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EXISTENCE OF INFINITELY MANY SOLUTIONS FOR ELLIPTIC BOUNDARY-VALUE PROBLEMS WITH NONSYMMETRICAL CRITICAL NONLINEARITY

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Abstract. In this paper, we study a semilinear elliptic boundary-value problem involving nonsymmetrical term with critical growth on a bounded smooth domain in $\mathbb{R}^n$. We show the existence of infinitely many weak solutions under the presence of some symmetric sublinear term, the corresponding critical values of the variational functional are negative and go to zero.

1. Introduction and Main Results

In the present paper, we consider the following Dirichlet problem, for the Laplace equation,

$$\begin{gathered} -\Delta u = g(x,u) + f(x,u), \quad \text{in } \Omega \\ u = 0, \quad \text{on } \partial\Omega\,, \end{gathered} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$ and $n \geq 3$.

We assume that the nonlinear term $g(x,u) \in C(\Omega \times \mathbb{R})$ is odd symmetric:

$$g(x,-u) = -g(x,u), \quad \text{for all } (x,u) \in \Omega \times (-\infty,+\infty). \tag{1.2}$$

The other nonlinear term $f(x,u)$ in (1.1) is a non-symmetric perturbation. When $f(x,u) = 0$, multiple solutions (usually infinitely many solutions) may be expected. As pointed out by many authors, the symmetry is not necessary to guarantee the multiplicity of solutions for (1.1); we refer to Rabinowitz [6], Struwe [7] and Dong & Li [5] and the references therein. Some relevant results can be found in [1], [2] and [4]. In these papers authors assumed that the nonlinear terms did not have critical growth, which enables the Palais-Smale condition to be verified in a simple method on a large scale. There arises a natural question whether there still exist multiple solutions even when the symmetric term is non-critical and the perturbed nonlinearity is critical. In this paper, we will partly answer this question.

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As a special case of (1.1), we consider the problem

$$\begin{aligned} -\Delta u = |u|^{p-1}u \pm |u|^{\frac{n+2}{n-2}}, \quad \text{in } \Omega \\ u = 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{1.3}$$

Then we have the following result.

Theorem 1.1. *Suppose* $1 > p > \max\{0, (n(n-2)-4)/(n(n-2)+4)\}$. *Then* (1.3) *possesses infinitely many (weak) solutions.*

In order to get a more general conclusion, we impose the following assumptions on $g(x,u)$ and $f(x,u)$.

(G1) There exist positive constants C_0, C_1 and a nonnegative constant C_2 such that
$$C_0|u|^p \le |g(x,u)| \le C_1|u|^p + C_2|u|^q,$$
where $p \in (0,1)$ and $q \in [p,1)$.

(G2) There exists a positive constant $\mu \in \left(\frac{1}{2}, \frac{1}{1+p}\right)$ such that
$$0 \le \mu u g(x,u) \le G(x,u) = \int_0^u g(x,\tau)d\tau, \quad \text{for all } (x,u) \in \Omega \times (-\infty, +\infty);$$

Moreover, we suppose that $f \in C^1(\Omega \times \mathbb{R})$ and satisfies

(F) There exists a positive constant C_3 such that for all $(x,u) \in \Omega \times (-\infty,+\infty)$,
$$|f(x,u)| \le C_3|u|^{(n+2)/(n-2)}, \quad |f'_u(x,u)| \le C_3(1 + |u|^{4/(n-2)}).$$

Remark. The growth of $f(x,u)$ is allowed to be critical, that is, when $1 \le s < 2^* = 2n/(n-2)$, the embedding of the Sobolev space $H_0^1(\Omega) \hookrightarrow L^s(\Omega)$ is compact; if $s = 2^*$, the embedding is only continuous but not compact. The best Sobolev embedding constant as $s = 2^*$ is denoted by S, namely,

$$S = \inf\{\|\nabla u\|_2^2; u \in H_0^1(\Omega) \quad \text{and} \quad \|u\|_{2^*} = 1\}. \tag{1.4}$$

We introduce a variational functional for (1.1) as

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u| dx - \int_\Omega G(x,u)dx - \int_\Omega F(x,u)dx\,,$$

where $\|u\| = \sqrt{\int_\Omega |\nabla u|^2 dx}$ is the norm in $H_0^1(\Omega)$, and $F(x,u) = \int_0^u f(x,\tau)d\tau$. The weak solutions of (1.1) are the critical points of the functional $I(u)$. Our main result in this paper is the following theorem.

Theorem 1.2. *Suppose that the exponent p in the assumption (G1) satisfies*

$$\max\{0, \frac{n(n-2)-4}{n(n-2)+4}\} < p < 1. \tag{1.5}$$

Then under hypotheses (G1), (G2) and (F), problem (1.1) *possesses infinitely many (weak) solutions. The corresponding critical values of* $I(u)$ *are negative and approach zero.*

The difficulty we have to overcome is that the functional $I(u)$ does not satisfy the Palais-Smale condition. On the other hand, the sublinear term in (1.1) suggests that the critical values of $I(u)$ should be negative. With those observations, following the ideas developed by Rabinowitz [6], we define a new functional $J(u)$ as a truncation of $I(u)$, The new functional verifies the compactness condition and possesses a series of negative critical values which coincide with those of $I(u)$.

This paper is organized as follows: In Section 2, we introduce the new functional $J(u)$ and show that $J(u)$ approaches, in some sense, to an even functional. By carefully determining the positive constants which appear in $J(u)$, we can show, in Section 3, that the new functional verifies the Palais-Smale condition. Section 4 is devoted to make out the structure of the series of critical values for the functional $J(u)$. A crucial step in this section is the proof that the series of these critical values are negative. With some growth estimates of the critical values we prove that most of the critical values of $J(u)$ near the origin are also the critical values of $I(u)$. This allows us to prove Theorem 1.1 and 1.2, which is done in Section 5.

2. A New Functional

If $u \in H_0^1(\Omega)$ is a critical point of the functional of $I(u)$, that is, $I'(u) = 0$, then $\langle I'(u), u\rangle = 0$. With this notation as above, it is not difficult to verify the following inequality

$$\|u\|^2 \le \frac{1}{\mu}\int_\Omega G(x,u)dx + C_3\|u\|_{2^*}^{2^*}. \tag{2.1}$$

Set $\Phi(t) \in C^\infty[0,+\infty)$ such that $0 \le \Phi(t) \le 1$ and $\Phi(t) = 1$ in $[0,1]$ and $\Phi(t) = 0$ in $[2\mu', +\infty)$, where $\mu' \in (1/2, \mu)$. (by assumption $\mu < \frac{1}{p+1} < 1$) Furthermore we can choose Φ satisfying

$$|\Phi'(t)| \le \frac{2}{2\mu' - 1}\,.$$

For $u \in H_0^1(\Omega)\backslash\{0\}$, we define

$$\phi(u) = \Phi\Big(\frac{\|u\|^2}{\frac{1}{\mu}\int_\Omega G(x,u)dx + A\|u\|_{2^*}^{2^*}}\Big),$$

where $A \ge C_3$ is a constant to be determined later.

