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BIFURCATION OF MULTI-BUMP HOMOCLINICS IN SYSTEMS WITH NORMAL AND SLOW VARIABLES

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Abstract. Bifurcation of multi-bump homoclinics is studied for a pair of ordinary differential equations with periodic perturbations when the first unperturbed equation has a manifold of homoclinic solutions and the second unperturbed equation is vanishing. Such ordinary differential equations often arise in perturbed autonomous Hamiltonian systems.

1. Introduction

Let us consider the system of ordinary differential equations

$$\begin{aligned} \dot{x} &= f(x,y) + \epsilon h(x,y,t,\epsilon)\,, \\ \dot{y} &= \epsilon\Big(g(y) + p(x,y,t,\epsilon) + \epsilon q(y,t,\epsilon)\Big)\,, \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $\epsilon \neq 0$ is sufficiently small, and all mappings are smooth, 1-periodic in the time variable $t \in \mathbb{R}$. Also assume that

(i) $f(0,\cdot) = 0$, $p(0,\cdot,\cdot,\cdot) = 0$.

(ii) The eigenvalues of $f_x(0,\cdot)$ lie off the imaginary axis. Here f_x means the derivative of f with respect to x. Similar notations are used below.

(iii) There exists a hyperbolic periodic solution $\xi(t)$ of $\dot{y} = g(y)$.

(iv) There exists a smooth mapping $\gamma(\theta,y,t) \neq 0$, where $\theta \in \mathbb{R}^{d-1}$, $d \geq 1$ and y is near the periodic solution ξ, such that

$$\begin{aligned} &\dot{\gamma}(\theta,y,t) = f(\gamma(\theta,y,t),y), \quad \gamma(\theta,y,t) = O\big(\mathrm{e}^{-c_1|t|}\big) \\ &\gamma_y(\theta,y,t) = O\big(\mathrm{e}^{-c_1|t|}\big), \quad \gamma_{yy}(\theta,y,t) = O\big(\mathrm{e}^{-c_1|t|}\big) \end{aligned}$$

for a constant $c_1 > 0$, and uniformly for θ, y. Moreover, we suppose

$$d = \dim W^s(y) \cap W^u(y) = \dim T_{\gamma(\theta,y,t)} W^s(y) \cap T_{\gamma(\theta,y,t)} W^u(y)\,.$$

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Here $W^{s(u)}(y)$ is the stable (unstable) manifold to $x = 0$ of $\dot{x} = f(x, y)$, respectively, and $T_z W^{s(u)}(y)$ is the tangent bundle of $W^{s(u)}(y)$ at $z \in W^{s(u)}(y)$, respectively.

Consequently, assumption (iv) means that equation $\dot{x} = f(x, y)$ has a nondegenerate homoclinic manifold [7, 12, 17]

$$W_h(y) = W^s(y) \cap W^u(y) = \left\{ \gamma(\theta, y, t) \mid \theta \in \mathbb{R}^{d-1},\, t \in \mathbb{R} \right\}.$$

We suppose that the closures $\overline{W}_h(y)$ are compact. We note that (θ, t) are the coordinates in $W_h(y)$.

This paper is a continuation of [6], where we study (1.1) under assumptions (i), (ii) and (iv), and instead of (iii) we suppose that $g(0) = 0$ and the eigenvalues of $g_y(0)$ lie off the imaginary axis. We derive conditions in [6] under which (1.1) possesses a transversal bounded solution on $\mathbb{R}$ for $\epsilon \neq 0$ sufficiently small. Consequently, (1.1) has a rich dynamics in [6]. The system (1.1) under assumptions (i)-(iv) is technically much more difficult than in [6] and so we use a different approach in this paper than in [6]. We find in this paper conditions under which (1.1) possesses for $\epsilon \neq 0$ sufficiently small certain multi-bump homoclinics near the set $W_\xi = \cup_{t\in\mathbb{R}} \big(W_h(\xi(t)), \xi(t) \big)$. By a multi-bump homoclinic solution of (1.1) for $\epsilon \neq 0$ sufficiently small near the set W_ξ we mean a solution which alternatively spends a certain amount of time near the periodic solution ξ and a certain amount of time near homoclinic orbits of W_ξ, and which is in addition homoclinic to a hyperbolic torus of (1.1) bifurcating from ξ.

Similar problems are studied also in [3, 9, 11, 12, 13]. Multi-bump and other types of solutions bifurcating from homoclinic manifolds are usually treated by geometric methods [9, 11, 12]. We propose in this paper an alternative method which is based on a shadowing lemma argument or Newton's method in function spaces like in [15], i.e. we construct some functions as pseudo orbits and correction terms are added to make true solutions. Furthermore, bifurcation functions are usually solved by using the implicit function theorem. In this paper we use Brouwer degree theory instead. Consequently, we assume that the so-called Melnikov mappings have nonzero Brouwer degrees on certain domains instead of assuming, like usually done, that these mappings have simple zeroes in the domains. For this reason we think that geometric methods like in [9, 11, 12] can not be applied in this case.

We use in derivation of our main theorem in Section 2 certain results and methods of the works [1, 2, 5, 8, 10, 14, 15, 16, 18]. Section 3 of this paper deals with a general problem which can be transformed to (1.1) by using the averaging method [16]. The final section 4 is devoted to examples for illustration of abstract results.

Finally we note that our method can be applied to the case when instead of the existence of one hyperbolic periodic solution $\xi(t)$ of $\dot{y} = g(y)$ there are several ones. Then like in [12], we can find conditions ensuring the existence of a homoclinic solution of (1.1) for $\epsilon \neq 0$ sufficiently small which is multi-bumping finitely many times between these hyperbolic periodic solutions. Moreover, when the period of $\xi(t)$ is a rational number then we can find by our method conditions for (1.1) that there are multi-bump periodics of (1.1) for $\epsilon \neq 0$ sufficiently small.

2. Multi-Bump Homoclinics

We consider in (1.1) a tubular coordinate system (v, φ), $v \in \mathbb{R}^{m-1}, \varphi \in \mathbb{R}$ near the periodic orbit ξ. Then (1.1) has the form

$$\begin{aligned}
\dot{x} &= \tilde{f}(x,v,\varphi) + \epsilon\tilde{h}(x,v,\varphi,t,\epsilon)\,,\\
\dot{v} &= \epsilon\Big(A(\varphi)v + \tilde{g}_1(v,\varphi) + \tilde{p}_1(x,v,\varphi,t,\epsilon) + \epsilon\tilde{q}_1(v,\varphi,t,\epsilon)\Big)\,,\\
\dot{\varphi} &= \epsilon\Big(1 + \tilde{g}_2(v,\varphi) + \tilde{p}_2(x,v,\varphi,t,\epsilon) + \epsilon\tilde{q}_2(v,\varphi,t,\epsilon)\Big)\,,
\end{aligned} \tag{2.1}$$

where all mappings are smooth, ω-periodic in φ and 1-periodic in t such that

$$\begin{aligned}
&\tilde{g}_1(0,\cdot) = 0\,, \quad \tilde{g}_{1v}(0,\cdot) = 0\,, \quad \tilde{g}_2(0,\cdot) = 0\,,\\
&\tilde{p}_1(0,\cdot,\cdot,\cdot,\cdot) = 0\,, \quad \tilde{p}_2(0,\cdot,\cdot,\cdot,\cdot) = 0\,.
\end{aligned}$$

Since A is ω-periodic, by the Floquet theorem [10] there is a 2ω-periodic real-valued regular matrix $P(t)$ and a constant matrix B such that $\dot{P} + PB = AP$. By making the change of variables $v \leftrightarrow P(\varphi)v$ in (2.1), we can assume that $A(\varphi) = B$ in (2.1). Moreover, since ξ is hyperbolic, the eigenvalues of B lie off the imaginary axis. Hence we study the system of equations

$$\begin{aligned}
\dot{x} &= f(x,v,\varphi) + \epsilon h(x,v,\varphi,t,\epsilon)\,,\\
\dot{v} &= \epsilon\Big(Bv + g_1(v,\varphi) + p_1(x,v,\varphi,t,\epsilon) + \epsilon q_1(v,\varphi,t,\epsilon)\Big)\,,\\
\dot{\varphi} &= \epsilon\Big(1 + g_2(v,\varphi) + p_2(x,v,\varphi,t,\epsilon) + \epsilon q_2(v,\varphi,t,\epsilon)\Big)\,,
\end{aligned} \tag{2.2}$$

where all mappings are smooth, 2ω-periodic in φ and 1-periodic in t such that

$$\begin{aligned}
&g_1(0,\cdot) = 0\,, \quad g_{1v}(0,\cdot) = 0\,, \quad g_2(0,\cdot) = 0\,,\\
&p_1(0,\cdot,\cdot,\cdot,\cdot) = 0\,, \quad p_2(0,\cdot,\cdot,\cdot,\cdot) = 0\,.
\end{aligned}$$

