.2. By Theorem 4.1 (i) and (ii), (1.1) has a positive solution at λ if and only if λ is in the range of $\lambda(\alpha)$, i.e., $\lambda \in \{\lambda(\alpha) : \alpha > 0\}$. This fact with (iii) in Theorem 4.1 shows the assertion (i). We shall show (ii). Let $\mu(p) = \mu(q)$. We draw the curve $(\lambda(\alpha),\alpha)$ in the plain, where we choose λ-axis and α-axis as axes of abscissa and ordinate, respectively. By (4.1), $\lambda(\alpha) > \mu(p) = \mu(q)$ for all α. Thus the bifurcation curve stays in the right side of the line $\lambda = \mu(p)$. We define

$$\begin{aligned}
\lambda^* :&= \sup\{\lambda(\alpha) : \alpha > 0\}\\
&= \sup\{\lambda : (1.1) \text{ has a positive solution}\}.
\end{aligned} \tag{4.14}$$

Then $\lambda^* > \mu(p)$ and Theorem 4.1 (iii) ensures the assertion (ii). □

Since $\lambda(\alpha)$ depends on p, q and L also, we write it as $\lambda(\alpha,p,q,L)$. Then λ^* given by (4.14) depends on p, q and L and we denote it by $\lambda^*(p,q,L)$.

Lemma 4.12. *The function $\lambda^*(p,q,L)$ is continuous.*

Proof. Let $1 < q_0 < p_0$, $0 < L_0$ and let (p_n, q_n, L_n) be a sequence converging to (p_0, q_0, L_0). We shall show that along a subsequence of (p_n, q_n, L_n),

$$\lim_{n\to\infty} \lambda^*(p_n, q_n, L_n) \le \lambda^*(p_0, q_0, L_0). \tag{4.15}$$

Choose $\alpha_n > 0$ such that

$$|\lambda(\alpha_n, p_n, q_n, L_n) - \lambda^*(p_n, q_n, L_n)| < 1/n. \tag{4.16}$$

After extracting a subsequence, we may assume that $\alpha_n \to \infty$, $\alpha_n \to 0$ or $\alpha_n \to \alpha_\infty > 0$. By Theorem 4.1 (iii), we have $\mu(p_0, L_0), \mu(q_0, L_0) \le \lambda^*(p_0, q_0, L_0)$. Observing the proof of Lemma 4.8, we can prove that if $\alpha_n \to \infty$, then $\lambda(\alpha_n, p_n, q_n, L_n)$ converges to $\mu(p_0, L_0)$. Hence

$$\lim_{n\to\infty} \lambda^*(p_n, q_n, L_n) = \mu(p_0, L_0) \le \lambda^*(p_0, q_0, L_0).$$

In the same way as in the proof of Lemma 4.11, we can show that if $\alpha_n \to 0$, then $\lambda(\alpha_n, p_n, q_n, L_n)$ converges to $\mu(q_0, L_0)$. Thus

$$\lim_{n\to\infty} \lambda^*(p_n, q_n, L_n) = \mu(q_0, L_0) \le \lambda^*(p_0, q_0, L_0).$$

If α_n converges to a limit $\alpha_\infty > 0$, the continuity of λ implies that

$$\lim_{n\to\infty} \lambda(\alpha_n, p_n, q_n, L_n) = \lambda(\alpha_\infty, p_0, q_0, L_0) \le \lambda^*(p_0, q_0, L_0),$$

which with (4.16) yields (4.15). Consequently, all the cases of α_n satisfy (4.15).

Since $\lambda(\alpha, p_n, q_n, L_n) \le \lambda^*(p_n, q_n, L_n)$ for $\alpha > 0$, we pass to the limit to obtain

$$\lambda(\alpha, p_0, q_0, L_0) \le \liminf_{n\to\infty} \lambda^*(p_n, q_n, L_n).$$

Taking the supremum on α, we have

$$\lambda^*(p_0, q_0, L_0) \le \liminf_{n\to\infty} \lambda^*(p_n, q_n, L_n).$$

By this inequality and (4.15), $\lambda^*(p_n, q_n, L_n)$ has a subsequence that converges to $\lambda^*(p_0, q_0, L_0)$. From the uniqueness of the limit, $\lambda^*(p_n, q_n, L_n)$ itself converges. Therefore $\lambda^*(p,q,L)$ is continuous. □

We define

$$m(p,q,L) := \min\{\mu(p,L), \mu(q,L)\}, \quad M(p,q,L) := \max\{\mu(p,L), \mu(q,L)\}.$$

Consider a small perturbation of (p,q,L) satisfying Corollary 4.2 (ii). Then we have the next result.

Theorem 4.13. *Let p_0, q_0 and L_0 satisfy that $1 < L_0$, $1 < q_0 < p_0$ and $\mu(p_0, L_0) = \mu(q_0, L_0)$. If (p,q,L) is sufficiently close to (p_0, q_0, L_0) and satisfies $\mu(p,L) \ne \mu(q,L)$, then $m(p,q,L) < M(p,q,L) < \lambda^*(p,q,L)$. Moreover (1.1) has no positive solutions if $\lambda \le m(p,q,L)$, at least one positive solution if $m(p,q,L) < \lambda \le M(p,q,L)$, at least two positive solutions if $M(p,q,L) < \lambda < \lambda^*(p,q,L)$, at least one positive solution if $\lambda = \lambda^*(p,q,L)$, and no positive solutions if $\lambda > \lambda^*(p,q,L)$.*

Proof. Note that $\mu(p,L)$ and $\mu(q,L)$ are continuous and so is $\lambda^*(p,q,L)$ by Lemma 4.12. It follows from Corollary 4.2 (ii) that $\lambda^*(p_0, q_0, L_0) > \mu(p_0, L_0)$. Therefore $\lambda^*(p,q,L)$ is greater than $M(p,q,L)$ when (p,q,L) is sufficiently close to (p_0, q_0, L_0). This shows the theorem. □

Remark 4.14. Recall that the five types (A)–(E) of the bifurcation diagrams have been defined in Section 1. Let $L_* = L_*(p,q)$ be defined by (1.6). As mentioned in Section 1, $\mu(p,L) = \mu(q,L)$ if and only if $L = L_*$. By (1.3), $\mu(p,L) > \mu(q,L)$ when $L < L_*$ and $\mu(p,L) < \mu(q,L)$ when $L > L_*$. This fact with Theorem 4.13 shows that if L is slightly less than L_*, then (B) occurs, and if it is slightly greater than L_*, then (D) occurs. Moreover, we shall prove in Section 6 that if L is small enough, then (A) appears and if it is large enough, (C) occurs. Therefore, as L increases, the bifurcation diagram changes in order of (A), (B), (E), (D), (C).

5. Derivative of the time map

Let $T(\lambda,\alpha)$ be defined by (3.10). We denote by T_α and T_λ the partial derivatives of $T(\lambda,\alpha)$ with respect to α and λ, respectively. Since $T_\lambda < 0$, (4.13) implies that $\lambda'(\alpha)$ and T_α have the same sign. To investigate the sign of $\lambda'(\alpha)$, we estimate T_α. In particular, we compute the limit of T_α as $\alpha \to \infty$ or as $\alpha \to +0$.

Proposition 5.1.
$$\lim_{\alpha\to\infty} \alpha^{p+1-q} T_\alpha(\lambda,\alpha) = T_\infty(\lambda), \tag{5.1}$$
where $T_\infty(\lambda)$ is given by
$$\begin{gathered}
T_\infty(\lambda) = T_\infty(\lambda,p,q) := c_1(p,q)\lambda^{-1/p} - c_2(p,q)\lambda^{(q-p-1)/p},\\
c_1(p,q) := q^{-1}(p-q)(p-1)^{1/p}\int_0^1 (1-s^q)(1-s^p)^{-(p+1)/p}ds,\\
c_2(p,q) := q^{-1}(p-q)(q-1)(p-1)^{-(q-1)/p}\int_0^1 (1-s^p)^{(q-p-1)/p}ds.
\end{gathered}$$
Furthermore, for any compact subset K in $(0,\infty)$*, the convergence of* (5.1) *is uniform on $\lambda \in K$.*

Proof. Let ξ be defined by (3.11). We differentiate $T(\lambda,\alpha)$ given by (3.10) with respect to α to obtain
$$T_\alpha(\lambda,\alpha) = \int_0^1 \frac{\Phi^{-1}(\xi)\Phi'(\Phi^{-1}(\xi)) - \alpha d\xi/d\alpha}{(\Phi^{-1}(\xi))^2\Phi'(\Phi^{-1}(\xi))}ds.$$
We denote the numerator and the denominator of the integrand by P and Q, respectively, i.e.,
$$T_\alpha(\lambda,\alpha) = \int_0^1 \frac{P(s,\lambda,\alpha)}{Q(s,\lambda,\alpha)}ds.$$
We observe that
$$\begin{gathered}
\Phi'(\Phi^{-1}(\xi)) = (p-1)\Phi^{-1}(\xi)^{p-1} + (q-1)\Phi^{-1}(\xi)^{q-1},\\
\frac{d\xi}{d\alpha} = \lambda(1-s^p)\alpha^{p-1} + \lambda(1-s^q)\alpha^{q-1}.
\end{gathered}$$
Therefore,
$$P := (p-1)\Phi^{-1}(\xi)^p + (q-1)\Phi^{-1}(\xi)^q - \lambda[(1-s^p)\alpha^p + (1-s^q)\alpha^q], \tag{5.2}$$
$$Q := (p-1)\Phi^{-1}(\xi)^{p+1} + (q-1)\Phi^{-1}(\xi)^{q+1}. \tag{5.3}$$
Put $\eta := \Phi^{-1}(\xi)$. Then we see that
$$\xi = \Phi(\eta) = \frac{p-1}{p}\eta^p + \frac{q-1}{q}\eta^q.$$

