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EXISTENCE OF POSITIVE SOLUTIONS FOR BREZIS-NIRENBERG TYPE PROBLEMS INVOLVING AN INVERSE OPERATOR

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ABSTRACT. This article concerns the existence of positive solutions for the second order equation involving a nonlocal term

$$-\Delta u = \gamma(-\Delta)^{-1}u + |u|^{p-1}u,$$

under Dirichlet boundary conditions. We prove the existence of positive solutions depending on the positive real parameter $\gamma > 0$, and up to the critical value of the exponent p, i.e. when $1 < p \le 2^* - 1$, where $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent. For $p = 2^* - 1$, this leads us to a Brezis-Nirenberg type problem, cf. [5], but, in our particular case, the linear term is a nonlocal term. The effect that this nonlocal term has on the equation changes the dimensions for which the classical technique based on the minimizers of the Sobolev constant, that ensures the existence of positive solution, going from dimensions $N \ge 4$ in the classical Brezis-Nirenberg problem, to dimensions $N \ge 7$ for this nonlocal problem.

1. Introduction

In this work, we analyze the existence of positive solutions of the second order elliptic equation under homogeneous Dirichlet boundary conditions and involving a non-local term,

$$\begin{gathered} -\Delta u = \gamma(-\Delta)^{-1}u + |u|^{p-1}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where γ is a positive real parameter and Ω is a smooth bounded domain of $\mathbb{R}^N$, with $N \ge 3$, $1 < p \le 2^* - 1$, where $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent. For $p = 2^* - 1$, this problem is critical and there is an important issue with the dimension N. Indeed, we will ascertain the existence of positive solutions at the critical exponent for dimensions $N \ge 7$, following similar arguments to those used by Brezis-Nirenberg [5]. Let us first observe that, at the critical exponent $p = 2^* - 1$, problem (1.1) can be seen as a linear perturbation of the critical problem

$$\begin{gathered} -\Delta u = |u|^{2^*-2}u \quad \text{in } \Omega \subset \mathbb{R}^N, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.2}$$

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for which, after applying the well-known result by Pohozaev [11], one can prove the non-existence of positive solutions under the star-shapeness assumption on the domain Ω. Moreover, the classical Brezis-Nirenberg problem

$$\begin{gathered} -\Delta u = \gamma u + |u|^{2^*-2}u \quad \text{in } \Omega \subset \mathbb{R}^N, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.3}$$

can be seen as well as a linear perturbation of problem (1.2). In their pioneering paper [5] it was proved that, for $N \geq 4$, there exists a positive solution of (1.3) if and only if the parameter γ belongs to the interval $(0, \lambda_1)$, being λ_1 the first eigenvalue for the Laplacian operator under homogeneous Dirichlet boundary conditions. Note that, in our situation, the non-local term $\gamma(-\Delta)^{-1}u$ plays actually the role of γu in (1.3). Here, we just arrive at the existence of positive solutions for $N \geq 7$.

Our main motivation to study (1.1) comes from the fourth-order equation, under homogeneous Navier boundary conditions,

$$\begin{gathered} (-\Delta)^2 u = \gamma u + (-\Delta)|u|^{p-1}u \quad \text{in } \Omega \subset \mathbb{R}^N, \\ u = 0 = \Delta u \quad \text{on } \partial\Omega. \end{gathered} \tag{1.4}$$

Thus, positive solutions of (1.4) can be seen as positive steady-state solutions of the fourth-order parabolic Cahn–Hilliard type equation,

$$\frac{\partial u}{\partial t} + (-\Delta)^2 u = \gamma u + (-\Delta)|u|^{p-1}u, \quad \text{in } \Omega \times \mathbb{R}_+,$$

assuming bounded smooth initial data $u(x, 0) = u_0(x)$. The latter equation has been previously studied in [2, 1] either for bounded domains or the whole $\mathbb{R}^N$, where the authors proved existence and multiplicity of solutions in the subcritical range $1 < p < 2^* - 1$. However, those solutions could change sign.

In this work we study the existence of positive solutions for both subcritical case, and the critical one with $p = 2^*-1$ which creates many more difficulties to deal with than the subcritical case. Let us recall that, thanks to the Sobolev's Embedding Theorem, we have the compact embedding

$$H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega), \tag{1.5}$$

for $1 \leq p < 2^* - 1$, being a continuous embedding up to the critical exponent $p = 2^* - 1$. Thus, there exists a positive constant $C := C(N, p)$ such that

$$\|u\|_{L^{p+1}(\Omega)} \leq C\|u\|_{H_0^1(\Omega)}, \tag{1.6}$$

for any $u \in H_0^1(\Omega)$ and $1 \leq p \leq 2^* - 1$.

Note that here, for the fourth-order elliptic problem (1.4), the Sobolev's critical exponent is $2^* = \frac{2N}{N-2}$, since this operator has the representation,

$$(-\Delta)^2 u - (-\Delta)|u|^{p-1}u = (-\Delta)((-\Delta)u - |u|^{p-1}u),$$

so that, the necessary embedding features are governed by a standard second-order equation,

$$-\Delta u = |u|^{p-1}u,$$

justifying the choice for our critical exponent and the relation between (1.1) and (1.4). We must also observe that this is different from the usual critical problems with a bi-Laplacian operator

$$(-\Delta)^2 u = \gamma u + |u|^{p-1}u, \tag{1.7}$$

analyzed by Pucci-Serrin [12], under homogeneous Dirichlet boundary conditions, where the Sobolev's critical exponent is $p_S = \frac{2N}{N-4}$; see also the interesting book by Gazzola-Grunau-Sweers [8].

Note that (1.4) is not a variational problem. Therefore, to arrive at the desired results of existence of positive solutions applying any variational method we might easily apply the inverse operator $(-\Delta)^{-1}$ of the Laplacian to problem (1.4), transforming it into the variational problem (1.1). Hence, establishing a direct connection between the fourth-order equation (1.4) under homogeneous Navier boundary conditions and the non-local elliptic Dirichlet problem (1.1). Since the second order problem with the inverse operator (1.1) is of variational type, it has an associated energy functional

$$\mathcal{F}_\gamma(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\gamma}{2}\int_\Omega u(-\Delta)^{-1}u\, dx - \frac{1}{p+1}\int_\Omega |u|^{p+1} dx, \tag{1.8}$$

so that solutions of (1.1) can be obtained as critical points of the Fréchet-differentiable functional $\mathcal{F}_\gamma$ defined by (1.8). Here, as customary $(-\Delta)^{-1}u = v$, if

$$-\Delta v = u \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega.$$

Thus, $(-\Delta)^{-1}$ is a positive linear integral compact operator from $L^2(\Omega)$ into itself, which is well-defined thanks to the Spectral Theorem.

Moreover, we can rewrite the functional (1.8) as

$$\mathcal{F}_\gamma(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\gamma}{2}\int_\Omega |(-\Delta)^{-1/2}u|^2 dx - \frac{1}{p+1}\int_\Omega |u|^{p+1} dx.$$

As a main result in this work, we prove the existence of positive solutions of problem (1.1) depending on the positive parameter γ. To do so, we first show the interval of the parameter γ for which there is the possibility of having positive solutions. Next, applying the well-known Mountain Pass Theorem (MPT for short) [4], we show that for the range $1 < p \leq 2^*-1$ there actually exists a positive solution to problem (1.1) provided

$$0 < \gamma < \lambda_1^*,$$

where λ_1^* is the first eigenvalue of the operator $(-\Delta)^2$ under homogeneous Navier boundary conditions, i.e., $\lambda_1^* = \lambda_1^2$ with λ_1 being the first eigenvalue for the Laplacian under homogeneous Dirichlet boundary conditions. In the subcritical case ($1 < p < 2^* - 1$) one might apply the MPT to (1.1) directly since, as we will show, our problem possesses the Mountain Pass (MP) geometry and, thanks to the compact embedding (1.5), the Palais-Smale condition is satisfied for the functional $\mathcal{F}_\gamma$ (see details below in Section 2). On the other hand, at the critical exponent $p = 2^* - 1$, the compactness of the Sobolev embedding is lost. Then, to check whether the Palais-Smale condition is satisfied becomes a delicate issue solving it. To overcome this lack of compactness we apply techniques based on the Concentration-Compactness Principle due to P.-L. Lions, [9], which allow us to prove the required Palais-Smale condition for $N \geq 7$. Now we state the main results of this paper.

Theorem 1.1. *For every $\gamma \in (0, \lambda_1^*)$ there exists a positive solution u of problem* (1.1) *if:*

- (i) $1 < p < 2^* - 1$ *and* $N \geq 3$,
- (ii) $p = 2^* - 1$ *and* $N \geq 7$.

Surprisingly, even though our problem (1.1) is non-local but also a linear perturbation of the problem (1.2), Theorem 1.1-(ii) addresses dimensions $N \geq 7$, in contrast to the existence result of Brezis and Nirenberg about the linear perturbation (1.3), that covers the wider range $N \geq 4$. In other words, the non-local term $\gamma(-\Delta)^{-1}u$, despite of being just a linear perturbation, has an important effect on the dimensions for which the classical Brezis-Nirenberg technique based on the minimizers of the Sobolev constant still works.

Considering problem (1.7), Pucci-Serrin [12] introduced the concept of critical dimensions as the dimensions for which there are no positive solutions of (1.7) under homogeneous Dirichlet boundary conditions with $\gamma > 0$ arbitrarily small. They also conjectured that the critical dimensions for the corresponding problem with a polyharmonic operator $(-\Delta)^m$ are $N = 2m+1, \dots, 4m-1$. Combining the result of Pucci and Serrin [12] with existence theorems of Edmunds, Fortunato, Jannelli [6] and Noussair, Swanson, Yang Jianfu [10], the conjecture is true for $m = 2$. Moreover, because the connection between problem (1.1) and the fourth order problem (1.4), through Theorem 1.1-(ii) combined with Remark 2.5, we also prove the existence of positive solutions for (1.4). Note that in our case $N = 7$ is not a critical dimension anymore, in contrast with what happens for problem (1.7). It seems that the term $(-\Delta)|u|^{p-1}u$ behaves better than the pure power $|u|^{p-1}u$ with respect to the critical dimensions, despite having a worst behavior in the second order problem with the nonlocal term, (1.1).

