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CLASSIFICATION AND EVOLUTION OF BIFURCATION CURVES FOR THE ONE-DIMENSIONAL PERTURBED GELFAND EQUATION WITH MIXED BOUNDARY CONDITIONS II

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Abstract. In this article, we study the classification and evolution of bifurcation curves of positive solutions for the one-dimensional perturbed Gelfand equation with mixed boundary conditions,

$$u''(x) + \lambda \exp\big(\frac{au}{a+u}\big) = 0, \quad 0 < x < 1,$$
$$u(0) = 0, \quad u'(1) = -c < 0,$$

where $4 \le a < a_1 \approx 4.107$. We prove that, for $4 \le a < a_1$, there exist two nonnegative $c_0 = c_0(a) < c_1 = c_1(a)$ satisfying $c_0 > 0$ for $4 \le a < a^* \approx 4.069$, and $c_0 = 0$ for $a^* \le a < a_1$, such that, on the $(\lambda, \|u\|_\infty)$-plane, (i) when $0 < c < c_0$, the bifurcation curve is strictly increasing; (ii) when $c = c_0$, the bifurcation curve is monotone increasing; (iii) when $c_0 < c < c_1$, the bifurcation curve is S-shaped; (iv) when $c \ge c_1$, the bifurcation curve is $\subset$-shaped. This work is a continuation of the work by Liang and Wang [8] where authors studied this problem for $a \ge a_1$, and our results partially prove a conjecture on this problem for $4 \le a < a_1$ in [8].

1. Introduction

In this article, we study the classification and evolution of bifurcation curves of positive solutions for the one-dimensional perturbed Gelfand equation with mixed (or more precisely, Dirichlet-Neumann) boundary conditions given by

$$\begin{gathered} u''(x) + \lambda \exp\big(\frac{au}{a+u}\big) = 0, \quad 0 < x < 1, \\ u(0) = 0, \quad u'(1) = -c < 0, \end{gathered} \tag{1.1}$$

where $\lambda > 0$ is treated as a bifurcation parameter, $c > 0$ is treated as an evolution parameter, and constant a satisfies $4 \le a < a_1 \approx 4.107$ where constant a_1 is defined in [4, (3.23)]. The bifurcation curve of positive solutions of (1.1) is defined by

$$\tilde{S}_c = \{(\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1.1)}\}.$$

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This work is a continuation of our previous work in [8] where we studied (1.1) for $a \geq a_1$. It is worthwhile noting that the classification and evolution of bifurcation curves $\tilde{S}_c$ of (1.1) is closely related to the one resulting from the same differential equation in (1.1) with zero Dirichlet boundary conditions [2, 5, 8], that is,

$$\begin{gathered} u''(x) + \lambda \exp\big(\frac{au}{a+u}\big) = 0, \quad 0 < x < 1, \\ u(0) = 0, \quad u(1) = 0. \end{gathered} \tag{1.2}$$

The bifurcation curve of positive solutions of (1.2) is defined by

$$S = \{(\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1.2)}\}.$$

Before going into further discussions on problems (1.1) and (1.2), we first give some terminologies in this paper for the shapes of bifurcation curves $\tilde{S}_c$ on the $(\lambda, \|u\|_\infty)$-plane (Following terminology also hold for S if $\tilde{S}_c$ is replaced by S.)

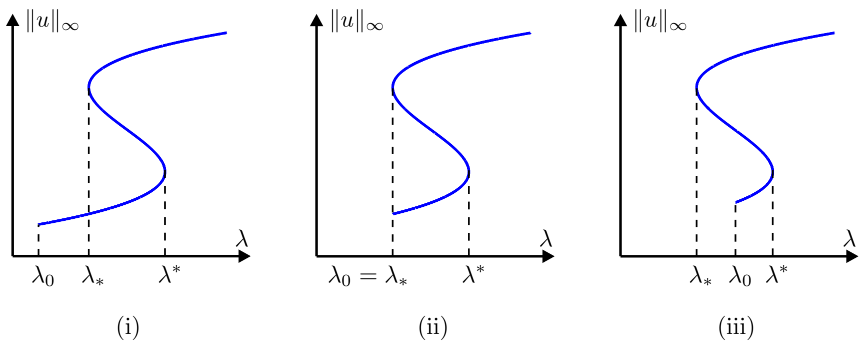

FIGURE 1. Three different types of exactly S-shaped bifurcation curves $\tilde{S}_c$ with $\lambda_0 > 0$ and $\|u_{\lambda_0}\|_\infty > 0$. (i) Type 1. (ii) Type 2. (iii) Type 3.

S-shaped: The bifurcation curve $\tilde{S}_c$ on the $(\lambda, \|u\|_\infty)$-plane is said to be S-*shaped* if $\tilde{S}_c$ has *at least two* turning points, say $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ and $(\lambda_*, \|u_{\lambda_*}\|_\infty)$, satisfying $\lambda_* < \lambda^*$ and $\|u_{\lambda^*}\|_\infty < \|u_{\lambda_*}\|_\infty$, and

(i) $\tilde{S}_c$ starts at some point $(\lambda_0, \|u_{\lambda_0}\|_\infty)$ and initially continues to the *right*,
(ii) at $(\lambda^*, \|u_{\lambda^*}\|_\infty)$, $\tilde{S}_c$ turns to the *left*,
(iii) at $(\lambda_*, \|u_{\lambda_*}\|_\infty)$, $\tilde{S}_c$ turns to the *right*,
(iv) $\tilde{S}_c$ tends to infinity as $\lambda \to \infty$. That is, $\lim_{\lambda\to\infty} \|u_\lambda\|_\infty = \infty$.

Exactly S-shaped: The bifurcation curve $\tilde{S}_c$ on the $(\lambda, \|u\|_\infty)$-plane is said to be *exactly S-shaped* if $\tilde{S}_c$ is S-shaped and it has *exactly two* turning points; see Figure 1.

