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EXISTENCE AND UNIQUENESS OF POSITIVE SOLUTIONS TO A QUASILINEAR ELLIPTIC PROBLEM IN $\mathbb{R}^N$

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Abstract. We prove the existence of a unique positive solution to the problem

$$-\Delta_p u = a(x)f(u)$$

in $\mathbb{R}^N$, $N > 2$. Our result extended previous works by Cirstea-Radulescu and Dinu, while the proofs are based on two theorems on bounded domains, due to Diaz-Saà and Goncalves-Santos.

1. Introduction

Our purpose in this paper is to study the problem

$$\begin{gathered} -\Delta_p u = a(x)f(u) \quad \text{in } \mathbb{R}^N \\ u > 0 \quad \text{in } \mathbb{R}^N \\ u(x) \to 0 \quad \text{as } |x| \to \infty\,, \end{gathered} \tag{1.1}$$

where $N > 2$, $\Delta_p u$, $(1 < p < \infty)$ is the p-Laplacian operator and the function $a(x)$ satisfies the following hypotheses:

- (A1) $a(x) \in C^{0,\alpha}_{\rm loc}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$;
- (A2) $a(x) > 0$ in $\mathbb{R}^N$;
- (A3) For $\Phi(r) = \max_{|x|=r} a(x)$ and $p < N$,

$$\begin{gathered} \int_0^\infty r^{1/(p-1)}\Phi^{1/(p-1)}(r)dr < \infty \quad \text{if } 1 < p \le 2 \\ \int_0^\infty r^{\frac{(p-2)N+1}{p-1}}\Phi(r)dr < \infty \quad \text{if } 2 \le p < \infty\,. \end{gathered}$$

This problem has been studied extensively in the case $p = 2$ and $f(u) = u^{-\gamma}$, with $\gamma > 0$. Lazer and McKenna [12] studied the special case when $\Omega \subset \mathbb{R}^N$ $(N \ge 1)$ is a bounded domain with smooth boundary. They proved the existence and the uniqueness of a positive solution $u \in C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$ with homogeneous Dirichlet boundary condition, provided that $a(x) \in C^\alpha(\overline{\Omega})$ and $a(x) > 0$ for all $x \in \overline{\Omega}$.

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The existence of entire positive solutions on $\mathbb{R}^N$ for $\gamma \in (0,1)$ and under certain additional hypotheses has been established by Edelson [7] and Kusano-Swanson [10].

Kusano-Swanson proved that the problem (1.1) has an entire positive solution in $\mathbb{R}^2$ with logarithmic growth at ∞ if $a(x) > 0$, $x > 0$, $a(x) \in C(0,\infty)$ and

$$\int_e^\infty t(Logt)^{-\gamma}\big(\max_{|x|=t} a(x)\big)dt < \infty.$$

Edelson proved the existence of a solution provided that

$$\int_1^\infty r^{N-1+\gamma(N-2)}(\max_{|x|=t} a(x))dt < \infty,$$

for some $\gamma \in (0,1)$. This result is generalized for any $\gamma > 0$ via the sub- and super solutions method in Shaker [13] and by other methods by Dalmasso [4].

Shaker proved that problem (1.1) with $p = 2$ and $f(u) = u^{-\gamma}$, $\gamma > 0$ has an entire positive solution $u(x)$ such that $c_1 \le u(x)|x|^{q|N-2|} \le c_2$ for some c_1, c_2 and $0 < q < 1$ as $x \to \infty$ if

(1) $a(x) \in C^\alpha_{\rm loc}(\mathbb{R}^N)$, $a(x) > 0$ for $x \in \mathbb{R}^N \backslash \{0\}$;
(2) There exists $0 < c < 1$ such that $c\Phi(|x|) \le a(x) \le \Phi(|x|)$ where $\Phi(r) := \max_{|x|=r} a(x)$, $r \in [0,\infty)$;
(3) $\int_1^\infty r^{N-1+\gamma(N-2)}(\max_{|x|=t} a(x))dt < \infty$.

Lair and Shaker continued in [11] the study of (1.1) for $p = 2$ and $f(u) = u^{-\gamma}$, $\gamma > 0$. Under the above conditions the authors proved the existence of a unique positive solution $u \in C^{2,\alpha}_{\rm loc}(\mathbb{R}^N)$ vanishing at infinity to this special problem.

Zhang [14], imposed the following condition to guarantee the existence of positive solutions to problem (1.1):

(A4) $f \in C^1((0,\infty),(0,\infty))$, $\lim_{s\searrow 0+} \lim f(s) = \infty$, and $f'(s) < 0$, for all $s \in (0,\infty)$, namely, f is strictly decreasing in $(0,\infty)$.

Under the above condition Zhang's proved that problem (1.1) has a unique positive solution, $u \in C^{2+\alpha}_{\rm loc}(\mathbb{R}^N)$, vanishing at infinity.

Cirstea-Radulescu [2] and Dinu [6] extended the results of Lair, Shaker and Zhang for the case of a nonlinearity that is not necessarily decreasing on $(0,\infty)$.

Our aim is to extend the results of Cirstea-Radulescu and Dinu in the sense that $1 < p < \infty$. More exactly, let $f : (0,\infty) \to (0,\infty)$ be a C^1 function that satisfies the following assumptions:

(F1) There exists $\beta > 0$ such that the mapping $u \mapsto f(u)/(u+\beta)^{p-1}$ is decreasing on $(0,\infty)$
(F2) $\lim_{u\searrow 0} f(u)/u^{p-1} = +\infty$ and f is bounded in a neighbourhood of $+\infty$.

Our main results are the following:

Theorem 1.1. *Under hypotheses (F1), (F2), (A1), (A2), (A3), problem* (1.1) *has a unique positive global solution vanishing at infinity.*

Theorem 1.2. *Suppose $a(r)$ is a positive radial function which is continuous on $\mathbb{R}^N$ and fulfills*

$$\int_0^\infty r^{1/(p-1)} a^{1/(p-1)}(r)dr = \infty \quad \text{if } 2 \le p < \infty$$

Then (1.1) *has no positive radial solution.*

Theorem 1.3. *Problem* (1.1) *has no positive radial solution if* $p \geq N$.

Theorem 1.4. *Suppose* $a(r)$ *is a positive radial function which is continuous on* $\mathbb{R}^N$ *and*

$$\int_0^\infty r^{\frac{(p-2)N+1}{p-1}} a(r)dr = \infty \quad \text{if } 1 < p \leq 2.$$

Then (1.1) *has no positive radial solution.*

2. Uniqueness

Suppose u and v are arbitrary solutions of problem (1.1). Let us show that $u \leq v$ or, equivalently, $\ln(u(x)+\beta) \leq \ln(v(x)+\beta)$, for any $x \in \mathbb{R}^N$. Assume the contrary. Since, we have

$$\lim_{|x|\to\infty}(\ln(u(x)+\beta) - \ln(v(x)+\beta)) = 0,$$

we deduce that

$$\max_{\mathbb{R}^N}(\ln(u(x)+\beta) - \ln(v(x)+\beta))$$

exists and is positive. At that point, say x_0, we have

$$\nabla(\ln(u(x_0)+\beta) - \ln(v(x_0)+\beta)) = 0,$$

so

$$\frac{1}{u(x_0)+\beta} \cdot \nabla u(x_0) = \frac{1}{v(x_0)+\beta} \cdot \nabla v(x_0),$$

and

$$\frac{1}{(u(x_0)+\beta)^{p-2}} \cdot |\nabla u(x_0)|^{p-2} = \frac{1}{(v(x_0)+\beta)^{p-2}} \cdot |\nabla v(x_0)|^{p-2}. \tag{2.1}$$

By (f1) we obtain

$$\frac{f(u(x_0))}{(u(x_0)+\beta)^{p-1}} < \frac{f(v(x_0))}{(v(x_0)+\beta)^{p-1}}. \tag{2.2}$$

