*Electronic Journal of Differential Equations*, Vol. 2004(2004), No. 08, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

# THE METHOD OF UPPER AND LOWER SOLUTIONS FOR CARATHEODORY N-TH ORDER DIFFERENTIAL INCLUSIONS

BUPURAO C. DHAGE, TARACHAND L. HOLAMBE, & SOTIRIS K. NTOUYAS

Abstract. In this paper, we prove an existence theorem for n-th order differential inclusions under Carathéodory conditions. The existence of extremal solutions is also obtained under certain monotonicity condition of the multi-function.

## 1. Introduction

Let $\mathbb{R}$ denote the real line and let $J = [0, a]$ be a closed and bounded interval in $\mathbb{R}$. Consider the initial value problem (in short IVP) of $n^{\text{th}}$ order differential inclusion

$$\begin{gathered} x^{(n)}(t) \in F(t, x(t)) \quad \text{a.e. } t \in J, \\ x^{(i)}(0) = x_i \in \mathbb{R} \end{gathered} \tag{1.1}$$

where $F : J \times \mathbb{R} \to 2^{\mathbb{R}}$, $i \in \{0, 1, \dots, n-1\}$ and $2^{\mathbb{R}}$ is the class of all nonempty subsets of $\mathbb{R}$.

By a solution of (1.1) we mean a function $x \in AC^{n-1}(J, \mathbb{R})$ whose $n^{\text{th}}$ derivative $x^{(n)}$ exists and is a member of $L^1(J, \mathbb{R})$ in $F(t, x)$, i.e. there exists a $v \in L^1(J, \mathbb{R})$ such that $v(t) \in F(t, x(t))$ a.e $t \in J$, and $x^{(i)}(0) = x_i \in \mathbb{R}, i = 0, 1, \dots, n-1$, where $AC^{n-1}(J, \mathbb{R})$ is the space of all continuous real-valued functions whose $(n-1)$ derivatives exist and are absolutely continuous on $J$.

The method of upper and lower solutions has been successfully applied to the problem of nonlinear differential equations and inclusions. For the first direction, we refer to Heikkila and Laksmikantham [8] and Bernfield and Laksmikantham [1] and for the second direction we refer to Halidias and Papageorgiou [7] and Benchohra [2]. In this paper we apply the multi-valued version of Schaefer's fixed point theorem due to Martelli [10] to the initial value problem (1.1) and prove the existence of solutions between the given lower and upper solutions, using the Carathéodory condition on $F$.

## 2. Preliminaries

Let $X$ be a Banach space and let $2^X$ be a class of all non- empty subsets of $X$. A correspondence $T : X \to 2^X$ is called a multi-valued map or simply multi and

2000 *Mathematics Subject Classification.* 34A60.
*Key words and phrases.* Differential inclusion, method of upper and lower solutions, existence theorem.

Submitted November 20, 2003. Published January 2, 2004.

$u \in Tu$ for some $u \in X$, then $u$ is called a fixed point of $T$. A multi $T$ is closed (resp. convex and compact) if $Tx$ is closed (resp. convex and compact) subset of $X$ for each $x \in X$. $T$ is said to be bounded on bounded sets if $T(B) = \bigcup_{x\in B} T(x) = \bigcup T(B)$ is a bounded subset of $X$ for all bounded sets $B$ in $X$. $T$ is called upper semi-continuous (u.s.c.) if for every open set $N \subset X$, the set $\{x \in X : Tx \subset N\}$ is open in $X$. $T$ is said to be totally bounded if for any bounded subset $B$ of $X$, the set $\cup T(B)$ is totally bounded subset of $X$.

Again $T$ is called completely continuous if it is upper semi-continuous and totally bounded on $X$. It is known that if the multi-valued map $T$ is totally bounded with non empty compact values, the $T$ is upper semi-continuous if and only if $T$ has a closed graph (that is $x_n \to x_*, y_n \to y_*, y_n \in Tx_n \Rightarrow y_* \in Tx_*$). By $KC(X)$ we denote the class of nonempty compact and convex subsets of $X$. We apply the following form of the fixed point theorem of Martelli [10] in the sequel.