Type 1/2/3 S-shaped: Assume that the bifurcation curve $\tilde{S}_c$ is S-shaped on the $(\lambda, \|u\|_\infty)$-plane. Let $(\lambda_0, \|u_{\lambda_0}\|_\infty)$ be the starting point of $\tilde{S}_c$, and

$$\bar{\lambda}_{\min} \equiv \min\{\lambda : (\lambda, \|u_\lambda\|_\infty) \text{ is a turning point of } \tilde{S}_c\}.$$

Then $\tilde{S}_c$ is said to be type 1 (resp., type 2 and type 3) S-shaped if $\lambda_0 < \bar{\lambda}_{\min}$ (resp., $\lambda_0 = \bar{\lambda}_{\min}$ and $\lambda_0 > \bar{\lambda}_{\min}$); see Figure 1(i) (resp., Figure 1(ii) and 1(iii)).

$\subset$-shaped: The bifurcation curve $\tilde{S}_c$ on the $(\lambda, \|u\|_\infty)$-plane is said to be $\subset$-*shaped* if $\tilde{S}_c$ has *at least one* turning point $(\lambda_*, \|u_{\lambda_*}\|_\infty)$, and

(i) $\tilde{S}_c$ starts at some point $(\lambda_0, \|u_{\lambda_0}\|_\infty)$ and initially continues to the *left*,
(ii) at $(\lambda_*, \|u_{\lambda_*}\|_\infty)$, $\tilde{S}_c$ turns to the *right*,
(iii) $\lambda_* < \lambda_0$ and $\|u_{\lambda_0}\|_\infty < \|u_{\lambda_*}\|_\infty$,
(iv) $\tilde{S}_c$ tends to infinity as $\lambda \to \infty$. That is, $\lim_{\lambda\to\infty} \|u_\lambda\|_\infty = \infty$.

Exactly $\subset$-shaped: The bifurcation curve $\tilde{S}_c$ on the $(\lambda, \|u\|_\infty)$-plane is said to be *exactly* $\subset$-*shaped* if $\tilde{S}_c$ is $\subset$-shaped and it has *exactly one* turning point; see Figure 2.

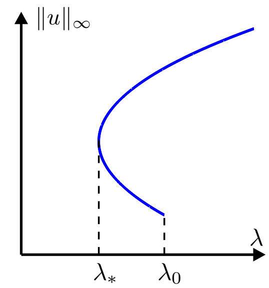

FIGURE 2. Exactly $\subset$-shaped bifurcation curve $\tilde{S}_c$ with $\lambda_0 > 0$ and $\|u_{\lambda_0}\|_\infty > 0$.

Strictly/Monotone increasing: The bifurcation curve $\tilde{S}_c$ on the $(\lambda, \|u\|_\infty)$-plane is said to be *strictly (resp., monotone) increasing* if $\lambda_1 < \lambda_2$ (resp., $\lambda_1 \le \lambda_2$) for any two points $(\lambda_i, \|u_{\lambda_i}\|_\infty)$, $i = 1, 2$, lying in $\tilde{S}_c$ with $\|u_{\lambda_1}\|_\infty < \|u_{\lambda_2}\|_\infty$.

For (1.2), it has been a long-standing conjecture [1, 6, 9] that there exists a positive critical bifurcation value $a^* \approx 4.07 > 4$ such that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve S is strictly increasing for $0 < a \le a^*$ and is exactly type 1 S-shaped for $a > a^*$. Very recently, Huang and Wang [3] gave a rigorous proof of this conjecture for (1.2). Their main result is stated in the next theorem.

Theorem 1.1 ([3, Theorem 4 and Fig. 1])**.** *Consider* (1.2) *with varying* $a > 0$. *Then, on the* $(\lambda, \|u\|_\infty)$*-plane, the bifurcation curve* S *of* (1.2) *is a continuous curve which starts at the origin and it tends to infinity as* $\lambda \to \infty$. *Moreover, there exists a critical bifurcation value* $a^* \approx 4.069$ *satisfying* $4 < a^* < a_1 \approx 4.107$ *such that the following assertions (i)–(iii) hold:*

(i) *For* $a > a^*$, *the bifurcation curve* S *is exactly type 1* S*-shaped on the* $(\lambda, \|u\|_\infty)$*-plane. Moreover, all positive solutions* u_λ *are nondegenerate except that* u_{λ_*} *and* u_{λ^*} *are degenerate for some positive* $\lambda_* < \lambda^*$.
(ii) *For* $a = a^*$, *the bifurcation curve* S *is strictly increasing on the* $(\lambda, \|u\|_\infty)$*-plane. Moreover, all positive solutions* u_λ *are nondegenerate except that* u_{λ_0} *is degenerate for some positive* λ_0.
(iii) *For* $0 < a < a^*$, *the bifurcation curve* S *is strictly increasing on the* $(\lambda, \|u\|_\infty)$*-plane. Moreover, all positive solutions* u_λ *are nondegenerate.*

For (1.1), Liang and Wang [8] proved the next theorem with any fixed $a > a_1 \approx 4.107$.

Theorem 1.2 ([8, Theorem 2.4] and see e.g., Figure 3 with $a = 5$)**.** *Consider* (1.1) *with any fixed* $a > a_1 \approx 4.107$*. Then, on the* $(\lambda, \|u\|_\infty)$*-plane, the bifurcation curve* $\tilde{S}_c$ *of* (1.1) *is a continuous curve which starts at some point* $(\lambda_0, \|u_{\lambda_0}\|_\infty)$ *with* $\lambda_0 > 0$ *and* $\|u_{\lambda_0}\|_\infty > 0$ *and it tends to infinity as* $\lambda \to \infty$*. Moreover, there exists* $c_1 = c_1(a) > 1.057$ *such that the following two assertions (i) and (ii) hold:*

- (i) *For* $0 < c < c_1$*, the bifurcation curve* $\tilde{S}_c$ *is* S*-shaped on the* $(\lambda, \|u\|_\infty)$*-plane. More precisely, there exist three positive* $c_{1,1} \le c_{1,2} \le c_{1,3}$ *on* $(0, c_1)$*, all depending on* a*, such that the* S*-shaped bifurcation curve* $\tilde{S}_c$ *belongs to type 1, type 2 and type 3 when* $0 < c < c_{1,1}$*,* $c = c_{1,2}$ *and* $c_{1,3} < c < c_1$*, respectively.*
- (ii) *For* $c \ge c_1$*, the bifurcation curve* $\tilde{S}_c$ *is* $\subset$*-shaped on the* $(\lambda, \|u\|_\infty)$*-plane.*