Since $0 \geq \Delta(\ln(u(x_0)+\beta) - \ln(v(x_0)+\beta))$, it follows that

$$\frac{\Delta u(x_0)}{u(x_0)+\beta} \leq \frac{\Delta v(x_0)}{v(x_0)+\beta},$$

so

$$\frac{1}{(u(x_0)+\beta)^{p-1}} \cdot |\nabla u(x_0)|^{p-2} \Delta u(x_0) \leq \frac{1}{(v(x_0)+\beta)^{p-1}} \cdot |\nabla v(x_0)|^{p-2} \Delta v(x_0)\,. \tag{2.3}$$

Since

$$|\nabla \ln(u(x_0)+\beta)|^{p-2} = \frac{1}{(u(x_0)+\beta)^{p-2}} \cdot |\nabla u(x_0)|^{p-2},$$

it follows that

$$\begin{aligned}&\nabla(|\nabla \ln(u(x_0)+\beta)|^{p-2})\\ &= -(p-2)\frac{|\nabla u(x_0)|^{p-2}(u(x_0)+\beta)^{p-3}}{(u(x_0)+\beta)^{2(p-2)}} \cdot \nabla u(x_0) + \frac{\nabla(|\nabla u(x_0)|^{p-2})}{(u(x_0)+\beta)^{p-2}}.\end{aligned}$$

Then

$$\begin{aligned}&\nabla(|\nabla \ln(u(x_0)+\beta)|^{p-2}) \cdot \nabla(\ln(u(x_0)+\beta))\\ &= -(p-2)\frac{|\nabla u(x_0)|^{p-2}|\nabla u(x_0)|^2}{(u(x_0)+\beta)^p} + \frac{\nabla(|\nabla u(x_0)|^{p-2}) \cdot \nabla u(x_0)}{(u(x_0)+\beta)^{p-1}}\end{aligned} \tag{2.4}$$

and

$$|\nabla \ln(u(x_0)+\beta)|^{p-2}\Delta(\ln(u(x_0)+\beta)) = \frac{|\nabla u(x_0)|^{p-2}\Delta u(x_0)}{(u(x_0)+\beta)^{p-1}} - \frac{|\nabla u(x_0)|^{p}}{(u(x_0)+\beta)^{p}}.$$

So, by (2.1), (2.2), (2.3) and (2.4) we have

$$\begin{aligned}
0 &\geq \Delta_p(\ln(u(x_0)+\beta)) - \Delta_p(\ln(v(x_0)+\beta))\\
&= \frac{\Delta_p u(x_0)}{(u(x_0)+\beta)^{p-1}} - (p-1)\frac{|\nabla u(x_0)|^p}{(u(x_0)+\beta)^p} - \frac{\Delta_p v(x_0)}{(v(x_0)+\beta)^{p-1}}\\
&\quad + (p-1)\frac{|\nabla v(x_0)|^p}{(v(x_0)+\beta)^p}\\
&= \frac{\Delta_p u(x_0)}{(u(x_0)+\beta)^{p-1}} - \frac{\Delta_p v(x_0)}{(v(x_0)+\beta)^{p-1}}\\
&= -a(x_0)\Big(\frac{f(u(x_0))}{(u(x_0)+\beta)^{p-1}} - \frac{f(v(x_0))}{(v(x_0)+\beta)^{p-1}}\Big) > 0
\end{aligned}$$

which is a contradiction. Hence $u \leq v$. By symmetry we also have $v \leq u$, and the proof is complete.

3. Existence of a solution

We first show that our hypothesis (F1) implies $\lim_{u\searrow 0} f(u)$ exists, finite or $+\infty$. Indeed, since $\frac{f(u)}{(u+\beta)^{p-1}}$ is decreasing, there exists $L := \lim_{u\searrow 0}\frac{f(u)}{(u+\beta)^{p-1}} \in (0,+\infty]$. It follows that $\lim_{u\searrow 0} f(u) = L\beta^{p-1}$.

To prove the existence of a solution to Problem (1.1), we need to employ a corresponding result by Diaz-Saà [5] for bounded domains. They considered the problem

$$\begin{gathered}
-\Delta_p u = g(x,u) \quad \text{in } \Omega\\
u \geq 0 \quad \text{in } \Omega\\
u(x) = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $g(x,u) : \Omega \times [0,\infty) \to \mathbb{R}$.

Assume that

-for a.e. $x \in \Omega$ the function $u \to g(x,u)$ is continuous on $[0,\infty)$

and the function $u \to g(x,u)/u^{p-1}$ is decreasing on $(0,\infty)$; (3.2)

-for each $u \geq 0$ the function $x \to g(x,u)$ belongs to $L^\infty(\Omega)$; (3.3)

-there exists $C > 0$ such that $g(x,u) \leq C(u^{p-1}+1)$ a.e. $x \in \Omega$, for all $u \geq 0$. (3.4)

Under these hypotheses on g, Diaz-Saà [5] proved that there is at most one solution of (1.1).

Let us consider the problem

$$\begin{gathered}
-\Delta_p u_k = a(x)f(u_k), \quad \text{if } |x| < k,\\
u_k(x) = 0, \quad \text{if } |x| = k.
\end{gathered} \tag{3.5}$$

The following two distinct situations may occur:
Case 1: f is bounded on $(0,+\infty)$. In this case, as we have initially observed, $\lim_{u\searrow 0} f(u)$ exists and is finite, so f can be extended by continuity at the origin.

To obtain a solution to (3.5), it is sufficinet to verify that the hypotheses of the Diaz-Saà theorem are fulfilled.

* Since $f \in C^1((0,\infty),(0,\infty))$ it follows that the mapping $u \to a(x)f(u)$ is continuous in $[0,\infty)$.

* From $a(x)\frac{f(u)}{u^{p-1}} = a(x)\frac{f(u)}{(u+\beta)^{p-1}} \cdot \frac{(u+\beta)^{p-1}}{u^{p-1}}$, using positivity of a and (F1) we deduce that the function $u \to a(x)\frac{f(u)}{u^{p-1}}$ is decreasing on $(0,\infty)$.

* For all $u \geq 0$, since $a(x) \in C^{0,\alpha}_{\rm loc}(\mathbb{R}^N)$, we obtain $x \to a(x)f(u)$ belongs to $L^\infty(\Omega)$.

* By $\lim_{u\to\infty} \frac{f(u)}{u^{p-1}+1} = \lim_{u\to\infty} \frac{f(u)}{u^{p-1}} \cdot \frac{u^{p-1}}{u^{p-1}+1} = 0$ and $f \in C^1((0,\infty),(0,\infty))$, there exists $C > 0$ such that $f(u) \leq C(u^{p-1}+1)$ for all $u \geq 0$. Therefore, $a(x)f(u) \leq C(u^{p-1}+1)$ for all $u \geq 0$.

* Observe that $a_0(x) = \lim_{u\searrow 0} \frac{p(x)f(u)}{u^{p-1}} = +\infty$ and $a_\infty(x) = \lim_{u\to+\infty} \frac{p(x)f(u)}{u^{p-1}} = 0$. Thus by Diaz-Saa, problem (3.5) has a unique solution u_k which, by the maximum principle, is positive in $|x| < k$.

Case 2. $\lim_{u\searrow 0} f(u) = +\infty$. We will apply the method of sub- and supersolutions in order to find a solution to the problem (3.5). We first observe that 0 is a subsolution for this problem.