We rewrite the relation above as

$$(p-1)\eta^p + (q-1)\eta^q = p\xi - \frac{(p-q)(q-1)}{q}\eta^q.$$

Substituting $\eta = \Phi^{-1}(\xi)$ into the relation above, we obtain

$$(p-1)\Phi^{-1}(\xi)^p + (q-1)\Phi^{-1}(\xi)^q = p\xi - \frac{(p-q)(q-1)}{q}\Phi^{-1}(\xi)^q.$$

Using this relation in (5.2) and replacing ξ by (3.11), we obtain

$$P = \frac{p-q}{q}\lambda(1-s^q)\alpha^q - \frac{(p-q)(q-1)}{q}\Phi^{-1}(\xi)^q.$$

Denote the first and the second terms on the right hand side by I and J, respectively, i.e.,

$$I := \frac{p-q}{q}\lambda(1-s^q)\alpha^q, \quad J := \frac{(p-q)(q-1)}{q}\Phi^{-1}(\xi)^q. \tag{5.4}$$

Then $P = I - J$. We compute the limits of $\alpha^{p+1-q}I/Q$ and $\alpha^{p+1-q}J/Q$ as $\alpha \to \infty$. For functions $f(t)$ and $g(t)$, we define the notation

$$f(t) \sim g(t) \quad \text{as } t \to \infty,$$

if $\lim_{t\to\infty} f(t)/g(t) = 1$. Fix $0 < s < 1$ arbitrarily. Then we see that

$$\Phi^{-1}(\xi) \sim \Big(\frac{p}{p-1}\Big)^{1/p}\xi^{1/p} \quad \text{as } \xi \to \infty, \tag{5.5}$$

$$\xi \sim \frac{\lambda}{p}(1-s^p)\alpha^p \quad \text{as } \alpha \to \infty. \tag{5.6}$$

Therefore as $\alpha \to \infty$,

$$\begin{aligned}
Q &= (p-1)\Phi^{-1}(\xi)^{p+1} + (q-1)\Phi^{-1}(\xi)^{q+1} \\
&\sim (p-1)\Phi^{-1}(\xi)^{p+1} \\
&\sim (p-1)(p/(p-1))^{(p+1)/p}\xi^{(p+1)/p} \\
&\sim (p-1)^{-1/p}\lambda^{(p+1)/p}(1-s^p)^{(p+1)/p}\alpha^{p+1}.
\end{aligned} \tag{5.7}$$

Accordingly, we have

$$\lim_{\alpha\to\infty} \frac{\alpha^{p+1-q}I}{Q} = \frac{p-q}{q}(p-1)^{1/p}\lambda^{-1/p}(1-s^q)(1-s^p)^{-(p+1)/p}. \tag{5.8}$$

By (5.4)–(5.6), J has an asymptotic formula

$$J \sim q^{-1}(p-q)(q-1)(p-1)^{-q/p}\lambda^{q/p}(1-s^p)^{q/p}\alpha^q.$$

The relation above with (5.7) implies

$$\begin{aligned}
&\lim_{\alpha\to\infty} \frac{\alpha^{p+1-q}J}{Q} \\
&= q^{-1}(p-q)(q-1)(p-1)^{-(q-1)/p}\lambda^{(q-p-1)/p}(1-s^p)^{(q-p-1)/p}.
\end{aligned} \tag{5.9}$$

We shall prove in the next Lemma 5.2 that there exists a constant $C > 0$ independent of α and s such that for $s \in (0,1)$ and $\alpha \geq 1$,

$$0 \leq \alpha^{p+1-q}I/Q \leq C(1-s^p)^{-1/q}, \tag{5.10}$$

$$0 \leq \alpha^{p+1-q}J/Q \leq C(1-s^p)^{(q-p-1)/p}. \tag{5.11}$$

The right hand sides of (5.10) and (5.11) are integrable on $(0,1)$. By the Lebesgue dominated convergence theorem with (5.8)–(5.11), we obtain

$$\begin{aligned}\lim_{\alpha\to\infty}\alpha^{p+1-q}\int_0^1\frac{I}{Q}\,ds &= \frac{p-q}{q}(p-1)^{1/p}\lambda^{-1/p}\int_0^1(1-s^q)(1-s^p)^{-(p+1)/p}\,ds\\ &= c_1(p,q)\lambda^{-1/p},\end{aligned}$$

and

$$\begin{aligned}&\lim_{\alpha\to\infty}\alpha^{p+1-q}\int_0^1\frac{J}{Q}\,ds\\ &= \frac{p-q}{q}(q-1)(p-1)^{-(q-1)/p}\lambda^{-(p+1-q)/p}\int_0^1(1-s^p)^{(q-p-1)/p}ds\\ &= c_2(p,q)\lambda^{-(p+1-q)/p}.\end{aligned}$$

Here $c_1(p,q)$ and $c_2(p,q)$ have been defined in the statement of Proposition 5.1. Thus $\alpha^{p+1-q}T_\alpha$ converges to T_∞.

Let K be any compact subset of $(0,\infty)$. Then (5.10) and (5.11) are valid for all $\lambda\in K$ and the constant C depends only on K. Therefore the convergence of (5.1) is uniform on $\lambda\in K$. The proof is complete. □

Let I, J and Q be defined in the proof above. We shall give estimates of I/Q and J/Q, which imply (5.10) and (5.11).

Lemma 5.2. *There exists a positive constant C such that*

$$\lambda^{1/p}\alpha^{p+1-q}I/Q\le C(1-s^p)^{-1/q}(1+\lambda^{-(p-q)/pq}\alpha^{-(p-q)/q}), \tag{5.12}$$

$$\lambda^{(p+1-q)/p}\alpha^{p+1-q}J/Q\le C(1-s^p)^{(q-p-1)/p}(1+\alpha^{q-p}) \tag{5.13}$$

for every $\lambda>0$, $\alpha>0$ and $s\in(0,1)$.

Proof. In this proof, we denote various positive constants independent of α, s, λ by c or C. From an easy computation, it follows that for all $t\ge 0$,

$$\Phi^{-1}(t)\le\Big(\frac{pt}{p-1}\Big)^{1/p},\quad \Phi^{-1}(t)\le\Big(\frac{qt}{q-1}\Big)^{1/q}. \tag{5.14}$$

Note that

$$\begin{gathered}\Phi^{-1}(t)\sim\Big(\frac{q}{q-1}\Big)^{1/q}t^{1/q}\quad\text{as } t\to+0,\\ \Phi^{-1}(t)\sim\Big(\frac{p}{p-1}\Big)^{1/p}t^{1/p}\quad\text{as } t\to\infty.\end{gathered}$$

By the relations above, there exists a constant $c>0$ such that

$$\Phi^{-1}(t)\ge ct^{1/q}\quad\text{when } 0\le t\le 1,\quad \Phi^{-1}(t)\ge ct^{1/p}\quad\text{when } t\ge 1. \tag{5.15}$$

Furthermore, for $s\in(0,1)$, we have

$$\xi=\frac{\lambda}{p}(1-s^p)\alpha^p+\frac{\lambda}{q}(1-s^q)\alpha^q\le C\lambda(1-s^p)(\alpha^p+\alpha^q). \tag{5.16}$$

We divide the proof into two cases: $0\le\xi\le 1$ and $\xi>1$.

Case 1. Let $0\le\xi\le 1$. Using the first inequality in (5.15), we have

$$Q\ge(q-1)\Phi^{-1}(\xi)^{q+1}\ge c\xi^{(q+1)/q}\ge c\lambda^{(q+1)/q}(1-s^p)^{(q+1)/q}\alpha^{p(q+1)/q}. \tag{5.17}$$

This inequality and (5.4) imply

$$\begin{aligned}\frac{\lambda^{1/p}\alpha^{p+1-q}I}{Q} &\le C\lambda^{-(p-q)/pq}\alpha^{-(p-q)/q}(1-s^q)(1-s^p)^{-(q+1)/q}\\ &\le C\lambda^{-(p-q)/pq}\alpha^{-(p-q)/q}(1-s^p)^{-1/q}.\end{aligned}$$

Thus we have (5.12). Let us estimate J. Using (5.4), the second inequality in (5.14) and (5.16), we have

$$J \le C\Phi^{-1}(\xi)^q \le C\xi \le C\lambda(1-s^p)(\alpha^p+\alpha^q),$$

which with (5.17) implies that

$$\frac{\lambda^{(p+1-q)/p}\alpha^{p+1-q}J}{Q} \le C(\lambda\alpha^p)^{(p-q)(q-1)/pq}(1-s^p)^{-1/q}(1+\alpha^{q-p}). \tag{5.18}$$

Since $\xi \le 1$, it holds that $(\lambda/p)(1-s^p)\alpha^p \le \xi \le 1$, and so $\lambda\alpha^p \le p(1-s^p)^{-1}$. Substituting this inequality into the right hand side of (5.18), we obtain

$$\frac{\lambda^{(p+1-q)/p}\alpha^{p+1-q}J}{Q} \le C(1-s^p)^{(q-p-1)/p}(1+\alpha^{q-p}).$$

Consequently, (5.13) holds.