According to [2], there is a global center manifold of (2.2) which is the graph of a mapping $x = \epsilon H(v,\varphi,t,\epsilon)$ for a smooth mapping H, periodic as above. Moreover $z(t) = H(v,\varphi,t,0)$ is the unique 1-periodic solution of the equation $\dot{z} = f_x(0,v,\varphi)z + h(0,v,\varphi,t,0)$. By making the change of variables $x = z + \epsilon H(v,\varphi,t,\epsilon)$ in (2.2) we arrive at the system

$$\begin{aligned}
\dot{z} &= f\big(z + \epsilon H(v,\varphi,t,\epsilon),v,\varphi\big) - f\big(\epsilon H(v,\varphi,t,\epsilon),v,\varphi\big)\\
&\quad + \epsilon\Big(h\big(z + \epsilon H(v,\varphi,t,\epsilon),v,\varphi,t,\epsilon\big) - h\big(\epsilon H(v,\varphi,t,\epsilon),v,\varphi,t,\epsilon\big)\Big)\\
&\quad - \epsilon^2 H_v(v,\varphi,t,\epsilon)\Big(p_1\big(z + \epsilon H(v,\varphi,t,\epsilon),v,\varphi,t,\epsilon\big) - p_1\big(\epsilon H(v,\varphi,t,\epsilon),v,\varphi,t,\epsilon\big)\Big)\\
&\quad - \epsilon^2 H_\varphi(v,\varphi,t,\epsilon)\Big(p_2\big(z + \epsilon H(v,\varphi,t,\epsilon),v,\varphi,t,\epsilon\big) - p_2\big(\epsilon H(v,\varphi,t,\epsilon),v,\varphi,t,\epsilon\big)\Big)\,,\\
\dot{v} &= \epsilon\Big(Bv + g_1(v,\varphi) + p_1\big(z + \epsilon H(v,\varphi,t,\epsilon),v,\varphi,t,\epsilon\big) + \epsilon q_1(v,\varphi,t,\epsilon)\Big)\,,\\
\dot{\varphi} &= \epsilon\Big(1 + g_2(v,\varphi) + p_2\big(z + \epsilon H(v,\varphi,t,\epsilon),v,\varphi,t,\epsilon\big) + \epsilon q_2(v,\varphi,t,\epsilon)\Big)\,.
\end{aligned} \tag{2.3}$$

Now let us consider the system

$$\begin{aligned}
\dot{v} &= \epsilon\Big(Bv + g_1(v,\varphi) + p_1\big(\epsilon H(v,\varphi,t,\epsilon),v,\varphi,t,\epsilon\big) + \epsilon q_1(v,\varphi,t,\epsilon)\Big)\,,\\
\dot{\varphi} &= \epsilon\Big(1 + g_2(v,\varphi) + p_2\big(\epsilon H(v,\varphi,t,\epsilon),v,\varphi,t,\epsilon\big) + \epsilon q_2(v,\varphi,t,\epsilon)\Big)\,.
\end{aligned} \tag{2.4}$$

According to [8, 18], there is a global center manifold (an invariant torus) for (2.4), which can be represented as the graph of a mapping $v = \epsilon G(\varphi, t, \epsilon)$ with the above periodicity. G is smooth in $\varphi, t, \epsilon \neq 0$ small and with uniformly bounded derivatives of (φ, t) as $\epsilon \to 0$. Moreover, one can check that $G_t = O(\epsilon)$. We make the change of variables $v \leftrightarrow \epsilon v + \epsilon G(\varphi, t, \epsilon)$ in (2.3) to get the system

$$\begin{aligned}\dot z &= f\big(z + \epsilon H(0,\varphi,t,0), \epsilon v + \epsilon G(\varphi,t,\epsilon),\varphi\big) - f\big(\epsilon H(0,\varphi,t,0), \epsilon v + \epsilon G(\varphi,t,\epsilon),\varphi\big)\\ &\quad + O(\epsilon^2) z + \epsilon\big(h(z,0,\varphi,t,0) - h(0,0,\varphi,t,0)\big)\,,\\ \dot v &= \epsilon\big(Bv + O(\epsilon)v + O(z)\big) + p_1(z,\epsilon v,\varphi,t,0)\,,\\ \dot\varphi &= \epsilon\Big(1 + g_2\big(\epsilon v + \epsilon G(\varphi,t,\epsilon),\varphi\big)\\ &\quad + p_2\big(z + \epsilon H\big(\epsilon v + \epsilon G(\varphi,t,\epsilon),\varphi,t,\epsilon\big), \epsilon v + \epsilon G(\varphi,t,\epsilon),\varphi,t,\epsilon\big)\\ &\quad + \epsilon q_2\big(\epsilon v + \epsilon G(\varphi,t,\epsilon),\varphi,t,\epsilon\big)\Big)\,.\end{aligned} \tag{2.5}$$

Remark 2.1. To simplify our writing, we identify points $(0,\varphi)$ with φ. So we drop the zeroes $v = 0$ in the formulas below.

Now we start with construction of multi-bump homoclinic solutions for (2.5). We need for this purpose the following notions. Let $\epsilon > 0$. Let $E = [1/\epsilon]$ and $F = [1/\sqrt{\epsilon}]$ be the integer parts of $1/\epsilon$ and $1/\sqrt{\epsilon}$, respectively. Let $p, N \in \mathbb{N}$ and take $i_1, i_2, \cdots, i_p \in \{1, 2, \cdots, N\}$. For convenience we put $i_0 = i_{p+1} = 1$. We define the Banach spaces

$$\begin{gathered} Y^n_{j,\epsilon} = C(J_{j,\epsilon}, \mathbb{R}^n), \quad 1 \le j \le p\,,\\ Y^n_{p+1,\epsilon} = C(J_{p+1,\epsilon}, \mathbb{R}^n), \quad Y^n_{0,\epsilon} = C(J_{0,\epsilon}, \mathbb{R}^n)\,,\\ Z^n_\epsilon = C([-F,F], \mathbb{R}^n)\end{gathered}$$

with the supremum norm $||\cdot||$, where $J_{j,\epsilon} = [-i_jE, i_jE], J_{p+1,\epsilon} = [-i_{p+1}E, \infty)$, $J_{0,\epsilon} = (-\infty, i_0E]$.

In (2.5) we now make the first set of change of variables

$$\begin{aligned} & z(t) = \epsilon z_0(t), \quad t \in J_{0,\epsilon}, \quad z_0 \in Y^n_{0,\epsilon}\,,\\ & z\big(t + (i_0 + 2(i_1 + \cdots + i_{j-1}) + i_j)E + 2jF\big) = \epsilon z_j(t)\\ & \quad t \in J_{j,\epsilon}, \quad z_j \in Y^n_{j,\epsilon}, \quad 1 \le j \le p+1\,,\\ & v\big(t + (i_0 + 2(i_1 + \cdots + i_{j-1}) + i_j)E + 2jF\big) = v_j(t),\\ & \quad t \in J_{j,\epsilon}, \quad v_j \in Y^m_{j,\epsilon}, \quad 0 \le j \le p+1\,,\\ & v\big(t + (i_0 + 2(i_1 + \cdots + i_j))E + (2j+1)F\big) = \tilde v_j(t),\\ & \quad t \in [-F,F], \quad \tilde v_j \in Z^m_\epsilon, \quad 0 \le j \le p\,,\\ & \varphi\big(t + (i_0 + 2(i_1 + \cdots + i_{j-1}) + i_j)E + 2jF\big) = \tau_j + \varphi_j(t),\\ & \quad t \in J_{j,\epsilon}, \quad \varphi_j \in Y^1_{j,\epsilon}, \quad 0 \le j \le p+1\,,\\ & \varphi\big(t + (i_0 + 2(i_1 + \cdots + i_j))E + (2j+1)F\big) = \tilde\tau_j + \tilde\varphi_j(t),\\ & \quad t \in [-F,F], \quad \tilde\varphi_j \in Z^1_\epsilon, \quad 0 \le j \le p\,,\end{aligned}$$

$$\begin{aligned}&\varphi_j(0)=0,\quad 1\le j\le p,\quad \tilde\varphi_j(0)=0,\quad 0\le j\le p\,,\\ &\varphi_0(i_0E)=\varphi_{p+1}(-i_{p+1}E)=0\,.\end{aligned}$$

Then we take $\alpha\in\mathbb{R}^{p+1}$, $\theta\in\mathbb{R}^{(p+1)(d-1)}$ and put:

$$\tilde\gamma_j(t)=\gamma\big(\theta_j,\epsilon\tilde v_j+\epsilon G(\tilde\tau_j+\tilde\varphi_j,t,\epsilon),\tilde\tau_j+\tilde\varphi_j,t-\alpha_j\big),\quad 0\le j\le p\,.$$

Here α are considered as the time shifts in the homoclinics γ. We define the functions $b_j^\pm$, $0\le j\le p$ by

$$b_j^+(r)=-\tilde\gamma_j(r),\quad b_j^-(r)=\tilde\gamma_j(-r)\,.$$

Note that there is a constant $M>0$ such that $|b_j^\pm(r)|=O\big(\mathrm{e}^{-Mr}\big)$ as $r\to+\infty$ uniformly with respect to other bounded parameters.