We construct in what follows a positive supersolution. By the boundedness of f in a neighbourhood of $+\infty$, there exists $A > 0$ such that $f(u) \leq A$, for any $u \in (1,+\infty)$. Let $f_0 : (0,1] \to (0,+\infty)$ be a continuous nonincreasing function such that $f_0 \geq f$ on $(0,1]$. We can assume without loss of generality that $f_0(1) = A$. Set

$$g(u) = \begin{cases} f_0(u), & \text{if } 0 < u \leq 1, \\ A, & \text{if } u > 1. \end{cases}$$

Then g is a continuous nonincreasing function on $(0,+\infty)$. Let $h : (0,+\infty) \to (0,+\infty)$ be a C^1 nonincreasing function such that $h \geq g$. Thus by in [8, Theorem 1.3] the problem

$$\begin{gathered} -\Delta_p U = p(x)h(U), \quad \text{if } |x| < k, \\ U = 0, \quad \text{if } |x| = k. \end{gathered}$$

has a positive solution. Now, since $h \geq f$ on $(0,+\infty)$, it follows that U is supersolution of (3.5).

In both cases studied above we define $u_k = 0$ for $|x| > k$. Using a comparision principle argument as already done above for proving the uniqueness, we can show that $u_k \leq u_{k+1}$ on $\mathbb{R}^N$.

We now justify the existence of a continuous function $v : \mathbb{R}^N \to \mathbb{R}$ such that $u_k \leq v$ in $\mathbb{R}^N$. We first construct a positive radially symmetric function w such that $-\Delta_p w = \Phi(r)$, ($r = |x|$) on $\mathbb{R}^N$ and $\lim_{r\to\infty} w(r) = 0$. A straightforward computation shows that

$$w(r) := K - \int_0^r \Big[\xi^{1-N} \int_0^\xi \sigma^{N-1}\Phi(\sigma)d\sigma\Big]^{1/(p-1)} d\xi,$$

where

$$K = \int_0^\infty \Big[\xi^{1-N} \int_0^\xi \sigma^{N-1}\Phi(\sigma)d\sigma\Big]^{1/(p-1)} d\xi.$$

We first show that (A3) implies that

$$\int_0^{+\infty}\Big[\xi^{1-N}\int_0^{\xi}\sigma^{N-1}\Phi(\sigma)d\sigma\Big]^{1/(p-1)}d\xi,$$

is finite.

Theorem 3.1. *If $j : I \subseteq R \to R$ is a locally integrable nonnegative function, then*

$$\Big(\frac{1}{b-a}\int_a^b j(x)dx\Big)^h \leq \ \textit{(resp. } \geq\textit{)}\ \frac{1}{b-a}\int_a^b j^h(x)dx$$

for all $a, b \in I$, $a < b$ and $1 \leq h < +\infty$ (resp $0 < h \leq 1$)

Case 1: Let $1 < p \leq 2$, so $0 < p-1 \leq 1$, follows that $1 \leq \frac{1}{p-1} < +\infty$. By Theorem 3.1 for any $r > 0$, we have

$$\begin{aligned}
&\int_0^r \xi^{\frac{1-N}{p-1}}\Big[\frac{\xi}{\xi}\int_0^{\xi}\sigma^{N-1}\Phi(\sigma)d\sigma\Big]^{1/(p-1)}d\xi \\
&= \int_0^r \xi^{\frac{1-N}{p-1}}\xi^{1/(p-1)}\Big[\frac{1}{\xi}\int_0^{\xi}\sigma^{N-1}\Phi(\sigma)d\sigma\Big]^{1/(p-1)}d\xi \\
&\leq \int_0^r \xi^{\frac{2-N}{p-1}}\frac{1}{\xi}\int_0^{\xi}\sigma^{\frac{N-1}{p-1}}\Phi^{1/(p-1)}(\sigma)d\sigma d\xi \\
&= \int_0^r \xi^{\frac{2-N}{p-1}-1}\int_0^{\xi}\sigma^{\frac{N-1}{p-1}}\Phi^{1/(p-1)}(\sigma)d\sigma d\xi \\
&= -\frac{p-1}{N-2}\int_0^r \frac{d}{d\xi}\xi^{\frac{2-N}{p-1}}\int_0^{\xi}\sigma^{\frac{N-1}{p-1}}\Phi^{1/(p-1)}(\sigma)d\sigma d\xi \\
&= \frac{p-1}{N-2}\Big[-r^{\frac{2-N}{p-1}}\int_0^r \sigma^{\frac{N-1}{p-1}}\Phi^{1/(p-1)}(\sigma)d\sigma + \int_0^r \xi^{1/(p-1)}\Phi^{1/(p-1)}(\xi)d\xi\Big].
\end{aligned}$$

Now, by L'Hôpital's rule, we have

$$\begin{aligned}
&\lim_{r\to\infty}\Big[-r^{\frac{2-N}{p-1}}\int_0^r \sigma^{\frac{N-1}{p-1}}\Phi^{1/(p-1)}(\sigma)d\sigma + \int_0^r \xi^{1/(p-1)}\Phi^{1/(p-1)}(\xi)d\xi\Big] \\
&= \lim_{r\to\infty}\frac{-\int_0^r \sigma^{\frac{N-1}{p-1}}\Phi^{1/(p-1)}(\sigma)d\sigma + r^{\frac{N-2}{p-1}}\int_0^r \xi^{1/(p-1)}\Phi^{1/(p-1)}(\xi)d\xi}{r^{\frac{N-2}{p-1}}} \\
&= \lim_{r\to\infty}\int_0^r \xi^{\frac{1}{p-1}}\Phi^{1/(p-1)}(\xi)d\xi \\
&= \int_0^{\infty}\xi^{1/(p-1)}\Phi^{1/(p-1)}(\xi)d\xi < \infty,
\end{aligned}$$

Case 2: Let $2 \leq p < +\infty$, so $1 \leq p-1$, it follows that $1 \geq \frac{1}{p-1} > 0$. Set

$$\int_0^{\xi}\sigma^{N-1}\Phi(\sigma)d\sigma \leq 1 \quad \text{for } \xi > 0, \quad \text{or}$$