**Theorem 2.1.** *Let $T : X \to KC(X)$ be a completely continuous multi-valued map. If the set*
$$\mathcal{E} = \{u \in X : \lambda u \in Tu \quad \text{for some } \lambda > 1\}$$
*is bounded, then $T$ has a fixed point.*

We also need the following definitions in the sequel.

**Definition 2.2.** A multi-valued map map $F : J \to KC(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function $t \to d(y, F(t)) = \inf\{\|y - x\| : x \in F(t)\}$ is measurable.

**Definition 2.3.** A multi-valued map $F : J \times \mathbb{R} \to 2^{\mathbb{R}}$ is said to be $L^1$-Carathéodory if

(i) $t \to F(t, x)$ is measurable for each $x \in \mathbb{R}$,
(ii) $x \to F(t, x)$ is upper semi-continuous for almost all $t \in J$, and
(iii) for each real number $k > 0$, there exists a function $h_k \in L^1(J, \mathbb{R})$ such that
$$\|F(t, x)\| = \sup\{|v| : v \in F(t, x)\} \le h_k(t), \quad a.e. \quad t \in J$$
for all $x \in \mathbb{R}$ with $|x| \le k$.

Denote
$$S^1_F(x) = \{v \in L^1(J, \mathbb{R}) : v(t) \in F(t, x(t)) \quad \text{a.e. } t \in J\}.$$
Then we have the following lemmas due to Lasota and Opial [9].

**Lemma 2.1.** *If $\dim(X) < \infty$ and $F : J \times X \to KC(X)$ then $S^1_F(x) \neq \emptyset$ for each $x \in X$.*

**Lemma 2.2.** *Let $X$ be a Banach space, $F$ an $L^1$-Carathéodory multi-valued map with $S^1_F \neq \emptyset$ and $\mathcal{K} : L^1(J, X) \to C(J, X)$ be a linear continuous mapping. Then the operator*
$$\mathcal{K} \circ S^1_F : C(J, X) \longrightarrow KC(C(J, X))$$
*is a closed graph operator in $C(J, X) \times C(J, X)$.*

We define the partial ordering $\le$ in $W^{n,1}(J, \mathbb{R})$ (the Sobolev class of functions $x : J \to \mathbb{R}$ for which $x^{(n-1)}$ are absolutely continuous and $x^{(n)} \in L^1(J, \mathbb{R})$) as follows. Let $x, y \in W^{n,1}(J, \mathbb{R})$. Then we define
$$x \le y \Leftrightarrow x(t) \le y(t), \ \forall t \in J.$$

If $a, b \in W^{n,1}(J, \mathbb{R})$ and $a \leq b$, then we define an order interval $[a, b]$ in $W^{n,1}(J, \mathbb{R})$ by

$$[a, b] = \{x \in W^{n,1}(J, \mathbb{R}) : a \leq x \leq b\}.$$

The following definition appears in Dhage *et al.* [3].

**Definition 2.4.** A function $\alpha \in W^{n,1}(J, \mathbb{R})$ is called a lower solution of IVP (1.1) if there exists $v_1 \in L^1(J, \mathbb{R})$ with $v_1(t) \in F(t, \alpha(t))$ a.e. $t \in J$ we have that $\alpha^{(n)}(t) \leq v_1(t)$ a.e. $t \in J$ and $\alpha^{(i)}(0) \leq x_i, i = 0, 1, \dots, n-1$. Similarly a function $\beta \in W^{n,1}(J, \mathbb{R})$ is called an upper solution of IVP (1.1) if there exists $v_2 \in L^1(J, \mathbb{R})$ with $v_2(t) \in F(t, \beta(t))$ a.e. $t \in J$ we have that $\beta^{(n)}(t) \geq v_2(t)$ a.e. $t \in J$ and $\beta^{(i)}(0) \geq x_i, i = 0, 1, \dots, n-1$.