Set $\Psi(t) \in C^\infty[0,+\infty)$ such that $0 \le \Psi(t) \le 1$ and $\Psi(t) = 1$ in $[0, R^2/2]$ and $\Psi(t) = 0$ in $[R^2, +\infty)$, where $R^2 \le 1$ and the positive constant R is to be determined later. moreover we can choose Ψ satisfying

$$|\Psi'(t)| \le \frac{4}{R^2}.$$

For $u \in H_0^1(\Omega)$, we define

$$\psi(u) = \Psi\left(\|u\|^2\right).$$

Before proceeding, we have the following estimates.

Lemma 2.1. *For $u \in H_0^1(\Omega) \setminus \{0\}$, we have*

$$|\langle \psi'(u), u\rangle| \le 8, \quad |\langle \phi'(u), u\rangle| \le M_0, \tag{2.2}$$

where M_0 is independent of u and A.

Proof. For $u \in H_0^1(\Omega)\backslash\{0\}$, let

$$t = \|u\|^2\Big(\frac{1}{\mu}\int_\Omega G(x,u)dx + A\|u\|_{2^*}^{2^*}\Big)^{-1}.$$

Then by the definition of $\phi(u)$, we have

$$\langle \phi'(u), u\rangle = \Phi'(t)t\Big[2 - \frac{t}{\|u\|^2}\int_\Omega\Big(\frac{g(x,u)u}{\mu} + 2^*A|u|^{\frac{4}{n-2}}u\Big)dx\Big]. \tag{2.3}$$

If $t \notin [1, 2\mu']$, then $\Phi'(t) = 0$, the conclusion of the lemma holds true. Without loss of generality, we can assume that $1 \le t \le 2\mu'$, that is,

$$\frac{1}{\mu}\int_\Omega G(x,u)dx + A\|u\|_{2^*}^{2^*} \le \|u\|^2 \le 2\mu'\Big(\frac{1}{\mu}\int_\Omega G(x,u)dx + A\|u\|_{2^*}^{2^*}\Big). \qquad (2.4)$$

From hypotheses (G1) and (G2),

$$|\langle \phi'(u), u\rangle| \le \frac{4\mu'}{2\mu'-1}[2+2^*] = M_0.$$

In a similar way, we have $\langle \psi'(u), u\rangle = 2\Psi'(\|u\|^2)\|u\|^2 = 2\Psi'(s)s$, where $s = \|u\|^2$. If $s \notin [R^2/2, R^2]$, then $\Psi'(s) = 0$, the conclusion of the lemma is true, so we can suppose that $R^2/2 \le s \le R^2$. Thus

$$|\langle \psi'(u), u\rangle| \le 2|\Psi'(s)|\|u\|^2 \le 8.$$

□

Remark. The significance of the lemma is that the bounds for the estimates in (2.2) are independent of R and A. Therefore, we can take $A = (M_0 + 9)C_3$ in advance and choose $R > 0$ small in the sequel.

Now, we introduce the new functional, which is a truncation of $I(u)$, as follows: For $u \in H_0^1(\Omega)\backslash\{0\}$, define

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|dx - \int_\Omega G(x,u)dx - \phi(u)\psi(u)\int_\Omega F(x,u)dx,$$

and $J(0) = 0$. The first fact about the functional $J(u)$ is that $J(u)$ is continuous differentiable.

Proposition 2.2. *$J \in C^1(H_0^1(\Omega), \mathbb{R})$.*

Proof. Since $I(u) \in C^1(H_0^1(\Omega), \mathbb{R})$, what we have to prove is that the last term in $J(u)$ is continuous differentiable at 0 in the sense of Frèchet's means. In fact, by denoting

$$\mathcal{F}(u) = \phi(u)\psi(u)\int_\Omega F(x,u)dx, \text{ for } u \in H_0^1(\Omega)\backslash\{0\},$$

it is easy to verify that 0 is a removable singular point of $\mathcal{F}(u)$ and by defining $\mathcal{F}(0) = 0$ the functional $\mathcal{F}(u)$ becomes continuous at 0. Moreover, for any $\eta \in H_0^1(\Omega)$ and $u \in H_0^1(\Omega)\backslash\{0\}$,

$$\langle \mathcal{F}'(u), \eta\rangle = [\langle \phi'(u), \eta\rangle\psi(u) + \langle \psi'(u), \eta\rangle\phi(u)]\int_\Omega F(x,u)dx + \phi(u)\psi(u)\int_\Omega f(x,u)\eta dx.$$

Then by the assumption (F), the Hölder inequality and (1.4), we have

$$|\langle \psi'(u), \eta\rangle\phi(u)|\int_\Omega |F(x,u)|dx \le 2C_3\|u\|_{2^*}^{2^*}|(u,\eta)_{H_0^1}| \le 2C_3 S^{(1+2^*)/2}\|u\|^{1+2^*}\|\eta\|;$$

$$\begin{aligned}\phi(u)\psi(u)\int_\Omega |f(x,u)\eta|dx &\le C_3\|u\|_{2^*}^{(n+2)/(n-2)}\|\eta\|_{2^*}\\ &\le C_3 S^{2^*/2}\|u\|^{(n+2)/(n-2)}\|\eta\|.\end{aligned}$$

Thus, we can estimate as follows:

$$\begin{aligned}
&|\langle\phi'(u),\eta\rangle|\phi(u)\int_\Omega |F(x,u)|dx\\
&\le \frac{2C_3\|u\|_{2^*}^{2^*}\psi(u)}{(2\mu'-1)t}\Big[|(u,\eta)_{H_0^1}|+\int_\Omega\Big(\frac{|g(x,u)|}{\mu}+2^*A|f(x,u)|\Big)|\eta|dx\Big]\\
&\le C\frac{\|u\|^{4/(n-2)}\psi(u)}{2\mu'-1}\Big[\|u\|+\|u\|^p+\|u\|^q+\|u\|^{(n+2)/(n-2)}\Big]\|\eta\|\\
&\le C\|u\|^{(n+2)/(n-2)}\|\eta\|,
\end{aligned}$$

where C depends only on S, $|\Omega|$, C_1, C_2, C_3 and μ'. Thus we can get the estimate of $\mathcal{F}'$:

$$\|\mathcal{F}'(u)\|_{H^{-1}(\Omega)}\le M_1\Big[\|u\|^{1+2^*}+\|u\|+\|u\|^p+\|u\|^q+\|u\|^{2^*-1}\Big],$$

which implies $\mathcal{F}'(u)\to 0$ as $u\to 0$. With the additional definition $\mathcal{F}'(0)=0$, the above limit implies that $\mathcal{F}'(u)$ is continuous at 0. □

Lemma 2.3. *If the positive constant R is small enough, for all $u\in\operatorname{supp}\phi\cap\operatorname{supp}\psi$ we have*

$$|J(u)|\ge M_2\|u\|_{1+p}^{1+p}, \tag{2.5}$$

where M_2 is a positive constant independent of u.