$$\int_0^{\xi}\sigma^{N-1}\Phi(\sigma)d\sigma > 1 \quad \text{for } \xi > 0,$$

In the first case

$$\Big[\int_0^{\xi}\sigma^{N-1}\Phi(\sigma)d\sigma\Big]^{1/(p-1)} \leq 1,$$

so
$$\int_0^r \xi^{\frac{1-N}{p-1}}\Big[\int_0^\xi \sigma^{N-1}\Phi(\sigma)d\sigma\Big]^{1/(p-1)}d\xi \le \int_0^r \xi^{\frac{1-N}{p-1}}d\xi$$
is finite as $r\to\infty$ and $N>p$. In the second case,
$$\Big[\int_0^\xi \sigma^{N-1}\Phi(\sigma)d\sigma\Big]^{1/(p-1)} \le \int_0^\xi \sigma^{N-1}\Phi(\sigma)d\sigma$$
for $\xi\ge 0$, so
$$\int_0^r \xi^{\frac{1-N}{p-1}}\Big[\int_0^\xi \sigma^{N-1}\Phi(\sigma)d\sigma\Big]^{1/(p-1)}d\xi \le \int_0^r \xi^{\frac{1-N}{p-1}}\int_0^\xi \sigma^{N-1}\Phi(\sigma)d\sigma d\xi\,.$$
Integration by parts gives
$$\begin{aligned}
&\int_0^r \xi^{\frac{1-N}{p-1}}\int_0^\xi \sigma^{N-1}\Phi(\sigma)d\sigma d\xi\\
&= -\frac{p-1}{N-p}\int_0^r \frac{d}{d\xi}\xi^{\frac{p-N}{p-1}}\int_0^\xi \sigma^{N-1}\Phi(\sigma)d\sigma d\xi\\
&= \frac{p-1}{N-p}(-r^{\frac{p-N}{p-1}}\int_0^r \sigma^{N-1}\Phi(\sigma)d\sigma + \int_0^r \xi^{\frac{(p-2)N+1}{p-1}}\Phi(\xi)d\xi)\,.
\end{aligned}$$
Now, by L' Hôpital's rule, we have
$$\begin{aligned}
&\lim_{r\to\infty}\Big[-r^{\frac{p-N}{p-1}}\int_0^r \sigma^{N-1}\Phi(\sigma)d\sigma + \int_0^r \xi^{\frac{(p-2)N+1}{p-1}}\Phi(\xi)d\xi\Big]\\
&= \lim_{r\to\infty}\frac{-\int_0^r \sigma^{N-1}\Phi(\sigma)d\sigma + r^{\frac{N-p}{p-1}}\int_0^r \xi^{\frac{(p-2)N+1}{p-1}}\Phi(\xi)d\xi}{r^{\frac{N-p}{p-1}}}\\
&= \lim_{r\to\infty}\int_0^r \xi^{\frac{(p-2)N+1}{p-1}}\Phi(\xi)d\xi\\
&= \int_0^\infty \xi^{\frac{(p-2)N+1}{p-1}}\Phi(\xi)d\xi < \infty,
\end{aligned}$$
From cases 1 and 2 above, it follows that
$$\begin{gathered}
K = \frac{p-1}{N-2}\cdot\int_0^\infty \xi^{\frac{1}{p-1}}\Phi^{1/(p-1)}(\xi)d\xi \text{ if } 1<p<2, \text{ or}\\
K = \frac{p-1}{N-p}\cdot\int_0^\infty \xi^{\frac{(p-2)N+1}{p-1}}\Phi(\xi)d\xi \quad \text{if } 2\le p<+\infty.
\end{gathered}$$
Clearly, for all $r>0$,
$$\begin{gathered}
w(r) < \frac{p-1}{N-2}\cdot\int_0^\infty \xi^{1/(p-1)}\Phi^{1/(p-1)}(\xi)d\xi \quad \text{if } 1<p\le 2, \text{ or}\\
w(r) < \frac{p-1}{N-p}\cdot\int_0^\infty \xi^{\frac{(p-2)N+1}{p-1}}\Phi(\xi)d\xi \quad \text{if } 2\le p<+\infty,
\end{gathered}$$

An upper-solution to (1.1) will be constructed. Consider the function $\overline{f}(u) = (f(u)+1)^{1/(p-1)}$, for $u>0$.

Note that the hypothesis $u\to f(u)/(u+\beta)^{p-1}$ is a decreasing function on $(0,\infty)$ implies that $u\to f(u)/u^{p-1}$ is a decreasing function on $(0,\infty)$, because $\frac{v+\beta}{v}\le\frac{u+\beta}{u}\Leftrightarrow vu+\beta u\le vu+v\beta\Leftrightarrow \beta(u-v)\le 0$, is true $\forall u\le v$ and $\beta>0$. We have

(F1') $\overline{f}(u)\ge f(u)^{1/(p-1)}$

(F2') $\lim_{u\searrow 0}\overline{f}(u)/u=\infty$ and $u\mapsto \overline{f}(u))/u^{p-1}$ is decreasing on $(0,\infty)$.

Let v be a positive function such that $w(r)=\frac{1}{C}\int_0^{v(r)}t^{p-1}/\overline{f}(t)\,dt$, where C is a positive constant such that $KC\leq\int_0^{C^{1/(p-1)}}t^{p-1}/\overline{f}(t)\,dt$. We prove that we can find $C>0$ with this property. From our hypothesis (F2') we obtain that $\lim_{x\to+\infty}\int_0^x t^{p-1}/\overline{f}(t)\,dt=+\infty$. Now using L'Hôpital's rule we have

$$\lim_{x\to\infty}\frac{1}{x^{p-1}}\int_0^x\frac{t^{p-1}}{\overline{f}(t)}dt=\lim_{x\to\infty}\frac{x}{(p-1)\overline{f}(x)}=+\infty.$$

This means that there exists $x_1>0$ such that $\int_0^x t^{p-1}/\overline{f}(t)\,dt\geq Kx^{p-1}$, for all $x\geq x_1$. It follows that for any $C\geq x_1$,

$$KC\leq\int_0^{C^{1/(p-1)}}\frac{t^{p-1}}{\overline{f}(t)}dt.$$

But w is a decreasing function, and this implies that v is a decreasing function too. Then

$$\int_0^{v(r)}\frac{t^{p-1}}{\overline{f}(t)}dt\leq\int_0^{v(0)}\frac{t^{p-1}}{\overline{f}(t)}dt=C\cdot w(0)=C\cdot K\leq\int_0^{C^{1/(p-1)}}\frac{t^{p-1}}{\overline{f}(t)}dt.$$

It follows that $v(r)\leq C^{1/(p-1)}$ for all $r>0$. From $w(r)\to 0$ as $r\to+\infty$ we deduce $v(r)\to 0$ as $r\to+\infty$. By the choice of v we have

$$\nabla w=\frac{1}{C}\cdot\frac{v^{p-1}}{\overline{f}(v)}\nabla v\quad\text{and}\quad\Delta w=\frac{1}{C}\cdot\frac{v^{p-1}}{\overline{f}(v)}\Delta v+\frac{1}{C}\Big(\frac{v^{p-1}}{\overline{f}(v)}\Big)'|\nabla v|^2.$$

so

$$|\nabla w|^{p-2}=\frac{1}{C^{p-2}}\Big(\frac{v^{p-1}}{\overline{f}(v)}\Big)^{p-2}|\nabla v|^{p-2}.$$

It follows that

$$\begin{aligned}|\nabla w|^{p-2}\Delta w&=\frac{1}{C^{p-2}}\Big(\frac{v^{p-1}}{\overline{f}(v)}\Big)^{p-2}|\nabla v|^{p-2}\Big(\frac{1}{C}\frac{v^{p-1}}{\overline{f}(v)}\Delta v+\frac{1}{C}\Big(\frac{v^{p-1}}{\overline{f}(v)}\Big)'|\nabla v|^2\Big)\\&=\frac{1}{C^{p-1}}\Big(\frac{v^{p-1}}{\overline{f}(v)}\Big)^{p-1}|\nabla v|^{p-2}\Delta v+\frac{1}{C^{p-1}}\Big(\frac{v^{p-1}}{\overline{f}(v)}\Big)^{p-2}\Big(\frac{v^{p-1}}{\overline{f}(v)}\Big)'|\nabla v|^p,\end{aligned}$$

so

$$\begin{aligned}&\nabla(|\nabla w|^{p-2})\cdot\nabla w\\&=\Big\{\frac{1}{C^{p-2}}\Big(\frac{v^{p-1}}{\overline{f}(v)}\Big)^{p-2}\nabla(|\nabla v|^{p-2})+\frac{1}{C^{p-2}}\Big[\Big(\frac{v^{p-1}}{\overline{f}(v)}\Big)^{p-2}\Big]'|\nabla v|^{p-2}\nabla v\Big\}\cdot\Big[\frac{1}{C}\cdot\frac{v^{p-1}}{\overline{f}(v)}\nabla v\Big]\\&=\frac{1}{C^{p-1}}\Big(\frac{v^{p-1}}{\overline{f}(v)}\Big)^{p-2}\nabla(|\nabla v|^{p-2})\cdot\nabla v+\frac{1}{C^{p-1}}\Big[\Big(\frac{v^{p-1}}{\overline{f}(v)}\Big)^{p-2}\Big]'\frac{v^{p-1}}{\overline{f}(v)}|\nabla v|^p,\end{aligned}$$

so that

$$\begin{aligned}\Delta_p w&=\frac{1}{C^{p-1}}\Big(\frac{v^{p-1}}{\overline{f}(v)}\Big)^{p-1}\Big[|\nabla v|^{p-2}\Delta v+\nabla(|\nabla v|^{p-2})\cdot\nabla v\Big]\\&\quad+\frac{1}{C^{p-1}}\Big(\frac{v^{p-1}}{\overline{f}(v)}\Big)^{p-2}|\nabla v|^p\Big(\frac{v^{p-1}}{\overline{f}(v)}\Big)'+\frac{1}{C^{p-1}}\Big[\Big(\frac{v^{p-1}}{\overline{f}(v)}\Big)^{p-2}\Big]'\frac{v^{p-1}}{\overline{f}(v)}|\nabla v|^p\qquad(3.6)\\&=\frac{1}{C^{p-1}}\Big(\frac{v^{p-1}}{\overline{f}(v)}\Big)^{p-1}\Delta_p v+(p-1)\frac{1}{C^{p-1}}|\nabla v|^p\Big(\frac{v^{p-1}}{\overline{f}(v)}\Big)^{p-2}\Big(\frac{v^{p-1}}{\overline{f}(v)}\Big)'.\end{aligned}$$