Case 2. Let $\xi > 1$. By the second inequality in (5.15), we have

$$Q \ge (p-1)\Phi^{-1}(\xi)^{p+1} \ge c\xi^{(p+1)/p} \ge c\lambda^{(p+1)/p}(1-s^p)^{(p+1)/p}\alpha^{p+1}. \tag{5.19}$$

Therefore

$$\frac{\lambda^{1/p}\alpha^{p+1-q}I}{Q} \le C(1-s^q)(1-s^p)^{-(p+1)/p} \le C(1-s^p)^{-1/p} \le C(1-s^p)^{-1/q},$$

which shows (5.12). We use the first inequality in (5.14) and (5.16) to obtain

$$J \le C\Phi^{-1}(\xi)^q \le C\xi^{q/p} \le C\lambda^{q/p}(1-s^p)^{q/p}(\alpha^p+\alpha^q)^{q/p}.$$

This inequality and (5.19) give us

$$\begin{aligned}\frac{\lambda^{(p+1-q)/p}\alpha^{p+1-q}J}{Q} &\le C(1-s^p)^{(q-p-1)/p}(1+\alpha^{q-p})^{q/p}\\ &\le C(1-s^p)^{(q-p-1)/p}(1+\alpha^{q-p}).\end{aligned}$$

Thus (5.13) holds and the proof is complete. □

In the following proposition, we compute the limit of T_α as $\alpha \to +0$.

Proposition 5.3.

$$\lim_{\alpha\to+0}\alpha^{q+1-p}T_\alpha(\lambda,\alpha) = T_0(\lambda), \tag{5.20}$$

where

$$\begin{gathered}T_0(\lambda) := T_0(\lambda,p,q) := d_2(p,q)\lambda^{(p-q-1)/q} - d_1(p,q)\lambda^{-1/q},\\ d_1(p,q) := p^{-1}(p-q)(q-1)^{1/q}\int_0^1 (1-s^p)(1-s^q)^{-(q+1)/q}\,ds,\\ d_2(p,q) := p^{-1}(p-q)(p-1)(q-1)^{-(p-1)/q}\int_0^1 (1-s^q)^{(p-q-1)/q}\,ds.\end{gathered}$$

Moreover, for any compact subset K of $(0,\infty)$, the convergence of (5.20) is uniform on $\lambda \in K$.

Remark 5.4. We point out the relation between $T_0(\lambda,p,q)$ and $T_\infty(\lambda,p,q)$ as in Proposition 5.1. First, we note that

$$c_1(p,q) = -d_1(q,p) \quad \text{and} \quad c_2(p,q) = -d_2(q,p), \tag{5.21}$$

where c_i and d_i $(i=1,2)$ are constants as in Proposition 5.1 or Proposition 5.3, respectively. This leads to

$$T_0(\lambda,p,q) = -c_2(q,p)\lambda^{(p-q-1)/q} + c_1(q,p)\lambda^{-1/q} = T_\infty(\lambda,q,p),$$

whence T_0 is obtained by replacing p and q each other in T_∞ as a matter of form.

Proof of Proposition 5.3. We use the same way as in the proof of Proposition 5.1. Replacing p and q each other in the proof of Proposition 5.1, we obtain

$$P = \frac{(p-q)(p-1)}{p}\Phi^{-1}(\xi)^p - \frac{p-q}{p}\lambda(1-s^p)\alpha^p.$$

Denote the first and the second terms by M and N, respectively, that is,

$$M := \frac{(p-q)(p-1)}{p}\Phi^{-1}(\xi)^p, \quad N := \frac{p-q}{p}\lambda(1-s^p)\alpha^p.$$

Then $P = M - N$. We compute the limits of $\alpha^{q+1-p}M/Q$ and $\alpha^{q+1-p}N/Q$ as $\alpha \to +0$. Fix $0 < s < 1$ arbitrarily. Observe that

$$\Phi^{-1}(\xi) \sim \Big(\frac{q}{q-1}\Big)^{1/q}\xi^{1/q}, \quad \xi \to +0, \tag{5.22}$$

$$\xi \sim \frac{\lambda}{q}(1-s^q)\alpha^q, \quad \alpha \to +0. \tag{5.23}$$

Therefore we find that as $\alpha \to +0$,

$$\begin{aligned} Q &= (p-1)\Phi^{-1}(\xi)^{p+1} + (q-1)\Phi^{-1}(\xi)^{q+1} \\ &\sim (q-1)\Phi^{-1}(\xi)^{q+1} \\ &\sim (q-1)(q/(q-1))^{(q+1)/q}\xi^{(q+1)/q} \\ &\sim (q-1)^{-1/q}\lambda^{(q+1)/q}(1-s^q)^{(q+1)/q}\alpha^{q+1}. \end{aligned} \tag{5.24}$$

Accordingly,

$$\lim_{\alpha\to+0}\frac{\alpha^{q+1-p}N}{Q} = \frac{p-q}{p}(q-1)^{1/q}\lambda^{-1/q}(1-s^p)(1-s^q)^{-(q+1)/q}.$$

By (5.22) and (5.23), the asymptotic formula of M is computed as

$$M \sim p^{-1}(p-q)(p-1)(q-1)^{-p/q}\lambda^{p/q}(1-s^q)^{p/q}\alpha^p.$$

This expression and (5.24) imply

$$\lim_{\alpha\to+0}\frac{\alpha^{q+1-p}M}{Q} = p^{-1}(p-q)(p-1)(q-1)^{-(p-1)/q}\lambda^{(p-q-1)/q}(1-s^q)^{(p-q-1)/q}.$$

Let λ_0 and λ_1 be any numbers satisfying $0 < \lambda_0 < \lambda_1$. We choose an $\alpha_0 \in (0,1)$ so small that $\xi \le 1$ for $\alpha \in (0,\alpha_0)$, $\lambda \in [\lambda_0,\lambda_1]$ and $s \in (0,1)$. According to the next Lemma 5.5, we can prove the existence of a constant $C > 0$ depending only on λ_0 and λ_1 such that for all $\alpha \in (0,\alpha_0)$, $\lambda \in [\lambda_0,\lambda_1]$ and $s \in (0,1)$,

$$0 \le \alpha^{q+1-p}M/Q \le C(1-s^q)^{(p-q-1)/q}(1+\alpha_0^{p-q})^{p/q}, \tag{5.25}$$

$$0 \le \alpha^{q+1-p}N/Q \le C(1-s^q)^{-1/q}. \tag{5.26}$$

Therefore, by the Lebesgue dominated convergence theorem, we obtain the assertion of the proposition. □

In Section 6, we need the next lemma, which ensures (5.25) and (5.26) also. Let M, N and Q be functions as in Proposition 5.3.

Lemma 5.5. *There exists a positive constant C such that*

$$\lambda^{1/p}\alpha^{q/p}M/Q \le C(1-s^q)^{-1/p}(1+\alpha^{p-q})^{p/q}, \tag{5.27}$$

$$\lambda^{1/p}\alpha^{-p+q(p+1)/p}N/Q \le C(1-s^q)^{-1/q}(1+(\lambda\alpha^q)^{-(p-q)/pq}), \tag{5.28}$$

for every $\lambda > 0$, $\alpha > 0$ and $s \in (0,1)$ and moreover,

$$\lambda^{(q+1-p)/q}\alpha^{q+1-p}M/Q \le C(1-s^q)^{(p-q-1)/q}(1+\alpha^{p-q})^{p/q}, \tag{5.29}$$

$$\lambda^{1/q}\alpha^{q+1-p}N/Q \le C(1-s^q)^{-1/q} \tag{5.30}$$

if $\lambda > 0$, $\alpha > 0$, $s \in (0,1)$ and $\xi \le 1$, where ξ is defined by (3.11)*.*

Proof. Observe that

$$1-s^p < (p/q)(1-s^q) \quad \text{for } 0 < s < 1. \tag{5.31}$$

By (5.14) and the inequality above, we have

$$M \le C\Phi^{-1}(\xi)^p \le C\xi^{p/q} \le C\lambda^{p/q}(1-s^q)^{p/q}\alpha^p(1+\alpha^{p-q})^{p/q}, \tag{5.32}$$

$$M \le C\Phi^{-1}(\xi)^p \le C\xi \le C\lambda(1-s^q)\alpha^q(1+\alpha^{p-q}). \tag{5.33}$$

In the same way as in the proof of Lemma 5.2, we divide the proof into two cases: $0 \le \xi \le 1$ and $\xi > 1$.