Then we make in (2.5) the second set of change of variables

$$\begin{aligned}&z\big(t+(i_0+2(i_1+\cdots+i_j))E+(2j+1)F\big)\\ &\quad=\tilde\gamma_j(t)+\epsilon\tilde z_j(t)+\frac{1}{2F}b_j^+(F)(t+F)+\frac{1}{2F}b_j^-(F)(t-F),\\ &\quad\ t\in[-F,F],\quad \tilde z_j\in Z_\epsilon^n,\quad 0\le j\le p\,.\end{aligned}$$

The functions $b_j^\pm$ are constructed so that if $z_j(i_jE)=\tilde z_j(-F)$, $z_{j+1}(-i_{j+1}E)=\tilde z_j(F)$, $0\le j\le p$ and $z_j,\tilde z_j$ are continuous then z is continuously extended on $\mathbb{R}$.

Summarizing we see that the expected multi-bump homoclinic solution

$$w(t)=\big(z(t),v(t),\varphi(t)\big)$$

of (2.5) is looked for as the union of the following sequence of orbits

$$\Big(w_0(t),\tilde w_0(t),w_1(t),\tilde w_1(t),\cdots,w_p(t),\tilde w_p(t),w_{p+1}(t)\Big)\,,$$

where $w_0(t)$ is defined for $t\in(-\infty,i_0E]$, $E=[1/\epsilon]$ and its z-component is small; $w_{p+1}(t)$ is defined for $t\in[-i_{p+1}E,\infty]$ and its z-component is small; $w_j(t)$, $j=1,2,\cdots,p$ are defined for $t\in[-i_jE,i_jE]$ and their z-components are small; $\tilde w_j(t)$, $j=0,1,\cdots,p$ are defined for $t\in[-F,F]$, $F=[1/\sqrt{\epsilon}\,]$ and their z-components are near $\tilde\gamma_j$, respectively. Of course, these orbits are smoothly connected and their definition intervals are suitablely shifted with respect to the original orbit $w(t)$.

Hence (2.5) splits into the following sequence of systems of ordinary differential equations

$$\begin{aligned}\dot z_j&=\big(f_x(0,\tau_j+\varphi_j)+O(\epsilon)\big)z_j\\ \dot v_j&=\epsilon\big(Bv_j+O(\epsilon)v_j+O(z_j)\big)\\ \dot\varphi_j&=\epsilon(1+O(\epsilon)),\quad j=0,\,p+1\,,\end{aligned}\tag{2.6.1}$$

$$\begin{aligned}\dot{\tilde z}_j&=f_x(\tilde\gamma_j,\tilde\tau_j+\tilde\varphi_j)\tilde z_j-\tilde\gamma_{jv}p_1(\tilde\gamma_j,\tilde\tau_j+\tilde\varphi_j,t,0)\\ &\quad+\big(f_x(\tilde\gamma_j,\tilde\tau_j+\tilde\varphi_j)-f_x(0,\tilde\tau_j+\tilde\varphi_j)\big)H(\tilde\tau_j+\tilde\varphi_j,t,0)\\ &\quad+h(\tilde\gamma_j,\tilde\tau_j+\tilde\varphi_j,t,0)-h(0,\tilde\tau_j+\tilde\varphi_j,t,0)\\ &\quad-\tilde\gamma_{j\varphi}\cdot\big(1+p_2(\tilde\gamma_j,\tilde\tau_j+\tilde\varphi_j,t,0)\big)+O(\epsilon)\,,\\ \dot{\tilde v}_j&=\epsilon\big(B\tilde v_j+O(\epsilon)\tilde v_j+p_{1v}(\tilde\gamma_j,\tilde\tau_1+\tilde\varphi_j,t,0)\tilde v_j\\ &\quad+O(\tilde\gamma_j)+O(\tilde z_j)+O(\epsilon)\big)+p_1(\tilde\gamma_j,\tilde\tau_j+\tilde\varphi_j,t,0)\,,\\ \dot{\tilde\varphi}_j&=\epsilon\big(1+p_2(\tilde\gamma_j,\tilde\tau_j+\tilde\varphi_j,t,0)+O(\epsilon)\big),\quad 0\le j\le p\,,\end{aligned}\tag{2.6.2}$$

$$\begin{aligned}\dot{z}_j &= \big(f_x(0,\tau_j+\varphi_j)+O(\epsilon)\big)z_j\\ \dot{v}_j &= \epsilon\big(Bv_j+O(\epsilon)v_j+O(z_j)\big)\\ \dot{\varphi}_j &= \epsilon(1+O(\epsilon)),\quad 1\le j\le p\,,\end{aligned}\tag{2.6.3}$$

where

$$\begin{aligned}\tilde{\gamma}_{jv}(t) &= \gamma_v\big(\theta_j,\epsilon\tilde{v}_j+\epsilon G(\tilde{\tau}_j+\tilde{\varphi}_j,t,\epsilon),\tilde{\tau}_j+\tilde{\varphi}_j,t-\alpha_j\big)\,,\\ \tilde{\gamma}_{j\varphi}(t) &= \gamma_\varphi\big(\theta_j,\epsilon\tilde{v}_j+\epsilon G(\tilde{\tau}_j+\tilde{\varphi}_j,t,\epsilon),\tilde{\tau}_j+\tilde{\varphi}_j,t-\alpha_j\big)\,.\end{aligned}$$

We recall Remark 2.1 for (2.6.1-2.6.3). Of course, the associated boundary conditions to (2.6.1-2.6.3) are as follows

$$\begin{gathered}z_j(i_jE)=\tilde{z}_j(-F),\quad \tilde{z}_j(F)=z_{j+1}(-i_{j+1}E)\\ \tau_j+\varphi_j(i_jE)=\tilde{\tau}_j+\tilde{\varphi}_j(-F),\quad \tilde{\tau}_j+\tilde{\varphi}_j(F)=\tau_{j+1}+\varphi_{j+1}(-i_{j+1}E)\\ v_j(i_jE)=\tilde{v}_j(-F),\quad \tilde{v}_j(F)=v_{j+1}(-i_{j+1}E),\quad 0\le j\le p\,.\end{gathered}\tag{2.7}$$

From (2.7) we immediately get

$$\begin{aligned}&\tau_0=\tilde{\tau}_0+O(\sqrt{\epsilon}),\quad \tau_{p+1}=\tilde{\tau}_p+O(\sqrt{\epsilon})\,,\\ &\tilde{\tau}_j=\tau_j+i_j+O(N\sqrt{\epsilon}),\quad 1\le j\le p\\ &\tilde{\tau}_j=\tau_{j+1}-i_{j+1}+O(N\sqrt{\epsilon}),\quad 0\le j\le p-1\,.\end{aligned}\tag{2.8}$$

Since

$$\begin{gathered}\tau_j+\varphi_j(t)=k_j-i_j+\tau_1+\epsilon t+O(jN\sqrt{\epsilon}),\quad 1\le j\le p\\ \tilde{\tau}_j+\tilde{\varphi}_j(t)=k_j+\tau_1+O((j+1)N\sqrt{\epsilon}),\quad 0\le j\le p\,,\end{gathered}$$

where $k_j=i_1+2(i_2+\cdots+i_j)$, $\quad 1\le j\le p$, $\quad k_0=-i_1$, the systems (2.6.2), (2.6.3) and (2.8) are equivalent to the systems

$$\begin{gathered}\dot{\tilde{z}}_j=f_x(\gamma_j,k_j+\tau_1)\tilde{z}_j-\gamma_{jv}p_1(\gamma_j,k_j+\tau_1,t,0)\\ +\big(f_x(\gamma_j,k_j+\tau_1)-f_x(0,k_j+\tau_1)\big)H(k_j+\tau_1,t,0)\\ +h(\gamma_j,k_j+\tau_1,t,0)-h(0,k_j+\tau_1,t,0)\\ -\gamma_{j\varphi}\cdot\big(1+p_2(\gamma_j,k_j+\tau_1,t,0)\big)+O((j+1)N\sqrt{\epsilon})\,,\\ \dot{\tilde{v}}_j=\epsilon\big(B\tilde{v}_j+O(\epsilon)\tilde{v}_j+p_{1v}(\gamma_j,k_j+\tau_1,t,0)\tilde{v}_j\\ +O(\tilde{z}_j)+O(\gamma_j)\big)+O((j+1)N\sqrt{\epsilon})+p_1(\gamma_j,k_j+\tau_1,t,0)\,,\\ \dot{\tilde{\varphi}}_j=\epsilon\big(1+p_2(\gamma_j,k_j+\tau_1,t,0)+O((j+1)N\sqrt{\epsilon})\big),\quad 0\le j\le p\,,\end{gathered}\tag{2.9.1}$$