Proof. Suppose that $u\in\operatorname{supp}\psi=\bar B_R(0)\subset H_0^1(\Omega)$. Then when R is small enough, from the Sobolev inequality it follows that

$$2\mu' A\|u\|_{2^*}^{2^*}\le\frac{1}{2}\|u\|^2.$$

Moreover, if $u\in\operatorname{supp}\phi$, by the assumption (G1), we have

$$\|u\|^2\le 2\mu'\Big(\frac{1}{\mu}\int_\Omega G(x,u)dx+A\|u\|_{2^*}^{2^*}\Big)\le 4\frac{\mu'}{\mu}\Big[C_1\|u\|_{1+p}^{1+p}+C_2\|u\|_{1+q}^{1+q}\Big]. \tag{2.6}$$

Without loss of generality, we can suppose that $q>p$ and $C_2>0$. According to the interpolation inequality, we have

$$\|u\|_{1+q}\le\|u\|_{1+p}^r\|u\|_{2^*}^{1-r}, \tag{2.7}$$

where $r=(1+p)(2n-(1+q)(n-2))/(1+q)(2n-(1+p)(n-2))$. Hence we get

$$\|u\|_{1+q}^{1+q}\le S^{\frac{1-r}{2}}\|u\|_{1+p}^{r(1+p)}\Big[\|u\|_{1+p}^{1+p}+\|u\|_{1+q}^{1+q}\Big]^{\frac{(1-r)(1+q)}{2}}\le C\Big[\|u\|_{1+p}^{\alpha}+\|u\|_{1+q}^{\beta}\Big], \tag{2.8}$$

where C is depend only on S, and $\alpha=(1+q)[r+(1-r)(1+p)/2]$ and $\beta=(1+q)[r+(1-r)(1+q)/2]$. It is clear that $\alpha<\beta$. On the other hand, a simple calculate shows that $\alpha=(1+p)(n-np+2q+2)/(n-np+2p+2)>1+p$. Then if R is small enough, (2.8) becomes

$$\|u\|_{1+q}^{1+q}\le\frac{1}{2C_2}\|u\|_{1+p}^{1+p}.$$

Thus we can write (2.6) as

$$\|u\|^2\le 8C_1\frac{\mu}{\mu'}\|u\|_{1+p}^{1+p}. \tag{2.9}$$

With (2.9) we can estimate $J(u)$ as follows

$$\begin{aligned}
|J(u)| &\geq \Big(1-\frac{\mu'}{\mu}\Big)\mu\int_\Omega ug(x,u)dx - (\mu' A + C_3)\|u\|_{2^*}^{2^*}\\
&\geq (\mu-\mu')C_0\|u\|_{1+p}^{1+p} - C\|u\|^{2^*}\\
&\geq (\mu-\mu')C_0\|u\|_{1+p}^{1+p} - C\|u\|_{1+p}^{(1+p)2^*/2}\\
&\geq \frac{1}{2}(\mu-\mu')C_0\|u\|_{1+p}^{1+p},
\end{aligned}$$

where C depends only on S, A and μ. Then the lemma follows with $M_2 = (\mu - \mu')C_0/2 > 0$. $\square$

Although $J(u)$ in generally is not an even functional, $J(u)$ approaches in some sense to such a functional as shown in the following result.

Proposition 2.4. *There exists a positive constant M_3 independent of u, such that*

$$|J(u) - J(-u)| \leq M_3|J(u)|^\theta, \text{ for all } u \in H_0^1(\Omega), \tag{2.10}$$

where $\theta = 2^*/2 = n/(n-2)$.

Proof. From the definition of $J(u)$, the embedding theorem, and (2.9), we have

$$\begin{aligned}
|J(u) - J(-u)| &= \phi(u)\psi(u)\Big|\int_\Omega F(x,u)dx - \int_\Omega F(-u)dx\Big|\\
&\leq 2\phi(u)\psi(u)C_3\|u\|_{2^*}^{2^*}\\
&\leq C\|u\|_{1+p}^{(1+p)2^*/2}\\
&\leq M_3|J(u)|^{2^*/2}.
\end{aligned}$$

The proposition follows with $\theta = 2^*/2$. Note that M_3 is depend only on C_3 and S. $\square$

3. Verification of Palais-Smale Condition

Because the functional $I(u)$ contains critical growth nonlinearity, a well-known fact is that the functional violates Palais-Smale condition. However, all energy values of the functional where this condition may fail can be characterized, we refer to Struwe [7]. The factors $\phi(u)$ and $\psi(u)$ in the new functional $J(u)$ will change the situation, that is, $J(u)$ remains most critical points of $I(u)$ and satisfies Palais-Smale condition as shown in this section.

Lemma 3.1. *There exists a suitable constant $R > 0$ such that for any $M > 0$, there exists $C(M) > 0$, if $|J(u)| \leq M$, then*

$$\|u\|^2 \leq C(M).$$

Proof. From the assumptions on $f(x,u)$ and $g(x,u)$ it follows that

$$|J(u)| \geq \frac{1}{2}\|u\|^2 - C\big[\|u\|^{1+p} + \|u\|^{1+q} + \phi(u)\psi(u)\|u\|^{2^*}\big] \tag{3.1}$$

If $\|u\|^2 > R^2$, then $\psi(u) = 0$. Without loss of generality, we can suppose that $\|u\|^2 \leq R^2$. Set $R > 0$ small enough such that

$$C\phi(u)\psi(u)\|u\|^{2^*} \leq \frac{1}{4}\|u\|^2 \quad \text{for all } u \in H_0^1(\Omega). \tag{3.2}$$

Then (3.1) becomes

$$|J(u)| \geq \frac{1}{4}\|u\|^2 - C\Big[\|u\|^{1+p} + \|u\|^{1+q}\Big]. \tag{3.3}$$

Therefore, from $|J(u)| \leq M$ it follows that there exists $C(M)$ such that $\|u\|^2 \leq C(M)$. □

Proposition 3.2. *For some suitable positive constants A and R, the functional $J(u)$ satisfies the Palais-Smale condition, that is, for any sequence $\{u_m\}$ in $H_0^1(\Omega)$ such that $|J(u_m)| \leq M$ and*

$$J'(u_m) \to 0 \quad \text{as } m \to \infty \quad \text{in } H^{-1}(\Omega),$$

then there exists subsequence of $\{u_m\}$ which is convergent in $H_0^1(\Omega)$.