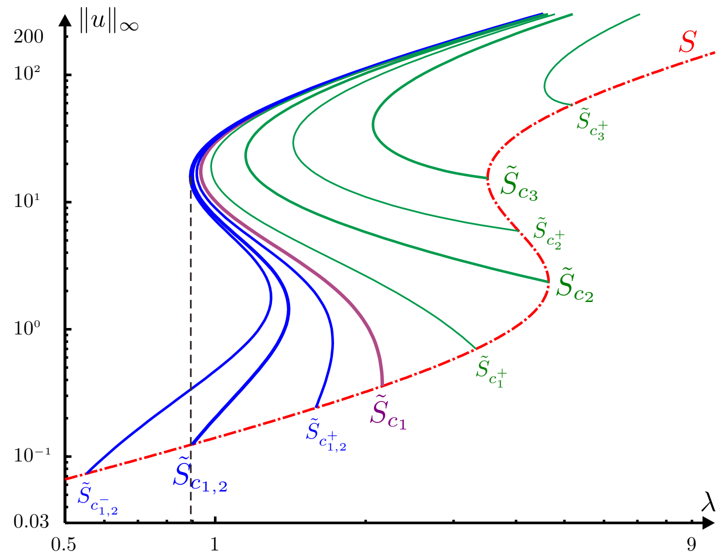

FIGURE 3. Numerical simulations of bifurcation curves S and $\tilde{S}_c$ for $a = 5$ and varying $c > 0$ on the $(\lambda, \|u\|_\infty)$-plane of the bi-logarithm coordinates. Here $c_{1,2}^- < c_{1,2} \approx 0.488 < c_{1,2}^+ < c_1 \approx 1.365 < c_1^+ < c_2 \approx 7.718 < c_2^+ < c_3 \approx 47.711 < c_3^+$ (adopted from [8, Fig. 4]).

This article is organized as follows: Section 2 contains statements of the main result. Section 3 contains the proof of the main result.

2. Main result

In this section, we give our main result (Theorem 2.1) for problem (1.1) with $4 \le a < a_1 \approx 4.107$, where classification and evolution of bifurcation curves $\tilde{S}_c$ for (1.1) with varying $c > 0$ are studied. Theorem 2.1 with $4 \le a < a_1$ extends Theorem 1.2 with $a \ge a_1$, and we obtain a more complicated evolution of bifurcation curves $\tilde{S}_c$ with varying $c > 0$. Note that some basic properties and ordering properties of bifurcation curves $\tilde{S}_c$ for positive a and c, on the $(\lambda, \|u\|_\infty)$-plane have been discussed in [8, Theorems 2.1 and 2.2].

Theorem 2.1 (See Figure 4). *Consider (1.1) for any fixed a satisfying $4 \le a < a_1 \approx 4.107$. Then there exist two nonnegative $c_0 = c_0(a) < c_1 = c_1(a)$ satisfying $c_0 > 0$ for $4 \le a < a^* approx 4.069$, $c_0 = 0$ for $a^* \le a < a_1$, and $c_1 > 1.057$ for $4 \le a < a_1$, such that the following assertions (I)–(IV) hold:*

(i) *For $0 < c < c_0$, the bifurcation curve $\tilde{S}_c$ is strictly increasing on the $(\lambda, \|u_\lambda\|_\infty)$-plane. Moreover, there exists a positive λ_0 such that (1.1) has no positive solution for $0 < \lambda < \lambda_0$, and exactly one positive solution for $\lambda \ge \lambda_0$.*

(ii) *For $c = c_0$, the bifurcation curve $\tilde{S}_c$ is monotone increasing on the $(\lambda, \|u_\lambda\|_\infty)$-plane. Moreover, there exists a positive λ_0 such that (1.1) has no positive solution for $0 < \lambda < \lambda_0$, and at least one positive solution for $\lambda \ge \lambda_0$.*

(iii) *(See Figure 1.) For $c_0 < c < c_1$, the bifurcation curve $\tilde{S}_c$ is S-shaped on the $(\lambda, \|u_\lambda\|_\infty)$-plane. More precisely, there exist three positive $c_{1,1} \le c_{1,2} \le c_{1,3}$ on (c_0, c_1), all depending on a, such that the following three assertions hold:*

  (a) *(See Figure 1(i)) If $c_0 < c < c_{1,1}$, then the bifurcation curve $\tilde{S}_c$ is* type 1 *S-shaped on the $(\lambda, \|u\|_\infty)$-plane. Moreover, there exist three positive $\lambda_0 < \lambda_* < \lambda^*$ which are all strictly increasing functions of c on $(c_0, c_{1,1})$ such that (1.1) has no positive solution for $0 < \lambda < \lambda_0$, at least one positive solution for $\lambda_0 \le \lambda < \lambda_*$ and $\lambda > \lambda^*$, at least two positive solutions for $\lambda = \lambda_*$ and $\lambda = \lambda^*$, and at least three positive solutions for $\lambda_* < \lambda < \lambda^*$.*

  (b) *(See Figure 1(ii)) If $c = c_{1,2}$, then the bifurcation curve $\tilde{S}_c$ is* type 2 *S-shaped on the $(\lambda, \|u\|_\infty)$-plane. Moreover, there exist three positive $\lambda_0 = \lambda_* < \lambda^*$ such that (1.1) has no positive solution for $0 < \lambda < \lambda_0$, at least one positive solution for $\lambda > \lambda^*$, at least two positive solutions for $\lambda = \lambda_*$ and $\lambda = \lambda^*$, and at least three positive solutions for $\lambda_* < \lambda < \lambda^*$.*