$$\begin{gathered}\tilde{\tau}_j=k_j+\tau_1+\sqrt{\epsilon}\tilde{\chi}_j,\quad 0\le j\le p\\ \tau_j=k_j-i_j+\tau_1+\sqrt{\epsilon}\chi_j,\quad 1\le j\le p\\ \tau_0=\tau_1-i_1+\sqrt{\epsilon}\chi_0,\quad \tau_{p+1}=k_p+\tau_1+\sqrt{\epsilon}\chi_{p+1}\,,\end{gathered}\tag{2.9.2}$$

$$\begin{aligned}\dot z_j &= \big(f_x(0,k_j-i_j+\tau_1+\epsilon t)+O(jN\sqrt{\epsilon})\big)z_j\\ \dot v_j &= \epsilon\big(Bv_j+O(jN\sqrt{\epsilon})v_j+O(z_j)\big)\\ \dot\varphi_j &= \epsilon(1+O(jN\sqrt{\epsilon})),\quad 1\le j\le p\,,\end{aligned}\tag{2.9.3}$$

where

$$\begin{aligned}\gamma_j &= \gamma(\theta_j,\tau_1+k_j,t-\alpha_j)\,,\\ \gamma_{jv} &= \gamma_v(\theta_j,\tau_1+k_j,t-\alpha_j)\,,\\ \gamma_{j\varphi} &= \gamma_\varphi(\theta_j,\tau_1+k_j,t-\alpha_j)\,.\end{aligned}$$

Since $f_x(\gamma_j,k_j+\tau_1)\to f_x(0,k_j+\tau_1)$ as $t\to\pm\infty$ uniformly for θ,τ_1,k_j and α bounded, and since $f_x(0,\cdot)$ satisfy (i), by results of [4, 14] the linear systems

$$\begin{aligned}\dot u &= A_j(t)u\,,\\ A_j(t) &= f_x(\gamma_j,k_j+\tau_1),\quad 0\le j\le p\end{aligned}\tag{2.10}$$

have exponential dichotomies both on $\mathbb{R}_+$ and $\mathbb{R}_-$. Consequently, we get the following result similar to [7].

Theorem 2.2. *There exist fundamental solutions U_j for (2.10) along with constants $M>0$, $K_0>0$ and projections P^j_{ss}, P^j_{su}, P^j_{us}, P^j_{uu}, $Q^{\tau_1+k_j}_{us}$, $Q^{\tau_1+k_j}_{su}$ such that $P^j_{ss}+P^j_{su}+P^j_{us}+P^j_{uu}=I$ and that the following hold:*

(i) $|U_j(t)(P^j_{ss}+P^j_{us})U_j(s)^{-1}|\le K_0\mathrm{e}^{2M(s-t)}$ *for* $0\le s\le t$,

(ii) $|U_j(t)(P^j_{su}+P^j_{uu})U_j(s)^{-1}|\le K_0\mathrm{e}^{2M(t-s)}$ *for* $0\le t\le s$,

(iii) $|U_j(t)(P^j_{ss}+P^j_{su})U_j(s)^{-1}|\le K_0\mathrm{e}^{2M(t-s)}$ *for* $t\le s\le 0$,

(iv) $|U_j(t)(P^j_{us}+P^j_{uu})U_j(s)^{-1}|\le K_0\mathrm{e}^{2M(s-t)}$ *for* $s\le t\le 0$,

(v) $\lim_{t\to+\infty}U_j(t)(P^j_{ss}+P^j_{us})U_j(t)^{-1}=Q^{\tau_1+k_j}_{us}$,

(vi) $\lim_{t\to+\infty}U_j(t)(P^j_{su}+P^j_{uu})U_j(t)^{-1}=Q^{\tau_1+k_j}_{su}$,

(vii) $\lim_{t\to-\infty}U_j(t)(P^j_{ss}+P^j_{su})U_j(t)^{-1}=Q^{\tau_1+k_j}_{su}$,

(viii) $\lim_{t\to-\infty}U_j(t)(P^j_{us}+P^j_{uu})U_j(t)^{-1}=Q^{\tau_1+k_j}_{us}$.

Also, rank P^j_{ss} = rank $P^j_{uu}=d$.

Let $u_{i,j}$ denote column i of U_j and assume these are numbered so that

$$P^j_{uu}=\begin{pmatrix}I_d&0_d&0\\0_d&0_d&0\\0&0&0\end{pmatrix},\qquad P^j_{ss}=\begin{pmatrix}0_d&0_d&0\\0_d&I_d&0\\0&0&0\end{pmatrix}.$$

Here, I_d denotes the $d\times d$ identity matrix and 0_d denotes the $d\times d$ zero matrix.

For each $i=1,\cdots,n$ we define $u^{\perp}_{i,j}(t)$ by $\langle u^{\perp}_{i,j}(t),u_{k,j}(t)\rangle=\delta_{ik}$, where $\langle\cdot,\cdot\rangle$ is the inner product. The vectors $u^{\perp}_{i,j}$ can be computed from the formula $U^{\perp t}_j=U^{-1}_j$ where $U^{\perp}_j$ denotes the matrix with $u^{\perp}_{i,j}$ as column i. We note that $U^{\perp}_j$ is the adjoint of U_j with respect to (2.10). Without loss of generality we can suppose that $u_{d+i,j}=\frac{\partial}{\partial\theta_i}\gamma_j(t)$, $i=1,\cdots,d-1$, $u_{2d,j}=\dot\gamma_j(t)$.

Furthermore, the linear equation $\dot u=f_x(0,k_j-i_j+\tau_1+\epsilon t)u$, $1\leq j\leq p$ has according to [4, 14] the exponential dichotomy on $\mathbb{R}$ with smooth projections $Q^{\epsilon,\tau_1+k_j-i_j}_{us}$ and $Q^{\epsilon,\tau_1+k_j-i_j}_{su}$.

We note that $Q^{\tau_1+k_j}_{us}$, $Q^{\tau_1+k_j}_{su}$ and $Q^{0,\tau_1+k_j-i_j}_{us}$, $Q^{0,\tau_1+k_j-i_j}_{su}$ are the projections of the exponential dichotomies on $\mathbb{R}$ of the linear equations $\dot u=f_x(0,\tau_1+k_j)u$ and $\dot u=f_x(0,\tau_1+k_j-i_j)u$, respectively. Moreover, the projections and fundamental solutions given by Theorem 2.2 depend smoothly on the parameters.

By using the method of [1, 2] we have the following result for system (2.6.1) with $j=0$.

Theorem 2.3. *Let $X_{u\tau_0},X_{uB}$ be the unstable spaces of the linear systems $\dot u=f_x(0,\tau_0)u$ and $\dot v=Bv$, respectively, with the projections Q^{0,τ_0}_{su} and P_{uB}. There are smooth mappings $Z_1(\psi_1,\psi_2,t,\epsilon)$, $V_1(\psi_1,\psi_2,t,\epsilon)$ for uniformly bounded $\psi_1\in X_{u\tau_0}$, $\psi_2\in X_{uB}$ and $\epsilon>0$ small such that there is a unique solution $z_0(t)=Z_1(\psi_1,\psi_2,t,\epsilon),v_0(t)=V_1(\psi_1,\psi_2,t,\epsilon)$, $\varphi_0(t)$ of (2.6.1) for $j=0$ satisfying the conditions $Q^{0,\tau_0}_{su}z_0(E)=\psi_1$, $P_{uB}v_0(E)=\psi_2$, $\varphi_0(E)=0$. Moreover, $z_0(t)\to 0,v_0(t)\to 0$ exponentially as $t\to-\infty$.*

We have a similar result for system (2.6.1) with $j=p+1$ again by using the method of [1, 2].