Proof. Suppose that $\{u_m\}$ is a sequence in $H_0^1(\Omega)$ with $|J(u_m)| \leq M$ and $J'(u_m) \to 0$ in $H^{-1}(\Omega)$. What we have to prove is that $\{u_m\}$ possesses a convergent subsequence. Without loss of generality, we can assume that there exists a positive constant ϵ such that $\|u_m\|^2 \geq \epsilon$. From lemma 3.1 it follows that the sequence $\{u_m\}$ is bounded in $H_0^1(\Omega)$ since $|J(u_m)| \leq M$. Thus there exist a subsequence (denoted still by $\{u_m\}$) and u in $H_0^1(\Omega)$ such that

$$\begin{gathered} u_m \rightharpoonup u \quad \text{in } H_0^1(\Omega), \\ u_m \to u \quad \text{strongly in } L^t(\Omega) \text{ for } t \in [1, 2^*), \\ u_m \to u \quad \text{almost everywhere in } \Omega. \end{gathered}$$

Denote

$$s_m = \frac{\|u_m\|^2}{\frac{1}{\mu}\int_\Omega G(x, u_m)dx + A\|u_m\|_{2^*}^{2^*}}, \quad t_m = \|u_m\|^2.$$

With this notation, for any $\eta \in C_0^\infty(\Omega)$ we have

$$\begin{aligned} \langle J'(u_m), \eta\rangle &= [1 - I_1(u_m)]\int_\Omega \nabla u_m \cdot \nabla\eta dx - [1 - I_2(u_m)]\int_\Omega g(x, u_m)\eta dx \\ &\quad - \phi(u_m)\psi(u_m)\int_\Omega f(x, u_m)\eta dx - I_3(u_m)\int_\Omega |u_m|^{\frac{4}{n-2}}u_m\eta dx, \end{aligned}$$

where

$$\begin{aligned} I_1(u_m) &= 2\Big[\frac{\psi(u_m)\Phi'(s_m)}{\frac{1}{\mu}\int_\Omega G(x, u_m)dx + A\|u_m\|_{2^*}^{2^*}} + \phi(u_m)\Psi'(t_m)\Big]\int_\Omega F(x, u_m)dx \\ I_2(u_m) &= \frac{\|u_m\|^2\Phi'(s_m)\psi(u_m)}{\mu\big(\frac{1}{\mu}\int_\Omega G(x, u_m)dx + A\|u_m\|_{2^*}^{2^*}\big)^2}\int_\Omega F(x, u_m)dx \\ I_3(u_m) &= 2^*A\frac{\|u_m\|^2\Phi'(s_m)\psi(u_m)}{\mu\big(\frac{1}{\mu}\int_\Omega G(x, u_m)dx + A\|u_m\|_{2^*}^{2^*}\big)^2}\int_\Omega F(x, u_m)dx. \end{aligned}$$

By Brezis-Lieb's result [3] (also see (3.9)), we can write

$$\|u_m\|^2 = \|u\|^2 + \delta' + o(1), \quad \|u_m\|_{2^*}^{2^*} = \|u\|_{2^*}^{2^*} + \delta + o(1), \tag{3.4}$$

where (subsequence, if necessary)

$$\delta' = \lim_{m\to\infty}\|u_m - u\|^2, \text{ and } \delta = \lim_{m\to\infty}\|u_m - u\|_{2^*}^{2^*}. \tag{3.5}$$

It is easy to verify that

$$\int_\Omega G(x,u_m)dx \to \int_\Omega G(x,u)dx, \int_\Omega f(x,u_m)\eta dx \to \int_\Omega f(x,u)\eta dx. \tag{3.6}$$

Denote

$$\|u\|^2+\delta' = t_0, \quad \frac{\|u\|^2+\delta'}{\frac{1}{\mu}\int_\Omega G(x,u)dx + A(\|u\|_{2^*}^{2^*}+\delta)} = s_0.$$

Moreover (subsequence, if necessary)

$$\lim_{m\to\infty}\int_\Omega F(x,u_m)dx = r_0.$$

With this notion, we obviously have

$$\begin{gathered}\psi(u_m) = \Psi(t_m) \to \Psi(t_0), \quad \Psi'(t_m) \to \Psi'(t_0),\\ \phi(u_m) = \Phi(s_m) \to \Phi(s_0), \quad \Phi'(s_m) \to \Phi'(s_0).\end{gathered}$$

Therefore, as m approaches infinity, we obtain

$$\begin{aligned}
I_1(u_m) &\to I_1 = 2\Big(\frac{\Psi(t_0)\Phi'(s_0)}{\frac{1}{\mu}\int_\Omega G(x,u)dx + A(\|u\|_{2^*}^{2^*}+\delta)} + \Phi(s_0)\Psi'(t_0)\Big) r_0\\
I_2(u_m) &\to I_2 = \frac{(\|u\|^2+\delta')\Phi'(s_0)\Psi(t_0)}{\mu\big(\frac{1}{\mu}\int_\Omega G(x,u)dx + A(\|u\|_{2^*}^{2^*}+\delta)\big)^2} r_0\\
I_3(u_m) &\to I_3 = 2^* A \frac{(\|u\|^2+\delta')\Phi'(s_0)\Psi(t_0)}{\mu\big(\frac{1}{\mu}\int_\Omega G(x,u)dx + A(\|u_m\|_{2^*}^{2^*}+\delta)\big)^2} r_0,
\end{aligned}$$

which implies

$$\langle J'(u_m),\eta\rangle \to \langle \tilde{J}(u),\eta\rangle,$$

where

$$\begin{aligned}
\langle \tilde{J}(u),\eta\rangle &= [1-I_1]\int_\Omega \nabla u\cdot\nabla\eta dx - [1-I_2]\int_\Omega g(x,u)\eta dx\\
&\quad - \Phi(s_0)\psi(t_0)\int_\Omega f(x,u)\eta dx - I_3\int_\Omega |u|^{\frac{4}{n-2}}u\eta dx.
\end{aligned}$$

From $\langle J'(u_m),\eta\rangle = o(1)$, we have $\langle \tilde{J}(u),v\rangle = 0$ for all v in $H_0^1(\Omega)$. It follows that

$$\langle J'(u_m)-\tilde{J}(u), u_m-u\rangle = \langle J'(u_m), u_m-u\rangle \to 0. \tag{3.7}$$