Case 1. Let $0 \le \xi \le 1$. We use the first inequality in (5.15) to obtain

$$Q \ge (q-1)\Phi^{-1}(\xi)^{q+1} \ge c\xi^{(q+1)/q} \ge c\lambda^{(q+1)/q}(1-s^q)^{(q+1)/q}\alpha^{q+1}. \tag{5.34}$$

This inequality and (5.32) show that

$$\lambda^{1/p}\alpha^{q/p}M/Q \le C(\lambda\alpha^q)^{(p-1)(p-q)/pq}(1+\alpha^{p-q})^{p/q}(1-s^q)^{(p-q-1)/q}.$$

Since $(\lambda/q)(1-s^q)\alpha^q \le \xi \le 1$, we have $\lambda\alpha^q \le q(1-s^q)^{-1}$. Substituting this inequality into the right hand side of the inequality above, we obtain

$$\lambda^{1/p}\alpha^{q/p}M/Q \le C(1-s^q)^{-1/p}(1+\alpha^{p-q})^{p/q}.$$

Therefore (5.27) holds. We use the definition of N and (5.34) to get

$$\lambda^{1/p}\alpha^{-p+q(p+1)/p}N/Q \le C(\lambda\alpha^q)^{-(p-q)/pq}(1-s^p)(1-s^q)^{-(q+1)/q},$$

which with (5.31) proves (5.28). Using (5.32), (5.34) and the definition of N, we have

$$\lambda^{(q+1-p)/q}\alpha^{q+1-p}M/Q \le C(1-s^q)^{(p-q-1)/q}(1+\alpha^{p-q})^{p/q},$$

$$\lambda^{1/q}\alpha^{q+1-p}N/Q \le C(1-s^p)(1-s^q)^{-(q+1)/q}.$$

These two inequalities with (5.31) prove (5.29) and (5.30).

Case 2. Let $\xi > 1$. We have only to prove (5.27) and (5.28). We use the second inequality in (5.15) to get

$$Q \ge (p-1)\Phi^{-1}(\xi)^{p+1} \ge c\xi^{(p+1)/p} \ge c\lambda^{(p+1)/p}(1-s^q)^{(p+1)/p}\alpha^{q(p+1)/p}.$$

This inequality, (5.33) and (5.31) show that

$$\lambda^{1/p}\alpha^{q/p}M/Q \le C(1-s^q)^{-1/p}(1+\alpha^{p-q}) \le C(1-s^q)^{-1/p}(1+\alpha^{p-q})^{p/q},$$

and

$$\lambda^{1/p}\alpha^{-p+q(p+1)/p}N/Q \le C(1-s^p)(1-s^q)^{-(p+1)/p}$$
$$\le C(1-s^q)^{-1/p} \le C(1-s^q)^{-1/q}.$$

Therefore we have (5.27) and (5.28). The proof is complete. □

6. Exact shape of the bifurcation curve

In this section, we investigate the shape of the bifurcation curve $(\lambda(\alpha),\alpha)$. The next theorem draws a whole bifurcation diagram $(\lambda(\alpha),\alpha)$ under the assumption,

$$\mu(q,L) \le (p-1)^{-q/(p-q)}(q-1)^{p/(p-q)}. \tag{6.1}$$

If p and q are fixed and L is large enough, the inequality above is fulfilled.

Theorem 6.1. *Assume* (6.1)*. Then it holds that $\mu(p,L)<\lambda(\alpha)<\mu(q,L)$ and $\lambda'(\alpha)<0$ for all $\alpha>0$. Therefore* (1.1) *possesses a unique positive solution if and only if $\mu(p,L)<\lambda<\mu(q,L)$.*

The theorem above says that the bifurcation curve starts from the initial point $(\mu(q),0)$ and goes monotonically to the left and reaches the final point $(\mu(p),\infty)$. This is of type (C) stated in Section 1. In the next theorem, we consider the opposite case where L is small enough and $\mu(q,L)<\mu(p,L)$.

Theorem 6.2. *Fix $1<q<p$. If $L>0$ is small enough, then $\mu(q,L)<\lambda(\alpha)<\mu(p,L)$ and $\lambda'(\alpha)>0$ for all $\alpha>0$. Therefore* (1.1) *possesses a unique positive solution if and only if $\mu(q,L)<\lambda<\mu(p,L)$.*

The theorem above shows that the bifurcation curve starts from $(\mu(q),0)$, goes monotonically to the right and reaches the final point $(\mu(p),\infty)$. This behavior is of type (A).

Proof of Theorem 6.1. Assumption (6.1) is equivalent to the inequality

$$(p-1)(\pi_q/(2L))^p \le (q-1)(\pi_q/(2L))^q = \mu(q). \tag{6.2}$$

This inequality implies that the right hand side of (4.1) is equal to $\mu(q)$.

On the other hand, observing (1.4), we see that π_p is decreasing on p. Then we use (6.2) to obtain

$$\mu(p) = (p-1)(\pi_p/(2L))^p < (p-1)(\pi_q/(2L))^p \le \mu(q).$$

Hence $\mu(p)<\mu(q)$. By Theorem 4.1 (v), $\mu(p)<\lambda(\alpha)<\mu(q)$ for all $\alpha>0$.

We shall show that $\lambda'(\alpha)<0$. Let I and J be as in (5.4). We shall prove that $I<J$, i.e.,

$$\frac{p-q}{q}\lambda(1-s^q)\alpha^q < \frac{(p-q)(q-1)}{q}\Phi^{-1}(\xi)^q$$

for all $\alpha>0$ and $s\in(0,1)$, where Φ and ξ are the functions defined by (3.4) and (3.11), respectively. Since Φ is increasing, this inequality is equivalent to

$$\Phi\left(\left[\lambda(q-1)^{-1}(1-s^q)\right]^{1/q}\alpha\right) < \xi.$$

By the definitions of Φ and ξ, this is rewritten as

$$\frac{p-1}{p}\left(\lambda(q-1)^{-1}(1-s^q)\right)^{p/q}\alpha^p < \frac{\lambda}{p}(1-s^p)\alpha^p,$$

or equivalently,

$$\lambda^{(p-q)/q}(1-s^q)^{p/q} < (p-1)^{-1}(q-1)^{p/q}(1-s^p). \tag{6.3}$$

Hence we have only to show the inequality above with $\lambda = \lambda(\alpha)$. Since $\lambda(\alpha) < \mu(q)$, (6.1) ensures that

$$\lambda(\alpha)^{(p-q)/q} < \mu(q)^{(p-q)/q} \le (p-1)^{-1}(q-1)^{p/q}.$$

Observe that $(1-s^q)^{p/q} < 1-s^q < 1-s^p$. Multiplying these inequalities, we obtain (6.3). Therefore $I < J$, and so

$$T_\alpha(\lambda(\alpha),\alpha) = \int_0^1 \frac{I-J}{Q}\,ds < 0.$$

Since $T_\lambda < 0$, we conclude from (4.13) that $\lambda'(\alpha) < 0$ for all α. The proof is complete. □

Since $T(\lambda,\alpha)$ depends on p and q also, we denote it by $T(\lambda,\alpha,p,q)$. To prove Theorem 6.2, we need the next lemma.

Lemma 6.3. *For any $1 < q < p$, there exists a $\Lambda(p,q) > 0$ such that*

$$T_\alpha(\lambda,\alpha,p,q) > 0 \quad \text{for } \alpha > 0 \text{ and } \lambda > \Lambda(p,q).$$

Proof. Fix $1 < q < p$ and denote $T_\alpha(\lambda,\alpha,p,q)$ by $T_\alpha(\lambda,\alpha)$ for simplicity. Suppose on the contrary that $T_\alpha(\lambda_n,\alpha_n) \le 0$ along some sequences α_n and λ_n, where $\lambda_n \to \infty$ as $n \to \infty$. After choosing a subsequence of α_n, we divide the proof into five cases below: (i) $\alpha_n \to \infty$, (ii) $\alpha_n \to \alpha_0 > 0$, (iii) $\alpha_n \to 0$ and $\lambda_n\alpha_n^q \to \infty$, (iv) $\alpha_n \to 0$ and $\lambda_n\alpha_n^q \to c_0 > 0$, (v) $\alpha_n \to 0$ and $\lambda_n\alpha_n^q \to 0$. We shall prove that all the cases lead to a contradiction. Let ξ_n, P_n, Q_n, I_n and J_n be defined by (3.11), (5.2), (5.3) and (5.4) with λ and α replaced by λ_n and α_n, respectively. Since $T_\alpha(\lambda_n,\alpha_n) \le 0$, we have

$$\int_0^1 P_n/Q_n\,ds \le 0 \quad \text{for all } n \in \mathbb{N}. \tag{6.4}$$

(i) Assume that $\alpha_n \to \infty$. Fix $0 < s < 1$ arbitrarily. Since $\xi_n \to \infty$, the same computation as in (5.7) shows that

$$Q_n \sim (p-1)^{-1/p}\lambda_n^{(p+1)/p}\alpha_n^{p+1}(1-s^p)^{(p+1)/p} \quad \text{as } n \to \infty.$$

Therefore

$$\lim_{n\to\infty} \lambda_n^{1/p}\alpha_n^{p+1-q} I_n/Q_n = q^{-1}(p-q)(p-1)^{1/p}(1-s^q)(1-s^p)^{-(p+1)/p}.$$

By (5.5) and (5.6), we see that

$$J_n \sim q^{-1}(p-q)(q-1)(p-1)^{-q/p}\lambda_n^{q/p}\alpha_n^q(1-s^p)^{q/p}.$$

Hence

$$\begin{aligned}
&\lim_{n\to\infty} \lambda_n^{(p+1-q)/p}\alpha_n^{p+1-q} J_n/Q_n \\
&= q^{-1}(p-q)(q-1)(p-1)^{-(q-1)/p}(1-s^p)^{(q-p-1)/p}.
\end{aligned}$$