Theorem 2.4. *Let $X_{s\tau_{p+1}},X_{sB}$ be the stable spaces of the linear systems $\dot u=f_x(0,\tau_{p+1})u$ and $\dot v=Bv$, respectively, with the projections $Q^{0,\tau_{p+1}}_{us}$ and P_{sB}. There are smooth mappings $Z_2(\psi_1,\psi_2,t,\epsilon)$, $V_2(\psi_1,\psi_2,t,\epsilon)$ for uniformly bounded $\psi_1\in X_{s\tau_{p+1}}$, $\psi_2\in X_{sB}$ and $\epsilon>0$ small such that there is a unique solution $z_{p+1}(t)=Z_2(\psi_1,\psi_2,t,\epsilon)$, $v_{p+1}(t)=V_2(\psi_1,\psi_2,t,\epsilon)$, $\varphi_{p+1}(t)$ of (2.6.1) for $j=p+1$ such that $Q^{0,\tau_{p+1}}_{us}z_{p+1}(-E)=\psi_1$, $P_{sB}v_{p+1}(-E)=\psi_2$, $\varphi_{p+1}(-E)=0$. Moreover, $z_{p+1}(t)\to 0,v_{p+1}(t)\to 0$ exponentially as $t\to+\infty$.*

Now we consider the non-homogeneous linear equations

$$\begin{gathered}\dot{\tilde z}_j=A_j(t)\tilde z_j+h_j,\quad h_j\in Z^n_\epsilon,\quad 0\leq j\leq p\\ \dot z_j=f_x(0,\tau_1+k_j-i_j+\epsilon t)z_j+m_j,\quad m_j\in Y^n_{j,\epsilon},\quad 1\leq j\leq p\\ z_j(i_jE)=\tilde z_j(-F),\quad \tilde z_j(F)=z_{j+1}(-i_{j+1}E),\quad 0\leq j\leq p\,,\end{gathered}\tag{2.11}$$

$$\begin{gathered}\dot{\tilde v}_j=\epsilon\bigl(B\tilde v_j+\tilde h_j\bigr)+\sqrt{\epsilon}\tilde r_j,\quad \tilde h_j,\,\tilde r_j\in Z^m_\epsilon,\quad 0\leq j\leq p\\ \dot v_j=\epsilon\bigl(Bv_j+\tilde m_j\bigr),\quad \tilde m_j\in Y^m_{j,\epsilon},\quad 1\leq j\leq p\\ v_j(i_jE)=\tilde v_j(-F),\quad \tilde v_j(F)=v_{j+1}(-i_{j+1}E),\quad 0\leq j\leq p\,.\end{gathered}\tag{2.12}$$

By combining the method of [5] together with Theorems 2.3 and 2.4, we get the following result.

Theorem 2.5. *Let the following transversality assumptions*

$$\begin{aligned}\operatorname{Im} Q_{su}^{\tau_1+k_j} \oplus \operatorname{Im} Q_{us}^{0,\tau_1+k_{j+1}-i_{j+1}} = \mathbb{R}^n\,,\\ \operatorname{Im} Q_{us}^{\tau_1+k_{j+1}} \oplus \operatorname{Im} Q_{su}^{0,\tau_1+k_{j+1}-i_{j+1}} = \mathbb{R}^n\end{aligned} \tag{2.13}$$

hold for any $0 \le j \le p-1$.

Then for any $K > 0$, there exist $\epsilon_0 > 0$, $M > 0$, $A > 0$, $B > 0$ such that for every j, $0 \le j \le p$, $0 < \epsilon < \epsilon_0$ and $\alpha \in \mathbb{R}^{p+1}$, $\theta \in \mathbb{R}^{(p+1)(d-1)}$ such that $|\alpha| \le K$, there exist functions $\mathcal{L}_{\epsilon,\alpha,\theta,j}\colon Z_\epsilon^n \to \mathbb{R}^n$ with $||P_{uu}^j \mathcal{L}_{\epsilon,\alpha,\theta,j}|| \le A\mathrm{e}^{-M/\sqrt{\epsilon}}$ and with the property that if $m_j \in Y_{j,\epsilon}^n$, $\tilde{m}_j \in Y_{j,\epsilon}^m$, $1 \le j \le p$, $h_j \in Z_\epsilon^n$, $\tilde{h}_j$, $\tilde{r}_j \in Z_\epsilon^m$, $0 \le j \le p$, $\max_{1\le j\le p} ||m_j|| \le K$, $\max_{1\le j\le p} ||\tilde{m}_j|| \le K$, $\max_{0\le j\le p} ||h_j|| \le K$, $\max_{0\le j\le p} ||\tilde{h}_j|| \le K$, $\max_{0\le j\le p} ||\tilde{r}_j|| \le K$ satisfy

$$\int_{-F}^{F} P_{uu}^j U_j(t)^{-1} h_j(t)\,dt + P_{uu}^j \mathcal{L}_{\epsilon,\alpha,\theta,j} h_j = 0, \quad \forall\, j,\, 0 \le j \le p\,, \tag{2.14}$$

then (2.11), (2.12) and (2.6.1) have solutions $\tilde{z}_j \in Z_\epsilon^n$, $\tilde{v}_j \in Z_\epsilon^m$, $z_j \in Y_{j,\epsilon}^n$, $v_j \in Y_{j,\epsilon}^m$, $\varphi_0 \in Y_{0,\epsilon}^1$, $\varphi_{p+1} \in Y_{p+1,\epsilon}^1$ satisfying

$$\begin{aligned}&P_{ss}^j U_j(0)^{-1} \tilde{z}_j(0) = 0\\ &z_0(i_0E) = \tilde{z}_0(-F), \quad \tilde{z}_p(F) = z_{p+1}(-i_{p+1}E), \quad v_0(i_0E) = \tilde{v}_0(-F)\,,\\ &\tilde{v}_p(F) = v_{p+1}(-i_{p+1}E), \quad \varphi_0(i_0E) = \varphi_{p+1}(-i_{p+1}E) = 0\\ &\max_{0\le j\le p+1} ||z_j|| \le B \max_{1\le j\le p} ||m_j||, \quad \max_{0\le j\le p+1} ||v_j|| \le B \max_{1\le j\le p} ||\tilde{m}_j||\\ &\max_{0\le j\le p} ||\tilde{z}_j|| \le B \max_{0\le j\le p} ||h_j||, \quad \max_{0\le j\le p} ||\tilde{v}_j|| \le B \max_{0\le j\le p} \big(||\tilde{h}_j|| + ||\tilde{r}_j||\big)\,.\end{aligned}$$

Moreover, these solutions depend smoothly on the parameters.

We note that (2.14) represents Fredholm-like solvability assumptions for (2.11).

Remark 2.6. We remark that (2.13) holds either when all $i_j, 1 \le j \le p$ are multiples of ω, since then $Q_{su}^{\cdot,\cdot}$ and $Q_{us}^{\cdot,\cdot}$ in (2.13) are independent of $i_j, 1 \le j \le p$, or if the family $f_x(0,\varphi), \varphi \in \mathbb{R}$ of matrices is uniformly diagonalized, i.e. there is a smooth, ω-periodic family $T(\varphi), \varphi \in \mathbb{R}$ of invertible matrices such that

$$T(\varphi) f_x(0,\varphi) T(\varphi)^{-1} = \begin{pmatrix} D_s(\varphi) & 0 \\ 0 & D_u(\varphi) \end{pmatrix}\,, \tag{2.15}$$

where $D_s(\varphi), D_u(\varphi)$ are smooth families of stable and unstable matrices, respectively. Then by making the change of variables $z \leftrightarrow T(\varphi)z$ in (2.1), assumption (2.13) is trivially satisfied, since $Q_{su}^{\cdot,\cdot}$ and $Q_{us}^{\cdot,\cdot}$ in (2.13) will be constant with respect to $i_j, 1 \le j \le p$ and τ_1 as well. We note that we can always find smooth $T(\varphi)$ on $\mathbb{R}$ satisfying (2.15), but the ω-periodicity of T is generally problematic.