On the other hand, we have

$$\begin{aligned}
&\langle J'(u_m)-\tilde{J}(u), u_m-u\rangle\\
&= (1-I_1)\|u_m-u\|^2 + o(1) - (1-I_2)\int_\Omega \Big[g(x,u_m)-g(x,u)\Big](u_m-u)dx\\
&\quad - \Psi(t_0)\Phi(s_0)\int_\Omega \Big[f(x,u_m)-f(x,u)\Big](u_m-u)dx\\
&\quad - I_3\int_\Omega \Big[|u_m|^{\frac{4}{n-2}}u_m - |u|^{\frac{4}{n-2}}u\Big](u_m-u)dx\\
&= (1-I_1)\|u_m-u\|^2 + o(1) - \Psi(t_0)\Phi(s_0)\int_\Omega f(x,u_m-u)(u_m-u)dx\\
&\quad - I_3\int_\Omega |u_m-u|^{2^*}dx.
\end{aligned} \tag{3.8}$$

where we have used the fact $\int_\Omega [f(x,u_m)-f(x,u)](u_m-u)dx = \int_\Omega f(x,u_m-u)(u_m-u)dx + o(1)$, which proof can be founded in the next lemma. Before proceeding furthermore, we first claim that $|I_1| \le 1/2$ and $|I_3| \le 1/4$. In fact, if $u_m \notin \operatorname{supp}\phi \cap \operatorname{supp}\psi$, then $I_1(u_m) = I_3(u_m) = 0$, we are done. In the contrary case, that is, $u_m \in \operatorname{supp}\phi \cap \operatorname{supp}\psi$, we can suppose that $u \in \operatorname{supp}\Psi'$. Otherwise, we have $I_3(u_m) = 0$ and $I_1(u_m) = \phi(u_m)\Psi'(t_m)\int_\Omega F(x,u_m)dx$, the desired result easy follows. Without loss generality, we can suppose that $t_m \in \Psi'$, namely

$$1 \le \|u_m\|^2 \Big(\frac{1}{\mu}\int_\Omega G(x,u_m)dx + A\|u_m\|_{1+p}^{1+p}\Big)^{-1} \le 2\mu',$$
$$\frac{R^2}{2} \le \|u_m\|^2 \le R^2.$$

By the choice of Φ and Ψ, we can estimate $I_1(u_m)$ as

$$\begin{aligned}
|I_1(u_m)| &\le \Big[\frac{4C_3(2\mu'-1)^{-1}}{\frac{1}{\mu}\int_\Omega G(x,u_m)dx + A\|u_m\|_{2^*}^{2^*}} + \frac{4C_3}{R^2}\Big]\|u_m\|_{2^*}^{2^*} \\
&\le 4C_3 S^{-\frac{2^*}{2}}\big(\frac{2\mu'}{2\mu'-1}+1\big)\|u\|^{\frac{4}{n-2}} \\
&\le \frac{4C_3 S^{\frac{n}{2-n}}}{2\mu'-1}R^{\frac{4}{n-2}}.
\end{aligned}$$

Let R be small enough, then $|I_1(u_m)| < 1/2$, which implies that $|I_1| \le 1/2$.

In a similar way, we can estimate I_3 as

$$\begin{aligned}
|I_3(u_m)| &\le 4\frac{2^* A C_3 \mu'}{2\mu'-1}\Big(\int_\Omega G(x,u_m)dx + A\|u_m\|_{2^*}^{2^*}\Big)^{-1}\|u_m\|_{2^*}^{2^*} \\
&\le 2^* A C_3 S^{\frac{n}{2-n}}\frac{8\mu'^2}{2\mu'-1}\|u\|^{\frac{n}{n-2}} \\
&\le 2^* A C_3 S^{\frac{n}{2-n}}\frac{8\mu'^2}{2\mu'-1}R^{\frac{n}{n-2}}.
\end{aligned}$$

After the constant A being fixed, we can set R be small enough such that $|I_3(u_m)| < 1/4$, which implies that $|I_3| \le 1/4$.

Let us go back to (3.7) and (3.8). If $u_m \in B_R(0) = \operatorname{supp}\psi$ and for R small enough

$$\begin{aligned}
o(1) &\ge \frac{1}{2}\|u_m-u\|^2 - \Psi(t_0)\Phi(s_0)S^{\frac{n}{2-n}}\|u_m-u\|^{2^*} - \frac{1}{4}\|u_m-u\|^{2^*} \\
&\ge \frac{1}{4}\|u_m-u\|^2.
\end{aligned}$$

Where we have used the fact

$$\|u_m-u\|^{2^*} \le (\|u_m\|^{\frac{n}{n-2}} + \|u\|^{\frac{n}{n-2}})\|u_m-u\|^2 \le 2R^{\frac{n}{n-2}}\|u_m-u\|^2.$$

If $u_m \notin \operatorname{supp}\psi$, we still have

$$o(1) \ge \frac{1}{2}\|u_m-u\|^2.$$

As a consequence, for a subsequence, $u_m \to u$ strongly in $H_0^1(\Omega)$. $\square$

Lemma 3.3. *Suppose that $h(x,u) \in C^1(\Omega \times (-\infty,+\infty))$ and $|h'_u(x,u)| \le C(1+|u|^{4/(n-2)})$. If $u_m \rightharpoonup u$ weakly in $H_0^1(\Omega)$, then*

$$\int_\Omega [h(x,u_m)-h(x,u)](u_m-u)dx = \int_\Omega h(x,u_m-u)(u_m-u)dx + o(1) \tag{3.9}$$

Proof. The hypothesis on the growth of h implies that $h(x,u_m)$ and $h'(x,u_m)u_m$ are bounded in $L^{2n/(n+2)}(\Omega)$. The compact embedding theorem for Sobolev spaces yields that for a subsequence,

$$h(x,u_m) \to h(x,u),\ h'_u(x,u_m)u_m \to h'_u(x,u)u \quad \text{strongly in } L^{\frac{2n}{n+2}}(\Omega). \tag{3.10}$$

Furthermore we can deduce that

$$\int_\Omega [h(x,u_m)-h(x,u)]u dx = o(1), \quad \text{and} \quad \int_\Omega h(x,u)(u_m-u)dx = o(1).$$

Consequently

$$\int_\Omega [h(x,u_m)-h(x,u)](u_m-u)dx = \int_\Omega [h(x,u_m)u_m - h(x,u)u]dx + o(1).$$

On the other hand, as $n \to \infty$,

$$\begin{aligned}
&\int_\Omega \Big[h(x,u_m)u_m - h(x,u_m-u)(u_m-u)\Big]dx \\
&= \int_\Omega \int_0^1 \frac{d}{dt}\{h(x,u_m-(1-t)u)(u_m-(1-t)u)\}\,dt\,dx \\
&= \int_\Omega \int_0^1 [h'_u(x,u_m-(1-t)u)(u_m-(1-t)u)u + h(x,u_m-(1-t)u)u]\,dt\,dx \\
&= \int_0^1 \int_\Omega [h'_u(x,u_m-(1-t)u)(u_m-(1-t)u)u + h(x,u_m-(1-t)u)u]\,dx\,dt \\
&\to \int_0^1 \int_\Omega [h'_u(x,tu)tu^2 + h(x,tu)u]dt\,dx \\
&= \int_0^1 \int_\Omega \frac{d}{dt}\{h(x,tu)(tu)\}\,dt\,dx = \int_\Omega h(x,u)u\,dx,
\end{aligned}$$

where the weak convergence limit is a consequence of (3.10). □

Remark. If f is convex, we can infer that the first inequality in (F) implies the second one. Moreover, a revised proof as in Brezis-Lieb's paper [3] can be used to establish that differentiability of f is not necessary.