  (c) *(See Figure 1(iii)) If $c_{1,3} < c < c_1$, then the bifurcation curve $\tilde{S}_c$ is* type 3 *S-shaped on the $(\lambda, \|u\|_\infty)$-plane. Moreover, there exist three positive $\lambda_* < \lambda_0 < \lambda^*$ which are all strictly increasing functions of c on $(c_{1,3}, c_1)$ such that (1.1) has no positive solution for $0 < \lambda < \lambda_*$, at least one positive solution for $\lambda = \lambda_*$ and $\lambda > \lambda^*$, at least two positive solutions for $\lambda^* < \lambda < \lambda_0$ and $\lambda = \lambda^*$, and at least three positive solutions for $\lambda_0 \le \lambda < \lambda^*$.*

(iv) *(See Figure 2) For $c \ge c_1$, the bifurcation curve $\tilde{S}_c$ is $\subset$-shaped on the $(\lambda, \|u\|_\infty)$-plane. Moreover, there exist two positive $\lambda_* < \lambda_0$ such that (1.1) has no positive solution for $0 < \lambda < \lambda_*$, at least one positive solution for $\lambda = \lambda_*$ and $\lambda > \lambda_0$, and at least two positive solutions for $\lambda_* < \lambda \le \lambda_0$.*

Remark 2.2. By Theorem 2.1, we conclude that, on the $(\lambda, \|u_\lambda\|_\infty)$-plane, (i) For $4.069 \approx a^* \le a < a_1 \approx 4.107$, since $c_0 = c_0(a) = 0$, the bifurcation curve $\tilde{S}_c$ evolves from an S-shaped curve to a $\subset$-shaped curve as the evolution parameter varies from 0^+ to ∞, which shows the same evolution for $a \ge a_1$, as claimed in Theorem 1.2. It then implies, by Theorem 1.1, that such evolution is persistent whenever the bifurcation curve S of (1.2) is exactly type 1 S-shaped on the $(\lambda, \|u_\lambda\|_\infty)$-plane; (ii) For $4 \le a < a^*$, since $c_0 > 0$, the bifurcation curve $\tilde{S}_c$ evolves from a strictly increasing curve to a monotone increasing curve, then to an S-shaped curve, and

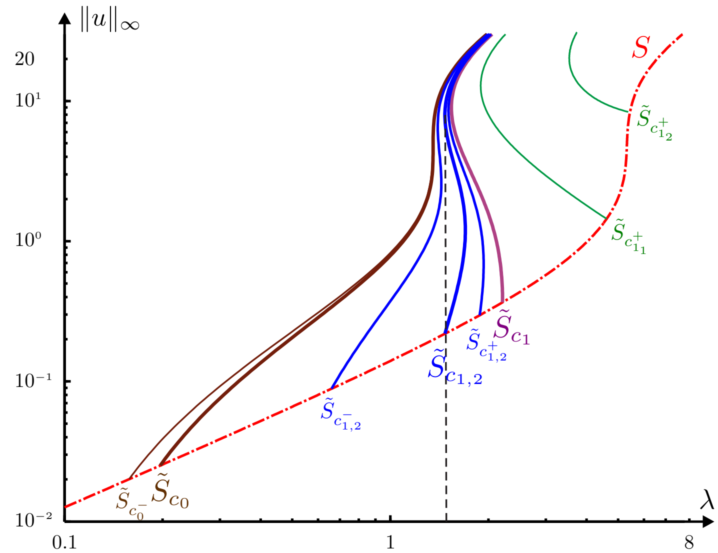

Figure 4. Numerical simulations of bifurcation curves S and $\tilde{S}_c$ for $a = 4$ and varying $c > 0$ on the $(\lambda, \|u\|_\infty)$-plane of the bi-logarithm coordinates. Here $0 < c_0^- < c_0 \approx 0.10 < c_{1,2}^- < c_{1,2} \approx 0.85 < c_{1,2}^+ < c_1 \approx 1.39 < c_{1_1}^+ < c_{1_2}^+$ (adopted from [8, Fig. 7]).

finally to a $\subset$-shaped curve when c varying from 0^+ to ∞. It partially verifies a conjecture on problem (1.1) for $4 \le a < a^*$ proposed in [8, Theorem 2.3] and shows the emergence of more complicated evolution of bifurcation curves $\tilde{S}_c$ with varying $c > 0$.

3. Proof of the main result

To prove our main result (Theorem 2.1) on problem (1.1), we modify time-map technique (the quadrature method) used in [2, 8]. We shall recall some well-developed results in [8]. First, for fixed $a, c > 0$, we define

$$\tilde{H}_c(\rho, q) = 2\int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} - \int_0^q \frac{ds}{\sqrt{F(\rho) - F(s)}} - \frac{c}{\sqrt{F(\rho) - F(q)}} \tag{3.1}$$

for $0 \le q < \rho$, where $f(s) = \exp\left(\frac{as}{a+s}\right)$ and $F(s) = \int_0^s f(t)dt$; see [8, (3.6)]. For fixed $a, c > 0$, let $\rho_0 = \rho_0(c)$ be the unique positive number such that $\tilde{H}_c(\rho_0, 0) = 0$, where the existence and uniqueness of ρ_0 are proved in [8, Lemma 3.2(ii)]. Then it can be proved that, for fixed $a, c > 0$ and $\rho \ge \rho_0$, $\tilde{H}_c(\rho, q)$ has a unique zero $q(\rho, c)$ on $[0, \rho)$; see [8, Lemma 3.2(iv)]. Moreover, the time map formula for mixed boundary value problem (1.1) is defined as

$$H_c(\rho, q(\rho, c)) \equiv \frac{c^2}{2\,[F(\rho) - F(q(\rho, c))]} \quad \text{for } \rho \ge \rho_0(c), \tag{3.2}$$

see [8, (3.26)]. Then it can be easily derived, by similar arguments as given in [2, Theorem 3.3] or [8, (3.26) and (3.27)], that positive solutions u of (1.1) correspond to

$$\|u\|_\infty = \rho \quad \text{and} \quad H_c(\rho, q(\rho, c)) = \lambda. \tag{3.3}$$

Thus studying the shape of the bifurcation curve $\tilde{S}_c$ of (1.1) for $a, c > 0$ is equivalent to studying the shape of the time map $H_c(\rho, q(\rho, c))$ for $\rho \geq \rho_0$.