According to Lemma 5.2 and noting $\lambda_n, \alpha_n \to \infty$, we obtain

$$\begin{gathered}
0 \le \lambda_n^{1/p}\alpha_n^{p+1-q} I_n/Q_n \le C(1-s^p)^{-1/q}, \\
0 \le \lambda_n^{(p+1-q)/p}\alpha_n^{p+1-q} J_n/Q_n \le C(1-s^p)^{(q-p-1)/p},
\end{gathered}$$

for $0 < s < 1$, where $C > 0$ is independent of s and n. From the Lebesgue dominated convergence theorem, it follows that

$$\lim_{n\to\infty} \lambda_n^{1/p}\alpha_n^{p+1-q}\int_0^1 I_n/Q_n\,ds = c, \quad \lim_{n\to\infty} \lambda_n^{(p+1-q)/p}\alpha_n^{p+1-q}\int_0^1 J_n/Q_n\,ds = d,$$

where

$$c := q^{-1}(p-q)(p-1)^{1/p}\int_0^1 (1-s^q)(1-s^p)^{-(p+1)/p}ds,$$
$$d := q^{-1}(p-q)(q-1)(p-1)^{-(q-1)/p}\int_0^1 (1-s^p)^{(q-p-1)/p}ds.$$

Using $P_n = I_n - J_n$ and combining the relations above, we obtain

$$\lim_{n\to\infty} \lambda_n^{1/p}\alpha_n^{p+1-q}\int_0^1 P_n/Q_n\,ds = c > 0.$$

This contradicts (6.4).

(ii) Assume that α_n converges to a positive limit α_0. Fix $0 < s < 1$ arbitrarily. Then $\xi_n \sim \lambda_n\xi_0(s)$ as $n \to \infty$, where $\xi_0(s)$ is defined by

$$\xi_0(s) := \frac{1}{p}(1-s^p)\alpha_0^p + \frac{1}{q}(1-s^q)\alpha_0^q.$$

Since $\xi_n \to \infty$, we have

$$\Phi^{-1}(\xi_n) \sim (p\xi_n/(p-1))^{1/p} \sim (p/(p-1))^{1/p}\lambda_n^{1/p}\xi_0(s)^{1/p},$$
$$J_n \sim q^{-1}(p-q)(q-1)(p/(p-1))^{q/p}\lambda_n^{q/p}\xi_0(s)^{q/p},$$
$$Q_n \sim (p-1)\Phi^{-1}(\xi_n)^{p+1} \sim p^{(p+1)/p}(p-1)^{-1/p}\lambda_n^{(p+1)/p}\xi_0(s)^{(p+1)/p}.$$

Moreover, due to Lemma 5.2 and by noting $\lambda_n \to \infty$ and $\alpha_n \to \alpha_0 > 0$, there is a constant $C > 0$ independent of n and s such that

$$\lambda_n^{1/p}I_n/Q_n \le C(1-s^p)^{-1/q}, \quad \lambda_n^{(p+1-q)/p}J_n/Q_n \le C(1-s^p)^{(q-p-1)/p},$$

for $0 < s < 1$. The Lebesgue dominated convergence theorem shows that

$$\lim_{n\to\infty} \lambda_n^{1/p}\int_0^1 I_n/Q_n\,ds = c, \quad \lim_{n\to\infty} \lambda_n^{(p+1-q)/p}\int_0^1 J_n/Q_n\,ds = d,$$

where

$$c := q^{-1}(p-q)p^{-(p+1)/p}(p-1)^{1/p}\alpha_0^q\int_0^1 (1-s^q)\xi_0(s)^{-(p+1)/p}ds,$$
$$d := q^{-1}(p-q)(q-1)p^{(q-p-1)/p}(p-1)^{-(q-1)/p}\int_0^1 \xi_0(s)^{(q-p-1)/p}ds.$$

By the relations above, we have

$$\lim_{n\to\infty} \lambda_n^{1/p}\int_0^1 P_n/Q_n\,ds = c > 0,$$

which contradicts (6.4).

(iii) Assume that $\alpha_n \to 0$ and $\lambda_n\alpha_n^q \to \infty$. We rewrite M and N given in the proof of Proposition 5.3 as M_n and N_n, after replacing λ and α by λ_n and α_n, respectively. Since $\alpha_n \to 0$ and $\lambda_n\alpha_n^q \to \infty$, we have, as $n \to \infty$

$$\xi_n \sim \frac{\lambda_n}{q}(1-s^q)\alpha_n^q,$$

$$\begin{aligned} \Phi^{-1}(\xi_n)^p &\sim p\xi_n/(p-1) \sim \frac{p}{(p-1)q}\lambda_n(1-s^q)\alpha_n^q, \\ Q_n &\sim (p-1)\Phi^{-1}(\xi_n)^{p+1} \\ &\sim (p/q)^{(p+1)/p}(p-1)^{-1/p}\lambda_n^{(p+1)/p}\alpha_n^{q(p+1)/p}(1-s^q)^{(p+1)/p}, \end{aligned}$$

which shows that

$$M_n \sim q^{-1}(p-q)\lambda_n\alpha_n^q(1-s^q).$$

By Lemma 5.5 with the facts that $p>q$, $\alpha_n \to 0$ and $\lambda_n\alpha_n^q \to \infty$, there exists a constant $C>0$ independent of n such that

$$\lambda_n^{1/p}\alpha_n^{q/p}M_n/Q_n \le C(1-s^q)^{-1/p},$$
$$\lambda_n^{1/p}\alpha_n^{-p+q(p+1)/p}N_n/Q_n \le C(1-s^q)^{-1/q}.$$

Therefore we obtain constants $c,d>0$ such that

$$\lim_{n\to\infty}\lambda_n^{1/p}\alpha_n^{q/p}\int_0^1 M_n/Q_n\,ds = c>0,$$
$$\lim_{n\to\infty}\lambda_n^{1/p}\alpha_n^{-p+q(p+1)/p}\int_0^1 N_n/Q_n\,ds = d>0.$$

Combining these identities, we obtain

$$\lim_{n\to\infty}\lambda_n^{1/p}\alpha_n^{q/p}\int_0^1 P_n/Q_n\,ds = c>0.$$

This contradicts (6.4).

(iv) Assume that $\alpha_n \to 0$ and $\lambda_n\alpha_n^q$ converges to a positive limit c_0. From this assumption, it follows that $\lambda_n\alpha_n^p \to 0$. Therefore, as $n\to\infty$,

$$\xi_n \to \xi_0(s) := \frac{c_0}{q}(1-s^q),$$
$$Q_n \to (p-1)\Phi^{-1}(\xi_0(s))^{p+1} + (q-1)\Phi^{-1}(\xi_0(s))^{q+1},$$
$$M_n \to p^{-1}(p-q)(p-1)\Phi^{-1}(\xi_0(s))^p,$$
$$N_n = \frac{p-q}{p}\lambda_n(1-s^p)\alpha_n^p \to 0.$$

All the convergences above are uniform on s. Hence we see that

$$\lim_{n\to\infty}\int_0^1 P_n/Q_n\,ds = c>0,$$

with some $c>0$. This contradicts (6.4).

(v) Assume that $\alpha_n \to 0$ and $\lambda_n\alpha_n^q \to 0$. Then $\xi_n \sim q^{-1}\lambda_n(1-s^q)\alpha_n^q$. Since ξ_n converges to 0 uniformly on s, we use (5.22) to obtain

$$\Phi^{-1}(\xi_n) \sim (q\xi_n/(q-1))^{1/q} \sim (q-1)^{-1/q}\lambda_n^{1/q}\alpha_n(1-s^q)^{1/q}.$$

Moreover,

$$M_n \sim p^{-1}(p-q)(p-1)(q-1)^{-p/q}\lambda_n^{p/q}\alpha_n^p(1-s^q)^{p/q},$$
$$Q_n \sim (q-1)\Phi^{-1}(\xi_n)^{q+1} \sim (q-1)^{-1/q}\lambda_n^{(q+1)/q}\alpha_n^{q+1}(1-s^q)^{(q+1)/q}.$$