Alternative approach. We have also a relevant connection between problem (1.4) and a second order elliptic system through problem (1.1) to be explored at the end of the paper. In particular, taking $w := (-\Delta)^{-1}u$, problem (1.1) provides us with the system

$$\begin{gathered} -\Delta u = \gamma w + |u|^{p-1}u \quad \text{in } \Omega, \\ -\Delta w = u \quad \text{in } \Omega, (u,w) = (0,0) \quad \text{on } \partial\Omega, \end{gathered} \tag{1.9}$$

which gives a different perspective to the problem in hand but providing similar results to the ones previously obtained in Theorem 1.1. However, since system (1.9) is not of variational type, as $\gamma > 0$, we can take $v := \sqrt{\gamma}w$ in (1.9) and obtain the variational system

$$\begin{gathered} -\Delta u = \sqrt{\gamma}v + |u|^{p-1}u \quad \text{in } \Omega,, \\ -\Delta v = \sqrt{\gamma}u \quad \text{in } \Omega, \\ (u,v) = (0,0) \quad \text{on } \partial\Omega, \end{gathered} \tag{1.10}$$

whose associated energy functional is

$$\mathcal{J}_\gamma(u,v) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \sqrt{\gamma}\int_\Omega uv\,dx - \frac{1}{p+1}\int_\Omega |u|^{p+1} dx. \tag{1.11}$$

Remark 1.2. Thanks to the Maximum Principle, for a given $u(x)$ as a positive solution of (1.1), and setting $v = \sqrt{\gamma}(-\Delta)^{-1}u$, it follows that $v > 0$. Thus, the pair $(u,v) = (u, \sqrt{\gamma}(-\Delta)^{-1}u)$ is a positive solution of (1.10). And vice versa, given (u,v) a positive solution of (1.10) it is trivial that $u(x)$ is a positive solution of (1.1).

Although the equivalence between system (1.10) and the non-local problem (1.1) provides us with existence results for system (1.10) by means of Theorem 1.1, we prove independently the following.

Theorem 1.3. *For every $\gamma \in (0, \lambda_1^*)$ there exists a positive solution (u, v) of system* (1.10) *if:*

(i) $1 < p < 2^* - 1$ *and* $N \geq 3$,
(ii) $p = 2^* - 1$ *and* $N \geq 7$.

In the last section of the paper we extend our study to a higher order problem and prove, under analogous hypotheses, that there exists a positive solution of problem

$$\begin{gathered} -\Delta u = \gamma(-\Delta)^{-m} u + |u|^{p-1} u \quad \text{in } \Omega \subset \mathbb{R}^N, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.12}$$

We recall the following well-known facts about polyharmonic operators of order $2m$ ($m \geq 1$ an integer number) in a smooth domain Ω. The Navier boundary conditions for the operator $(-\Delta)^m$ are defined as

$$u = \Delta u = \Delta^2 u = \ldots = \Delta^{k-1} u = 0, \quad \text{on } \partial\Omega.$$

Clearly, the operator $(-\Delta)^m$ is the m-th power of the classical Dirichlet Laplacian in the sense of the spectral theory and it can be defined as the operator whose action on a function u is given by

$$\langle (-\Delta)^m u, u \rangle = \sum_{j \geq 1} \lambda_j^m |\langle u_1, \varphi_j \rangle|^2,$$

where (φ_i, λ_i) are the eigenfunctions and eigenvalues of the Laplace operator $(-\Delta)$ with homogeneous Dirichlet boundary data. Thus, the operator $(-\Delta)^m$ is well defined in the space of functions that vanish on the boundary,

$$H_0^m(\Omega) = \Big\{ u = \sum_{j=1}^{\infty} a_j \varphi_j \in L^2(\Omega) : \|u\|_{H_0^m(\Omega)} = \Big(\sum_{j=1}^{\infty} a_j^2 \lambda_j^m \Big)^{1/2} < \infty \Big\}.$$

Because of the lack of a comparison principle for higher-order equations, to obtain the existence results dealing with (1.12) we can not tackle this problem directly, and we need to use a similar correspondence to the one performed above for the problem (1.4), now with an elliptic system of $m + 1$ equations.

This article is organized as follows: In Section Section 2, we prove results for problem (1.1); and using similar ideas, for system (1.10) in Section 3. To finish, in Section 4 we study system (1.12).

2. Existence of positive solutions for (1.4) via problem (1.1)

In this section we prove Theorem 1.1. First, we establish a condition on the range of values of γ necessary for the existence of positive solutions of (1.1). Let us consider the generalized eigenvalue problem associated with (1.1),

$$\begin{gathered} -\Delta u = \lambda(-\Delta)^{-1} u \quad \text{in } \Omega \subset \mathbb{R}^N, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.1}$$

Then, we find that for the first eigenfunction φ_1 associated with the first eigenvalue λ_1^* in (2.1),

$$\int_\Omega |\nabla \varphi_1|^2 dx = \lambda_1^* \int_\Omega |(-\Delta)^{-1/2} \varphi_1|^2 dx, \quad \text{with } \varphi_1 \in H_0^1(\Omega),$$

and, hence,

$$\lambda_1^* = \inf_{u\in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |(-\Delta)^{-1/2}u|^2 dx}. \tag{2.2}$$

On the other hand, it is clear that substituting the first eigenfunction of the Laplace operator under homogeneous Dirichlet boundary conditions, φ_1, into (2.1), it follows that $\lambda_1^* = \lambda_1^2$. Thus, by definition of the powers of the Laplace operator, λ_1^* coincides with the first eigenvalue of the operator $(-\Delta)^2$ under homogeneous Navier boundary conditions. Moreover, the first eigenfunction of (2.1) coincides with the first eigenfunction of the Laplace operator under homogeneous Dirichlet boundary conditions. Now, we prove the following result.

Lemma 2.1. *Problem* (1.1) *does not possess a positive solution when* $\gamma \geq \lambda_1^*$.

Proof. Assume that u is a positive solution to (1.1) and let φ_1 be the principal eigenfunction of the Laplacian operator in Ω under homogeneous Dirichlet boundary conditions. Taking φ_1 as a test function for the equation of (1.1) we obtain

$$\int_\Omega \varphi_1(-\Delta)u\,dx = \gamma\int_\Omega \varphi_1(-\Delta)^{-1}u\,dx + \int_\Omega |u|^{p-1}u\varphi_1\,dx > \gamma\int_\Omega \varphi_1(-\Delta)^{-1}u\,dx.$$

Then, integrating by parts on both sides,

$$\lambda_1\int_\Omega u\varphi_1\,dx > \gamma\int_\Omega u(-\Delta)^{-1}\varphi_1\,dx = \frac{\gamma}{\lambda_1}\int_\Omega u\varphi_1\,dx.$$

Hence, $\gamma < \lambda_1^2 = \lambda_1^*$. □

Lemma 2.2. *The functional* $\mathcal{F}_\gamma$ *defined by* (1.8) *has the mountain pass geometry.*

Proof. Without loss of generality we can take a function $g \in H_0^1(\Omega)$ such that $\|g\|_{L^{p+1}(\Omega)} = 1$. Then, taking a real number $t > 0$ and applying the Sobolev inequality (1.6) together with (2.2), we find that

$$\begin{aligned}
\mathcal{F}_\gamma(tg) &= \frac{t^2}{2}\int_\Omega |\nabla g|^2 dx - \frac{t^2\gamma}{2}\int_\Omega |(-\Delta)^{-\frac12}g|^2 dx - \frac{t^{p+1}}{p+1} \\
&\geq \frac{t^2}{2}\Big(1-\frac{\gamma}{\lambda_1^*}\Big)\int_\Omega |\nabla g|^2 dx - \frac{t^{p+1}}{p+1} \\
&\geq \Big(\frac12\Big(1-\frac{\gamma}{\lambda_1^*}\Big)t^2 - \frac{C}{(p+1)}t^{p+1}\Big)\int_\Omega |\nabla g|^2 dx > 0,
\end{aligned}$$

for t small enough, i.e., $0 < t^{p-1} < \frac{p+1}{2C}\big(1-\frac{\gamma}{\lambda_1^*}\big)$. Thus, the functional $\mathcal{F}_\gamma$ has a local minimum at $u = 0$, i.e.

$$\mathcal{F}_\gamma(tg) > \mathcal{F}_\gamma(0) = 0,$$

for any $g \in H_0^1(\Omega)$ provided $t > 0$ is small enough. Also, it is clear that

$$\begin{aligned}
\mathcal{F}_\gamma(tg) &= \frac{t^2}{2}\int_\Omega |\nabla g|^2 dx - \frac{\gamma t^2}{2}\int_\Omega |(-\Delta)^{-1/2}g|^2 dx - \frac{t^{p+1}}{p+1} \\
&\leq \frac{t^2}{2}\|g\|^2_{H_0^1(\Omega)} - \frac{t^{p+1}}{p+1}.
\end{aligned}$$

Then, $\mathcal{F}_\gamma(tg) \to -\infty$ as $t \to \infty$ and, thus, there exists $\hat{u} \in H_0^1(\Omega)$ such that $\mathcal{F}_\gamma(\hat{u}) < 0$. □

Now we turn our attention to the so-called Palais-Smale condition.

Definition 2.3. Let V be a Banach space. We say that a sequence $\{u_n\} \subset V$ is a PS sequence for a functional $\mathfrak{F}$ if

$$\mathfrak{F}(u_n) \text{ is bounded and } \mathfrak{F}'(u_n) \to 0 \text{ in } V' \text{ as } n \to \infty, \tag{2.3}$$

where V' is the dual space of V. Moreover, we say that a PS sequence $\{u_n\} \subset V$ satisfies a PS condition if

$$\{u_n\} \text{ has a convergent subsequence.} \tag{2.4}$$

In particular, given a PS sequence $\{u_n\} \subset V$ such that $\mathfrak{F}(u_n) \to c$, if (2.4) is satisfied, we will say that the PS sequence satisfies the PS condition at level c for the functional $\mathfrak{F}$. Moreover, we say that the functional $\mathfrak{F}$ satisfies the PS condition at level c if every PS sequence at level c for $\mathfrak{F}$ possesses a convergent subsequence in V.

For our problem, in the subcritical range the PS condition is always satisfied at any level c because of the compact Sobolev embedding (1.5). However, at the critical exponent 2^* the problem is further complicated because of the lack of compactness in the Sobolev embedding. We will overcome this issue applying an argument based on the Concentration-Compactness Principle developed by P.-L. Lions, [9], proving that the functional $\mathcal{F}_\gamma$ satisfies the PS condition for levels c below a certain critical value c^* (to be determined).