To prove Theorem 2.1, we need the following Lemmas 3.1–3.4. First, in Lemma 3.1, we record some results on the time map formula $H_c(\rho, q(\rho, c))$ in [8].

Lemma 3.1. *Fix $a \geq 4$ and consider $H_c(\rho, q(\rho, c))$ for $c > 0$ and $\rho \geq \rho_0$. Then the following assertions (i)–(ix) hold:*

(i) [8, Lemma 3.2(iv)] *For $c > 0$, if $0 < \rho < \rho_0(c)$, then $\tilde{H}_c(\rho, q)$ has no zero q on $[0, \rho)$, while if $\rho \geq \rho_0(c)$, then $\tilde{H}_c(\rho, q)$ has a unique zero $q(\rho, c)$ on $[0, \rho)$, that is,*

$$\tilde{H}_c(\rho, q(\rho, c)) = 0. \tag{3.4}$$

Moreover, $q(\rho, c) = 0$ if and only if $\rho = \rho_0(c)$.

(ii) [8, Lemma 3.2(vi)] *For $c > 0$ and $\rho \geq \rho_0$,*

$$0 < \rho - q(\rho, c) \leq \frac{c^2 e^a}{4\rho}. \tag{3.5}$$

(iii) [8, Lemma 3.2(vii)] *$\rho_0(c) \in C(0, \infty)$ is a strictly increasing function of c on $(0, \infty)$.*

(iv) [8, Lemma 3.2(viii)] *For $\rho > 0$, $q(\rho, c) \in C(0, \hat{c}] \cap C^1(0, \hat{c})$ is a strictly decreasing function of c on $(0, \hat{c}]$. Here $\hat{c} = \sqrt{2F(\rho)}\, G(\rho)$.*

(v) [8, Lemma 3.4(i)] *For any two positive numbers $\tilde{c}_1 < \tilde{c}_2$, $H_{\tilde{c}_1}(\rho, q(\rho, \tilde{c}_1)) < H_{\tilde{c}_2}(\rho, q(\rho, \tilde{c}_2))$ for $\rho \geq \rho_0(\tilde{c}_2)$.*

(vi) [8, Lemma 3.5(i)] *There exists a unique positive $c_1 = c_1(a)$ such that*

$$\lim_{\rho \to \rho_0(c)^+} \frac{d}{d\rho} H_c(\rho, q(\rho, c)) \begin{cases} > 0 & \text{when } c \in (0, c_1), \\ = 0 & \text{when } c = c_1, \\ < 0 & \text{when } c \in (c_1, \infty). \end{cases} \tag{3.6}$$

(vii) [8, Lemma 3.5(ii)] *For $c \geq c_1$, there exists $\bar{\rho}(c) > \rho_0(c)$ such that $\frac{d}{d\rho} H_c(\rho, q(\rho, c)) < 0$ for $\rho_0(c) < \rho < \bar{\rho}(c)$.*

(viii) [8, Lemma 3.5(iii)] *For $0 < c < c_1$ and $\rho_0(c) < \rho < \rho_0(c_1)$, $\frac{d}{d\rho} H_c(\rho, q(\rho, c)) > 0$.*

On the other hand, for zero Dirichlet boundary value problem (1.2), its time map formula is defined as

$$G(\rho) \equiv \sqrt{2} \int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} \quad \text{for } \rho > 0, \tag{3.7}$$

see [1, 4, 7]. Then positive solutions u of (1.2) correspond to

$$\|u\|_\infty = \rho \quad \text{and} \quad G(\rho) = \sqrt{\lambda}. \tag{3.8}$$

Thus studying the shape of the bifurcation curve of (1.2) for $a > 0$ is equivalent to studying the shape of the time map $G(\rho)$ on $[0, \infty)$. It is worthwhile to point out that the first term of $\tilde{H}_c(\rho, q)$ defined in the right hand side of (3.1) is equal to $\sqrt{2} G(\rho)$, which implies that $G(\rho)$ has an influence on $H_c(\rho, q(\rho, c))$ (or say that the shape of the bifurcation curve $\tilde{S}_c$ of (1.1) is correlated with the shape of the bifurcation curve S of (1.2).)

In the next Lemma 3.2, we record some results on the relationship between $H_c(\rho, q(\rho, c))$ and $G(\rho)$ in [8].

Lemma 3.2. *Fix $a > 0$ and consider $G(\rho)$ for $\rho > 0$ and $H_c(\rho, q(\rho, c))$ for $\rho \geq \rho_0$ and $c > 0$. Then the following two assertions hold:*

(i) [8, Lemma 3.3(i)] *For $c > 0$ and $\rho \geq \rho_0$, $H_c(\rho, q(\rho, c)) \leq [G(\rho)]^2$, and the equality holds if and only if $\rho = \rho_0$.*
(ii) [8, Lemma 3.6] *If $G'(\rho) \leq 0$ for some $\rho > 0$, then $\frac{d}{d\rho} H_c(\rho, q(\rho, c)) < 0$ for $0 < c < \hat{c}$.*

In the next lemma we record the sign of derivatives of the time map formula $G(\rho)$ for $\rho > 0$ in [3].