4. The Construction of Critical Values

In this section, we establish a series of minimax sequences of the functional $J(u)$ and prove at last that there is a subsequence of them which is the infinitely many critical values of the functional $I(u)$.

Denote the eigenvalue of $-\Delta$ with vanish boundary value by λ_k, $k=1,2,\dots$, and the normalized eigenfunction corresponding to λ_k by e_k. Set

$$\begin{gathered}
E_k = \operatorname{span}\{e_1,e_2,\dots,e_k\}, \\
S_k = \{u \in E_k; \|u\| = 1\} \\
S^+_{k+1} = \{u = te_{k+1} + w; \|u\| = 1, w \in E_k, t \ge 0\}.
\end{gathered}$$

Define the map sets as follows:

$$\Lambda_k = \{h \in C(S_k, H_0^1(\Omega)); h \text{ is odd map}\},$$
$$\Gamma_k = \{h \in C(S_{k+1}^+, H_0^1(\Omega)); h|_{S_k} \in \Lambda_k\}.$$

With these sets of maps, we can define minimax sequence of $J(u)$ as follows:

$$b_k = \inf_{h\in\Lambda_k} \max_{u\in S_k} J(h(u)), \quad c_k = \inf_{h\in\Gamma_k} \max_{u\in S_{k+1}^+} J(h(u)).$$

For $\delta > 0$, we set

$$\Gamma_k(\delta) = \{h \in \Gamma_k; J(h(u)) \le b_k + \delta, u \in S_k\}$$
$$c_k(\delta) = \inf_{h\in\Gamma_k(\delta)} \max_{u\in S_{k+1}^+} J(h(u)).$$

It is easy to prove that the above notation are well-defined and $b_k \le c_k \le b_{k+1}$, $c_k \le c_k(\delta)$.

From the definition of $J(u)$ and the assumptions (G1) and (F), it follows that, if $R > 0$ is small,

$$J(u) \le \frac{1}{2}\|u\|^2 - C_0\|u\|_{p+1}^{p+1} + \phi(u)\psi(u)C_3\|u\|_{2^*}^{2^*} \le \|u\|^2 - C_0\|u\|_p^p.$$

By setting $H(u) = \rho u$, we obviously have $H \in \Lambda_k$. Since $H(S_k) \subset E_k$, we can find out $\rho = \rho_k$ and $C_k > 0$ such that for any $u \in S_k$,

$$J(H(u)) = J(\rho_k u) \le \rho_k^2 - C_0C_k\rho_k^p < 0, \tag{4.1}$$

which implies that $J(u) < 0$ for all u in $(B_{\rho_k}(0) \cap E_k)\backslash\{0\}$ and $b_k < 0$ for $k = 1, 2, \dots$. Furthermore, for any $\delta > 0$, it is clear that $c_k(\delta') \le c_k + \delta$, where $\delta' = c_k - b_k + \delta$. However, an important fact is that $c_k(\delta) < 0$ for each k and each $\delta > 0$. Before giving the proof of the fact, we first claim that the functional $J(u)$ possesses no critical point with nonnegative critical value except the origin. In fact, suppose $u \in K = \{u \in H_0^1(\Omega); J'(u) = 0\}$ and $J(u) \ge 0$, then

$$\begin{aligned}
0 &\le J(u) = J(u) - \frac{1}{2}\langle J'(u), u\rangle \\
&\le (\frac{1}{2} - \mu)\int_\Omega g(x,u)u dx + \phi(u)\psi(u)\Big|\int_\Omega \Big(F(x,u) + f(x,u)u\Big)dx\Big| \\
&\quad + \big[|\langle\phi'(u), u\rangle|\psi(u) + |\langle\psi'(u), u\rangle|\phi(u)\big]\int_\Omega |F(x,u)|dx \\
&\le (\frac{1}{2} - \mu)C_0\|u\|_{1+p}^{1+p} + C_3\big[\phi(u)\psi(u) + M_0\psi(u) + 8\phi(u)\big]\|u\|_{2^*}^{2^*} \\
&\le \frac{1}{2}(\frac{1}{2} - \mu)C_0\|u\|_{1+p}^{1+p} \le 0,
\end{aligned}$$

for suitable small R, which leads to $u = 0$.

Lemma 4.1. *For $k = 1, 2, \dots,$ and $\delta > 0$ we have $c_k(\delta) < 0$.*

Proof. Without loss of generality, we suppose that $b_k + \delta < 0$. From the definition of $c_k(\delta)$ it follows that there exits $h \in \Lambda_k$ such that

$$\max_{u\in S_k} J(h(u)) \le b_k + \frac{\delta}{2}. \tag{4.2}$$

Denote the orthogonal projective operator from $H_0^1(\Omega)$ to E_m by P_m. Since $h(S_k)$ is a compact set in $H_0^1(\Omega)$, it is not difficult to show that there exists a positive integer m ($m \geq k$) such that

$$\max_{u\in S_k} J(P_m h(u)) \leq b_k + \delta. \tag{4.3}$$

Note that $P_m h \in \Lambda_k$. The fact that the functional $J(u)$ possesses no critical point with nonnegative critical value except the origin implies that $B_{\rho/2}(0) \subset H_0^1(\Omega)$ is a neighborhood of K_0, where $K_a = \{u \in H_0^1(\Omega); J(u) = a, J'(u) = 0\}$. Let $\bar{\epsilon} = -(b_k + \delta)/2$ and $\rho = \min\{\rho_{m+1}, \operatorname{dist}(0, J_{-\bar{\epsilon}})\}$, where ρ_{m+1} is determined as in (4.1). Therefore, the deformation theorem can be used to find out a positive $\epsilon \in (0, \bar{\epsilon})$ and a continuous map $\eta \in C(H_0^1(\Omega) \times [0,1], H_0^1(\Omega))$, such that

(i) $\eta(u, 1) = u$, for all $u \notin J^{-1}(-\bar{\epsilon}, \bar{\epsilon})$

(ii) $\eta(J_\epsilon \backslash B_{\rho/2}(0), 1) \subset J_{-\epsilon}$.

The idea in the following is to seek a contractible subset of J_ϵ and to expend the map $P_m h$ in the subset before it can be deformed into $J_{-\epsilon}$ with the deformation map η.