According to Theorem 2.2, the projection P_{uu}^j and the fundamental solution U_j correspond to the linear system

$$\dot{u} = f_x\big(\gamma(\theta_j, \tau_1 + k_j, t - \alpha_j), \tau_1 + k_j\big)u\,.$$

We note that

$$P^j_{uu}U_j(t)^{-1}w = \Big(\langle u^\perp_{1,j}(t), w\rangle, \cdots, \langle u^\perp_{d,j}(t), w\rangle\Big)\,.$$

The functions $\Big\{u^\perp_{1,j}(t), u^\perp_{2,j}(t), \cdots, u^\perp_{d,j}(t)\Big\}$ represents the complete family of bounded solutions to the adjoint equation

$$\dot u = -f_x\big(\gamma(\theta_j, \tau_1+k_j, t-\alpha_j), \tau_1+k_j\big)^t u\,.$$

We can take instead of $\Big\{u^\perp_{1,j}(t), u^\perp_{2,j}(t), \cdots, u^\perp_{d,j}(t)\Big\}$ any family

$$\Big\{w_i(\theta_j, \tau_1+k_j, t-\alpha_j) \mid i = 1, 2, \cdots d\Big\}\,,$$

where $\Big\{w_i(\theta, \tau, t) \mid i = 1, 2, \cdots d\Big\}$ forms a smooth family of bounded solutions of the adjoint equation

$$\dot w = -f_x\big(\gamma(\theta, \tau, t), \tau\big)^t w\,.$$

By using properties (i)-(viii) of Theorem 2.2 to $P^j_{uu}U_j(t)^{-1}$ from (2.14) along with Theorem 2.5 we can simultaneously solve all systems (2.6.1) and (2.9.1), (2.9.2), (2.9.3) together for $\epsilon > 0$ sufficiently small by applying the Lyapunov-Schmidt procedure like in [5]. The values $\tilde\chi_0, \tilde\chi_1, \cdots, \tilde\chi_p, \chi_0, \chi_1, \cdots, \chi_{p+1}$ can be recursively computed from (2.9.2) by using (2.7). In this way we get from (2.14) and the first equations of (2.9.1), the limit bifurcation equations (see [5, p. 2868])

$$M_j(\theta_j, \alpha_j, \tau_1) = 0, \quad 0 \le j \le p\,, \tag{2.16}$$

$$\begin{aligned}M_j(\theta_j, \alpha_j, \tau_1) = \int_{-\infty}^{\infty} P^j_{uu}U_j(t)^{-1}\Big\{ -\gamma_{jv}p_1(\gamma_j, \tau_1+k_j, t, 0)\\ +\big(f_x(\gamma_j, \tau_1+k_j) - f_x(0, \tau_1+k_j)\big)H(\tau_1+k_j, t, 0)\\ +h(\gamma_j, \tau_1+k_j, t, 0) - h(0, \tau_1+k_j, t, 0) - \gamma_{j\varphi}\cdot\big(1+p_2(\gamma_j, \tau_1+k_j, t, 0)\big)\Big\}\,dt\,.\end{aligned}$$

Since

$$\begin{aligned}&\big(f_x(\gamma_j, \tau_1+k_j) - f_x(0, \tau_1+k_j)\big)H(\tau_1+k_j, t, 0) - h(0, \tau_1+k_j, t, 0)\\ &= -H_t(\tau_1+k_j, t, 0) + f_x(\gamma_j, \tau_1+k_j)H(\tau_1+k_j, t, 0)\,,\end{aligned}$$

then according to [7, 15] from (2.16) we get

$$M_j = \big(M_{j1}, M_{j2}, \cdots, M_{jd}\big) = 0, \quad 0 \le j \le p\,, \tag{2.17}$$

$$\begin{aligned}M_{jk}(\theta_j, \alpha_j, \tau_1) = \int_{-\infty}^{\infty} \Big\langle w_k(\theta_j, \tau_1+k_j, t), h\big(\gamma(\theta_j, \tau_1+k_j, t), \tau_1+k_j, t+\alpha_j, 0\big)\\ -\gamma_v(\theta_j, \tau_1+k_j, t)p_1\big(\gamma(\theta_j, \tau_1+k_j, t), \tau_1+k_j, t+\alpha_j, 0\big)\\ -\gamma_\varphi(\theta_j, \tau_1+k_j, t)\big(1+p_2\big(\gamma(\theta_j, \tau_1+k_j, t), \tau_1+k_j, t+\alpha_j, 0\big)\big)\Big\rangle\,dt\,.\end{aligned}$$

We derived the above results for $\epsilon > 0$ small in (2.5). But for $\epsilon < 0$ small, we can change t to $-t$ in (2.2) and we again arrive at (2.17) when α_j are changed to $-\alpha_j$.

Now we can state the main result of this paper.

Theorem 2.7. *If there are $\tau_1 \in \mathbb{R}$, $i_1, i_2, \cdots, i_p \in \mathbb{N}$ and open bounded subsets $\Omega_j \subset \mathbb{R}^d$ for any j, $0 \le j \le p$ such that (2.13) holds along with the following assumptions:*

a) $M_j(\theta_j, \alpha_j, \tau_1) \neq 0$, $\forall\, (\theta_j, \alpha_j) \in \partial\Omega_j$, $\forall\, j$, $0 \le j \le p$.

b) $\deg\big(M_j, \Omega_j, 0\big) \neq 0$, $\forall\, j$, $0 \le j \le p$. *Here deg is the Brouwer degree of mappings.*

Then there is a solution of (1.1) for any $\epsilon \neq 0$ sufficiently small which is $p+1$-times bumping near the homoclinic manifold $\big\{\big(W_h(y), y\big) \mid y \in \xi(\cdot)\big\}$ and which is homoclinic to the hyperbolic torus of (1.1) lying near ξ.

Proof. We already know that the solvability of the problem stated in Theorem 2.7 is reduced to the solvability of (2.17). On the other hand, the assumptions of Theorem 2.7 imply the solvability of (2.17) with respect to (θ_j, α_j) by using the Brouwer degree theory of mappings [8].

It is not difficult to check that after reversing all the changes of variables made above, including Remark 2.6 as well if it is applicable, then the mapping M_j has for the original equation (1.1) the form

$$
\begin{aligned}
&M_{jk}(\theta_j, \alpha_j, \tau_1) = \\
&\int_{-\infty}^{\infty} \Big\langle w_k\big(\theta_j, \xi(\tau_1 + k_j), t\big), h\big(\gamma\big(\theta_j, \xi(\tau_1 + k_j), t\big), \xi(\tau_1 + k_j), t + \alpha_j, 0\big) \\
&-\gamma_y\big(\theta_j, \xi(\tau_1 + k_j), t\big) p\big(\gamma\big(\theta_j, \xi(\tau_1 + k_j), t\big), \xi(\tau_1 + k_j), t + \alpha_j, 0\big) \Big\rangle \, dt\,,
\end{aligned}
\tag{2.18}
$$

where $\Big\{w_i(\theta, y, t) \mid i = 1, 2, \cdots d\Big\}$ forms a smooth family of linearly independent bounded solutions of the adjoint equation $\dot{w} = -f_x\big(\gamma(\theta, y, t), y\big)^t w$.

Note that Theorem 2.7 is not valid uniformly for p and $N = \max\{i_1, i_2, \cdots, i_p\}$. This means that the larger p and N, the smaller ϵ (see (2.9.1)).

Remark 2.8. When $p = 0$ then (2.13) is irrelevant and we get only one mapping $\bar{M}_0(\theta_0, \alpha_0, \tau_0)$ given by

$$
\begin{aligned}
&\bar{M}_{0k}(\theta_0, \alpha_0, \tau_0) = \\
&\int_{-\infty}^{\infty} \Big\langle w_k\big(\theta_0, \xi(\tau_0), t\big), h\big(\gamma\big(\theta_0, \xi(\tau_0), t\big), \xi(\tau_0), t + \alpha_0, 0\big) \\
&-\gamma_y\big(\theta_0, \xi(\tau_0), t\big) p\big(\gamma\big(\theta_0, \xi(\tau_0), t\big), \xi(\tau_0), t + \alpha_0, 0\big) \Big\rangle \, dt\,.
\end{aligned}
$$

So the equation $\bar{M}_0(\theta_0, \alpha_0, \tau_0) = 0$ can be solved either for (θ_0, α_0) like in Theorem 2.7 or for (θ_0, τ_0) and then we get new conditions for the statement of Theorem 2.7. When $p \ge 1$ then we can solve one equation $M_{j_1}(\theta_{j_1}, \alpha_{j_1}, \tau_1) = 0$ for (θ_{j_1}, τ_1) and the rest ones for (θ_j, α_j), $j \neq j_1$.

Remark 2.9. Assumptions a), b) of Theorem 2.7 hold when $(\theta_j, \alpha_j,\, \tau_1)$ is a simple root of (2.17), i.e. $M_j(\theta_j, \alpha_j, \tau_1) = 0$ and the linearization $M_{j(\theta,\alpha)}(\theta_j, \alpha_j, \tau_1)$ is invertible as a linear mapping from $\mathbb{R}^d$ to $\mathbb{R}^d$.

Remark 2.10. When ω is a rational number then we can find by the above method conditions for (1.1) that there are multi-bump periodics of (1.1) for $\epsilon \neq 0$ sufficiently small.

Remark 2.11. Equations (2.6.2) and (2.6.3) are considered on the intervals of the orders $O(1/\sqrt{\epsilon})$ and $O(1/\epsilon)$, respectively. Hence $(\tilde{v}_j, \tilde{\varphi}_j)$ and (v_j, φ_j) move with magnitudes of the orders $O(\sqrt{\epsilon})$ and $O(1)$, respectively. This means that multi-bump homoclinics of Theorem 2.7 have shapes of spikes. It is possible to consider (2.6.2) also on intervals of the order $O(1/\epsilon)$. Then $(\tilde{v}_j, \tilde{\varphi}_j)$ move with the order $O(1)$ as well. The difference with the above theory is only that in (2.10) instead of $A_j(t)$ we get

$$A_j^\epsilon(t) = f_x(\gamma_j, \bar{k}_j + \tau_1 + \epsilon t)$$

for suitable integers $\bar{k}_j$. Hence $A_j^\epsilon(t)$ in (2.10) depends also slowly on t. Then there is no general result analogous to Theorem 2.2 uniformly for $\epsilon \neq 0$ sufficiently small. On the other hand, by methods of [1, 2, 14] together with [5] this problem can be investigated like above. When $f(x, y)$ is independent of y then of course the above approach can be directly applied together with Theorem 2.2.