Lemma 3.3 ([3, Theorem 4])**.** *Consider (1.2) with varying $a > 0$. There exists a critical bifurcation value $a^* \approx 4.069$ satisfying $4 < a^* < a_1 \approx 4.107$ such that the following three assertions hold:*

(i) *For $0 < a < a^*$, $G'(\rho) > 0$ for all $\rho > 0$.*
(ii) *For $a = a^*$, there exist a unique positive ρ^* such that $G'(\rho^*) = 0$ and $G'(\rho) > 0$ for all $\rho > 0$ and $\rho \neq \rho^*$.*
(iii) *For $a > a^*$, there exist two positive $\bar{\rho}_1 < \bar{\rho}_2$ such that*

$$G'(\rho) \begin{cases} < 0 & \text{when } \rho \in (\bar{\rho}_1, \bar{\rho}_2), \\ = 0 & \text{when } \rho = \bar{\rho}_1 \text{ or } \bar{\rho}_2, \\ > 0 & \text{when } \rho \in (0, \bar{\rho}_1) \cup (\bar{\rho}_2, \infty). \end{cases} \tag{3.9}$$

Lemma 3.4. *Fix $a \geq 4$ and consider $H_c(\rho, q(\rho, c))$ for $\rho \geq \rho_0$ and $c > 0$. Then the following three assertions hold:*

(i) *For any $c > 0$, there exists a positive $\rho_M = \rho_M(a, c) \geq \rho_0$ such that $\frac{d}{d\rho} H_c(\rho, q(\rho, c)) > 0$ for $\rho \geq \rho_M$.*
(ii) *For any two positive numbers $\tilde{c}_1 < \tilde{c}_2$ and $\rho \geq \rho_0(\tilde{c}_2)$, if $\frac{d}{d\rho} H_{\tilde{c}_2}(\rho, q(\rho, \tilde{c}_2)) \geq 0$, then $\frac{d}{d\rho} H_{\tilde{c}_1}(\rho, q(\rho, \tilde{c}_1)) > 0$.*
(iii) *If there exist two positive numbers $\tilde{\rho}_1 < \tilde{\rho}_2$ such that $G'(\rho) > 0$ for $\tilde{\rho}_1 \leq \rho \leq \tilde{\rho}_2$, then there exists a positive $\tilde{c} = \tilde{c}(a)$ such that $\frac{d}{d\rho} H_c(\rho, q(\rho, c)) > 0$ for $\tilde{\rho}_1 \leq \rho \leq \tilde{\rho}_2$ and $0 < c < \tilde{c}$.*

Proof. Note first that, as computed in [8, (3.3), (3.30), (3.31) and the last equation in the proof of Lemma 3.6],

$$\begin{aligned} &\frac{d}{d\rho} H_c(\rho, q(\rho, c)) \\ &= \frac{c^2 f(q(\rho, c))}{2\left[F(\rho) - F(q(\rho, c))\right]^{1/2} \left\{2\left[F(\rho) - F(q(\rho, c))\right] + cf(q(\rho, c))\right\}} \Psi(\rho, q(\rho, c)) \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} \Psi(\rho, q(\rho, c)) &= \sqrt{2} G'(\rho) - 2 \int_{q(\rho,c)}^{\rho} \frac{f'(s) f(\rho)}{[f(s)]^2 \sqrt{F(\rho) - F(s)}} ds \\ &= \int_0^{\rho} \frac{\theta(\rho) - \theta(s)}{\rho \left[F(\rho) - F(s)\right]^{3/2}} ds - 2 \int_{q(\rho,c)}^{\rho} \frac{f'(s) f(\rho)}{[f(s)]^2 \sqrt{F(\rho) - F(s)}} ds \end{aligned}$$

and $\theta(\rho) = 2F(\rho) - \rho f(\rho)$. Hence studying the sign of $\frac{d}{d\rho}H_c(\rho, q(\rho, c))$ is equivalent to studying that of $\Psi(\rho, q(\rho, c))$.

(I) We prove Lemma 3.4(i). For fixed $c > 0$, it can be verified easily that there exists a sufficiently large $\rho_M > c^2e^a$ such that, for $\rho > \rho_M$, the following three inequalities hold:

$$\theta(\rho) - \theta(s) > 0 \quad \text{for } 0 \le s < \rho, \tag{3.11}$$

$$\Big[\frac{3}{2}F(\rho) - \rho f(\rho)\Big] - \Big[\frac{3}{2}F(s) - sf(s)\Big] > 0 \quad \text{for } 0 \le s < \rho, \tag{3.12}$$

$$\rho f(\rho)\frac{f'(s)}{[f(s)]^2} < \frac{1}{4} \quad \text{for } \rho - 1 < s < \rho. \tag{3.13}$$

The proofs of (3.11)–(3.13) are omitted since they are trivial. Then, for $\rho > \rho_M$, we have that $\rho - q(\rho, c) < 1$ by (3.5), and

$$\begin{aligned}
&\Psi(\rho, q(\rho, c)) \\
&= \int_0^\rho \frac{\theta(\rho) - \theta(s)}{\rho\,[F(\rho) - F(s)]^{3/2}}ds - 2\int_{q(\rho,c)}^\rho \frac{f'(s)f(\rho)}{[f(s)]^2\sqrt{F(\rho) - F(s)}}\,ds \\
&> \int_{q(\rho,c)}^\rho \frac{2[1 - \rho f(\rho)\frac{f'(s)}{[f(s)]^2}][F(\rho) - F(s)] - [\rho f(\rho) - sf(s)]}{\rho[F(\rho) - F(s)]^{3/2}}\,ds \quad \text{(by (3.11))} \\
&> \int_{q(\rho,c)}^\rho \frac{\frac{3}{2}[F(\rho) - F(s)] - [\rho f(\rho) - sf(s)]}{\rho\,[F(\rho) - F(s)]^{3/2}}\,ds \ \ \text{(by (3.13))} \\
&> 0
\end{aligned}$$

by (3.12). So Lemma 3.4(i) holds.

(II) We prove Lemma 3.4(ii). Let $\tilde{c}_1 < \tilde{c}_2$ be arbitrary two positive numbers and suppose that $\frac{d}{d\rho}H_{\tilde{c}_2}(\rho, q(\rho, \tilde{c}_2)) \ge 0$ for some $\rho \ge \rho_0(\tilde{c}_2)$. Then, since

$$\frac{\partial}{\partial q}\Psi(\rho, q) = 2\frac{f'(q)f(\rho)}{[f(q)]^2\sqrt{F(\rho) - F(q)}} > 0$$

and $q(\rho, \tilde{c}_1) > q(\rho, \tilde{c}_2)$ for all $\rho \ge \rho_0(\tilde{c}_2)$ by Lemma 3.1(iv), we have

$$\Psi(\rho, q(\rho, \tilde{c}_1)) > \Psi(\rho, q(\rho, \tilde{c}_2)) \ge 0.$$

Consequently, $\frac{d}{d\rho}H_{\tilde{c}_1}(\rho, q(\rho, \tilde{c}_1)) > \frac{d}{d\rho}H_{\tilde{c}_2}(\rho, q(\rho, \tilde{c}_2))$ by (3.10). So Lemma 3.4(ii) holds.