Indeed, in view of $P_m h(S_k) \subset E_m$, it is natural for us to consider the functional $\tilde{J}$, the restriction of J on E_m, namely $\tilde{J} = J|_{E_m}$. It is clear that $\tilde{J} \in C^1(E_m, \mathbb{R})$ and by the same argument as previous shown, one can obtain that $\tilde{J}$ possesses no critical point with nonnegative critical value except the origin. Thus that fact implies that the level set $\tilde{J}_\sigma = \{u \in E_m; \tilde{J}(u) < \sigma\}$ is a deformation retract of E_m for any $\sigma > 0$, so is $\tilde{J}_\epsilon$ contractible, for the positive ϵ found in the previous deformation theorem. Hence the map $P_m h$ can be extended as

$$\widetilde{P_m h} : S_{k+1}^+ \to \tilde{J}_\epsilon.$$

Let T be a map from E_m to E_{m+1} defined by

$$T(u) = \begin{cases} u, & u \notin \bar{B}_\rho(0) \cap E_m \\ u + \sqrt{\rho^2 - \|u\|^2} e_{m+1}, & u \in \bar{B}_\rho(0) \cap E_m. \end{cases}$$

It is clear that T is continuous, and

$$(T \circ \widetilde{P_m h})[S_{k+1}^+] \cap B_{\rho/2}(0) = \emptyset.$$

Since $\widetilde{P_m h}[S_{k+1}^+] \subset \tilde{J}_\epsilon \subset J_\epsilon$, we also have $T(\widetilde{P_m h}[S_{k+1}^+]) \subset J_\epsilon$.

Denote $H(\cdot) = \eta(T \circ \widetilde{P_m h}(\cdot), 1)$ and $A = H(S_{k+1}^+)$. From (i) it follows that $H|_{S_{k+1}^+} = P_m h \in \Lambda_k$ which implies that $H \in \Gamma_k(\delta)$. Moreover from (ii) it follows that $A = H(S_{k+1}^+) \subset J_{-\epsilon}$. Therefore,

$$\max_{u\in A} J(u) \leq -\epsilon < 0$$

which implies that $c_k(\delta) < 0$. $\square$

Lemma 4.2. *Set $d_k = |b_k|$. Then there exists a positive constant M_4 such that for all k large enough*

$$d_k \leq M_4 k^{-\frac{2}{n}\frac{1+p}{1-p}}, \tag{4.4}$$

where M_4 is independent of k.

Proof. For any fixed $h \in \Lambda_k$ ($k \geq 2$), from Borsuk-Ulam Theorem it follows that
$$h(S_k) \cap E_{k-1}^{\perp} \neq \emptyset.$$
Take $w \in h(S_k)$, denote $\rho = \|w\|$. In $E_{k-1}^{\perp}$, we have
$$\|u\|^2 \geq \lambda_k \|u\|_2^2.$$
With the properties of $\phi(u)$ and $\psi(u)$, we can estimate $J(w)$ as follows:
$$\begin{aligned} J(w) &\geq \frac{1}{2}\|w\|^2 - C_1\|w\|_{1+p}^{1+p} - C_2\|w\|_{1+q}^{1+q} - C_3\phi(w)\psi(w)\|w\|_{2^*}^{2^*} \\ &\geq \frac{1}{4}\|w\|^2 - C\|w\|_2^{1+p} \\ &\geq \frac{1}{4}\rho^2 - C\lambda_k^{-\frac{1+p}{2}}\rho^{1+p} = Q(\rho). \end{aligned}$$
Hence
$$\max_{u \in h(S_k)} J(u) \geq J(w) \geq \inf_{u \in \partial B_\rho(0) \cap E_{k-1}^{\perp}} J(u) \geq \inf_{\rho \geq 0} Q(\rho) \geq -M_5 \lambda_k^{-\frac{1+p}{1-p}}.$$
On the other hand, $\lambda_k \geq M_6 k^{2/n}$; see for example [6]). By the arbitrariness of h, we get $b_k \geq -M_4 k^{2(1+p)/n(p-1)}$. □

Lemma 4.3. *If $c_k = b_k$ for all k large enough, then*
$$d_k \geq M_7 k^{-\frac{1}{\theta-1}},$$
where $\theta = 2^*/2 = n/(n-2)$ and $1/(\theta-1) = (n-2)/2$ and M_7 is independent of k.

Proof. Suppose that for some k_0, we have $b_k = c_k$ ($k = k_0, k_0+1, \dots$). Then for any $\epsilon \in (0, -b_k)$, there exists a map $H \in \Gamma_k$ such that
$$b_k \leq J(H(u_0)) = \max_{u \in S_{k+1}^+} J(H(u)) \leq c_k + \epsilon = b_k + \epsilon. \tag{4.5}$$
Define $\bar{H} : S_{k+1} \to H_0^1(\Omega)$ as
$$\bar{H} = \begin{cases} H(u), & u \in S_{k+1}^+ \\ -H(-u), & -u \in S_{k+1}^+. \end{cases}$$
Since $S_{k+1} = S_{k+1}^+ \cup (-S_{k+1}^+)$, it is clear that $\bar{H} \in \Lambda_{k+1}$, that is,
$$b_{k+1} \leq \max_{u \in S_{k+1}} J(\bar{H}(u)) = \max\big\{\max_{u \in S_{k+1}^+} J(H(u)), \max_{-u \in S_{k+1}^+} J(-H(-u))\big\}. \tag{4.6}$$
We claim that
$$b_{k+1} \leq b_k + \epsilon + M_3|b_{k+1}|^\theta. \tag{4.7}$$
In fact, if
$$\max_{u \in S_{k+1}^+} J(H(u)) \geq \max_{-u \in S_{k+1}^+} J(-H(-u)),$$
then (4.6) implies $b_{k+1} \leq b_k + \epsilon$ which leads to (4.7). On the contrary, we can use Lemma 2.10
$$J(-v) \leq J(v) + M_3|J(-v)|^\theta, \text{ for all } v \in H_0^1(\Omega), \tag{4.8}$$
where $\theta = 2^*/2$. Let v_0 be in $H(S_{k+1}^+)$ such that
$$J(-v_0) = \max_{v \in H(S_{k+1}^+)} J(-v),$$

then (4.8) becomes $J(-v_0) \le b_k + \epsilon + M_3|J(-v_0)|^\theta$, or

$$b_{k+1} \le b_k + \epsilon + M_3\big|\max_{v\in H(S_{k+1}^+)} J(-v)\big|^\theta. \tag{4.9}$$

If $\max_{v\in H(S_{k+1}^+)} J(-v) \le 0$, then (4.7) can be drawn directly from the above inequality. Otherwise, we can take some $v = v_0 \in H(S_{k+1}^+)$ such that

$$b_{k+1} \le J(-v_0) \le 0,$$

we still get (4.7) in the same way. Now by letting $\epsilon \to 0$ we get

$$b_{k+1} \le b_k + M_3|b_{k+1}|^\theta, \text{ for } k \ge k_0. \tag{4.10}$$

With induction, we can show the lemma from (4.10), we omit the details. $\square$

5. Proof of The Main Theorems

Lemma 5.1. *If $c_k > b_k$, then for $\delta \in (0, c_k - b_k)$, $c_k(\delta)$ is a critical value of the functional $J(u)$.*

Proof. If this is not the case, that is, $c_k > b_k$, but for $\delta \in (0, c_k - b_k)$, $c_k(\delta)$ is not a critical value of the functional $J(u)$. Set $\bar\epsilon = c_k - b_k - \delta$. Then $\bar\epsilon > 0$. From the deformation theorem it follows that there exist a positive constant $\epsilon \in (0, \bar\epsilon)$ and a continuous map $\eta(\cdot,\cdot) \in C(H_0^1(\Omega) \times [0,1], H_0^1(\Omega))$ such that

(i) $\eta(u,t) = u$, for all $u \notin J^{-1}(c_k(\delta) - \bar\epsilon, c_k(\delta) + \bar\epsilon)$
(ii) $\eta(J_{c_k(\delta)+\epsilon}, 1) \subset J_{c_k(\delta)-\epsilon}$.