From (3.6) we deduce that

$$\Delta_p w = \frac{1}{C^{p-1}}\Big(\frac{v^{p-1}}{\overline{f}(v)}\Big)^{p-1}\Delta_p v + (p-1)\frac{1}{C^{p-1}}|\nabla v|^p\Big(\frac{v^{p-1}}{\overline{f}(v)}\Big)^{p-2}\Big(\frac{v^{p-1}}{\overline{f}(v)}\Big)'. \tag{3.7}$$

From (3.7) and the fact that $u \to \frac{\overline{f}(u)}{u^{p-1}}$ is a decreasing function on $(0, +\infty)$, we deduce that

$$\Delta_p v \le C^{p-1}\Big(\frac{\overline{f}(v)}{v^{p-1}}\Big)^{p-1}\Delta_p w = -C^{p-1}\Big(\frac{\overline{f}(v)}{v^{p-1}}\Big)^{p-1}\Phi(r) \le -f(v)\Phi(r). \tag{3.8}$$

By (3.7) and (3.8) and using in an essential manner the hypothesis (F1), as already done for proving the uniqueness, we obtain that $u_k \le v$ for $|x| \le k$ and, hence, for all $\mathbb{R}^N$. Now we have a bounded increasing sequence $u_1 \le u_2 \le \dots \le u_k \le dots \le v$ with v vanishing at infinity. Thus there exists a function, say $u \le v$ such that $u_k \to u$ pointwise in $\mathbb{R}^N$. Using

$$\begin{gathered}
u'(r) = \Big[r^{1-N}\int_0^r \sigma^{N-1}p(\sigma)f(u(\sigma))d\sigma\Big]^{1/(p-1)},\\
u''(r) = -\frac{p(r)f(u(r)) + (1-N)r^{-N}\int_0^r \sigma^{N-1}p(\sigma)f(u(\sigma))d\sigma}{p-1}\\
\times\Big[r^{1-N}\int_0^r \sigma^{N-1}p(\sigma)f(u(\sigma))d\sigma\Big]^{\frac{2-p}{p-1}},\\
\frac{2-p}{p-1} \ge 0 \iff 1 < p \le 2\\
\lim_{r\to 0}\frac{\int_0^r \sigma^{N-1}p(\sigma)f(u(\sigma))d\sigma}{r^N} = 0\\
\lim_{r\to 0}\frac{\int_0^r \sigma^{N-1}p(\sigma)f(u(\sigma))d\sigma}{r^{N-1}} = 0
\end{gathered}$$

it is easy to prove that $u(r) \in C^2(\mathbb{R}^N)$ if $1 < p \le 2$ because $\lim_{r\to\infty} u''(r)$ is finite and $u(r) \in C^1(\mathbb{R}^N)$ if $2 < p < \infty$ because $\lim_{r\to\infty} u'(r)$ is finite.

4. Proof of Theorem 1.2

Suppose (1.1) has a solution $u(r)$, then

$$(r^{N-1}|u'(r)|^{p-2}u'(r))' = -r^{N-1}f(u(r))a(r),$$

integrating from 0 to r, we have

$$|u'(r)|^{p-2}u'(r) = -r^{1-N}\int_0^r \sigma^{N-1}f(u(\sigma))a(\sigma)d\sigma,$$

hence $u'(r) < 0$. We put $\ln(u(r)+1) := \overline{u}(r) > 0$ for all $r > 0$. Then we have

$$\Delta_p\overline{u}(r) = \frac{\Delta_p u(r)}{(u(r)+1)^{p-1}} - (p-1)\frac{|\nabla u(r)|^p}{(u(r)+1)^p}.$$

Then $\overline{u}(r)$ satisfies

$$\frac{1}{r^{N-1}}\Big(r^{N-1}(-\overline{u}'(r))^{p-2}\overline{u}'(r)\Big)' + (p-1)\frac{|\nabla u(r)|^p}{(u(r)+1)^p} = -\frac{f(u(r))a(r)}{(u(r)+1)^{p-1}}. \tag{4.1}$$

Multiplying (4.1) by r^{N-1} and integrating on $(0,\xi)$ yield

$$\begin{aligned}&\int_0^\xi \Big((-\overline{u}'(\sigma))^{p-1}\sigma^{N-1}\Big)' d\sigma - (p-1)\int_0^\xi \frac{\sigma^{N-1}|\nabla u(\sigma)|^p}{(u(\sigma)+1)^p} d\sigma \\ &= \int_0^\xi \frac{f(u(\sigma))a(\sigma)\sigma^{N-1}}{(u(\sigma)+1)^{p-1}} d\sigma,\end{aligned}$$

equivalently

$$(-\overline{u}'(\xi))^{p-1}\xi^{N-1} - \int_0^\xi (p-1)\frac{\sigma^{N-1}|\nabla u(\sigma)|^p}{(u(\sigma)+1)^p} d\sigma = \int_0^\xi \frac{f(u(\sigma))a(\sigma)\sigma^{N-1}}{(u(\sigma)+1)^{p-1}} d\sigma\,. \tag{4.2}$$

Multiplying equation (4.2) by ξ^{1-N}, we deduce

$$(-\overline{u}'(\xi))^{p-1} - \xi^{1-N}(p-1)\int_0^\xi \frac{\sigma^{N-1}|\nabla u(\sigma)|^p}{(u(\sigma)+1)^p} d\sigma = \xi^{1-N}\int_0^\xi \frac{f(u(\sigma))a(\sigma)\sigma^{N-1}}{(u(\sigma)+1)^{p-1}} d\sigma. \tag{4.3}$$

From (4.3), we have

$$\big(-\overline{u}'(\xi)\big)^{p-1} \geq \xi^{1-N}\int_0^\xi \frac{f(u(\sigma))a(\sigma)\sigma^{N-1}}{(u(\sigma)+1)^{p-1}} d\sigma,$$

so

$$-\overline{u}'(\xi) \geq \xi^{\frac{1-N}{p-1}}\Big[\int_0^\xi \frac{f(u(\sigma))a(\sigma)\sigma^{N-1}}{(u(\sigma)+1)^{p-1}} d\sigma\Big]^{1/(p-1)}, \tag{4.4}$$

integrating (4.4) on $(0,r)$, we have

$$\overline{u}(0) - \overline{u}(r) \geq \int_0^r \xi^{\frac{1-N}{p-1}}\Big[\int_0^\xi \frac{f(u(\sigma))a(\sigma)\sigma^{N-1}}{(u(\sigma)+1)^{p-1}} d\sigma\Big]^{1/(p-1)} d\xi.$$

We observe that $\overline{u}(r) < \overline{u}(0)$, for all $r > 0$ implies $u(r) < u(0)$, for all $r > 0$.