(III) We prove Lemma 3.4(ii). Suppose there exist two positive numbers $\tilde{\rho}_1 < \tilde{\rho}_2$ such that $G'(\rho) > 0$ for $\tilde{\rho}_1 \le \rho \le \tilde{\rho}_2$. Then there exists $\epsilon > 0$ such that $G'(\rho) \ge \epsilon$ for $\tilde{\rho}_1 \le \rho \le \tilde{\rho}_2$. By (3.5), there exists $\tilde{c} > 0$ such that $\rho - q(\rho, c) < \frac{\epsilon^2}{16e^{4a}}$ for $\tilde{\rho}_1 \le \rho \le \tilde{\rho}_2$ and $0 < c \le \tilde{c}$. This implies that

$$\Psi(\rho, q(\rho, c)) \ge \sqrt{2}\epsilon - 2\int_{q(\rho,c)}^\rho \frac{e^{2a}}{\sqrt{\rho - s}}\,ds = \sqrt{2}\epsilon - 4e^{2a}\sqrt{\rho - q(\rho, c)} > 0$$

for $\tilde{\rho}_1 \le \rho \le \tilde{\rho}_2$ and $0 < c \le \tilde{c}$. So Lemma 3.4(iii) holds. The proof is complete. □

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1. **Case 1.** $4 \le a < a^* \approx 4.069$. Define set

$$I = \{c > 0 : \frac{d}{d\rho}H_c(\rho, q(\rho, c)) > 0 \ \text{ on } (\rho_0(c), \infty)\}. \tag{3.14}$$

We first show that I is nonempty. In fact, let c_1 be defined in (3.6) and $\tilde{\rho}_1 = \rho_0(c_1)$. Then, by Lemma 3.1(viii), we have that, for $0 < c < c_1$,

$$\frac{d}{d\rho} H_c(\rho, q(\rho, c)) > 0 \quad \text{on } (\rho_0(c), \tilde{\rho}_1). \tag{3.15}$$

On the other hand, by Lemma 3.4(i)–(ii) and letting $\tilde{\rho}_2 = \rho_M(a, c_1)$, we have that, for $0 < c < c_1$,

$$\frac{d}{d\rho} H_c(\rho, q(\rho, c)) > 0 \quad \text{on } [\tilde{\rho}_2, \infty)\,. \tag{3.16}$$

Moreover, by Lemma 3.3(i) and Lemma 3.4(iii), there exists a positive $\tilde{c}_0 < c_1$ such that, for $0 < c < \tilde{c}_0$, $\frac{d}{d\rho} H_c(\rho, q(\rho, c)) > 0$ on $[\tilde{\rho}_1, \tilde{\rho}_2]$. Hence, for $0 < c < \tilde{c}_0$, $\frac{d}{d\rho} H_c(\rho, q(\rho, c)) > 0$ on $(\rho_0(c), \infty)$ and hence $(0, \tilde{c}_0) \subset I$. So I is nonempty.

Next, we show that I is a finite connected interval. Note that, by Lemma 3.1(vii), when $c \geq c_1$, $\frac{d}{d\rho} H_c(\rho, q(\rho, c)) < 0$ for ρ slightly larger than $\rho_0(c)$. Hence $I \subset (0, c_1)$. Moreover, if there exist $\bar{c} \in (0, c_1)$ such that $\bar{c} \notin I$, then there exists $\bar{\rho} > \rho_0(\bar{c})$ such that $\frac{d}{d\rho} H_{\bar{c}}(\bar{\rho}, q(\bar{\rho}, \bar{c})) \leq 0$. Then, by (3.15), we have that $\bar{\rho} > \tilde{\rho}_1$. It implies, by Lemma 3.4(ii), that, for $c \in (\bar{c}, c_1)$, $\bar{\rho}\,(> \tilde{\rho}_1 = \rho_0(c_1)) > \rho_0(c)$ and $\frac{d}{d\rho} H_c(\bar{\rho}, q(\bar{\rho}, c)) < 0$. Consequently, $(\bar{c}, c_1) \notin I$ and hence I is a finite connected interval.

By the definition of I, above arguments and Lemma 3.1(vii), we obtain that there exists a positive $c_0 < c_1$ such that

$$I = (0, c_0). \tag{3.17}$$

Moreover, when $c = c_0$,

$$\frac{d}{d\rho} H_{c_0}(\rho, q(\rho, c_0)) \geq 0 \ \text{ on } (\rho_0(c_0), \infty), \tag{3.18}$$

and there exists $\tilde{\rho} > \rho_0(c_0)$ such that $\frac{d}{d\rho} H_{c_0}(\tilde{\rho}, q(\tilde{\rho}, c_0)) = 0$. Indeed, such $\tilde{\rho} > \tilde{\rho}_1$ by (3.15). It follows that, by Lemma 3.4(ii), for $c_0 < c < c_1$, $\tilde{\rho}\,(> \tilde{\rho}_1) > \rho_0(c)$ and

$$\frac{d}{d\rho} H_c(\tilde{\rho}, q(\tilde{\rho}, c)) < 0. \tag{3.19}$$

By the relationship between bifurcation curves $\tilde{S}_c$ and the time map H_c from (3.2) and (3.3), we have the following conclusions:

Case (I). For $0 < c < c_0$, that is, $c \in I$, the bifurcation curve $\tilde{S}_c$ is strictly increasing on the $(\lambda, \|u\|_\infty)$-plane since $\frac{d}{d\rho} H_c(\rho, q(\rho, c)) > 0$ on $(\rho_0(c), \infty)$.

Case (II). For $c = c_0$, the bifurcation curve $\tilde{S}_c$ is monotone increasing on the $(\lambda, \|u\|_\infty)$-plane by (3.18).