Lemma 2.4. *Let $\{u_n\}$ be a PS sequence at level c for the functional $\mathcal{F}_\gamma$, i.e. $\mathcal{F}_\gamma(u_n) \to c$ and $\mathcal{F}'_\gamma(u_n) \to 0$ as $n \to \infty$. Then $\{u_n\}$ is bounded in $H_0^1(\Omega)$.*

Proof. Since $\mathcal{F}'_\gamma(u_n) \to 0$ in $\big(H_0^1(\Omega)\big)'$, in particular $\big\langle \mathcal{F}'_\gamma(u_n) | \frac{u_n}{\|u_n\|_{H_0^1(\Omega)}} \big\rangle \to 0$. Thus, for any $\varepsilon > 0$ there exists a subsequence, denoted again by $\{u_n\}$, such that

$$\int_\Omega |\nabla u_n|^2 dx - \gamma \int_\Omega |(-\Delta)^{-1/2} u_n|^2 dx - \int_\Omega |u_n|^{p+1} dx = \|u_n\|_{H_0^1(\Omega)} \cdot o(1).$$

Moreover, since $\mathcal{F}_\gamma(u_n) \to c$,

$$\frac{1}{2} \int_\Omega |\nabla u_n|^2 dx - \frac{\gamma}{2} \int_\Omega |(-\Delta)^{-1/2} u_n|^2 dx - \frac{1}{p+1} \int_\Omega |u_n|^{p+1} dx = c + o(1),$$

for n large enough. Therefore, for a positive constant μ (to be determined below) we find that

$$\mathcal{F}_\gamma(u_n) - \mu \big\langle \mathcal{F}'_\gamma(u_n) | \frac{u_n}{\|u_n\|_{H_0^1(\Omega)}} \big\rangle = c + \|u_n\|_{H_0^1(\Omega)} \cdot o(1).$$

That is

$$\begin{aligned}&\big(\frac{1}{2} - \mu\big) \int_\Omega |\nabla u_n|^2 dx - \gamma \big(\frac{1}{2} - \mu\big) \int_\Omega |(-\Delta)^{-1/2} u_n|^2 dx - \big(\frac{1}{p+1} - \mu\big) \int_\Omega |u_n|^{p+1} dx \\ &= c + \|u_n\|_{H_0^1(\Omega)} \cdot o(1).\end{aligned}$$

Hence, taking μ such that $\frac{1}{p+1} < \mu < \frac{1}{2}$,

$$\big(\frac{1}{2} - \mu\big) \int_\Omega |\nabla u_n|^2 dx - \big(\frac{1}{2} - \mu\big) \gamma \int_\Omega |(-\Delta)^{-1/2} u_n|^2 dx \le c + \|u_n\|_{H_0^1(\Omega)} \cdot o(1),$$

and using (2.2),

$$\big(\frac{1}{2} - \mu\big) \big(1 - \frac{\gamma}{\lambda_1^*}\big) \int_\Omega |\nabla u_n|^2 dx \le \big(\frac{1}{2} - \mu\big) \int_\Omega |\nabla u_n|^2 dx$$

$$
\begin{aligned}
&+ 2mu - \big(\frac{1}{2}-\mu\big)\gamma\int_\Omega |(-\Delta)^{-1/2}u_n|^2 dx\\
&\le c + \|u_n\|_{H_0^1(\Omega)}\cdot o(1).
\end{aligned}
$$

From here, we conclude that

$$
\big(\frac{1}{2}-\mu\big)\big(1-\frac{\gamma}{\lambda_1^*}\big)\|u_n\|^2_{H_0^1(\Omega)} \le c + \|u_n\|_{H_0^1(\Omega)}\cdot o(1).
$$

Since $0<\gamma<\lambda_1^*$, it follows that $\big(\frac{1}{2}-\mu\big)\big(1-\frac{\gamma}{\lambda_1^*}\big)>0$ and, thus, because of the former inequality we conclude that the sequence $\{u_n\}$ is bounded in $H_0^1(\Omega)$. $\square$

Proof of Theorem 1.1-(i). Let us consider the subcritical case $1<p<2^*-1$. Given a PS sequence $\{u_n\}\subset H_0^1(\Omega)$ at level c, by Lemma 2.4 and the Rellich-Kondrachov Theorem the PS condition is satisfied. Hence, the functional $\mathcal{F}_\gamma$ satisfies the PS condition. Moreover, by Lemma 2.2 the functional $\mathcal{F}_\gamma$ possesses the MP geometry. Therefore, the hypotheses of the mountain pass theorem are fulfilled and we conclude that the functional $\mathcal{F}_\gamma$ possesses a critical point $u\in H_0^1(\Omega)$. Moreover, if we define the set of paths

$$
\Gamma := \{g\in C([0,1],H_0^1(\Omega)) : g(0)=0,\ g(1)=\hat{u}\},
$$

with $\hat{u}$ given as in the proof of Lemma 2.2, then

$$
\mathcal{F}_\gamma(u) = c := \inf_{g\in\Gamma}\max_{\theta\in[0,1]}\mathcal{F}_\gamma(g(\theta)).
$$

To show that $u>0$, let us consider the functional,

$$
\mathcal{F}_\gamma^+(u) = \mathcal{F}_\gamma(u^+),
$$

where $u^+=\max\{u,0\}$. Repeating the arguments carried out above, with minor changes, one readily shows that what was proved for the functional $\mathcal{F}_\gamma$ still holds for the functional $\mathcal{F}_\gamma^+$. Therefore, $u\ge 0$ and thanks to the Maximum Principle, $u>0$. $\square$

Remark 2.5. Assuming that $\partial\Omega$ is a $\mathcal{C}^2$ manifold, by standard elliptic regularity theory, [7, Sec. 8.3, Theorem 1], it follows that $u\in H_0^1(\Omega)\cap H^2(\Omega)$ and thus, u is a positive weak solution of (1.4).

2.1. **Concentration-compactness for the non-local problem** (1.1)**.** In this subsection we focus on the critical exponent case, $p=2^*-1$, and our aim is to prove the PS condition for the functional $\mathcal{F}_\gamma$. We carry out this task by means of a concentration-compactness argument based on the following lemma.

Lemma 2.6 (P.-L. Lions,[9])**.** *Let $\{u_n\}$ be a weakly convergent sequence to u in $H_0^1(\Omega)$. Let μ, and ν be two nonnegative measures such that*

$$
|\nabla u_n|^2 \to \mu \quad \text{and} \quad |u_n|^{2^*}\to\nu \quad \text{as } n\to\infty.
$$

Then, there exist a countable set I of points $\{x_j\}_{j\in I}\subset\overline{\Omega}$ and some positive numbers μ_j, and ν_j such that

$$
\begin{aligned}
|\nabla u_n|^2 &\rightharpoonup \mu = |\nabla u_0|^2 + \sum_{j\in I}\mu_j\delta_{x_j},\\
|u_n|^{2^*} &\rightharpoonup \nu = |u_0|^{2^*} + \sum_{j\in I}\nu_j\delta_{x_j},
\end{aligned}
\tag{2.5}
$$

where δ_{x_j} is the Dirac's delta centered at x_j and satisfying

$$\mu_j \geq S_N \nu_j^{2/2^*}. \tag{2.6}$$

Lemma 2.7. *Assume $p = 2^* - 1$. Then, the functional $\mathcal{F}_\gamma$ satisfies the Palais-Smale condition for any level c such that,*

$$c < c^* = \frac{1}{N} S_N^{N/2}.$$

Proof. Although the proof is rather standard we include it for the sake of completeness. Let $\{u_n\} \subset H_0^1(\Omega)$ be a PS sequence of level $c < c^*$ for the functional $\mathcal{F}_\gamma$. Thanks to Lemma 2.4, the sequence $\{u_n\}$ is uniformly bounded and, as a consequence, we can assume that, up to a subsequence,

$$\begin{gathered} u_n \rightharpoonup u_0 \quad \text{weakly in } \; H_0^1(\Omega), \\ u_n \to u_0 \quad \text{strongly in } L^q(\Omega), 1 \leq q < 2^*, \\ u_n \to u_0 \quad \text{a.e. in } \Omega. \end{gathered} \tag{2.7}$$

Next, for $j \in I$ and $\varepsilon > 0$, let $\varphi_{j,\varepsilon} \in C_0^\infty(\Omega)$ be a cut-off function such that

$$\varphi_{j,\varepsilon} = 1 \text{ in } B_\varepsilon(x_j), \quad \varphi_{j,\varepsilon} = 0 \text{ in } B_{2\varepsilon}^c(x_j), \quad |\nabla \varphi_{j,\varepsilon}| \leq \frac{2}{\varepsilon}, \tag{2.8}$$

where $B_r(x_j)$ is the ball of radius $r > 0$, centered at a point $x_j \in \overline{\Omega}$. Thus, using $\varphi_{j,\varepsilon} u_n$ as a test function we find that

$$\begin{aligned} &\langle \mathcal{F}'_\gamma(u_n) | \varphi_{j,\varepsilon} u_n \rangle \\ &= \int_\Omega \nabla u_n \cdot \nabla(\varphi_{j,\varepsilon} u_n) dx - \gamma \int_\Omega \varphi_{j,\varepsilon} u_n (-\Delta)^{-1} u_n \, dx - \int_\Omega \varphi_{j,\varepsilon} |u_n|^{2^*} dx \\ &= \int_\Omega \varphi_{j,\varepsilon} |\nabla u_n|^2 dx - \int_\Omega \varphi_{j,\varepsilon} |u_n|^{2^*} dx + \int_\Omega u_n \nabla u_n \cdot \nabla \varphi_{j,\varepsilon} dx \\ &\quad - \gamma \int_\Omega \varphi_{j,\varepsilon} u_n (-\Delta)^{-1} u_n \, dx. \end{aligned}$$

Moreover, from (2.5) and (2.7), we hve

$$\begin{aligned} \lim_{n \to \infty} \langle \mathcal{F}'_\gamma(u_n) | \varphi_{j,\varepsilon} u_n \rangle = &\int_\Omega \varphi_{j,\varepsilon} d\mu - \int_\Omega \varphi_{j,\varepsilon} d\nu - \gamma \int_\Omega \varphi_{j,\varepsilon} u_0 (-\Delta)^{-1} u_0 dx \\ &+ \int_\Omega u_0 \nabla u_0 \cdot \nabla \varphi_{j,\varepsilon} dx. \end{aligned}$$

By construction,

$$\lim_{\varepsilon \to 0} \Big[- \gamma \int_\Omega \varphi_{j,\varepsilon} u_0 (-\Delta)^{-1} u_0 dx + \int_\Omega u_0 \nabla u_0 \cdot \nabla \varphi_{j,\varepsilon} dx \Big] = 0.$$

Then, as $\mathcal{F}'_\gamma(u_n) \to 0$ in $\big(H_0^1(\Omega)\big)'$, we obtain that

$$\lim_{\varepsilon \to 0} \Big(\int_\Omega \varphi_{j,\varepsilon} d\mu - \int_\Omega \varphi_{j,\varepsilon} d\nu \Big) = \mu_j - \nu_j = 0,$$

and we conclude that

$$\nu_j = \mu_j. \tag{2.9}$$

Finally, we have two options either the PS sequence has a convergent subsequence or it concentrates around some of the points x_j. In other words, $\nu_j = \mu_j = 0$, or there exists some $\nu_j > 0$ such that, by (2.6) and (2.9), $\nu_j \geq S_N^{N/2}$. In case of having concentration, we find that

$$\begin{aligned} c &= \lim_{n\to\infty} \mathcal{F}_\gamma(u_n) = \lim_{n\to\infty} \mathcal{F}_\gamma(u_n) - \frac{1}{2}\langle \mathcal{F}'_\gamma(u_n)|u_n\rangle \\ &= \big(\frac{1}{2} - \frac{1}{2^*}\big)\int_\Omega |u_0|^{2^*}dx + \big(\frac{1}{2} - \frac{1}{2^*}\big)\nu_j \\ &\geq \frac{1}{N} S_N^{N/2} = c^*, \end{aligned}$$

in contradiction with the hypotheses $c < c^*$. Therefore, the PS sequence has a convergent subsequence and the PS condition is satisfied. □