Note that $p>q$, $\alpha_n\to 0$ and $\xi_n \to 0$ uniformly on s. Then by Lemma 5.5, there exists a constant $C>0$ independent of n such that

$$\lambda_n^{(q+1-p)/q}\alpha_n^{q+1-p}M_n/Q_n \le C(1-s^q)^{(p-q-1)/q},$$

$$\lambda_n^{1/q}\alpha_n^{q+1-p}N_n/Q_n \le C(1-s^q)^{-1/q},$$

for sufficiently large n. By the Lebesgue dominated convergence theorem, we have constants $c, d > 0$ such that

$$\lim_{n\to\infty}\lambda_n^{(q+1-p)/q}\alpha_n^{q+1-p}\int_0^1 M_n/Q_n\,ds = c,$$

$$\lim_{n\to\infty}\lambda_n^{1/q}\alpha_n^{q+1-p}\int_0^1 N_n/Q_n\,ds = d.$$

From these identities, we obtain

$$\lim_{n\to\infty}\lambda_n^{(q+1-p)/q}\alpha_n^{q+1-p}\int_0^1 P_n/Q_n\,ds = c > 0,$$

which contradicts (6.4). The proof is complete. □

Proof of Theorem 6.2. Fix $1 < q < p$. If $L \le 1$, then $\mu(q,L) < \mu(p,L)$ by Theorem 1.1. Recall that $\lambda(\alpha) > \min\{\mu(p,L), \mu(q,L)\}$ for all $\alpha > 0$. Therefore $\mu(q,L) < \lambda(\alpha)$ for $\alpha > 0$. Let $\Lambda(p,q)$ be as in Lemma 6.3. Observe that $\mu(q,L) \to \infty$ as $L \to +0$ by the definition of $\mu(q,L)$. Thus, if L is small enough, then $\mu(q,L) > \Lambda(p,q)$ and hence $\lambda(\alpha) > \Lambda(p,q)$. Lemma 6.3 with (4.13) ensures that

$$\lambda'(\alpha) = -\frac{T_\alpha(\lambda(\alpha),\alpha)}{T_\lambda(\lambda(\alpha),\alpha)} > 0 \quad \text{for all } \alpha > 0.$$

Since $\lambda(\alpha)$ converges to $\mu(p,L)$ as $\alpha \to \infty$, it holds that $\mu(q,L) < \lambda(\alpha) < \mu(p,L)$ for $\alpha > 0$. The proof is complete. □

7. Bifurcation curve near the initial and final points

In this section, we investigate the direction in which the bifurcation curve moves near the initial point and near the final point. Using these results, we shall construct the bifurcation diagrams (B) and (D) stated in Section 1 .

Let $c_i(p,q)$ and $d_i(p,q)$ with $i = 1, 2$ be the constants given in Propositions 5.1 and 5.3. We define

$$z_\infty(p,q) := (c_2(p,q)/c_1(p,q))^{p/(p-q)}, \tag{7.1}$$

$$z_0(p,q) := (d_1(p,q)/d_2(p,q))^{q/(p-q)}. \tag{7.2}$$

Then z_∞ and z_0 are unique zeros of $T_\infty(\lambda)$ and $T_0(\lambda)$ in $(0,\infty)$, respectively. Moreover, we easily see that $z_\infty(p,q) = z_0(q,p)$ as a matter of form. Indeed, it follows from (5.21) in Remark 5.4 that

$$\begin{aligned} z_\infty(p,q) &= \Big(\frac{c_2(p,q)}{c_1(p,q)}\Big)^{p/(p-q)} = \Big(\frac{d_2(q,p)}{d_1(q,p)}\Big)^{p/(p-q)} \\ &= \Big(\frac{d_1(q,p)}{d_2(q,p)}\Big)^{p/(q-p)} = z_0(q,p). \end{aligned}$$

By the definitions of T_∞ and T_0, we have

Lemma 7.1. (i) *$T_\infty(\lambda) > 0$ when $\lambda > z_\infty$ and $T_\infty(\lambda) < 0$ when $\lambda < z_\infty$.*
(ii) *$T_0(\lambda) > 0$ when $\lambda > z_0$ and $T_0(\lambda) < 0$ when $\lambda < z_0$.*

In the next proposition, we shall prove that the sign of $\lambda'(\alpha)$ for α large enough (or small enough) is determined by the order relation between the first eigenvalue $\mu(p,L)$ (or $\mu(q,L)$) and the zero z_∞ (or z_0, respectively).

Proposition 7.2. (i) *If $\mu(p, L) > z_\infty(p, q)$, then $\lambda'(\alpha) > 0$ for $\alpha > 0$ large enough.*
(ii) *If $\mu(p, L) < z_\infty(p, q)$, then $\lambda'(\alpha) < 0$ for $\alpha > 0$ large enough.*
(iii) *If $\mu(q, L) > z_0(p, q)$, then $\lambda'(\alpha) > 0$ for $\alpha > 0$ small enough.*
(iv) *If $\mu(q, L) < z_0(p, q)$, then $\lambda'(\alpha) < 0$ for $\alpha > 0$ small enough.*

Proof. Since $\lim_{\alpha\to\infty} \lambda(\alpha) = \mu(p, L)$, we use Proposition 5.1 to obtain

$$\lim_{\alpha\to\infty} \alpha^{p+1-q} T_\alpha(\lambda(\alpha), \alpha) = T_\infty(\mu(p, L)).$$

Assume that $\mu(p, L) > z_\infty(p, q)$. Then $T_\infty(\mu(p, L)) > 0$ by Lemma 7.1 (i). For $\alpha > 0$ large enough, we use (4.13) to obtain

$$\alpha^{p+1-q}\lambda'(\alpha) = -\frac{\alpha^{p+1-q} T_\alpha(\lambda(\alpha), \alpha)}{T_\lambda(\lambda(\alpha), \alpha)} > 0,$$

because $T_\lambda < 0$. Therefore the assertion (i) holds. The other assertions can be proved in the same method. □

The proposition above indicates which direction the bifurcation curve goes to near the initial point $(\mu(q), 0)$ and near the final point $(\mu(p), \infty)$. If $\lambda'(\alpha) > 0$, then the curve $(\lambda(\alpha), \alpha)$ goes to the right in the (λ, α) plain. If $\lambda'(\alpha) < 0$, then it moves to the left.

Theorem 7.3. *Let $q > 1$ and $L > 0$.*
(i) *If $\pi_q > 2L$, then for p large enough, $\lambda'(\alpha)$ is positive for $\alpha > 0$ small enough.*
(ii) *If $\pi_q < 2L$, then for p large enough, it holds that $\mu(p, L) < \mu(q, L)$ and $\lambda'(\alpha)$ is negative for $\alpha > 0$ small enough.*

The value $2L$ is the length of the interval $[-L, L]$. The theorem above means that the relation between the length of the interval and π_q determines the direction in which the bifurcation curve starts up. If the length is less (or greater) than π_q, the curve grows to the right (or left, respectively).

Using Theorem 7.3, we show the uniqueness of positive solutions for λ slightly greater than $\mu(q, L)$ in the next corollary.

Corollary 7.4. *Let $1 < q$ and $0 < L \le 1$. If p is large enough, there exists an $\varepsilon > 0$ such that a positive solution of* (1.1) *is unique when $\mu(q, L) < \lambda < \mu(q, L) + \varepsilon$.*

Proof. Let $1 < q < p$ and $0 < L \le 1$. Then Theorem 1.1 ensures that $\mu(q, L) < \mu(p, L)$. It follows easily from (1.4) that π_q is strictly decreasing in q, $\lim_{q\to 1+0} \pi_q = \infty$ and $\lim_{q\to\infty} \pi_q = 2$. Therefore $\pi_q > 2 \ge 2L$. If p is large enough, then Theorem 7.3 (i) shows that $\lambda'(\alpha) > 0$ for $\alpha \in (0, a)$ with an $a > 0$ small enough. We define $\lambda_0 := \inf_{a<\alpha<\infty} \lambda(\alpha)$. Then $\lambda_0 > \mu(q, L)$ because $\lambda(\alpha) > \mu(q, L)$ for any $\alpha > 0$ and $\lambda(\alpha) \to \mu(p, L)$ as $\alpha \to \infty$ by Theorem 4.1 (iii) and (v) (note $\mu(q, L) < \mu(p, L)$ also). For each $\mu \in (\mu(q, L), \lambda_0)$, there exists a unique α which satisfies $\mu = \lambda(\alpha)$, that is, a positive solution is unique. The proof is complete. □

Using Theorem 7.3, we have another corollary.

Corollary 7.5. *Let q and L satisfy that $\pi_q > 2L > 2$. If p is large enough, then $\mu(p, L) < \mu(q, L) < \lambda^*$, where λ^* is given by* (4.14)*. Moreover, the following assertions hold.*
(i) *If $\lambda \le \mu(p, L)$ or $\lambda > \lambda^*$,* (1.1) *has no positive solutions.*

(ii) *If* $\mu(p,L) < \lambda \le \mu(q,L)$ *or* $\lambda = \lambda^*$, (1.1) *has at least one positive solution.*
(iii) *If* $\mu(q,L) < \lambda < \lambda^*$, (1.1) *has at least two positive solutions.*

Proof. Since $L > 1$, $\mu(p,L) \to 0$ as $p \to \infty$ by (2.8). Therefore $\mu(p,L) < \mu(q,L)$ for p large enough. Since $\pi_q > 2L$, the bifurcation curve $(\lambda(\alpha),\alpha)$ grows to the right from the initial point $(\mu(q,L),0)$ by Theorem 7.3 (i). Therefore $\mu(p,L) < \mu(q,L) < \lambda^*$ and the assertions (i)–(iii) follow. □

Corollary 7.5 gives an example of type (D). In theorem 7.3, we fixed q and then took p large enough. In the next theorem, we consider the opposite case where p is fixed and then q is sufficiently close to 1.