Now we are ready to prove in the next section our main existence result for the IVP (1.1).

## 3. Existence Result

We consider the following assumptions:

(H1) The multi $F(t, x)$ has compact and convex values for each $(t, x) \in J \times \mathbb{R}$.
(H2) $F(t, x)$ is $L^1$-Carathéodory.
(H3) The IVP (1.1) has a lower solution $\alpha$ and an upper solution $\beta$ with $\alpha \leq \beta$.

**Theorem 3.1.** *Assume that (H1)–(H3) hold. Then the IVP (1.1) has at least one solution $x$ such that*

$$\alpha(t) \leq x(t) \leq \beta(t), \quad \textit{for all} \quad t \in J.$$

*Proof.* First we transform (1.1) into a fixed point inclusion in a suitable Banach space. Consider the IVP

$$\begin{gathered} x^{(n)}(t) \in F(t, \tau x(t)) \quad \text{a.e. } t \in J, \\ x^{(i)}(0) = x_i \in \mathbb{R} \end{gathered} \tag{3.1}$$

for all $i \in \{0, 1, \dots, n-1\}$, where $\tau : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ is the truncation operator defined by

$$(\tau x)(t) = \begin{cases} \alpha(t), & \text{if } x(t) < \alpha(t) \\ x(t), & \text{if } \alpha(t) \leq x(t) \leq \beta(t) \\ \beta(t), & \text{if } \beta(t) < x(t). \end{cases} \tag{3.2}$$

The problem of existence of a solution to (1.1) reduces to finding the solution of the integral inclusion

$$x(t) \in \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} F(s, \tau x(s)) ds, \quad t \in J. \tag{3.3}$$

We study the integral inclusion (3.3) in the space $C(J, \mathbb{R})$ of all continuous real-valued functions on $J$ with a supremum norm $\| \cdot \|_C$. Define a multi-valued map $T : C(J, \mathbb{R}) \to 2^{C(J, \mathbb{R})}$ by

$$Tx = \Big\{ u \in C(J, \mathbb{R}) : u(t) = \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v(s)\, ds, \quad v \in \overline{S_F^1}(\tau x) \Big\} \tag{3.4}$$

where

$$\overline{S_F^1}(\tau x) = \{v \in S_F^1(\tau x) : v(t) \geq \alpha(t) \text{ a.e. } t \in A_1 \text{ and } v(t) \leq \beta(t), \text{ a.e. } t \in A_2\}$$

and

$$A_1 = \{t \in J : x(t) < \alpha(t) \le \beta(t)\},$$
$$A_2 = \{t \in J : \alpha(t) \le \beta(t) < x(t)\},$$
$$A_3 = \{t \in J : \alpha(t) \le x(t) \le \beta(t)\}.$$

By Lemma 2.1, $S^1_F(\tau x) \neq \emptyset$ for each $x \in C(J,\mathbb{R})$ which further yields that $\overline{S^1_F}(\tau x) \neq \emptyset$ for each $x \in C(J,\mathbb{R})$. Indeed, if $v \in S^1_F(x)$ then the function $w \in L^1(J,\mathbb{R})$ defined by

$$w = \alpha\chi_{A_1} + \beta\chi_{A_2} + v\chi_{A_3},$$

is in $\overline{S^1_F}(\tau x)$ by virtue of decomposability of $w$.

We shall show that the multi $T$ satisfies all the conditions of Theorem 3.1.