It remains to show that we can obtain a path for $\mathcal{F}_\gamma$ under the critical level c^*. To obtain such a path we take test functions of the form

$$\tilde{u}_\varepsilon = M\phi_\varepsilon,$$

where

$$\phi_\varepsilon = \varphi_{j,R}\, u_{j,\varepsilon}, \tag{2.10}$$

with $\varphi_{j,R}$ a cut-off function defined as (2.8) for some $R > 0$ small enough, $M > 0$ a large enough constant such that $\mathcal{F}_\gamma(\tilde{u}_\varepsilon) < 0$ and $u_{j,\varepsilon}$ are the family of functions

$$u_{j,\varepsilon}(x) = \Big(\frac{\varepsilon}{\varepsilon^2 + |x - x_j|^2}\Big)^{\frac{N-2}{2}}, \tag{2.11}$$

for $\varepsilon > 0$. Let us notice that the functions $u_{j,\varepsilon}$ are the extremal functions for the Sobolev's inequality in $\mathbb{R}^N$, where the constant S_N is achieved (see [13]). Then

$$\int_{\mathbb{R}^N} |\nabla u_{j,\varepsilon}|^2 dx = S_N \Big(\int_{\mathbb{R}^N} |u_{j,\varepsilon}|^{p+1} dx\Big)^{2/2^*}.$$

For the sake of simplicity we will consider $x_j = 0$, we will denote $\varphi_{j,R} = \varphi$ under the construction (2.8) and $u_{j,\varepsilon} = u_\varepsilon$. We will also assume the normalization

$$\|u_\varepsilon\|_{L^{2^*}(\Omega)} = 1, \tag{2.12}$$

so that the Sobolev constant is given by

$$S_N = \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 dx.$$

Then, under the previous considerations we define the set of paths

$$\Gamma_\varepsilon := \{g \in C([0,1], H_0^1(\Omega)) : g(0) = 0,\ g(1) = \tilde{u}_\varepsilon\},$$

and consider the minimax values

$$c_\varepsilon = \inf_{g\in\Gamma_\varepsilon} \max_{t\in[0,1]} \mathcal{F}_\gamma(g(t)).$$

The final issue we must solve now is the fact that the levels c_ε are always below c^* for ε small enough. To do so, we recall the following.

Lemma 2.8 ([5, Lemma 1.1]). *Let ϕ be the function denoted by* (2.10) *around the point $x_j = 0$. Then*

$$\int_{\mathbb{R}^N} \phi_\varepsilon^2 dx = \begin{cases} C\varepsilon + O(\varepsilon^2) & \text{if } N = 3, \\ \frac{C\varepsilon^2}{2}|\log\varepsilon| + O(\varepsilon^2) & \text{if } N = 4, \\ C\varepsilon^2 + O(\varepsilon^{N-2}) & \text{if } N \geq 5. \end{cases} \tag{2.13}$$

Moreover,

$$\|\nabla\phi_\varepsilon\|_2^2 = S_N + O(\varepsilon^{N-2}). \tag{2.14}$$

Remark 2.9. Using similar arguments one could also estimate $\|\phi_\varepsilon\|_{L^{2^*}(\Omega)} \sim C$, however it is simpler if we normalize it as done in (2.12).

To carry out the analysis of the levels c_ε we need some estimates dealing with the term $\int_\Omega \phi_\varepsilon(-\Delta)^{-1}\phi_\varepsilon dx$. To this end, we prove the following result.

Lemma 2.10. *Let ϕ_ε be the function denoted by* (2.10) *around the point $x_j = 0$. Then, there exists a constant $C > 0$ independent of ε such that*

$$\int_\Omega \phi_\varepsilon(-\Delta)^{-1}\phi_\varepsilon dx > C\varepsilon^{1+\frac{N}{N-4}} \quad \textit{and} \quad N \geq 7.$$

Proof. Let $v_\varepsilon(x) = (-\Delta)^{-1}\phi_\varepsilon(x)$ and note that because of the definition of the cut-off function (2.8), we can choose $v_\varepsilon(x)$ such that

$$\begin{gathered}(-\Delta)v_\varepsilon = \phi_\varepsilon \quad \text{in } B_{2R}(0),\\ v_\varepsilon = 0 \quad \text{in } \partial B_{2R}(0).\end{gathered}$$

Moreover, since $\phi_\varepsilon > 0$ in $B_{2R}(0)$, thanks to the Maximum Principle, it follows that $v_\varepsilon > 0$ in $B_{2R}(0)$. Now, let us notice that for any $x \in B_R(0)$ we have $\phi_\varepsilon(x) = u_\varepsilon(x)$ as well as

$$\frac{\varepsilon^{-\frac{N-2}{2}}}{\left(1+(\frac{R}{\varepsilon})^2\right)^{\frac{N-2}{2}}} \leq u_\varepsilon(x) \leq \varepsilon^{-\frac{N-2}{2}}.$$

Next, take $\rho < R/2$ and consider the function $\widetilde{v}(x) = \frac{2}{N}\left(1-(\frac{|x|}{2\rho})^2\right)_+$, where $(\cdot)_+$ stands for the positive part. Then, $\widetilde{v}$ satisfies the problem

$$\begin{gathered}(-\Delta)\widetilde{v} = \frac{1}{\rho^2} \quad \text{in } B_{2\rho}(0),\\ \widetilde{v} = 0 \quad \text{in } \partial B_{2\rho}(0).\end{gathered}$$

To apply a comparison principle we choose $\rho = \rho(\varepsilon) > 0$, with $\rho \to 0$ as $\varepsilon \to 0$, such that

$$(-\Delta)\widetilde{v} \leq (-\Delta)v_\varepsilon \quad \text{in } B_{2\rho}(0). \tag{2.15}$$

Then, given $\varepsilon > 0$ arbitrarily small, we distinguish two cases depending upon $\rho < \varepsilon$ or $\rho > \varepsilon$. In the first case, since

$$u_\varepsilon(x)\big|_{x\in B_{2\rho}(0)} \geq \frac{\varepsilon^{-\frac{N-2}{2}}}{\left(1+(\frac{2\rho}{\varepsilon})^2\right)^{\frac{N-2}{2}}} > \left(\frac{1}{5}\right)^{\frac{N-2}{2}}\varepsilon^{-\frac{N-2}{2}}.$$

As a consequence if

$$\frac{1}{\rho^2} \leq \left(\frac{1}{5}\right)^{\frac{N-2}{2}}\varepsilon^{-\frac{N-2}{2}},$$

then (2.15) holds. Since $\rho < \varepsilon$, we have

$$5^{\frac{N-2}{2}}\varepsilon^{\frac{N-2}{2}} \leq \rho^2 < \varepsilon^2, \tag{2.16}$$

from which we conclude that $N > 6$. In the second case, $\rho > \varepsilon$, since

$$u_\varepsilon(x)\big|_{x\in B_{2\rho}(0)} \geq \frac{\varepsilon^{-\frac{N-2}{2}}}{\left(1+(\frac{2\rho}{\varepsilon})^2\right)^{\frac{N-2}{2}}} = \frac{\varepsilon^{\frac{N-2}{2}}}{\rho^{N-2}\left(4+(\frac{\varepsilon}{\rho})^2\right)^{\frac{N-2}{2}}} > \left(\frac{1}{5}\right)^{\frac{N-2}{2}}\varepsilon^{\frac{N-2}{2}}\rho^{2-N}.$$

Once again, if

$$\frac{1}{\rho^2} \leq \big(\frac{1}{5}\big)^{\frac{N-2}{2}} \varepsilon^{\frac{N-2}{2}} \rho^{2-N}. \tag{2.17}$$

Then (2.15) holds. By $\rho > \varepsilon$ and (2.17), we have

$$\frac{1}{\varepsilon^{4-N}} < \frac{1}{\rho^{4-N}} \leq \big(\frac{1}{5}\big)^{\frac{N-2}{2}} \varepsilon^{\frac{N-2}{2}}, \tag{2.18}$$

from which we deduce $N > 6$. Finally, by construction,

$$0 = \widetilde{v}(x)\big|_{x\in\partial B_{2\rho}(0)} < v_\varepsilon(x)\big|_{x\in\partial B_{2\rho}(0)}.$$

Because of the Maximum Principle, we conclude that $v_\varepsilon(x) > \widetilde{v}(x)$ for $x \in B_{2\rho}(0)$, thus

$$\begin{aligned}
\int_\Omega \phi_\varepsilon(-\Delta)^{-1}\phi_\varepsilon dx &\geq \int_{B_R(0)} u_\varepsilon(x) v_\varepsilon(x) dx > \int_{B_{2\rho}(0)} u_\varepsilon(x)\widetilde{v}(x) dx \\
&\geq \int_{B_\rho(0)} u_\varepsilon(x)\widetilde{v}(x) dx \\
&= \frac{2}{N}\int_{B_\rho(0)} u_\varepsilon(x)\Big(1 - \big(\frac{|x|}{2\rho}\big)^2\Big) dx \\
&\geq \frac{3}{2N}\int_{B_\rho(0)} u_\varepsilon(x) dx.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_{B_\rho(0)} u_\varepsilon(x) dx &= \varepsilon^{-\frac{N-2}{2}} \int_{B_\rho(0)} \frac{1}{\big(1 + (\frac{|x|}{\varepsilon})^2\big)^{\frac{N-2}{2}}} dx \\
&= \varepsilon^{-\frac{N-2}{2}} \int_0^\rho \frac{r^{N-1}}{\big(1 + (\frac{r}{\varepsilon})^2\big)^{\frac{N-2}{2}}} dr \\
&= \varepsilon^{\frac{N}{2}} \int_0^\rho \frac{\big(r/\varepsilon\big)^{N-1}}{\big(1 + (\frac{r}{\varepsilon})^2\big)^{\frac{N-2}{2}}} dr \\
&= \varepsilon^{\frac{N}{2}+1} \int_0^{\rho/\varepsilon} \frac{s^{N-1}}{\big(1 + s^2\big)^{\frac{N-2}{2}}} ds \\
&\geq c\varepsilon^{\frac{N}{2}+1} \int_0^{\rho/\varepsilon} s^{N-1} ds = c\varepsilon^{\frac{N}{2}+1}\big(\frac{\rho}{\varepsilon}\big)^N,
\end{aligned}$$

for a positive constant c. Then, we arrive at the estimate

$$\int_\Omega \phi_\varepsilon(-\Delta)^{-1}\phi_\varepsilon dx > C\varepsilon^{-\frac{N-2}{2}}\rho^N \quad \text{and} \quad N > 6. \tag{2.19}$$

Now, from (2.16) and (2.18), we have the bounds

$$\rho < \varepsilon \quad \text{or} \quad \rho \leq \big(\frac{1}{5}\big)^{\frac{N-2}{2(N-4)}} \varepsilon^{\frac{N-2}{2(N-4)}}. \tag{2.20}$$

Since $N > 6$ it follows that $\frac{N-2}{2(N-4)} < 1$ so that, thanks to (2.20) and (2.19), we conclude

$$\int_\Omega \phi_\varepsilon(-\Delta)^{-1}\phi_\varepsilon dx > C\varepsilon^{1+\frac{N}{N-4}} \quad \text{and} \quad N > 6.$$

$\square$

Remark 2.11. Looking at the proof of Lemma 2.10 and considering either $\rho < \varepsilon$ or $\rho > \varepsilon$ we are forced to work with dimensions $N \geq 7$. On the other hand, we might consider $\rho = c_0\varepsilon$ for a suitable constant in order to analyze if the case $N = 6$ could be included in the previous result. Then, following the previous argument in Lemma 2.10, such that $\rho = c_0\varepsilon$ and $N = 6$, we have

$$u_\varepsilon(x)\big|_{x\in B_{2\rho}(0)} \geq \frac{\varepsilon^{-2}}{\big(1+(\frac{2\rho}{\varepsilon})^2\big)^2} = \Big(\frac{1}{1+4c_0^2}\Big)^2\varepsilon^{-2}.$$

As a consequence (2.15) holds if,

$$\frac{1}{c_0^2\varepsilon^2} \leq \Big(\frac{1}{1+4c_0^2}\Big)^2\varepsilon^{-2},$$

or equivalently if $(1+4c_0^2)^2 \leq c_0^2$, which is not possible. Therefore, as a consequence of that discussion, dimension $N = 6$ will be analyzed in a forthcoming paper by the use of different methods.