If $\beta \geq 1$, then the function $u \mapsto \frac{f(u)}{(u+\beta)^{p-1}}$ is decreasing on $(0,+\infty)$. This implies

$$\frac{f(u(\sigma))}{(u(\sigma)+1)^{p-1}} > \frac{f(u(0))}{(u(0)+1)^{p-1}}. \tag{4.5}$$

Since $\overline{u}$ is positive, we have

$$\int_0^r \xi^{\frac{1-N}{p-1}}\Big[\int_0^\xi \frac{f(u(\sigma))a(\sigma)\sigma^{N-1}}{(u(\sigma)+1)^{p-1}} d\sigma\Big]^{1/(p-1)} d\xi \leq \overline{u}(0), \quad \forall r > 0,$$

substituting (4.5) into this expression, we obtain

$$\int_0^r \xi^{\frac{1-N}{p-1}}\Big[\int_0^\xi a(\sigma)\sigma^{N-1} d\sigma\Big]^{1/(p-1)} d\xi \leq \frac{u(0)+1}{f(u(0))^{\frac{1}{p-1}}}\overline{u}(0) < \infty.$$

Let $2 \leq p < +\infty$, so $1 \leq p-1$, follows that $1 \geq \frac{1}{p-1} > 0$. We have

$$\begin{aligned}
&\int_0^r \xi^{\frac{1-N}{p-1}} \Big[\frac{\xi}{\xi}\int_0^\xi \sigma^{N-1} a(\sigma)d\sigma\Big]^{1/(p-1)} d\xi \\
&= \int_0^r \xi^{\frac{1-N}{p-1}} \xi^{1/(p-1)} \Big[\frac{1}{\xi}\int_0^\xi \sigma^{N-1} a(\sigma)d\sigma\Big]^{1/(p-1)} d\xi \\
&\geq \int_0^r \xi^{\frac{2-N}{p-1}} \frac{1}{\xi}\int_0^\xi \sigma^{\frac{N-1}{p-1}} a^{1/(p-1)}(\sigma)d\sigma d\xi \\
&= \int_0^r \xi^{\frac{2-N}{p-1}-1} \int_0^\xi \sigma^{\frac{N-1}{p-1}} a^{1/(p-1)}(\sigma)d\sigma d\xi \\
&= -\frac{p-1}{N-2}\int_0^r \frac{d}{d\xi}\xi^{\frac{2-N}{p-1}} \int_0^\xi \sigma^{\frac{N-1}{p-1}} a^{1/(p-1)}(\sigma)d\sigma d\xi \\
&= \frac{p-1}{N-2}(-r^{\frac{2-N}{p-1}} \int_0^r \sigma^{\frac{N-1}{p-1}} a^{1/(p-1)}(\sigma)d\sigma + \int_0^r \xi^{1/(p-1)} a(\xi)^{1/(p-1)} d\xi) \\
&\geq \frac{p-1}{N-2}\frac{1}{r^{\frac{N-2}{p-1}}}\int_0^r \Big[r^{\frac{N-2}{p-1}} - (t)^{\frac{N-2}{p-1}}\Big] t^{1/(p-1)} a^{1/(p-1)}(t)dt \\
&\geq \frac{p-1}{N-2}\frac{1}{r^{\frac{N-2}{p-1}}}\Big(r^{\frac{N-2}{p-1}} - (\frac{r}{2})^{\frac{N-2}{p-1}}\Big)\int_0^{r/2} t^{1/(p-1)} a^{1/(p-1)}(t)dt \\
&= \frac{p-1}{N-2}(1-(\frac{1}{2})^{\frac{N-2}{p-1}})\int_0^{r/2} t^{1/(p-1)} a^{1/(p-1)}(t)dt \to \infty \quad \text{as } r \to \infty.
\end{aligned}$$

So

$$\infty > \frac{u(0)+1}{f(u(0))^{1/(p-1)}}\overline{u}(0) \geq \infty,$$

which is a contradiction.

If $\beta < 1$ then the function $u \mapsto \frac{(u+\beta)^{p-1}}{(u+1)^{p-1}}$ is increasing on $(0,+\infty)$. In this case

$$\begin{aligned}
\overline{u}(0) &\geq \int_0^r \xi^{\frac{1-N}{p-1}} \Big[\int_0^\xi \frac{f(u(\sigma))a(\sigma)\sigma^{N-1}}{(u(\sigma)+1)^{p-1}} d\sigma\Big]^{1/(p-1)} d\xi \\
&= \int_0^r \xi^{\frac{1-N}{p-1}} \Big[\int_0^\xi \frac{f(u(\sigma))a(\sigma)(u(\sigma)+\beta)^{p-1}\sigma^{N-1}}{(u(\sigma)+\beta)^{p-1}(u(\sigma)+1)^{p-1}} d\sigma\Big]^{1/(p-1)} d\xi \\
&\geq \frac{f(u(0))^{1/(p-1)}}{u(0)+\beta}\beta \int_0^r \xi^{\frac{1-N}{p-1}} \Big[\int_0^\xi \sigma^{N-1} a(\sigma)d\sigma\Big]^{1/(p-1)} d\xi,
\end{aligned}$$

which implies

$$\infty > \frac{u(0)+\beta}{f(u(0))^{1/(p-1)}\beta}\overline{u}(0) \geq \int_0^r \xi^{\frac{1-N}{p-1}} \Big[\int_0^\xi \sigma^{N-1} a(\sigma)d\sigma\Big]^{1/(p-1)} d\xi \geq \infty,$$

which is a contradiction.

5. Proof of Theorem 1.3

Assume u is positive for $r > 0$ and satisfies

$$(r^{N-1}|u'(r)|^{p-2}u'(r))' = -r^{N-1} f(u(r))a(r).$$

Since $f(u(r))a(r)$ is positive for $r > 0$, follows that

$$(r^{N-1}|u'(r)|^{p-2}u'(r))' < 0, \quad \text{for } r > 0,$$

and that $r^{N-1}|u'(r)|^{p-2}u'(r)$ is a decreasing function. Because this function is decreasing and $u'<0$,

$$r^{N-1}|u'(r)|^{p-2}u'(r) \le -C, \quad \text{for } r \ge R,$$

where C is positive constant. As a consequence

$$-u'(r) \ge C_1 r^{-\frac{1-N}{p-1}}, \quad \text{with } C_1>0.$$

Integrating this inequality from R to r we have

$$u(R)-u(r) \ge C_1 \int_R^r r^{-\frac{1-N}{p-1}} dr, \quad \text{for } r \ge R.$$

Letting $r \to \infty$, we arrive at a contradiction.

6. Proof of Theorem 1.4

As in proof of Theorem 1.2, we have

$$\overline{u}(0)-\overline{u}(r) \ge \int_0^r \xi^{\frac{1-N}{p-1}} \Big[\int_0^\xi \frac{f(u(\sigma))a(\sigma)\sigma^{N-1}}{(u(\sigma)+1)^{p-1}} d\sigma\Big]^{1/(p-1)} d\xi,$$

We observe that $\overline{u}(r)<\overline{u}(0)$, for all $r>0$ implies $u(r)<u(0)$, for all $r>0$. If $\beta \ge 1$ then the function $u \mapsto \frac{f(u)}{(u+\beta)^{p-1}}$ is decreasing on $(0,+\infty)$. This implies

$$\frac{f(u(\sigma))}{(u(\sigma)+1)^{p-1}} > \frac{f(u(0))}{(u(0)+1)^{p-1}}, \tag{6.1}$$

Since $\overline{u}$ is positive, we have

$$\int_0^r \xi^{\frac{1-N}{p-1}} \Big[\int_0^\xi \frac{f(u(\sigma))a(\sigma)\sigma^{N-1}}{(u(\sigma)+1)^{p-1}} d\sigma\Big]^{1/(p-1)} d\xi \le \overline{u}(0), \quad \forall r>0$$

substituting 6.1 into this expression we obtain

$$\int_0^r \xi^{\frac{1-N}{p-1}} \Big[\int_0^\xi a(\sigma)\sigma^{N-1} d\sigma\Big]^{1/(p-1)} d\xi \le \frac{u(0)+1}{f(u(0))^{1/(p-1)}}\overline{u}(0) < \infty.$$