3. General Problems

We usually start with a system of the form

$$\begin{aligned} \dot{x} &= f_1(x, y) + \epsilon h_1(x, y, t, \epsilon)\,, \\ \dot{y} &= \epsilon g_1(x, y, t, \epsilon)\,, \end{aligned} \tag{3.1}$$

where (3.1) is 1-periodic in t and nonlinearities are smooth. Then we suppose:

(I) $f_1(x, y) = 0$ has a smooth solution $x = \psi(y)$.

(II) The eigenvalues of $f_{1x}\big(\psi(y), y\big)$ lie off the imaginary axis.

By changing the variables $x \leftrightarrow x + \psi(y)$, we get the system

$$\begin{aligned} \dot{x} &= f_1\big(x + \psi(y), y\big) \\ &\quad + \epsilon\Big(h_1\big(x + \psi(y), y, t, \epsilon\big) - \psi_y(y) g_1\big(x + \psi(y), y, t, \epsilon\big)\Big) \\ &= \tilde{f}_1(x, y) + \epsilon \tilde{h}_1(x, y, t, \epsilon)\,, \\ \dot{y} &= \epsilon g_1\big(x + \psi(y), y, t, \epsilon\big) = \epsilon \tilde{g}_1(x, y, t, \epsilon)\,. \end{aligned} \tag{3.2}$$

Hence $\tilde{f}_1(0, y) = 0$. Then we consider the equation $\dot{y} = \epsilon g_1\big(\psi(y), y, t, \epsilon\big)$ and we take its averaged equation $\dot{y} = \epsilon \int_0^1 g_1\big(\psi(y), y, t, 0\big)\, dt$ (see [16]). We assume:

(III) Let $\xi(t)$ be a hyperbolic periodic solution of the equation
$$\dot{y} = \int_0^1 g_1\big(\psi(y), y, t, 0\big)\, dt.$$

By making in (3.2) the usual averaging change of variables of the form $y \leftrightarrow y + \epsilon S(y,t,\epsilon)$, where S is smooth and 1-periodic in t, we arrive at the system like (1.1). So let us take $y(t) = v(t) + \epsilon S(v(t),t,\epsilon)$ in (1.1). Then we get

$$\begin{aligned}\dot{x} &= f(x,v) + \epsilon\big(f_y(x,v)S(v,t,0) + h(x,v,t,0)\big) + O(\epsilon^2)\\ &= \tilde{f}_1(x,v) + \epsilon\tilde{h}_1(x,v,t,\epsilon)\,,\\ \dot{v} &= \epsilon\big(I + \epsilon S_v(v,t,\epsilon)\big)^{-1}\Big(g\big(v+\epsilon S(v,t,\epsilon)\big) - S_t(v,t,\epsilon)\\ &+ p\big(x, v+\epsilon S(v,t,\epsilon),t,\epsilon\big) + \epsilon q\big(v+\epsilon S(v,t,\epsilon),t,\epsilon\big)\Big)\\ &= \epsilon\tilde{g}_1(x,v,t,\epsilon)\,.\end{aligned} \tag{3.3}$$

We note that $\int_0^1 \tilde{g}_1(0,v,t,\epsilon)\,dt = g(v)$ in (3.3). The unperturbed equation of (3.3) has the same form as for (1.1). For the mapping (2.18) in terms of (3.3), we have

$$\begin{aligned}M_{jk}(\theta_j,\alpha_j,\tau_1) = -\int_{-\infty}^{\infty}\Big\langle & w_k(\theta_j,\xi(\tau_1+k_j),t),\\ & f_y\big(\gamma(\theta_j,\xi(\tau_1+k_j),t),\xi(\tau_1+k_j)\big)S(\xi(\tau_1+k_j),t+\alpha_j,0)\\ & +\gamma_y(\theta_j,\xi(\tau_1+k_j),t)\big(S_t(\xi(\tau_1+k_j),t+\alpha_j,0) - g(\xi(\tau_1+k_j))\big)\Big\rangle\,dt\\ +\int_{-\infty}^{\infty}\Big\langle & w_k(\theta_j,\xi(\tau_1+k_j),t),\tilde{h}_1\big(\gamma(\theta_j,\xi(\tau_1+k_j),t),\xi(\tau_1+k_j),t+\alpha_j,0\big)\\ & -\gamma_y(\theta_j,\xi(\tau_1+k_j),t)\tilde{g}_1\big(\gamma(\theta_j,\xi(\tau_1+k_j),t),\xi(\tau_1+k_j),t+\alpha_j,0\big)\Big\rangle\,dt\,.\end{aligned} \tag{3.4}$$

Assumption (iv) for $\Gamma(t) = \gamma_y(\theta_j,\xi(\tau_1+k_j),t)S(\xi(\tau_1+k_j),t+\alpha_j,0)$ gives

$$\begin{aligned}\dot{\Gamma}(t) &= f_x\big(\gamma(\theta_j,\xi(\tau_1+k_j),t),\xi(\tau_1+k_j)\big)\Gamma(t)\\ &+ f_y\big(\gamma(\theta_j,\xi(\tau_1+k_j),t),\xi(\tau_1+k_j)\big)S(\xi(\tau_1+k_j),t+\alpha_j,0)\\ &+ \gamma_y(\theta_j,\xi(\tau_1+k_j),t)S_t(\xi(\tau_1+k_j),t+\alpha_j,0)\,.\end{aligned} \tag{3.5}$$

Since $\Gamma(t)$ and $\dot{\Gamma}(t)$ are bounded on $\mathbb{R}$, according to (3.5) and [7, 15] we see that (3.4) is simplified in terms of (3.1) (see also (3.2)) to

$$\begin{aligned}&M_{jk}(\theta_j,\alpha_j,\tau_1) =\\ &\int_{-\infty}^{\infty}\Big\langle w_k(\theta_j,\xi(\tau_1+k_j),t),\gamma_y(\theta_j,\xi(\tau_1+k_j),t)g(\xi(\tau_1+k_j))\Big\rangle\,dt\\ &+\int_{-\infty}^{\infty}\Big\langle w_k(\theta_j,\xi(\tau_1+k_j),t),h_1\big(\gamma(\theta_j,\xi(\tau_1+k_j),t),\xi(\tau_1+k_j),t+\alpha_j,0\big)\\ &-\gamma_y(\theta_j,\xi(\tau_1+k_j),t)g_1\big(\gamma(\theta_j,\xi(\tau_1+k_j),t),\xi(\tau_1+k_j),t+\alpha_j,0\big)\Big\rangle\,dt\,.\end{aligned} \tag{3.6}$$

We recall that $g(y) = \int_0^1 g_1\big(\psi(y), y, t, 0\big)\, dt$ and $\gamma(\theta, y, t)$ in (3.6) is the $(d-1)$-parametric smooth family of solutions of $\dot{x} = f(x, y)$ homoclinic to $\psi(y)$ satisfying assumption (iv), while $w_k(\theta, y, t)$, $k = 1, 2, \cdots, d$ are linearly independent smooth bounded solutions to the adjoint equation

$$\dot{w} = -f_x\big(\gamma(\theta, y, t), y\big)^t w \,.$$

Consequently, (3.6) is the Melnikov-type mapping of Theorem 2.7 for the general problem (3.1) under the above assumptions (I), (II), (III) and (iv).

4. Examples

To illustrate our abstract results, let us consider the following simple example

$$\begin{aligned} \ddot{x} &= x - \frac{1}{(y_1^2 + y_2^2)^2} x^3 \\ \dot{y}_1 &= \epsilon\big(y_2 + y_1(1 - y_1^2 - y_2^2) + x\cos 2\pi t\big) \\ \dot{y}_2 &= \epsilon\big(- y_1 + y_2(1 - y_1^2 - y_2^2)\big) \,. \end{aligned} \tag{4.1}$$

System (4.1) has the form of (1.1). The equation $\dot{y} = g(y)$ for assumption (iii) has now the form

$$\begin{aligned} \dot{y}_1 &= y_2 + y_1(1 - y_1^2 - y_2^2) \\ \dot{y}_2 &= -y_1 + y_2(1 - y_1^2 - y_2^2) \,. \end{aligned} \tag{4.2}$$

It is not hard to see by introducing the polar coordinates in (4.2) that $\xi(t) = (\cos t, -\sin t)$ is a hyperbolic 2π-periodic solution of (4.2).