**Step I.** First we prove that $T(x)$ is a convex subset of $C(J,\mathbb{R})$ for each $x \in C(J,\mathbb{R})$. Let $u_1, u_2 \in T(x)$. Then there exists $v_1$ and $v_2$ in $\overline{S^1_F}(\tau x)$ such that

$$u_j(t) = \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v_j(s)ds, \quad j = 1, 2.$$

Since $F(t,x)$ has convex values, one has for $0 \le k \le 1$

$$[kv_1 + (1-k)v_2](t) \in S^1_F(\tau x)(t), \quad \forall t \in J.$$

As a result we have

$$[ku_1 + (1-k)u_2](t) = \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} [kv_1(s) + (1-k)v_2(t)]ds.$$

Therefore $[ku_1 + (1-k)u_2] \in Tx$ and consequently $T$ has convex values in $C(J,\mathbb{R})$.

**Step II.** $T$ maps bounded sets into bounded sets in $C(J,\mathbb{R})$. To see this, let $B$ be a bounded set in $C(J,\mathbb{R})$. Then there exists a real number $r > 0$ such that $\|x\| \le r, \forall x \in B$.

Now for each $u \in Tx$, there exists a $v \in \overline{S^1_F}(\tau x)$ such that

$$u(t) = \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v(s)ds.$$

Then for each $t \in J$,

$$\begin{aligned}
|u(t)| &\le \sum_{i=0}^{n-1} \frac{|x_i| a^i}{i!} + \int_0^t \frac{a^{n-1}}{(n-1)!} |v(s)|ds \\
&\le \sum_{i=0}^{n-1} \frac{|x_i| a^i}{i!} + \int_0^t \frac{a^{n-1}}{(n-1)!} h_r(s)ds \\
&= \sum_{i=0}^{n-1} \frac{|x_i| a^i}{i!} + \frac{a^{n-1}}{(n-1)!} \|h_r\|_{L^1}.
\end{aligned}$$

This further implies that

$$\|u\|_C \le \sum_{i=0}^{n-1} \frac{|x_i a^i}{i!} + \frac{a^{n-1}}{(n-1)!} \|h_r\|_{L^1}$$

for all $u \in Tx \subset \bigcup T(B)$. Hence $\bigcup T(B)$ is bounded.

**Step III.** Next we show that $T$ maps bounded sets into equicontinuous sets. Let $B$ be a bounded set as in step II, and $u \in Tx$ for some $x \in B$. Then there exists $v \in \overline{S^1_F}(\tau x)$ such that

$$u(t) = \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v(s) ds.$$

Then for any $t_1, t_2 \in J$ we have

$$\begin{aligned}
&|u(t_1) - u(t_2)| \\
&\leq \Big| \sum_{i=0}^{n-1} \frac{x_i t_1^i}{i!} - \sum_{i=0}^{n-1} \frac{x_i t_2^i}{i!} \Big| + \Big| \int_0^{t_1} \frac{(t_1-s)^{n-1}}{(n-1)!} v(s) ds - \int_0^{t_2} \frac{(t_2-s)^{n-1}}{(n-1)!} v(s) ds \Big| \\
&\leq |q(t_1) - q(t_2)| + \Big| \int_0^{t_1} \frac{(t_1-s)^{n-1}}{(n-1)!} v(s) ds - \int_0^{t_1} \frac{(t_2-s)^{n-1}}{(n-1)!} v(s) ds \Big| \\
&\quad + \Big| \int_0^{t_1} \frac{(t_2-s)^{n-1}}{(n-1)!} v(s) ds - \int_0^{t_2} \frac{(t_2-s)^{n-1}}{(n-1)!} v(s) ds \Big| \\
&\leq |q(t_1) - q(t_2)| + \int_0^{t_1} \left| \frac{(t_1-s)^{n-1}}{(n-1)!} - \frac{(t_2-s)^{n-1}}{(n-1)!} \right| |v(s)| ds \\
&\quad + \Big| \int_{t_1}^{t_2} \left| \frac{(t_2-s)^{n-1}}{(n-1)!} \right| |v(s)| ds \Big| \\
&\leq |q(t_1) - q(t_2)| + |p(t_1) - p(t_2)| \\
&\quad + \frac{1}{(n-1)!} \int_0^{t_1} \big| (t_1-s)^{n-1} - (t_2-s)^{n-1} \big| \, \|F(s, u(s))\| \, ds \\
&\leq |q(t_1) - q(t_2)| + |p(t_1) - p(t_2)| \\
&\quad + \frac{1}{(n-1)!} \int_0^{a} \big| (t_1-s)^{n-1} - (t_2-s)^{n-1} \big| \, h_r(s) \, ds
\end{aligned}$$