Let $1<p<2$, so $0<p-1<1$, it follows that $1<\frac{1}{p-1}<+\infty$. Set

$$\int_0^\xi r^{N-1}a(r)dr < 1 \quad \text{for } \xi>0, \quad \text{or}$$
$$\int_0^\xi r^{N-1}a(r)dr \ge 1 \quad \text{for } \xi>0,$$

In the second case, we have

$$\Big[\int_0^\xi \sigma^{N-1}a(\sigma)d\sigma\Big]^{1/(p-1)} \ge \int_0^\xi \sigma^{N-1}a(\sigma)d\sigma,$$

so

$$\int_0^r \xi^{\frac{1-N}{p-1}} \Big[\int_0^\xi \sigma^{N-1}a(\sigma)d\sigma\Big]^{1/(p-1)} d\xi \ge \int_0^r \xi^{\frac{1-N}{p-1}} \int_0^\xi \sigma^{N-1}a(\sigma)d\sigma d\xi\,.$$

Integration by parts gives

$$\begin{aligned}
&\int_0^r \xi^{\frac{1-N}{p-1}} \int_0^\xi \sigma^{N-1} a(\sigma) d\sigma d\xi \\
&= -\frac{p-1}{N-p} \int_0^r \frac{d}{d\xi} \xi^{\frac{p-N}{p-1}} \int_0^\xi \sigma^{N-1} a(\sigma) d\sigma d\xi \\
&= \frac{p-1}{N-p} (-r^{\frac{p-N}{p-1}} \int_0^r \sigma^{N-1} a(\sigma) d\sigma + \int_0^r \xi^{\frac{(p-2)N+1}{p-1}} a(\xi) d\xi) \\
&\geq \frac{p-1}{N-p} \frac{1}{r^{\frac{N-p}{p-1}}} \int_0^r \Big[r^{\frac{N-p}{p-1}} - (t)^{\frac{N-p}{p-1}} \Big] t^{\frac{(p-2)N+1}{p-1}} p(t) dt \\
&\geq \frac{p-1}{N-p} \frac{1}{r^{\frac{N-p}{p-1}}} (r^{\frac{N-p}{p-1}} - (\frac{r}{2})^{\frac{N-p}{p-1}}) \int_0^{r/2} t^{\frac{(p-2)N+1}{p-1}} a(t) dt \\
&= \frac{p-1}{N-p} (1 - (\frac{1}{2})^{\frac{N-p}{p-1}}) \int_0^{r/2} t^{\frac{(p-2)N+1}{p-1}} a(t) dt \\
&= \infty \quad \text{as } r \to \infty.
\end{aligned}$$

Then

$$\infty > \frac{u(0)+1}{f(u(0))^{1/(p-1)}} \overline{u}(0) \geq \infty,$$

which is a contradiction.

If $\beta < 1$ we have $\frac{u+\beta}{u+1} > \beta \iff u + \beta > \beta u + \beta \iff (1-\beta)u > 0$ is true. In this case we have

$$\begin{aligned}
\overline{u}(0) &\geq \int_0^r \xi^{\frac{1-N}{p-1}} \Big[\int_0^\xi \frac{f(u(\sigma)) a(\sigma) \sigma^{N-1}}{(u(\sigma)+1)^{p-1}} d\sigma \Big]^{1/(p-1)} d\xi \\
&= \int_0^r \xi^{\frac{1-N}{p-1}} \Big[\int_0^\xi \frac{f(u(\sigma)) a(\sigma) (u(\sigma)+\beta)^{p-1} \sigma^{N-1}}{(u(\sigma)+\beta)^{p-1} (u(\sigma)+1)^{p-1}} d\sigma \Big]^{1/(p-1)} d\xi \\
&\geq \frac{f(u(0))^{1/(p-1)}}{u(0)+\beta} \beta \int_0^r \xi^{\frac{1-N}{p-1}} \Big[\int_0^\xi \sigma^{N-1} a(\sigma) d\sigma \Big]^{1/(p-1)} d\xi,
\end{aligned}$$

which implies

$$\infty > \frac{u(0)+\beta}{f(u(0))^{1/(p-1)} \beta} \overline{u}(0) \geq \int_0^r \xi^{\frac{1-N}{p-1}} \Big[\int_0^\xi \sigma^{N-1} a(\sigma) d\sigma \Big]^{1/(p-1)} d\xi \geq \infty,$$

which is a contradiction.

In the first case we observe that we can not have $\int_0^\xi r^{\frac{(p-2)N+1}{p-1}} a(r) dr = \infty$ because

$$\begin{aligned}
\int_0^r \xi^{\frac{1-N}{p-1}} d\xi &> \int_0^r \xi^{\frac{1-N}{p-1}} \int_0^\xi \sigma^{N-1} a(\sigma) d\sigma d\xi \\
&\geq \frac{p-1}{N-p} (1 - (\frac{1}{2})^{\frac{N-p}{p-1}}) \int_0^{r/2} t^{\frac{(p-2)N+1}{p-1}} a(t) dt \to \infty \quad \text{as } r \to \infty
\end{aligned}$$

which is a contradiction.

Remark 6.1. Let $2 \leq p < +\infty$. Then $1 \geq \frac{1}{p-1} > 0$. From the above proofs we observe if that

$$\Big(\int_0^\xi \sigma^{N-1} a(\sigma) d\sigma \Big)^{1/(p-1)} \leq 1$$

then

$$\begin{aligned}
&\frac{p-1}{N-2}(1-(\frac{1}{2})^{\frac{N-2}{p-1}})\int_0^{r/2} t^{1/(p-1)}a^{1/(p-1)}(t)dt\\
&\le \int_0^r \xi^{\frac{1-N}{p-1}}\Big[\int_0^\xi \sigma^{N-1}a(\sigma)d\sigma\Big]^{1/(p-1)}d\xi\\
&\le \int_0^r \xi^{\frac{1-N}{p-1}}d\xi.
\end{aligned}$$

As $r\to\infty$, we have $\int_0^\infty t^{\frac{1}{p-1}}a^{1/(p-1)}(t)dt\neq\infty$.

On the other hand, if

$$\Big(\int_0^\xi \sigma^{N-1}a(\sigma)d\sigma\Big)^{1/(p-1)}>1,$$

then

$$\begin{aligned}
&\frac{p-1}{N-2}(1-(\frac{1}{2})^{\frac{N-2}{p-1}})\int_0^{r/2} t^{1/(p-1)}a^{1/(p-1)}(t)dt\\
&\le \int_0^r \xi^{\frac{1-N}{p-1}}\Big[\int_0^\xi \sigma^{N-1}a(\sigma)d\sigma\Big]^{1/(p-1)}d\xi\\
&\le \frac{p-1}{N-p}\int_0^{r/2} t^{\frac{(p-2)N+1}{p-1}}a(t)dt.
\end{aligned}$$

Then if $\int_0^\infty t^{1/(p-1)}a^{1/(p-1)}(t)dt=\infty$ we have $\int_0^\infty t^{\frac{(p-2)N+1}{p-1}}a(t)dt=\infty$.

Remark 6.2. Let$1<p\le 2$. Then $1\le\frac{1}{p-1}<+\infty$. From the above proofs we observe that

$$\int_0^r \xi^{\frac{1-N}{p-1}}\Big[\int_0^\xi \sigma^{N-1}a(\sigma)d\sigma\Big]^{1/(p-1)}d\xi\le\frac{p-1}{N-2}\int_0^\infty t^{1/(p-1)}a^{1/(p-1)}(t)dt.$$

If $\int_0^\xi \sigma^{N-1}a(\sigma)d\sigma\ge 1$, then

$$\begin{aligned}
&\frac{N-1}{N-p}(1-(\frac{1}{2})^{\frac{N-p}{p-1}})\int_0^{r/2} t^{\frac{(p-2)N+1}{p-1}}a(t)dt\\
&\le \int_0^r \xi^{\frac{1-N}{p-1}}\Big[\int_0^\xi \sigma^{N-1}a(\sigma)d\sigma\Big]^{1/(p-1)}d\xi\,.
\end{aligned}$$

Then if $\int_0^\infty t^{\frac{(p-2)N+1}{p-1}}a(t)dt=\infty$ we have $\int_0^\infty t^{1/(p-1)}a^{1/(p-1)}(t)dt=\infty$.