The equation

$$\ddot{x} = x - \frac{1}{(y_1^2 + y_2^2)^2} x^3 \tag{4.3}$$

is the Duffing equation [17], so we have $d = 1$ and

$$\gamma(y, t) = \sqrt{2}\big(y_1^2 + y_2^2\big)\big(r(t), \dot{r}(t)\big) \,,$$

where $r(t) = \operatorname{sech} t$ and $y = (y_1, y_2)$. Now condition (2.13) is trivially satisfied since the linearization of (4.3) at the zero equilibrium has the form $\ddot{u} = u$. Hence Remark 2.6 holds. Furthermore, since the linearization of (4.3) at the homoclinic solution $\gamma(y, t)$ is given by $\ddot{u} = \big(1 - 6r(t)^2\big)u$, it is also known [7, 15] that now we can take

$$w_1(y, t) = \big(- \ddot{r}(t), \dot{r}(t)\big) \,.$$

By applying formula (2.18) to (4.1), after several computations we get

$$\begin{aligned} M_j(\alpha_j, \tau_1) &= -4\cos(\tau_1 + k_j) \int_{-\infty}^{\infty} \cos 2\pi(t + \alpha_j)\, r(t)^5\, dt \\ &= -4\cos(\tau_1 + k_j)\cos 2\pi\alpha_j \int_{-\infty}^{\infty} \cos 2\pi t\, r(t)^5\, dt \\ &= -\frac{1}{6}\pi\big(4\pi^2 + 1\big)\big(4\pi^2 + 9\big)\operatorname{sech}\pi^2 \cos(\tau_1 + k_j)\cos 2\pi\alpha_j \,. \end{aligned}$$

We see that when $\cos(\tau_1 + k_j) \neq 0$, $\forall j$, $0 \leq j \leq p$ then $\alpha_j = 1/4$ satisfies the assumptions of Theorem 2.7. Consequently, we can get any type of multi-bump homoclinics, and the smaller $\epsilon \neq 0$, the more such solutions there are. Moreover, the value τ_1 provides a certain 1-parametric family such of solutions. Unfortunately, we can not run with τ_1 around $[0, 2\pi]$ uniformly for fixed k_j, $0 \leq j \leq p$, since the functions $\tau_1 \to \cos(\tau_1 + k_j)$ change the signs. This means that these sets of multi-bump homoclinics seem to be not foliated around the periodic solution $\xi(t)$.

Now we study a more general example than (4.1) of the form

$$\begin{aligned} \ddot z_1 &= a(y)^2 z_1 - z_1\big(z_1^2 + z_2^2\big) + \epsilon z_1 \\ \ddot z_2 &= a(y)^2 z_2 - z_2\big(z_1^2 + z_2^2\big) - \epsilon z_2 \\ \dot y &= \epsilon\big(g(y) + p(z_1, z_2)\cos 2\pi t\big)\,, \end{aligned} \tag{4.4}$$

where $y \in \mathbb{R}^m$, g is smooth satisfying assumption (iii), a is smooth and positive in a neighbourhood of ξ and p is smooth such that $p(0,0) = 0$.

The equation

$$\begin{aligned} \ddot z_1 &= a(y)^2 z_1 - z_1\big(z_1^2 + z_2^2\big) \\ \ddot z_2 &= a(y)^2 z_2 - z_2\big(z_1^2 + z_2^2\big) \end{aligned} \tag{4.5}$$

for a fixed y near ξ, is the known equation [7, 17], so we have $d = 2$ along with

$$\gamma(\theta, y, t) =$$
$$a(y)\sqrt{2}\Big(\sin\theta r\big(a(y)t\big), a(y)\sin\theta \dot r\big(a(y)t\big), \cos\theta r\big(a(y)t\big), a(y)\cos\theta \dot r\big(a(y)t\big)\Big)\,,$$
$$w_1(\theta, y, t) =$$
$$\Big(-\sin\theta a(y)^2 \ddot r\big(a(y)t\big), a(y)\sin\theta \dot r\big(a(y)t\big), -\cos\theta a(y)^2 \ddot r\big(a(y)t\big), a(y)\cos\theta \dot r\big(a(y)t\big)\Big)\,,$$
$$w_2(\theta, y, t) =$$
$$\Big(-\cos\theta a(y)\dot r\big(a(y)t\big), \cos\theta r\big(a(y)t\big), \sin\theta a(y)\dot r\big(a(y)t\big), -\sin\theta r\big(a(y)t\big)\Big)\,.$$

The linearization of (4.5) at the zero equilibrium is decoupled on the two equal 2-dimensional equations given by

$$\dot z = \begin{pmatrix} 0 & 1 \\ a(y)^2 & 0 \end{pmatrix} z\,.$$

Since

$$\begin{pmatrix} 1 & 1 \\ a(y) & -a(y) \end{pmatrix}^{-1} \circ \begin{pmatrix} 0 & 1 \\ a(y)^2 & 0 \end{pmatrix} \circ \begin{pmatrix} 1 & 1 \\ a(y) & -a(y) \end{pmatrix} = \begin{pmatrix} a(y) & 0 \\ 0 & -a(y) \end{pmatrix}\,,$$

we see that assumption (2.13) always holds according to Remark 2.6 (see (2.15)). The mapping (2.18) has after several computations now the form

$$M_{j1}\big(\theta_j, \alpha_j, \tau_1\big) = -2\sqrt{2}a_j \cos 2\pi\alpha_j$$
$$\times \int_0^\infty a_y\big(\xi(\tau_1 + k_j)\big) \circ p\big(a_j\sqrt{2}\sin\theta r(t), a_j\sqrt{2}\cos\theta r(t)\big) \cdot r(t)^2 \cos\frac{2\pi t}{a_j}\, dt\,,$$
$$M_{j2}\big(\theta_j, \alpha_j, \tau_1\big) = 2\sqrt{2}\sin 2\theta_j\,,$$

where $a_j = a\big(\xi(\tau_1 + k_j)\big)$. We see that we can take $\theta_j = k\pi/2$, $k \in \mathbb{Z}$, $\alpha_j = 1/4$ as simple roots of $\big(M_{j1}, M_{j2}\big) = 0$ provided that one of the following conditions holds

$$a_y\big(\xi(\tau_1 + k_j)\big) \circ \int_0^\infty p\big(0, (-1)^{k/2} a_j \sqrt{2} r(t)\big) \cdot r(t)^2 \cos \frac{2\pi t}{a_j}\, dt \neq 0\,, \qquad (4.6)$$
$$\forall\, j,\ 0 \leq j \leq p \quad \text{for} \quad k \quad \text{even}\,,$$

$$a_y\big(\xi(\tau_1 + k_j)\big) \circ \int_0^\infty p\big((-1)^{(k-1)/2} a_j \sqrt{2} r(t), 0\big) \cdot r(t)^2 \cos \frac{2\pi t}{a_j}\, dt \neq 0\,, \qquad (4.7)$$
$$\forall\, j,\ 0 \leq j \leq p \quad \text{for} \quad k \quad \text{odd}\,.$$

If (4.6) or (4.7) holds then Theorem 2.7 can be applied for (4.4) to get multi-bump homoclinics in (4.4). When in addition the period of ξ is a rational number then there are also conditions for the existence of multi-bump periodics of (4.4) according to Remark 2.10: For $p \geq 1$, one can choose k_p such that it is a natural multiple of the period of ξ.

Finally, we note that in the above examples we have looked for simple roots of the corresponding Melnikov mappings. To use the Brouwer degree argument, we consider in (4.4) instead of the term $\cos 2\pi t$, the term $R(\cos 2\pi t)$ for a real polynomial R. Then the mappings M_{j2} remain and M_{j1} become polynomials of $\cos 2\pi\alpha_j$, respectively. Consequently, we take again $\theta_j = k\pi/2$, $k \in \mathbb{Z}$ and then $M_{j1}(k\pi/2, \alpha_j, \tau_1) = R_{kj}(\tau_1, \cos 2\pi\alpha_j)$ where $R_{kj}(\tau_1, x)$ are real polynomials of x. If for some fixed k, τ_1 the polynomials $x \to R_{kj}(\tau_1, x)$ are changing the signs on the interval $[-1, 1]$, then the Brouwer degrees of $M_j = (M_{j1}, M_{j2})$ are nonzero on certain domains, and so Theorem 2.7 can be applied. This happens if each of polynomials $R_{kj}(\tau_1, x)$ has a root in $(-1, 1)$ with an odd order for some fixed k, τ_1. If one of this order is greater than 1 then geometric methods like in [9, 11, 12] seem to be not applicable for this case.

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