where

$$q(t) = \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} \quad \text{and} \quad p(t) = \int_0^t \frac{(a-s)^{n-1}}{(n-1)!} h_r(s) ds.$$

Now the functions $p$ and $q$ are continuous on the compact interval $J$, hence they are uniformly continuous on $J$. Hence we have

$$|u(t_1) - u(t_2)| \to 0 \quad \text{as } t_1 \to t_2.$$

As a result $\bigcup T(B)$ is an equicontinuous set in $C(J, \mathbb{R})$. Now an application of Arzelá-Ascoli theorem yields that the multi $T$ is totally bounded on $C(J, \mathbb{R})$.

**Step IV.** Next we prove that $T$ has a closed graph. Let $\{x_n\} \subset C(J, \mathbb{R})$ be a sequence such that $x_n \to x_*$ and let $\{y_n\}$ be a sequence defined by $y_n \in Tx_n$ for each $n \in \mathbb{N}$ such that $y_n \to y_*$. We just show that $y_* \in Tx_*$. Since $y_n \in Tx_n$, there exists a $v_n \in \overline{S^1_F}(\tau x_n)$ such that

$$y_n(t) = \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v_n(s) ds.$$

Consider the linear and continuous operator $\mathcal{K} : L^1(J,\mathbb{R}) \to C(J,\mathbb{R})$ defined by

$$\mathcal{K}v(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v(s)ds.$$

Now

$$\begin{aligned}
\Big| y_n(t) - \sum_{i=0}^{n-1} \frac{|x_i| t^i}{i!} - y_*(t) - \sum_{i=0}^{n-1} \frac{|x_i| t^i}{i!} \Big| \\
&\le |y_n(t) - y_*(t)| \\
&\le \|y_n - y_*\|_C \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

From Lemma 2.2 it follows that $(\mathcal{K} \circ \overline{S^1_F})$ is a closed graph operator and from the definition of $\mathcal{K}$ one has

$$y_n(t) - \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} \in (\mathcal{K} \circ \overline{S^1_F}(\tau x_n)).$$

As $x_n \to x_*$ and $y_n \to y_*$, there is a $v_* \in \overline{S^1_F}(\tau x_*)$ such that

$$y_* = \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v_*(s)ds.$$

Hence the multi $T$ is an upper semi-continuous operator on $C(J,\mathbb{R})$.

**Step V.** Finally we show that the set

$$\mathcal{E} = \{x \in C(J,\mathbb{R}) : \lambda x \in Tx \quad \text{for some } \lambda > 1\}$$

is bounded. Let $u \in \mathcal{E}$ be any element. Then there exists a $v \in \overline{S^1_F}(\tau x)$ such that

$$u(t) = \lambda^{-1} \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \lambda^{-1} \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v(s)ds.$$

Then

$$|u(t)| \le \sum_{i=0}^{n-1} \frac{|x_i| a^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} |v(s)|ds.$$