Case (III). For $c_0 < c < c_1$, the bifurcation curve $\tilde{S}_c$ is S-shaped on the $(\lambda, \|u\|_\infty)$-plane since $\lim_{\rho \to \rho_0(c)^+} \frac{d}{d\rho} H_c(\rho, q(\rho, c)) > 0$ by (3.6), $\frac{d}{d\rho} H_c(\rho, q(\rho, c)) > 0$ on $[\tilde{\rho}_2, \infty)$ by (3.16), and $\frac{d}{d\rho} H_c(\tilde{\rho}, q(\tilde{\rho}, c)) < 0$ by (3.19).

We next show that the S-shaped bifurcation curve $\tilde{S}_c$ could be of either type 1, type 2 or type 3 for some value c on (c_0, c_1).

Case (III)(a). The existence of type 1 S-shaped bifurcation curves $\tilde{S}_c$. Since $\frac{d}{d\rho}H_c(\rho,q(\rho,c))>0$ on $[\tilde{\rho}_2,\infty)$ by (3.16), we have that, for $c_0<c<c_1$,

$$\begin{aligned} \min_{\rho\geq\tilde{\rho}_1} H_c(\rho,q(\rho,c)) &= \min_{\tilde{\rho}_1\leq\rho\leq\tilde{\rho}_2} H_c(\rho,q(\rho,c)) \\ &> \min_{\tilde{\rho}_1\leq\rho\leq\tilde{\rho}_2} H_{c_0}(\rho,q(\rho,c_0)) \quad \text{(by Lemma 3.1(v))} \\ &= H_{c_0}(\tilde{\rho}_1,q(\tilde{\rho}_1,c_0)) \end{aligned} \tag{3.20}$$

by (3.18). On the other hand, by (3.15) and Lemma 3.1(v), we have that

$$\begin{aligned} H_{c_0}(\rho_0(c),q(\rho_0(c),c_0)) &< H_{c_0}(\tilde{\rho}_1,q(\tilde{\rho}_1,c_0)) \\ &< H_{c_1}(\tilde{\rho}_1,q(\tilde{\rho}_1,c_1)) = H_{c_1}(\rho_0(c),q(\rho_0(c),c_1)). \end{aligned}$$

Consequently, by the intermediate value theorem, there exists $c_{1,1}\in(c_0,c_1)$ such that

$$H_{c_{1,1}}(\rho_0(c_{1,1}),q(\rho_0(c_{1,1}),c_{1,1})) = H_{c_0}(\tilde{\rho}_1,q(\tilde{\rho}_1,c_0)). \tag{3.21}$$

Hence, for $0<c<c_{1,1}$,

$$\begin{aligned} H_c(\rho_0(c),q(\rho_0(c),c)) &= G(\rho_0(c)) \quad \text{(by Lemma 3.2(i))} \\ &< G(\rho_0(c_{1,1})) \quad \text{(by Lemma 3.3(i) and Lemma 3.1(iii))} \\ &= H_{c_{1,1}}(\rho_0(c_{1,1}),q(\rho_0(c_{1,1}),c_{1,1})) \quad \text{(by Lemma 3.2(i))} \\ &= H_{c_0}(\tilde{\rho}_1,q(\tilde{\rho}_1,c_0)) \quad \text{(by (3.21))} \\ &< \min_{\rho\geq\tilde{\rho}_1} H_c(\rho,q(\rho,c)) \end{aligned}$$

by (3.20). It then follows, by (3.15), that

$$H_c(\rho_0(c),q(\rho_0(c),c)) < H_c(\rho,q(\rho,c))$$

for $\rho>\rho_0(c)$. It implies that, for $0<c\leq c_{1,1}$, the S-shaped bifurcation curve $\tilde{S}_c$ is of type 1 on the $(\lambda,\|u\|_\infty)$-plane.

Case (III)(b). The existence of type 3 S-shaped bifurcation curves $\tilde{S}_c$. The proof of this part is the same as that given in [8, Proof of Theorem 2.4, Cases (i)(b)] and hence the proof is omitted.

Case (III)(c). The existence of a type 2 S-shaped bifurcation curve $\tilde{S}_c$. The proof of this part is the same as that given in [8, Proof of Theorem 2.4, Case (i)(c)] and hence the proof is omitted.

Case (IV). For $c>c_1$, the bifurcation curve $\tilde{S}_c$ is $\subset$-shaped on the $(\lambda,\|u\|_\infty)$-plane since $\lim_{\rho\to\rho_0(c)^+}\frac{d}{d\rho}H_c(\rho,q(\rho,c))<0$ by (3.6) and since $\frac{d}{d\rho}H_c(\rho,q(\rho,c))>0$ for $\rho\geq\rho_M(a,c)$ by Lemma 3.4(i).

Case 2. $a=a^*\approx 4.069$. Let ρ^* be the unique positive number such that $G'(\rho^*)=0$ as defined in Lemma 3.3(ii). Then, for $c>0$, $\frac{d}{d\rho}H_c(\rho^*,q(\rho^*,c))<0$ by Lemma 3.2(ii). Hence the bifurcation curve $\tilde{S}_c$ must not be monotone increasing on the $(\lambda,\|u\|_\infty)$-plane. Or equivalently, $c_0=0$ if we similarly define $I=(0,c_0)$ as in (3.14) and (3.17) in Case 1. The remaining parts of the proof in this case followed by similar arguments stated in above Case 1 and hence they are omitted here.

Case 3. $a^*<a<a_1$. Note that, by Lemma 3.3(iii), Equation (3.9) holds for all $a>a^*$. Thus the proof of this part followed by same arguments given as in [8, Proof of Theorem 2.4] and hence the proof is omitted here.

Finally, we remark that the proof of the estimation of $c_1>1.057$ for $4\leq a<a_1$ is the same as the one computed in [8, Proof of Theorem 2.4, part (III)] and the

multiplicity result of positive solutions for (1.1) in each case follows immediately from the definition of shapes of bifurcations curves, see e.g., Figures 1 and 2. The proof is complete. □

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