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Corrigendum posted on October 8, 2007

The author wants to correct some misprints and to clarify the existence and the non-existence of solutions to the problem considered in this article.

Page 2, line 8. In the formula

$$\int_1^\infty r^{N-1+\gamma(N-2)}(\max_{|x|=t} a(x))dt < \infty,$$

replace the factor $r^{N-1+\gamma(N-2)}$ with $t^{N-1+\gamma(N-2)}$.

Page 2, line 17. In the formula

$$\int_1^\infty r^{N-1+\gamma(N-2)}(\max_{|x|=t} a(x))dt < \infty,$$

replace the factor $r^{N-1+\gamma(N-2)}$ with $t^{N-1+\gamma(N-2)}$.

Page 3, line 1. Replace "*Problem* (1.1) *has no positive radial solution if* $p \geq N$." with "*Suppose* $a(r)$ *is a positive radial function which is continuous on* $\mathbb{R}^N$. *Then problem* (1.1) *has no positive radial solution if* $p \geq N$."

Page 4, line 16. Replace "bounded domain" with "smooth open bounded domain".

Page 4, line 21. Replace "bounded domain with smooth boundary" with "smooth open bounded domain".

Page 5, line 25-26. Replace "[8, Theorem 1.3]" with "[8, Theorem 1.3 and lower-upper solutions method]"

Page 5, line 27. In the equation

$$-\Delta_p U = p(x)h(U), \quad \text{if } |x| < k,$$

replace $p(x)$ with $a(x)$.

Page 7, line 17. In the formula

$$K = \frac{p-1}{N-2} \cdot \int_0^\infty \xi^{\frac{1}{p-1}} \Phi^{1/(p-1)}(\xi) d\xi$$

replace the symbol " $=$ " with " $\leq$ ".

Page 7, line 18. Replace

$$K = \frac{p-1}{N-p} \cdot \int_0^\infty \xi^{\frac{(p-2)N+1}{p-1}} \Phi(\xi) d\xi \quad \text{if } 2 \leq p < +\infty.$$

with "

$$K \leq \frac{p-1}{N-p} \cdot \int_0^\infty \xi^{\frac{(p-2)N+1}{p-1}} \Phi(\xi) d\xi \quad \text{or} \quad K \leq \text{Const.} + \int_1^\infty \xi^{\frac{1-N}{p-1}} d\xi,$$

if $2 \leq p < +\infty$ and for the above considered cases."

Page 7, line 21. Replace

$$w(r) < \frac{p-1}{N-p} \cdot \int_0^\infty \xi^{\frac{(p-2)N+1}{p-1}} \Phi(\xi) d\xi \quad \text{if } 2 \leq p < +\infty.$$

with "

$$w(r) \leq \frac{p-1}{N-p} \cdot \int_0^\infty \xi^{\frac{(p-2)N+1}{p-1}} \Phi(\xi) d\xi \quad \text{or} \quad w(r) \leq \text{Const.} + \int_1^\infty \xi^{\frac{1-N}{p-1}} d\xi.$$

if $2 \leq p < +\infty$ and for the above considered cases."

Page 8, line 1. "(F2')" must be replaced by "(F2') $\lim_{u \searrow 0} \overline{f}(u)/u = \infty$, $\lim_{u \nearrow \infty} \overline{f}(u)/u = 0$ and $u \mapsto \overline{f}(u)/u^{p-1}$ is decreasing on $(0, \infty)$."

Page 9. line 10. Replace "Using" with "If $u(x)$ is radially symmetric solution (see [8] for conditions to $a(x)$ and $f(u(x))$) then, using"

Page 9. line 15. Replace

$$\lim_{r \to 0} \frac{\int_0^r \sigma^{N-1} p(\sigma) f(u(\sigma)) d\sigma}{r^N} = 0$$

with

$$\lim_{r \to 0} \frac{\int_0^r \sigma^{N-1} a(\sigma) f(u(\sigma)) d\sigma}{r^N} = \frac{a(0) f(u(0))}{N}$$

Page 9. line 16. Replace $p(\sigma)$ with $a(\sigma)$.

Page 9. After line 18, insert "If $u(x)$ is a weak solution, then applying the regularity theory for quasilinear elliptic equations (see for example [16] or [15, Theorem 1.3]) we find that $u \in C^{1,\alpha}(\mathbb{R}^N)$."

Page 12. line 5. Replace $r^{-\frac{1-N}{p-1}}$ with $r^{\frac{1-N}{p-1}}$.

Page 12. line 7. Replace $r^{-\frac{1-N}{p-1}}$ with $r^{\frac{1-N}{p-1}}$.

Page 13. line 5. Replace $p(t)$ with $a(t)$.

Page 14. After line 18 insert "On the other hand, if

$$\Big(\int_0^\xi \sigma^{N-1} a(\sigma) d\sigma \Big)^{1/(p-1)} < 1,$$

we observe that $\int_0^\infty t^{\frac{(p-2)N+1}{p-1}} a(t) dt \neq \infty$ as in Case 2, Theorem 1.1."

Also the author wants to present an alternative proof for the existence of solutions (see section 3. Existence of a solution).

Proof. The existence of solutions will be established by solving the approximate problems

$$\begin{gathered} -\Delta_p u = a(x) f(u+\varepsilon), \quad \text{if } |x| < k, \\ u(x) = 0, \quad \text{if } |x| = k, \end{gathered} \tag{6.2}$$

for $\varepsilon > 0$ and then showing the convergence of u_ε as $\varepsilon \to +0$ to a solution u. It is clear that the problems (6.2) has a unique solution which is due to Diaz-Saà. In the next steps we established some properties for such solution. For this, let $\varepsilon := \varepsilon_n$ be a decreasing sequence converging to 0 and set $u_n := u_{\varepsilon_n}$ with $n > k \geq 1$ in (6.2). By [15] we see that $u_n \geq c_{0,B_k}\varphi_{1,B_k}$ and there exists some function $u_k \in C(\overline{B}_k)$ such that

((i) $u_n \to u_k$ a.e. in B_k as $n \to \infty$,
(ii) $u_k \geq c_{0,k}\varphi_1$ a.e. in B_k,

where $\varphi_1 := \varphi_{1,B_k}$ is the first eigenfunction for the eigenvalue λ_1 of $(-\Delta_p)$ in $W_0^{1,p}(B_k)$ and $B_k := \{x \in \mathbb{R}^N : |x|\}$. Moreover using Diaz-Saà's comparison lemma we have a sequence $\{u_k\}$ (which is 0 for $|x| > k$), as in the present paper, such that

$$u_1 \leq u_2 \leq \cdots \leq u_k \leq \cdots \leq v \quad \text{in } \mathbb{R}^N,$$

where v is the same function as above and so the existence of solution u to the problem (1.1) is proved. □

This alternative proof is treated more generally in [15]. With this alternative proof we observe that it is sufficient to apply the rest the reference [5]. This technique is inspired by [3] and by another results due to Goncalves and Santos, which are treated more generally in [15].

References

[15] D. P. Covei, Existence and asymptotic behavior of positive solution to a quasilinear elliptic problem in R^N, Nonlinear Analysis: TMA, DOI 10.1016/j.na.2007.08.039.
[16] E. DiBenedetto, $C^{1,\alpha}$- local regularity of weak solutions of degenerate elliptic equations, Nonlinear Anal. 7 (1983), 827-850.