Next we perform the analysis of the levels c_ε, proving that, in fact, the levels c_ε are always below the critical level c^* provided $\varepsilon > 0$ is small enough.

Lemma 2.12. *Assume $p = 2^* - 1$ and $N \geq 7$. Then, there exists $\varepsilon > 0$ small enough such that*

$$\sup_{0\leq t\leq 1} \mathcal{F}_\gamma(t\tilde{u}_\varepsilon) < \frac{1}{N}S_N^{N/2}.$$

Proof. Using (2.14) in Lemma 2.8 and assuming the normalization (2.12), we find that

$$\begin{aligned} g(t) := \mathcal{F}_\gamma(t\tilde{u}_\varepsilon) &= \frac{t^2M^2}{2}\|\nabla\phi_\varepsilon\|^2_{L^2(\Omega)} - \frac{t^2M^2\gamma}{2}\int_\Omega \phi_\varepsilon(-\Delta)^{-1}\phi_\varepsilon dx - \frac{t^{2^*}M^{2^*}}{2^*} \\ &= \frac{M^2}{2}\left(S_N + O(\varepsilon^{N-2}) - \gamma F(\varepsilon)\right)t^2 - \frac{M^{2^*}}{2^*}t^{2^*}, \end{aligned}$$

where $F(\varepsilon) = \int_\Omega \phi_\varepsilon(-\Delta)^{-1}\phi_\varepsilon dx$. It is clear that $\lim_{t\to\infty} g(t) = -\infty$ as well as that $g(t) > 0$ for $t > 0$ small enough. Therefore, the function $g(t)$ possesses a maximum value at the point,

$$t_\varepsilon := \Big(\frac{M^2\big(S_N + O(\varepsilon^{N-2}) - \gamma F(\varepsilon)\big)}{M^{2^*}}\Big)^{\frac{1}{2^*-2}}.$$

Moreover, at the point t_ε we have

$$g(t_\varepsilon) = \frac{1}{N}\big(S_N + O(\varepsilon^{N-2}) - \gamma F(\varepsilon)\big)^{N/2}.$$

Then, the proof will be complete if the inequality

$$\frac{1}{N}\left(S_N + O(\varepsilon^{N-2}) - \gamma F(\varepsilon)\right)^{N/2} < \frac{1}{N}S_N^{N/2},$$

or, equivalently, the inequality

$$O(\varepsilon^{N-2}) < \gamma F(\varepsilon), \tag{2.21}$$

holds provided ε is small enough. Thanks to Lemma 2.10, we have $F(\varepsilon) > C\varepsilon^{1+\frac{N}{N-4}}$, so that (2.21) is equivalent to

$$O(\varepsilon^{N-2}) < C\varepsilon^{1+\frac{N}{N-4}}, \tag{2.22}$$

for $\varepsilon > 0$ small enough. So that it is sufficient to observe that (2.22) requires $N-2 > 1 + \frac{N}{N-4}$ which is equivalent to $(N-2)(N-6) > 0$, and that is obviously satisfied. $\square$

Proof of Theorem 1.1-(ii). Thanks to Lemmas 2.2 and 2.12, we find that

$$0 < c_\varepsilon \le \sup_{0\le t\le 1} \mathcal{F}_\gamma(t\tilde{u}_\varepsilon) < \frac{1}{N} S_N^{N/2},$$

provided $\varepsilon > 0$ is small enough. Because of Lemma 2.2 the functional $\mathcal{F}_\gamma$ has the mountain pass geometry. Moreover, because of Lemma 2.7 the functional $\mathcal{F}_\gamma$ satisfies the PS condition for any level c_ε provided $\varepsilon > 0$ is small enough. Therefore, we can apply the mountain pass theorem to obtain the existence of a critical point $u \in H_0^1(\Omega)$. The rest follows as in the subcritical case. $\square$

3. Existence of positive solutions for (1.10)

In this section we provide the existence result for the system (1.10). We start by stating the analogous results of those obtained for the functional $\mathcal{F}_\gamma$.

Lemma 3.1. *The functional $\mathcal{J}_\gamma$ denoted by* (1.11) *has the mountain pass geometry.*

Proof. Let us consider, without loss of generality, a pair $(g,h) \in H_0^1(\Omega)\times H_0^1(\Omega)$ such that $\|g\|_{L^{p+1}(\Omega)} = 1$. Then, taking a real number $t > 0$ and using the Young's inequality together with the Poincaré's inequality and the Sobolev inequality (1.6), we find that

$$\begin{aligned}
\mathcal{J}_\gamma(tg,th) &= \frac{t^2}{2}\int_\Omega |\nabla g|^2 dx + \frac{t^2}{2}\int_\Omega |\nabla h|^2 dx - t^2\sqrt{\gamma}\int_\Omega gh\,dx - \frac{t^{p+1}}{p+1} \\
&\ge \frac{t^2}{2}\Big(\|g\|^2_{H_0^1(\Omega)} + \|h\|^2_{H_0^1(\Omega)} - \sqrt{\gamma}\int_\Omega g^2 dx - \sqrt{\gamma}\int_\Omega h^2 dx\Big) - \frac{t^{p+1}}{p+1} \\
&\ge \frac{t^2}{2}\Big(1-\frac{\sqrt{\gamma}}{\lambda_1}\Big)\Big(\|g\|^2_{H_0^1(\Omega)} + \|h\|^2_{H_0^1(\Omega)}\Big) - \|g\|^2_{H_0^1(\Omega)}\frac{C}{p+1}t^{p+1} \\
&\ge \Big(\frac{1}{2}\Big(1-\frac{\sqrt{\gamma}}{\lambda_1}\Big)t^2 - \frac{C}{p+1}t^{p+1}\Big)\Big(\|g\|^2_{H_0^1(\Omega)} + \|h\|^2_{H_0^1(\Omega)}\Big),
\end{aligned} \tag{3.1}$$

where λ_1 is the first eigenvalue of the Laplace operator under Dirichlet boundary conditions. Since $0 < \gamma < \lambda_1^* = \lambda_1^2$ it follows that $\sqrt{\gamma} < \lambda_1$ and we obtain $\Big(1-\frac{\sqrt{\gamma}}{\lambda_1}\Big) > 0$. Therefore, taking $t > 0$ such that

$$0 < t^{p-1} < \frac{p+1}{2C}\Big(1-\frac{\sqrt{\gamma}}{\lambda_1}\Big),$$

from (3.1) we conclude that $\mathcal{J}_\gamma(tg,th) > 0$. Thus, the functional $\mathcal{J}_\gamma$ has a local minimum at $(u,v) = (0,0)$, i.e.,

$$\mathcal{J}_\gamma(tg,th) > \mathcal{J}_\gamma(0,0) = 0,$$

for any pair $(g,h) \in H_0^1(\Omega)\times H_0^1(\Omega)$ provided $t > 0$ is small enough. Also, it is clear that, because of the Poincaré's inequality,

$$\begin{aligned}
\mathcal{J}_\gamma(tg,th) &= \frac{t^2}{2}\int_\Omega |\nabla g|^2 dx + \frac{t^2}{2}\int_\Omega |\nabla h|^2 dx - t^2\sqrt{\gamma}\int_\Omega gh\,dx - \frac{t^{p+1}}{p+1} \\
&\le \frac{t^2}{2}\Big(\|g\|^2_{H_0^1(\Omega)} + \|h\|^2_{H_0^1(\Omega)} + \sqrt{\gamma}\int_\Omega g^2 dx + \sqrt{\gamma}\int_\Omega h^2 dx\Big) - \frac{t^{p+1}}{p+1}
\end{aligned}$$

$$\le \frac{t^2}{2}\Big(1+\frac{\sqrt{\gamma}}{\lambda_1}\Big)\Big(\|g\|^2_{H^1_0(\Omega)}+\|h\|^2_{H^1_0(\Omega)}\Big)-\frac{t^{p+1}}{p+1}.$$

Then $\mathcal{J}_\gamma(tg,th)\to-\infty$ as $t\to\infty$, and thus, there exists a pair $(\hat{u},\hat{v})$ such that $\mathcal{J}_\gamma(\hat{u},\hat{v})<0$. □

Lemma 3.2. *Let $\{(u_n,v_n)\}\subset H^1_0(\Omega)\times H^1_0(\Omega)$ be a PS sequence at level c for the functional $\mathcal{J}_\gamma$, i.e. $\mathcal{J}_\gamma(u_n,v_n)\to c$ and $\mathcal{J}'_\gamma(u_n,v_n)\to 0$, as $n\to\infty$. Then $\{(u_n,v_n)\}$ is bounded in $H^1_0(\Omega)\times H^1_0(\Omega)$.*

Proof. Since $\mathcal{J}'_\gamma(u_n,v_n)\to 0$ in $\big(H^1_0(\Omega)\times H^1_0(\Omega)\big)'$, in particular

$$\Big\langle \mathcal{J}'_\gamma(u_n,v_n)\Big|\frac{(u_n,v_n)}{\|u_n\|_{H^1_0(\Omega)}+\|v_n\|_{H^1_0(\Omega)}}\Big\rangle\to 0.$$

Thus, for any $\varepsilon>0$, there exists a subsequence, denoted again by $\{(u_n,v_n)\}$, such that

$$\begin{aligned}&\int_\Omega|\nabla u_n|^2dx+\int_\Omega|\nabla v_n|^2dx-2\sqrt{\gamma}\int_\Omega u_nv_n\,dx-\int_\Omega|u_n|^{p+1}dx\\&=\big[\|u_n\|_{H^1_0(\Omega)}+\|v_n\|_{H^1_0(\Omega)}\big]\cdot o(1).\end{aligned}$$

Moreover, since $\mathcal{J}_\gamma(u_n,v_n)\to c$,

$$\frac12\int_\Omega|\nabla u_n|^2dx+\frac12\int_\Omega|\nabla v_n|^2dx-\sqrt{\gamma}\int_\Omega u_nv_n\,dx-\frac{1}{p+1}\int_\Omega|u_n|^{p+1}dx=c+o(1),$$

for $n>0$ big enough. Therefore, for a positive constant μ (to be determined below) we find that

$$\mathcal{J}_\gamma(u_n,v_n)-\mu\Big\langle\mathcal{J}'_\gamma(u_n,v_n)\Big|\frac{1}{\|u_n\|_{H^1_0(\Omega)}}(u_n,v_n)\Big\rangle=c+[\|u_n\|_{H^1_0(\Omega)}+\|v_n\|_{H^1_0(\Omega)}]\cdot o(1).$$