By the definition of $c_k(\delta)$, there exists $H \in \Gamma_k(\delta)$ such that

$$\max_{u\in S_{k+1}^+} J(H(u)) \le c_k(\delta) + \epsilon. \tag{5.1}$$

For u in S_k, we also have $J(H(u)) \le b_k + \delta = c_k - \bar\epsilon \le c_k(\delta) - \bar\epsilon$. Set

$$\bar H(u) = \eta(H(u), 1), \text{ for all } u \in S_k.$$

From (i) it follows that $\bar H \in \Gamma_k(\delta)$, but (ii) implies that $\bar H(J_{c_k(\delta)+\epsilon}) \subset J_{c_k(\delta)-\epsilon}$, which arises a contradiction. $\square$

Lemma 5.2. *Suppose that $1/(\theta-1) = (n-2)/2 > 2(1+p)/(1-p)n$. Then there exists a subsequence c_{k_j} of c_k ($j = 1,2,\dots$) such that*

$$c_{k_j} > b_{k_j}, j = 1,2,\dots. \tag{5.2}$$

Proof. With Lemma 4.2 and Lemma 4.3, if for all k large enough $c_k = b_k$, then we have

$$M_7 k^{-\frac{1}{\theta-1}} \le d_k \le M_4 k^{-\frac{2}{n}\frac{1+p}{1-p}}. \tag{5.3}$$

Since $1/(\theta-1) < 2(1+p)/n(1-p)$, the above inequality will lead to a contradiction for large k. Hence the subsequence c_{k_j} satisfying (5.2) must exist. $\square$

Lemma 5.3. *Suppose that p satisfies $p > \max\{0, (n(n-2)-4)/(n(n-2)+4)\}$. Let u_{k_j} be the critical points of $J(u)$ corresponding to the critical values $c_{k_j}(\delta_j)$, where $\delta_j = (c_{k_j} - b_{k_j})/2$. Then we have*

$$\|u_{k_j}\| \to 0 \quad \text{as } j \to \infty.$$

Proof. By the previous lemmas, for the subsequence $\{c_{k_j}\}_{j=1}^{\infty}$, we have

$$-M_4 k_j^{-\frac{2}{n}\frac{1+p}{1-p}} \le b_{k_j} < c_{k_j} \le c_{k_j}(\delta_j) = J(u_{k_j}) < 0, \tag{5.4}$$

where $\delta_j = (c_{k_j} - b_{k_j})/2 \in (0, c_{k_j} - b_{k_j})$.

Since u_{k_j} is the critical point of the functional $J(u)$, corresponding to $J(u_{k_j}) = c_{k_j}(\delta_j)$, that is, $u_{k_j} \in K_{c_{k_j}(\delta_j)}$, we have $\langle J'(u_{k_j}), u_{k_j}\rangle = 0$. By the result of Lemma 2.1, we can estimate $\|u_{k_j}\|$ as follows:

$$\begin{aligned}
\|u_{k_j}\|^2 &\le \frac{1}{\mu}\int_\Omega G(x,u_{k_j})dx + \Big[\big|\langle \phi'(u_{k_j}), u_{k_j}\rangle\big|\psi(u_{k_j}) \\
&\quad + \big|\langle \psi'(u_{k_j}), u_{k_j}\rangle\big|\phi(u_{k_j}) + \psi(u_{k_j})\phi(u_{k_j})\Big] C_3\|u_{k_j}\|_{2^*}^{2^*} \\
&\le \frac{1}{\mu}\int_\Omega G(x,u_{k_j})dx + A\|u_{k_j}\|_{2^*}^{2^*},
\end{aligned}$$

where we have used $A = (M_0+9)C_3$. The above inequality implies that

$$\phi(u_{k_j}) = 1 \quad \text{and} \quad \langle \phi'(u_{k_j}), u_{k_j}\rangle = 0. \tag{5.5}$$

With this fact and $\langle J'(u_{k_j}), u_{k_j}\rangle = 0$ we infer that

$$\begin{aligned}
&\|u_{k_j}\|^2 \\
&= \int_\Omega g(x,u_{k_j})u_{k_j}dx + \langle \psi(u_{k_j}), u_{k_j}\rangle \int_\Omega F(x,u_{k_j})dx + \psi(u_{k_j})\int_\Omega f(x,u_{k_j})u_{k_j}dx.
\end{aligned}$$

Therefore, $J(u_{k_j})$ becomes

$$\begin{aligned}
J(u_{k_j}) &\le \big(\frac{1}{2\mu} - 1\big)\int_\Omega G(x,u_{k_j})dx + \Big(\big|\langle \psi'(u_{k_j}), u_{k_j}\rangle\big|\phi(u_{k_j}) + \frac{\psi(u_{k_j})}{2}\Big) C_3\|u_{k_j}\|_{2^*}^{2^*} \\
&\le \frac{1}{2}\Big(\frac{1}{2\mu} - 1\Big)c_0\|u_{k_j}\|_{1+p}^{1+p} < 0.
\end{aligned}$$

By (5.4) we have

$$\|u_{k_j}\|_{1+p}^{1+p} \to 0 \quad \text{as } j \to \infty. \tag{5.6}$$

Then the desired result follows easily from (5.4). □

Proof of Theorem 1.2 and Theorem 1.1. We have found a sequence $\{u_{k_j}\}_{j=1}^{\infty}$ of critical points of the functional $J(u)$ with the critical values $\{c_{k_j}((c_{k_j} - b_{k_j})/2)\}_{j=1}^{\infty}$. The facts that $c_k(\delta) < 0$ (Lemma 4.1) and $b_k < 0$ and $b_k \to 0$ imply that there are infinitely many critical values of $J(u)$. With Lemma 5.3 we conclude that the corresponding critical points $u_{k_j} \to 0$ as $j \to \infty$, thus, as j is large enough, we have $\phi(u_{k_j}) = 1$ and $\psi(u_{k_j}) = 1$, which means that those u_{k_j} are also the critical points of the functional $I(u)$ for all large j, and hence, they are also the weak solutions of (1.1), which conclude the Theorem 1.2. The Theorem 1.1 easily follows from Theorem 1.2. □

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