Theorem 7.6. *Let* $p > 1$ *and* $L > 0$*. If* $q \in (1,\infty)$ *is sufficiently close to* 1*, then* $\lambda'(\alpha) > 0$ *for* $\alpha > 0$ *small enough. Therefore the bifurcation curve* $(\lambda(\alpha),\alpha)$ *grows to the right from the initial point* $(\mu(q),0)$*.*

Since $\pi_q \to \infty$ as $q \to 1+0$, it holds that $\pi_q > 2L$ for q sufficiently close to 1. Hence Theorem 7.6 perhaps follows from Theorem 7.3 (i). However this is not true. Indeed, in Theorem 7.3 (i) we need to choose p large enough, but in Theorem 7.6 we can take p as any number greater than 1 and then choose q sufficiently close to 1. Therefore non of these theorems follows from another.

Consider the case where $L > 1$ and $\mu(p,L) < 1/L$. For example, fix $L > 1$ and then choose p large enough. Then this inequality holds. Recall (2.6), i.e., $\mu(q,L) \to 1/L$ as $q \to 1+0$. Therefore if q is sufficiently close to 1, then $\mu(p,L) < \mu(q,L)$. By Theorem 7.6, the bifurcation curve $(\lambda(\alpha),\alpha)$ grows to the right from $(\mu(q,L),0)$, and hence $\lambda^* > \mu(q,L) > \mu(p,L)$. This case gives an example of type (D). Furthermore, we have the next corollary.

Corollary 7.7. *Let* L *and* p *satisfy that* $L > 1$ *and* $\mu(p,L) < 1/L$*. If* q *is sufficiently close to* 1*, then* $\mu(p,L) < \mu(q,L) < \lambda^*$ *and the assertions of Corollary 7.5 are valid.*

To prove Theorems 7.3 and 7.6, we need the two lemmas below.

Lemma 7.8. *Define*
$$K(p,q) := \int_0^1 (1-s^q)^p\, ds.$$
Then for each $q > 1$*,*
$$\lim_{p\to\infty} p^{1/q} q K(p,q) = \int_0^\infty t^{-(q-1)/q} e^{-t} dt.$$

Proof. We use the change of variables $s^q = t/p$. Since
$$s = (t/p)^{1/q}, \quad ds = p^{-1/q} q^{-1} t^{-(q-1)/q} dt,$$
K is rewritten as
$$K = p^{-1/q} q^{-1} \int_0^p (1-t/p)^p t^{-(q-1)/q} dt,$$
or equivalently,
$$p^{1/q} q K = \int_0^p (1-t/p)^p t^{-(q-1)/q} dt.$$
We denote the integrand by $g(t,p)$. Then
$$\lim_{p\to\infty} g(t,p) = t^{-(q-1)/q} e^{-t},$$

$$0 \le g(t,p) \le t^{-(q-1)/q} e^{-t} \quad \text{for } p > 1 \text{ and } t \in (0,p).$$

By the Lebesgue dominated convergence theorem, as $p \to \infty$,

$$\int_0^p g(t,p)dt \to \int_0^\infty t^{-(q-1)/q} e^{-t} dt.$$

This completes the proof. □

Lemma 7.9. *Let $z_0(p,q)$ be the constant as in* (7.2). *Then*

$$\lim_{p\to\infty} z_0(p,q) = q-1, \tag{7.3}$$

$$\lim_{q\to 1+0} z_0(p,q) = 0. \tag{7.4}$$

Proof. By the definitions of d_1 and d_2 in Proposition 5.3, we have

$$\frac{d_1(p,q)}{d_2(p,q)} = (p-1)^{-1}(q-1)^{p/q}(V/W), \tag{7.5}$$

where

$$V = V(p,q) := \int_0^1 (1-s^p)(1-s^q)^{-(q+1)/q} ds,$$

$$W = W(p,q) := \int_0^1 (1-s^q)^{(p-q-1)/q} ds.$$

Using $K(p,q)$ in Lemma 7.8, we write $W(p,q) = K((p-q-1)/q, q)$ and obtain

$$\begin{aligned} \lim_{p\to\infty} p^{1/q} W(p,q) &= \lim_{p\to\infty} q^{1/q}((p-q-1)/q)^{1/q} K((p-q-1)/q, q) \\ &= q^{-(q-1)/q} \int_0^\infty t^{-(q-1)/q} e^{-t} dt. \end{aligned}$$

We denote the right hand side by W_q. For p large enough, it holds that

$$(1/2) p^{-1/q} W_q \le W(p,q) \le 2p^{-1/q} W_q. \tag{7.6}$$

Since $1 - s < 1 - s^p < p(1-s)$ for $s \in (0,1)$, it follows that

$$\int_0^1 (1-s)(1-s^q)^{-(q+1)/q} ds \le V \le p \int_0^1 (1-s)(1-s^q)^{-(q+1)/q} ds.$$

Writing the left hand side as V_q, we have

$$V_q \le V(p,q) \le pV_q. \tag{7.7}$$

Note that the constants V_q and W_q depend only on q. By (7.5)–(7.7), we obtain

$$\begin{aligned} &(V_q W_q^{-1}/2)^{q/(p-q)} (p-1)^{-q/(p-q)} p^{1/(p-q)} (q-1)^{p/(p-q)} \\ &\le (d_1/d_2)^{q/(p-q)} \\ &\le (2V_q W_q^{-1})^{q/(p-q)} (p-1)^{-q/(p-q)} p^{(q+1)/(p-q)} (q-1)^{p/(p-q)}. \end{aligned}$$

Recall that $z_0(p,q) = (d_1/d_2)^{q/(p-q)}$. Letting $p \to \infty$, we obtain (7.3).

We shall show (7.4). Since $1 - s^p < p(1-s)$ and $1 - s^q > 1 - s$ for $s \in (0,1)$, we have

$$V(p,q) \le p \int_0^1 (1-s)^{-1/q} ds = pq/(q-1).$$

Hence

$$\limsup_{q\to 1+0} (q-1)V(p,q) \le p. \tag{7.8}$$

By the Lebesgue dominated convergence theorem, we see easily that

$$\lim_{q\to 1+0} W(p,q) = \int_0^1 (1-s)^{p-2} ds = \frac{1}{p-1}. \tag{7.9}$$

From (7.5), (7.8) and (7.9), it follows that

$$\limsup_{q\to 1+0}(d_1/d_2) = \limsup_{q\to 1+0}(p-1)^{-1}(q-1)^{p/q}(V/W) = 0.$$

Since $z_0 = (d_1/d_2)^{q/(p-q)}$, (7.4) holds. □

Using the two lemmas above, we shall prove Theorems 7.3 and 7.6.

Proof of Theorem 7.3. Assume that $\pi_q > 2L$. We use Lemma 7.9 to get

$$\mu(q,L) = (q-1)(\pi_q/(2L))^q > q-1 = \lim_{p\to\infty} z_0(p,q).$$

Therefore $\mu(q,L) > z_0(p,q)$ for p large enough. By Proposition 7.2 (iii), $\lambda'(\alpha) > 0$ for $\alpha > 0$ small enough. Thus the assertion (i) holds.

Let us show (ii). As stated in the proof of Corollary 7.4, π_q is greater than 2. Therefore if $\pi_q \le 2L$, then $L > 1$. Hence (2.8) implies that $\mu(p,L) < \mu(q,L)$ for p large enough. The negativity of $\lambda'(\alpha)$ can be proved in the same way as in (i). □

Proof of Theorem 7.6. By (2.6) and (7.4), $\mu(q,L) > z_0(p,q)$ for q sufficiently close to 1. From Proposition 7.2 (iii), the conclusion follows. □

In Theorems 7.3 and 7.6, we studied the behavior of the bifurcation curve $(\lambda(\alpha),\alpha)$ near the initial point $(\mu(q),0)$. We shall investigate it near the final point $(\mu(p),\infty)$. To this end, we compute $z_\infty(p,q)$ in the next lemma.

Lemma 7.10. *Let $z_\infty(p,q)$ be the constant as in* (7.1)*. Then*

$$\lim_{p\to\infty} z_\infty(p,q) = q+1, \tag{7.10}$$

$$\lim_{q\to 1+0} z_\infty(p,q) = (p-1)^{-1/(p-1)}\Big(\int_0^1 (1-s)(1-s^p)^{-(p+1)/p} ds\Big)^{-p/(p-1)}. \tag{7.11}$$

Proof. We define

$$\begin{gathered} X := X(p,q) := \int_0^1 (1-s^q)(1-s^p)^{-(p+1)/p} ds, \\ Y := Y(p,q) := \int_0^1 (1-s^p)^{(q-p-1)/p} ds. \end{gathered} \tag{7.12}$$

Fix $q > 1$ arbitrarily. We apply the Lebesgue dominated convergence theorem to obtain

$$\lim_{p\to\infty} X(p,q) = \int_0^1 (1-s^q) ds = \frac{q}{q+1}. \tag{7.13}$$

We shall compute the limit of $Y(p,q)$ as $p\to\infty$. Changing the variables $s^p = t$, we have

$$Y(p,q) = (1/p)B(1/p,(q-1)/p),$$

where $B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}dt$ is the beta function. Using the relation $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$, where $\Gamma(x)$ is the gamma function, we rewrite $Y(p,q)$ as

$$Y(p,q) = \frac{\Gamma(1/p)\Gamma((q-1)/p)}{p\Gamma(q/p)}.$$

Since $\lim_{x\to+0} x\Gamma(x) = \lim_{x\to+0}\Gamma(x+1) = \Gamma(1) = 1$, we have

$$\begin{aligned} \lim_{p\to\infty} Y(p,q) &= \lim_{p\to\infty} \frac{\Gamma(1/p)\Gamma((q-1)/p)}{p\Gamma(q/p)} \\ &= \lim_{p\to\infty} \frac{[(1/p)\Gamma(1/p)][((q-1)/p)\Gamma((q-1)/p)]}{(q/p)\Gamma(q/p)} \frac{q}{q-1} \\ &= \frac{q}{q-1}. \end{aligned} \tag{7.14}$$

By the definitions of z_∞, c_1 and c_2, we find that

$$z_\infty(p,q) = (c_2/c_1)^{p/(p-q)} = (q-1)^{p/(p-q)}(p-1)^{-q/(p-q)}(Y/X)^{p/(p-q)}.$$

Using (7.13) and (7.14), we obtain $\lim_{p\to\infty} z_\infty(p,q) = q+1$. Thus (7.10) holds.