That is,

$$\begin{aligned}&\big(\frac12-\mu\big)\Big[\int_\Omega|\nabla u_n|^2dx+\int_\Omega|\nabla v_n|^2dx-2\sqrt{\gamma}\int_\Omega u_nv_n\,dx\Big]\\&-\big(\frac{1}{p+1}-\mu\big)\int_\Omega|u_n|^{p+1}dx\\&=c+[\|u_n\|_{H^1_0(\Omega)}+\|v_n\|_{H^1_0(\Omega)}]\cdot o(1).\end{aligned}$$

Hence, taking μ such that $\frac{1}{p+1}<\mu<\frac12$,

$$\begin{aligned}&\big(\frac12-\mu\big)\Big[\int_\Omega|\nabla u_n|^2dx+\int_\Omega|\nabla v_n|^2dx\Big]-(1-2\mu)\sqrt{\gamma}\int_\Omega u_nv_n\,dx\\&\le c+[\|u_n\|_{H^1_0(\Omega)}+\|v_n\|_{H^1_0(\Omega)}]\cdot o(1),\end{aligned}$$

and using Young's inequality,

$$\begin{aligned}&\big(\frac12-\mu\big)\Big[\int_\Omega|\nabla u_n|^2dx+\int_\Omega|\nabla v_n|^2dx-\sqrt{\gamma}\int_\Omega u_n^2dx-\sqrt{\gamma}\int_\Omega v_n^2dx\Big]\\&\le c+[\|u_n\|_{H^1_0(\Omega)}+\|v_n\|_{H^1_0(\Omega)}]\cdot o(1).\end{aligned}$$

Then, because of the Poincaré inequality, we conclude that

$$\big(\frac12-\mu\big)\big(1-\frac{\sqrt{\gamma}}{\lambda_1}\big)[\|u_n\|^2_{H^1_0(\Omega)}+\|v_n\|^2_{H^1_0(\Omega)}]\le c+[\|u_n\|_{H^1_0(\Omega)}+\|v_n\|_{H^1_0(\Omega)}]\cdot o(1), \quad (3.2)$$

where λ_1 is the first eigenvalue of the Laplace operator under Dirichlet boundary conditions. Since $0<\gamma<\lambda_1^*=\lambda_1^2$, it follows that

$$\big(\frac{1}{2}-\mu\big)\big(1-\frac{\sqrt{\gamma}}{\lambda_1}\big)>0,$$

and thus, by (3.2), we conclude that the sequence $\{(u_n,v_n)\}$ is bounded in $H_0^1(\Omega)\times H_0^1(\Omega)$. $\square$

Proof of Theorem 1.3-(i). If $1<p<2^*-1$, given a PS sequence $\{(u_n,v_n)\}\subset H_0^1(\Omega)\times H_0^1(\Omega)$ at level c, by Lemma 3.1, the functional $\mathcal{J}_\gamma$ has the MP geometry. Moreover, due to Lemma 3.2 and the compact inclusion

$$H_0^1(\Omega)\times H_0^1(\Omega)\hookrightarrow L^{p+1}(\Omega)\times L^{p+1}(\Omega),\quad \text{for } 1\le p<2^*-1,$$

provided by Rellich-Kondrachov Theorem, the functional $\mathcal{J}_\gamma$ satisfies the PS condition at any level c. Therefore, the hypotheses of the Mountain Pass Theorem are fulfilled and we conclude that the functional $\mathcal{J}_\gamma$ possesses a critical point $(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega)$. Moreover, if we define the set of the paths

$$\Gamma:=\left\{g\in C\left([0,1],H_0^1(\Omega)\times H_0^1(\Omega)\right): g(0)=(0,0),\ g(1)=(\hat{u},\hat{v})\right\},$$

with $(\hat{u},\hat{v})$ given as in the proof of Lemma 3.1, then

$$\mathcal{J}_\gamma(u,v)=c:=\inf_{g\in\Gamma}\max_{\theta\in[0,1]}\mathcal{J}_\gamma(g(\theta)).$$

To show the positivity of the pair (u,v) we argue as in the proof of Theorem 1.1-(i). Let us consider the functional

$$\mathcal{J}_\gamma^+(u,v)=\mathcal{J}_\gamma(u^+,v^+),$$

where, as before, $u^+=\max\{u,0\}$. Repeating with minor changes the arguments carried out above for the functional $\mathcal{J}_\gamma$ we conclude that the functional $\mathcal{J}_\gamma^+$ has a critical point $(\tilde{u},\tilde{v})$ such that $\tilde{u}\ge 0$ and $\tilde{v}\ge 0$. Moreover, by the Maximum Principle, it follows that $\tilde{u}>0$ and $\tilde{v}>0$, then $(\tilde{u},\tilde{v})$ is a positive solution of (1.10). $\square$

To prove the PS condition when $p=2^*-1$ we must apply once again a concentration-compactness argument.

Lemma 3.3. *Assume $p=2^*-1$. Then, the functional $\mathcal{J}_\gamma$ satisfies the Palais-Smale condition for any level c such that*

$$c<c^*=\frac{1}{N}S_N^{N/2}.$$

Proof. Let $\{(u_n,v_n)\}\subset H_0^1(\Omega)\times H_0^1(\Omega)$ be a PS sequence of level $c<c^*$ for the functional $\mathcal{J}_\gamma$. Thanks to Lemma 3.2, the sequence $\{(u_n,v_n)\}$ is uniformly bounded and, as a consequence, we can assume that there exists a subsequence still denoted by $\{(u_n,v_n)\}$, such that

$$\begin{gathered}(u_n,v_n)\rightharpoonup(u_0,v_0)\quad\text{weakly in } H_0^1(\Omega)\times H_0^1(\Omega),\\ (u_n,v_n)\to(u_0,v_0)\quad\text{strongly in } L^q(\Omega)\times L^q(\Omega), 1\le q<2^*,\\ (u_n,v_n)\to(u_0,v_0)\quad\text{a.e. in }\Omega.\end{gathered}\tag{3.3}$$

Moreover, we can assume that, up to a subsequence, there exist three measures μ, $\tilde{\mu}$ and ν such that $|\nabla u_n|^2$, $|\nabla v_n|^2$ and $|u_n|^{2^*}$, converge in the sense of the measures

μ, $\tilde{\mu}$ and ν respectively. Thus, because of Lemma 2.6, there is a countable set I of points $\{x_j\}_{j\in I} \subset \overline{\Omega}$, and some positive numbers μ_j, $\tilde{\mu}_j$ and ν_j such that

$$\begin{aligned} |\nabla u_n|^2 \rightharpoonup d\mu &= |\nabla u_0|^2 + \sum_{j\in I} \mu_j \delta_{x_j}, \\ |\nabla v_n|^2 \rightharpoonup d\tilde{\mu} &= |\nabla v_0|^2 + \sum_{j\in I} \tilde{\mu}_j \delta_{x_j}, \\ |u_n|^{2^*} \rightharpoonup d\nu &= |u_0|^{2^*} + \sum_{j\in I} \nu_j \delta_{x_j}, \end{aligned} \tag{3.4}$$

where δ_{x_j} is the Dirac's delta centered at x_j with $j \in I$ and satisfying

$$\mu_j \geq S_N \nu_j^{2/2^*}. \tag{3.5}$$

Next, for $j \in I$, let $\varphi_{j,\varepsilon} \in C_0^\infty(\Omega)$ be a cut-off function satisfying (2.8) centered at $x_j \in \overline{\Omega}$. Thus, using $(\varphi_{j,\varepsilon} u_n, \varphi_{j,\varepsilon} v_n)$ as a test function, we find,

$$\begin{aligned} &\langle \mathcal{J}_\gamma'(u_n, v_n) | (\varphi_{j,\varepsilon} u_n, \varphi_{j,\varepsilon} v_n)\rangle \\ &= \int_\Omega \nabla u_n \cdot \nabla(\varphi_{j,\varepsilon} u_n) dx + \int_\Omega \nabla v_n \cdot \nabla(\varphi_{j,\varepsilon} v_n) dx - 2\sqrt{\gamma} \int_\Omega \varphi_{j,\varepsilon} u_n v_n \, dx \\ &\quad - \int_\Omega \varphi_{j,\varepsilon} u_n^{2^*} dx \\ &= \int_\Omega \varphi_{j,\varepsilon} |\nabla u_n|^2 dx + \int_\Omega \varphi_{j,\varepsilon} |\nabla v_n|^2 dx - \int_\Omega \varphi_{j,\varepsilon} u_n^{2^*} dx + \int_\Omega u_n \langle \nabla u_n, \nabla \varphi_{j,\varepsilon}\rangle dx \\ &\quad + \int_\Omega v_n \langle \nabla v_n, \nabla \varphi_{j,\varepsilon}\rangle dx - 2\sqrt{\gamma} \int_\Omega \varphi_{j,\varepsilon} u_n v_n dx. \end{aligned}$$

Moreover, from (3.3) and (3.4), we have

$$\begin{aligned} &\lim_{n\to\infty} \langle \mathcal{J}_\gamma'(u_n, v_n) | (\varphi_{j,\varepsilon} u_n, \varphi_{j,\varepsilon} v_n)\rangle \\ &= \int_\Omega \varphi_{j,\varepsilon} d\mu + \int_\Omega \varphi_{j,\varepsilon} d\tilde{\mu} - \int_\Omega \varphi_{j,\varepsilon} d\nu - 2\sqrt{\gamma} \int_\Omega \varphi_{j,\varepsilon} u_0 v_0 dx \\ &\quad + \int_\Omega u_0 \langle \nabla u_0, \nabla \varphi_{j,\varepsilon}\rangle dx + \int_\Omega v_0 \langle \nabla v_0, \nabla \varphi_{j,\varepsilon}\rangle dx. \end{aligned}$$

By construction,

$$\lim_{\varepsilon\to 0} \Big[- 2\sqrt{\gamma} \int_\Omega \varphi_{j,\varepsilon} u_0 v_0 dx + \int_\Omega u_0 \left\langle \nabla u_0, \nabla \varphi_{j,\varepsilon}\right\rangle dx + \int_\Omega v_0 \left\langle \nabla v_0, \nabla \varphi_{j,\varepsilon}\right\rangle dx \Big] = 0.$$

Then, as $\mathcal{J}_\gamma'(u_n) \to 0$ in $\big(H_0^1(\Omega) \times H_0^1(\Omega)\big)'$, we obtain that

$$\lim_{\varepsilon\to 0} \Big(\int_\Omega \varphi_{j,\varepsilon} d\mu + \int_\Omega \varphi_{j,\varepsilon} d\tilde{\mu} - \int_\Omega \varphi_{j,\varepsilon} d\nu \Big) = \mu_j + \tilde{\mu}_j - \nu_j = 0,$$

and we conclude that

$$\nu_j = \mu_j + \tilde{\mu}_j. \tag{3.6}$$

Finally, we have two options either the PS sequence has a convergent subsequence or it concentrates around some of the points x_j. In other words, $\nu_j = \mu_j = \tilde{\mu}_j = 0$, or there exists some $\nu_j > 0$ such that, by (3.5) and (3.6), $\nu_j \geq S_N^{N/2}$. In case of having concentration, we find that

$$c = \lim_{n\to\infty} \mathcal{J}_\gamma(u_n, v_n)$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \mathcal{J}_\gamma(u_n,v_n) - \frac{1}{2}\langle \mathcal{J}_\gamma(u_n,v_n)|(u_n,v_n)\rangle \\
&= \big(\frac{1}{2}-\frac{1}{2^*}\big)\int_\Omega |u_0|^{2^*}dx + \big(\frac{1}{2}-\frac{1}{2^*}\big)\nu_j \\
&\geq \frac{1}{N}S_N^{N/2} = c^*,
\end{aligned}
$$

in contradiction with the hypotheses $c < c^*$. Therefore, the PS sequence has a convergent subsequence and the PS condition is satisfied. $\square$

Next we show that we can obtain a path for $\mathcal{J}_\gamma$ under the critical level c^*. To obtain such a path we will assume test functions of the form

$$
(\tilde{u}_\varepsilon,\tilde{v}_\varepsilon) = (M\phi_\varepsilon, M\rho\phi_\varepsilon), \quad \text{where } \phi_\varepsilon = \varphi_{j,R}\, u_{j,\varepsilon},
$$

with $\varphi_{j,R}$ is a cut-off function defined by (2.8), for some $R > 0$ small enough, $M > 0$ a sufficiently large constant such that $\mathcal{J}_\gamma(\tilde{u}_\varepsilon,\tilde{v}_\varepsilon) < 0$, ρ is a positive term to be determined below and $u_{j,\varepsilon}$ are the family of functions defined by (2.11). For the sake of simplicity, in the sequel we will consider $x_j = 0$ as well as the normalization (2.12).

Then, under the previous construction, we define the set of paths

$$
\Gamma_\varepsilon := \big\{g \in C\left([0,1], H_0^1(\Omega)\times H_0^1(\Omega)\right) : g(0) = (0,0),\ g(1) = (\tilde{u}_\varepsilon,\tilde{v}_\varepsilon)\big\},
$$

and consider the minimax value

$$
c_\varepsilon = \inf_{g\in\Gamma_\varepsilon}\max_{t\in[0,1]} \mathcal{J}_\gamma(g(t)).
$$

Now we prove that, in fact, the levels c_ε are always below c^* for $\varepsilon > 0$ small enough.

Lemma 3.4. *Assume $p = 2^* - 1$. Then, there exists $\varepsilon > 0$ small enough such that*

$$
\sup_{0\leq t\leq 1} \mathcal{J}_\gamma(t\tilde{u}_\varepsilon, t\tilde{v}_\varepsilon) < \frac{1}{N}S_N^{N/2},
$$

provided $N \geq 7$.

Proof. Let us denote by $F(\varepsilon)$ the estimate (2.13) in Lemma 2.8. Then, assuming the normalization (2.12),

$$
\begin{aligned}
g(t) :=& \mathcal{J}_\gamma(t\tilde{u}_\varepsilon, t\tilde{v}_\varepsilon) \\
=& \Big(\frac{t^2M^2}{2} + \frac{\rho^2t^2M^2}{2}\Big)\|\nabla\phi_\varepsilon\|^2_{L^2(\Omega)} - t^2M^2\rho\sqrt{\gamma}\int_\Omega \phi_\varepsilon^2 dx - \frac{t^{2^*}M^{2^*}}{2^*} \\
=& \frac{t^2M^2}{2}\Big((1+\rho^2)[S_N + O(\varepsilon^{N-2})] - 2\rho\sqrt{\gamma}F(\varepsilon)\Big) - \frac{t^{2^*}M^{2^*}}{2^*}.
\end{aligned}
$$

It is clear that $\lim_{t\to\infty} g(t) = -\infty$, therefore, the function $g(t)$ possesses a maximum value at the point

$$
t_\varepsilon = \Big(\frac{M^2[(1+\rho^2)[S_N + O(\varepsilon^{N-2})] - 2\rho\sqrt{\gamma}F(\varepsilon)]}{M^{2^*}}\Big)^{\frac{1}{2^*-2}}.
$$

Moreover, at the point t_ε,

$$
g(t_\varepsilon) = \frac{1}{N}\left[\left(1+\rho^2\right)[S_N + O(\varepsilon^{N-2})] - 2\rho\sqrt{\gamma}F(\varepsilon)\right]^{N/2}.
$$

Then, the proof will be complete if we can choose $\rho > 0$ such that the inequality,

$$
[(1+\rho^2)[S_N + O(\varepsilon^{N-2})] - 2\rho\sqrt{\gamma}F(\varepsilon)] < S_N, \tag{3.7}
$$

holds provided $\varepsilon > 0$ is small enough. Indeed, if we take $\rho = \varepsilon^\alpha$, with $\alpha > 0$ (to be determined), inequality (3.7) is equivalent to

$$S_N\varepsilon^{2\alpha} + O(\varepsilon^{N-2+2\alpha}) + O(\varepsilon^{N-2}) < 2\sqrt{\gamma}\varepsilon^\alpha F(\varepsilon),$$

Since $S_N\varepsilon^{2\alpha} + O(\varepsilon^{N-2+2\alpha}) + O(\varepsilon^{N-2}) = O(\varepsilon^\tau)$ with $\tau = \min\{2\alpha, N-2+2\alpha, N-2\} = \min\{2\alpha, N-2\}$, we are left to prove $\alpha > 0$ can be chosen such that

$$O(\varepsilon^\tau) < 2\sqrt{\gamma}\varepsilon^\alpha \cdot \begin{cases} C\varepsilon + O(\varepsilon^2), & \text{if } N = 3, \\ \frac{C\varepsilon^2}{2}|\log\varepsilon| + O(\varepsilon^2), & \text{if } N = 4, \\ C\varepsilon^2 + O(\varepsilon^{N-2}), & \text{if } N \geq 5. \end{cases} \tag{3.8}$$

provided $\varepsilon > 0$ is small enough.

If $N = 3$, the corresponding inequality in (3.8) holds true if $\tau = \min\{2\alpha, 1\} > \alpha + 1$ that is not possible.

If $N = 4$, the corresponding inequality (3.8) holds true if

$$O(\varepsilon^\tau) < C\sqrt{\gamma}\varepsilon^{2\alpha+2}|\log\varepsilon| \;\Rightarrow\; O(\varepsilon^{\tau-2-\alpha}) < C\sqrt{\gamma}|\log\varepsilon|,$$

and thus, necessarily $\tau = \min\{2\alpha, 2\} > 2 + \alpha$, that, once again, is not possible.

If $N \geq 5$, the corresponding inequality (3.8) holds true if $\tau = \min\{2\alpha, N-2\} > 2 + \alpha$. Let us observe that $\min\{a, b\} = \frac{1}{2}(a + b - |a - b|)$, hence, inequality (3.8) will be satisfied if we can choose $\alpha > 0$ such that

$$N - |2\alpha - (N-2)| > 6. \tag{3.9}$$

Now we have two options, either $2\alpha > N - 2$ or $2\alpha < N - 2$.

- In the first case, thanks to inequality (3.9), we find the condition $\frac{N}{2} + 1 > N - \alpha > 4$, that can be fulfilled only for $N > 6$.
- In the second case, thanks to inequality (3.9), we find the condition $N-2 > 2\alpha > 4$, that can be fulfilled, once again, only for $N > 6$.

Thus, if $N \geq 7$ we can choose $\alpha > 2$ such that (3.8) is satisfied. Finally, note that with the assumption $\rho = \varepsilon^\alpha$ we have

$$t_\varepsilon = \Big(\frac{M^2[(1+\rho^2)[S_N + O(\varepsilon^{N-2})] - 2\rho\sqrt{\gamma}F(\varepsilon)]}{M^{2^*}}\Big)^{\frac{1}{2^*-2}} \geq \delta > 0,$$

provided $\varepsilon > 0$ is small enough. $\square$

Proof of Theorem 1.3-(ii). Applying Lemmas 3.1 and 3.4, we find that

$$0 < c_\varepsilon \leq \sup_{0\leq t\leq 1} \mathcal{J}_\gamma(t\tilde{u}_\varepsilon, t\tilde{v}_\varepsilon) < \frac{1}{N}S_N^{N/2},$$

provided $\varepsilon > 0$ is small enough. Indeed, due to Lemma 3.1 the functional $\mathcal{J}_\gamma$ has the MP geometry. Moreover, thanks to Lemma 3.3 the functional $\mathcal{J}_\gamma$ satisfies the PS condition for any level c_ε with $\varepsilon > 0$ small enough. Therefore, we can apply the Mountain Pass Theorem and conclude the existence of a critical point $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$. The rest follows as in the subcritical case. $\square$

4. Polyharmonic problems

Let us consider the higher order problem with generalized Navier boundary conditions,

$$\begin{gathered} (-\Delta)^{m+1}u = \gamma u + (-\Delta)^m|u|^{p-1}u \quad \text{in } \Omega \subset \mathbb{R}^N \\ (-\Delta)^j u = 0 \quad \text{for } 0 \leq j \leq m, \quad \text{on } \partial\Omega \end{gathered} \tag{4.1}$$

with m a natural number bigger than 1, and the variational problem obtained applying the operator $(-\Delta)^{-m}$ to (4.1),

$$\begin{gathered} -\Delta u=\gamma(-\Delta)^{-m}u+|u|^{p-1}u \quad \text{in } \Omega\subset\mathbb{R}^N \\ u=0 \quad \text{on } \partial\Omega \end{gathered} \tag{4.2}$$

associated with the energy functional

$$\mathcal{F}_{\gamma,m}(u)=\frac{1}{2}\int_\Omega|\nabla u|^2dx-\frac{\gamma}{2}\int_\Omega|(-\Delta)^{-m/2}u|^2dx-\frac{1}{p+1}\int_\Omega|u|^{p+1}dx.$$

Note that, as it happens for $m=1$, the embedding features for problem (4.2) are governed by the standard second-order equation,

$$-\Delta u=|u|^{p-1}u,$$

thus, the variational framework coincides with the one of the case $m=1$, so that we also consider $1<p\le 2^*-1$.