Let us prove (7.11). In the same way as in (7.14), we have

$$\begin{aligned} \lim_{q\to 1+0}(q-1)Y(p,q) &= \lim_{q\to 1+0} \frac{(q-1)\Gamma(1/p)\Gamma((q-1)/p)}{p\Gamma(q/p)} \\ &= \lim_{q\to 1+0} \frac{\Gamma(1/p)[((q-1)/p)\Gamma((q-1)/p)]}{\Gamma(q/p)} = 1. \end{aligned}$$

As $q \to 1+0$, we find easily that

$$X(p,q) = \int_0^1 (1-s^q)(1-s^p)^{-(p+1)/p} ds \to \int_0^1 (1-s)(1-s^p)^{-(p+1)/p} ds.$$

Therefore

$$\begin{aligned} \frac{c_2}{c_1} &= (q-1)(p-1)^{-q/p} Y(p,q)X(p,q)^{-1} \\ &\to (p-1)^{-1/p}\Big(\int_0^1 (1-s)(1-s^p)^{-(p+1)/p}\, ds\Big)^{-1} \quad \text{as } q \to 1+0. \end{aligned}$$

Since $z_\infty = (c_2/c_1)^{p/(p-q)}$, we obtain (7.11). The proof is complete. □

In the next theorem, we shall investigate how the bifurcation curve behaves as it approaches the final point $(\mu(p), \infty)$.

Theorem 7.11. (i) *Let $q > 1$ and $L \le 1$. If p is large enough, then $\mu(q,L) < \mu(p,L)$ and $\lambda'(\alpha) > 0$ for α large enough.*

(ii) *Let $q > 1$ and $L > 1$. If p is large enough, then $\mu(p,L) < \mu(q,L)$ and $\lambda'(\alpha) < 0$ for α large enough. Moreover, a positive solution of* (1.1) *is unique when $\mu(p,L) < \lambda < \mu(p,L) + \varepsilon$ with a small $\varepsilon > 0$.*

The theorem above says that if $L \le 1$ and p is large enough, the bifurcation curve $(\lambda(\alpha), \alpha)$ approaches the final point $(\mu(p), \infty)$ from the left. On the other hand, if $L > 1$ and p is large enough, then the curve approaches the final point from the right. If $\pi_q > 2L$, we combine Theorem 7.3 (i) with Theorem 7.11 (ii). Then this gives Type (D).

Proof of Theorem 7.11. Fix $q > 1$ and $L \le 1$ arbitrarily. We use (7.10) with the fact that $\mu(p,L) \to \infty$ as $p \to \infty$ by (2.7). Then $\mu(p,L) > z_\infty(p,q)$ for p large enough. By Proposition 7.2 (i), $\lambda'(\alpha) > 0$ for $\alpha > 0$ large enough.

Let us show (ii). Fix $q > 1$ and $L > 1$. Since $\lim_{p\to\infty}\mu(p,L) = 0$ by (2.8), it follows from (7.10) that $\mu(p,L) < \min\{z_\infty(p,q), \mu(q,L)\}$ for p large enough. By Proposition 7.2 (ii), there exists an $A > 0$ such that $\lambda'(\alpha) < 0$ for $\alpha \ge A$. Consequently, $\lambda(\alpha)$ is decreasing and converges to $\mu(p,L)$ as $\alpha \to \infty$. We put

$\lambda_0 := \inf_{0<\alpha\le A} \lambda(\alpha)$. Then $\lambda_0 > \mu(p, L)$ holds because $\lim_{\alpha\to+0} \lambda(\alpha) = \mu(q, L)$ and $\lambda(\alpha) > \mu(p, L)$ for any $\alpha > 0$. For each $\mu \in (\mu(p, L), \lambda_0)$, there exists a unique α which satisfies $\mu = \lambda(\alpha)$. This shows the uniqueness of positive solutions. The proof is complete. □

Let us consider type (B), which has been obtained as a small perturbation from type (E) in Remark 4.14. We shall construct type (B) by a different method in the next theorem.

Theorem 7.12. *If $L \in (1, \infty)$ is sufficiently close to 1, there exist p and q such that $1 < q < p$ and*

$$\mu(q, L) < \mu(p, L) < z_\infty(p, q). \tag{7.15}$$

Therefore $\lambda'(\alpha) < 0$ for $\alpha > 0$ large enough and $\mu(q, L) < \mu(p, L) < \lambda^*$.

The theorem above says that the bifurcation curve reaches the final point from the right and gives an example of type (B). The next corollary follows immediately from the theorem above.

Corollary 7.13. *Let p, q and L satisfy $L > 1$ and (7.15). Then the following assertions hold.*

(i) *If $\lambda \le \mu(q, L)$ or $\lambda > \lambda^*$, there exist no positive solutions.*
(ii) *If $\mu(q, L) < \lambda \le \mu(p, L)$ or $\lambda = \lambda^*$, there exists at least one positive solution.*
(iii) *If $\mu(p, L) < \lambda < \lambda^*$, there exist at least two positive solutions.*

In Lemma 7.10, we investigated the behavior of $z_\infty(p, q)$ as $q \to 1 + 0$. Using this result, we prove Theorem 7.12.

Proof of Theorem 7.12. Denote the right hand side of (7.11) by $Z_\infty(p)$, i.e.,

$$Z_\infty(p) := (p-1)^{-1/(p-1)} \Big(\int_0^1 (1-s)(1-s^p)^{-(p+1)/p} ds \Big)^{-p/(p-1)}.$$

A simple computation shows that $\lim_{p\to\infty} Z_\infty(p) = 2$. Fix a satisfying $1 < a < 2$. Then there exists a $p(a) > 1$ such that if $p > p(a)$, then $Z_\infty(p) > a$. By (2.9) and (2.10), we observe that

$$\lim_{L\to1+0} p_*(L) = \infty, \quad \lim_{L\to1+0} \mu(p_*(L), L) = \infty.$$

Accordingly, if L is slightly greater than 1, then

$$\mu(p_*(L), L) > a, \quad p_*(L) > p(a). \tag{7.16}$$

We fix such an $L > 1$. Since $\mu(p, L)$ is strictly decreasing in $p \in [p_*(L), \infty)$ and converges to 0 as $p \to \infty$, we can choose $p \in (p_*(L), \infty)$ satisfying $1 < \mu(p, L) < a$. We fix such a p. Since $\lim_{q\to1+0} \mu(q, L) = 1/L < 1$, $\mu(q, L)$ is less than 1 for q sufficiently close to 1. Thus we have

$$\mu(q, L) < 1 < \mu(p, L) < a.$$

Since $p > p_*(L)$, (7.16) ensures that $p > p(a)$. Hence $Z_\infty(p) > a$. Therefore we obtain

$$\lim_{q\to1+0} z_\infty(p, q) = Z_\infty(p) > a.$$

Consequently, if q is sufficiently close to 1, it holds that

$$z_\infty(p, q) > a > \mu(p, L) > 1 > \mu(q, L).$$

Thus (7.15) is obtained. By Proposition 7.2, $\lambda'(\alpha) < 0$ for α large enough. Since $\lambda(\alpha)$ is decreasing and converges to $\mu(p, L)$ as $\alpha \to \infty$, it holds that $\lambda^* > \mu(p, L)$. The proof is complete. □

Observing all our results and using L_* defined by (1.6), we propose the next conjecture.

Conjecture 7.14. Fix $1 < q < p$. Then there exist constants $\underline{L}, \overline{L}$ satisfying $0 < \underline{L} < L_* < \overline{L}$ such that the types (A), (B), (E), (D) and (C) occur when $0 < L \le \underline{L}$, $\underline{L} < L < L_*$, $L = L_*$, $L_* < L < \overline{L}$ and $\overline{L} \le L < \infty$, respectively. The types (A) and (C) have no turning points and (B), (D), (E) have exactly one turning point.

The conjecture above is partially solved in the present paper. Indeed, we have already proved the following theorem (see Remark 4.14 and Theorems 6.1 and 6.2).

Theorem 7.15. *There exists a constant $\varepsilon > 0$ such that* (A), (B), (E), (D), (C) *occur when $0 < L < \varepsilon$, $L_* - \varepsilon < L < L_*$, $L = L_*$, $L_* < L < L_* + \varepsilon$, $1/\varepsilon < L < \infty$, respectively, and there exist no turning points for* (A) *with $0 < L < \varepsilon$ and* (C) *with $1/\varepsilon < L < \infty$.*

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