Let us observe that if we try to prove the existence of a positive solution problem (4.2) directly, as performed for the problem (1.1) in Section (2), we immediately run into complications. Because of the lack of a comparison principle, we can not use a similar argument to Lemma (2.10) when dealing with the operator $(-\Delta)^{-m}$. Thus, we will use the correspondence between problem (4.2) and the elliptic system

$$\left.\begin{aligned} -\Delta u&=\gamma^{\frac{1}{m+1}}v_1+|u|^{p-1}u,\\ -\Delta v_1&=\gamma^{\frac{1}{m+1}}v_2,\\ -\Delta v_2&=\gamma^{\frac{1}{m+1}}v_3,\\ &\dots\\ -\Delta v_m&=\gamma^{\frac{1}{m+1}}u \end{aligned}\right\} \quad \text{in } \Omega \\ (u,v_1,\dots,v_m)=(0,0,\dots,0) \quad \text{on } \partial\Omega \tag{4.3}$$

whose associated energy functional is

$$\begin{aligned} &\mathcal{J}_{\gamma,m}(\mathcal{U})\\ &=\frac{1}{2}\int_\Omega|\nabla u|^2dx+\frac{1}{2}\sum_{i=1}^m\int_\Omega|\nabla v_i|^2dx-\frac{\gamma^{\frac{1}{m+1}}}{m+1}\Big(\int_\Omega uv_1\,dx+\int_\Omega uv_m\,dx\\ &\quad+\sum_{i=1}^{m-1}\int_\Omega v_iv_{i+1}dx\Big)-\frac{1}{p+1}\int_\Omega|u|^{p+1}dx, \end{aligned} \tag{4.4}$$

where $\mathcal{U}=(u,v_1,\dots,v_m)$. The functional $\mathcal{J}_{\gamma,m}$ has the same structure as the functional $\mathcal{J}_\gamma$ thus, the ideas developed in Section 3 will fit, with slight variations, in this scenario.

Let us denote by Λ_1^* the first eigenvalue of the operator $(-\Delta)^{m+1}$ under the homogeneous generalized Navier boundary conditions given in (4.1). It is clear from the spectral definition of the operator $(-\Delta)^{m+1}$ that $\Lambda_1^*=\lambda_1^{m+1}$ with λ_1 the first eigenvalue of the Laplace operator under homogeneous Dirichlet boundary conditions.

The aim of this last section is then to prove the following result.

Theorem 4.1. *Assume $1<p<2^*-1$. Then, for every $\gamma\in(0,\Lambda_1^*)$, there exists a positive solution to system* (4.3).

Theorem 4.2. *Assume $p = 2^* - 1$. Then, for every $\gamma \in (0, \Lambda_1^*)$, there exists a positive solution to system* (4.3) *provided $N \geq 7$.*

We start determining the interval of values of the parameter $\gamma > 0$ compatible with existence of positive solutions related to problem (4.2).

Lemma 4.3. *Equation* (4.2) *does not possess a positive solution when $\gamma \geq \Lambda_1^*$.*

Proof. Using as a test function in (4.2) the first eigenfunction φ_1 associated with the first eigenvalue λ_1 for the Laplacian operator $(-\Delta)$ with homogeneous Dirichlet boundary conditions together with $\Lambda_1^* = \lambda_1^{m+1}$ the result follows. □

Next we deal with the mountain pass conditions. We state the analogous results to those of the case $m = 1$. Since the proofs of the next results rely on the ideas developed for the case $m = 1$, we will only remark the main differences, if any.

Lemma 4.4. *The functional $\mathcal{J}_{\gamma,m}(\mathcal{U})$ has the mountain pass geometry.*

The proof of the above lemma is similar to that of Lemma 3.1 so we omit it.

Lemma 4.5. *Let $\mathbb{E}_m := H_0^1(\Omega) \times H_0^1(\Omega) \times \ldots \times H_0^1(\Omega)$ and*

$$\{\mathcal{U}_n\} = \{(u_n, v_{1,n}, \ldots, v_{m,n})\} \subset \mathbb{E}_m$$

be a PS sequence for the functional $\mathcal{J}_{\gamma,m}$, i.e., $\mathcal{J}_{\gamma,m}(\mathcal{U}_n) \to c$ and $\mathcal{J}'_{\gamma,m}(\mathcal{U}_n) \to 0$ as $n \to \infty$. Then $\{\mathcal{U}_n\}$ is bounded in $\mathbb{E}_m$.

Proof. Arguing as in the proof of Lemma 3.2 we find that

$$\begin{aligned}
&(m+1)\big(\frac{1}{2} - \mu\big)\Big(1 - \frac{2\gamma^{\frac{1}{m+1}}}{(m+1)\lambda_1}\Big)\Big(\|u_n\|^2_{H_0^1(\Omega)} + \sum_{i=1}^m \|v_{i,n}\|^2_{H_0^1(\Omega)}\Big) \\
&\leq (m+1)c + \Big(\|u_n\|_{H_0^1(\Omega)} + \sum_{i=1}^m \|v_{i,n}\|_{H_0^1(\Omega)}\Big) \cdot o(1).
\end{aligned}$$

Keeping in mind Lemma 4.3, it follows that

$$\big(\frac{1}{2} - \mu\big)\Big(1 - \frac{2\gamma^{\frac{1}{m+1}}}{(m+1)\lambda_1}\Big) > 0,$$

and we conclude the boundedness of the sequence $\{\mathcal{U}_n\}$ in $\mathbb{E}_m$. □

Proof of Theorem 4.1. Combining Lemmas 4.4, 4.5, the Rellich-Kondrachov Theorem, the hypotheses of the MPT are fulfilled and we arrive at the same conclusion as in the proof of Theorem 1.3-(i). □

To finish, we deal with the critical case $p = 2^* - 1$. As it was done in previous sections, with the aid of a concentration-compactness argument we will prove that the PS condition is satisfied for any level below the critical level

$$c^* = \frac{1}{N} S_N^{N/2}.$$

Let us observe that the critical level c^* is independent of the order of the inverse operator involved in problem (4.2) as it coincides with the critical level for problem (1.1).

Lemma 4.6. *The functional $\mathcal{J}_{\gamma,m}$ defined by* (4.4) *satisfies the Palais-Smale condition for any level c below the critical level c^*.*

Proof. Let $\{\mathcal{U}_n\} = \{(u_n, v_{1,n}, \dots, v_{m,n})\} \subset \mathbb{E}_m$ be a PS sequence of level $c < c^*$. Because of Lemma 4.5 and Lemma 2.6, we can replicate the steps of the proof of Lemma 3.3 incorporating the slight difference that, instead (3.6), we find now

$$\nu_j = \mu_j + \sum_{i=1}^{m} \tilde{\mu}_{i,j}. \tag{4.5}$$

with

$$\mu_j \geq S_N \nu_j^{2/2^*}. \tag{4.6}$$

Then, either the PS sequence has a convergent subsequence or it concentrates around some of the points x_j. In other words, $\nu_j = \mu_j = \tilde{\mu}_{i,j} = 0$, or there exists some $\nu_j > 0$ such that, thanks to (4.5) and (4.6), $\nu_j \geq S_N^{N/2}$. In case of having concentration,

$$\begin{aligned} c &= \lim_{n\to\infty} \mathcal{J}_{\gamma,m}(\mathcal{U}_n) \\ &= \lim_{n\to\infty} \mathcal{J}_{\gamma,m}(\mathcal{U}_n) - \frac{1}{2}\langle \mathcal{J}_{\gamma,m}(\mathcal{U}_n) | \mathcal{U}_n \rangle \\ &= \big(\frac{1}{2} - \frac{1}{2^*}\big) \int_\Omega |u_0|^{2^*} dx + \big(\frac{1}{2} - \frac{1}{2^*}\big)\nu_j \\ &\geq \frac{1}{N} S_N^{N/2} = c^*, \end{aligned}$$

in contradiction with the hypotheses $c < c^*$. □

Finally, we show that we can obtain a path for the functional $\mathcal{J}_{\gamma,m}$ under the critical level c^*. Following the ideas of the previous sections, we will assume test functions of the form

$$\tilde{\mathcal{U}}_\varepsilon = (\tilde{u}_\varepsilon, \tilde{v}_{1,\varepsilon}, \dots, \tilde{v}_{m,\varepsilon}) = (M\phi_\varepsilon, M\rho\phi_\varepsilon, \dots, M\rho\phi_\varepsilon), \tag{4.7}$$

with $M > 0$ a sufficiently large constant so that $\mathcal{J}_{\gamma,m}(\tilde{\mathcal{U}}_\varepsilon) < 0$, ρ is positive term to be determined as in the case $m = 1$, and $u_{j,\varepsilon}$ are the family of functions defined by (2.11). As performed above we will consider $x_j = 0$. Then, under the previous construction, let us define the set of paths

$$\Gamma_\varepsilon := \{g \in C([0,1], \mathbb{E}_m) : g(0) = \overline{0},\ g(1) = \tilde{\mathcal{U}}_\varepsilon\},$$

and consider the minimax value

$$c_\varepsilon = \inf_{g\in\Gamma_\varepsilon} \max_{t\in[0,1]} \mathcal{J}_{\gamma,m}(g(t)).$$

Next, we check that any level c_ε is always below c^* provided $\varepsilon > 0$ is small enough. This is done applying Lemma 2.8.

Lemma 4.7. *Assume $p = 2^* - 1$ and $N \geq 7$. Then, there exists $\varepsilon > 0$ small enough such that*

$$\sup_{0\leq t\leq 1} \mathcal{J}_{\gamma,m}(t\tilde{\mathcal{U}}_\varepsilon) < \frac{1}{N} S_N^{N/2}.$$

Proof. Let us denote by $F(\varepsilon)$ estimate (2.13) in Lemma 2.8. Then, assuming the normalization (2.12), we obtain

$$\begin{aligned} g(t) :=& \mathcal{J}_{\gamma,m}(t\tilde{\mathcal{U}}_\varepsilon) \\ =& \Big(\frac{1}{2}(1+m\rho^2)[S_N + O(\varepsilon^{N-2})] - \frac{\gamma^{\frac{1}{m+1}}}{m+1}(2\rho + (m-1)\rho^2)F(\varepsilon)\Big) M^2 t^2 \end{aligned}$$

$$- \frac{M^{2^*}t^{2^*}}{2^*}.$$

Proceeding as in the proof of Lemma 3.4, the proof will be completed if we can choose $\rho > 0$ such that the inequality

$$O(\varepsilon^{N-2}) + m\rho^2 S_N + m\rho^2 O(\varepsilon^{N-2}) < 2\frac{\gamma^{\frac{1}{m+1}}}{m+1}(2\rho + (m-1)\rho^2)F(\varepsilon),$$

holds provided $\varepsilon > 0$ is small enough. We take $\rho = \varepsilon^\alpha$ with $\alpha > 0$ (to be determined) and $\tau = \min\{N-2, 2\alpha, 2\alpha + N - 2\} = \min\{N-2, 2\alpha\}$. Then, since $O(\varepsilon^\alpha + \varepsilon^{2\alpha}) = O(\varepsilon^\alpha)$, we are left to prove that for a constant $C > 0$ the inequality

$$O(\varepsilon^\tau) < C\varepsilon^\alpha F(\varepsilon), \tag{4.8}$$

holds provided $\varepsilon > 0$ is small enough. Since inequality (4.8) coincides with (3.8) the arguments performed in Lemma 3.4 allow us to complete the proof. □

Proof of Theorem 4.2. Thanks to Lemmas 3.1 and 3.4, we find that

$$c_\varepsilon \le \sup_{t \ge 0} \mathcal{J}_\gamma(t\tilde{\mathcal{U}}_\varepsilon) < \frac{1}{N} S_N^{N/2},$$

provided $\varepsilon > 0$ is sufficiently small. Hence, combining Lemmas 4.4 and 4.6 we can apply the mountain pass theorem and conclude the existence of a critical point $\mathcal{U} \in \mathbb{E}_m$. The rest follows as in the former cases. □

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