Since $\tau x \in [\alpha,\beta], \forall x \in C(J,\mathbb{R})$, we have

$$\|\tau x\|_C \le \|\alpha\|_C + \|\beta\|_C := l.$$

By (H2) there is a function $h_l \in L^1(J,\mathbb{R})$ such that

$$\|F(t,\tau x)\| = \sup\{|u| : u \in F(t,\tau x)\} \le h_l(t) \quad \text{a.e. } t \in J$$

for all $x \in C(J,\mathbb{R})$. Therefore

$$\|u\|_C \le \sum_{i=0}^{n-1} \frac{|x_i| a^i}{i!} + \frac{a^{n-1}}{(n-1)!} \int_0^a h_l \, ds = \sum_{i=0}^{n-1} \frac{|x_i| a^i}{i!} + \frac{a^{n-1}}{(n-1)!} \|h_l\|_{L^1}$$

and so, the set $\mathcal{E}$ is bounded in $C(J,\mathbb{R})$.

Thus $T$ satisfies all the conditions of Theorem 2.1 and so an application of this theorem yields that the multi $T$ has a fixed point. Consequently (3.2) has a solution $u$ on $J$.

Next we show that $u$ is also a solution of (1.1) on $J$. First we show that $u \in [\alpha,\beta]$. Suppose not. Then either $\alpha \not\le u$ or $u \not\le \beta$ on some subinterval$J'$ of $J$. If $u \not\ge \alpha$, then there exist $t_0, t_1 \in J, t_0 < t_1$ such that $u(t_0) = \alpha(t_0)$ and $\alpha(t) > u(t)$ for all $t \in (t_0, t_1) \subset J$. From the definition of the operator $\tau$ it follows that

$$u^{(n)}(t) \in F(t, \alpha(t)) \quad \text{a.e. } t \in J.$$

Then there exists a $v(t) \in F(t, \alpha(t))$ such that $v(t) \geq v_1(t), \forall t \in J$ with

$$u^{(n)}(t) = v(t) \quad \text{a.e. } t \in J.$$

Integrating from $t_0$ to $t$ $n$ times yields

$$u(t) - \sum_{i=0}^{n-1} \frac{u_i(0)(t-t_0)^i}{i!} = \int_{t_0}^{t} \frac{(t-s)^{n-1}}{(n-1)!} v(s) ds.$$

Since $\alpha$ is a lower solution of (1.1), we have

$$\begin{aligned} u(t) &= \sum_{i=0}^{n-1} \frac{u_i(0)(t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{n-1}}{(n-1)!} v(s) ds \\ &\geq \sum_{i=0}^{n-1} \frac{\alpha_i(0)(t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{n-1}}{(n-1)!} \alpha(s) ds \\ &= \alpha(t) \end{aligned}$$

for all $t \in (t_0, t_1)$. This is a contradiction. Similarly if $u \not\leq \beta$ on some subinterval of $J$, then also we get a contradiction. Hence $\alpha \leq u \leq \beta$ on $J$. As a result (3.2) has a solution $u$ in $[\alpha, \beta]$. Finally since $\tau x = x, \forall x \in [\alpha, \beta]$, $u$ is a required solution of (1.1) on $J$. This completes the proof. $\square$

## 4. Existence of Extremal Solutions

In this section we establish the existence of extremal solutions to (1.1) when the multi-map $F(t, x)$ is isotone increasing in $x$. Here our technique involves combining method of upper and lower solutions with an algebraic fixed point theorem of Dhage [6] on ordered Banach spaces.

Define a cone $K$ in $C(J, \mathbb{R})$ by

$$K = \{x \in C(J, \mathbb{R}) : x(t) \geq 0, \forall t \in J\}. \tag{4.1}$$

Then the cone $K$ defines an order relation, $\leq$, in $C(J, \mathbb{R})$ by

$$x \leq y \quad \text{iff} \quad x(t) \leq y(t), \quad \forall t \in J. \tag{4.2}$$

It is known that the cone $K$ is normal in $C(J, \mathbb{R})$. See Heikkila and Laksmikantham [8] and the references therein. For any $A, B \in 2^{C(J,\mathbb{R})}$ we define the order relation, $\leq$, in $2^{C(J,\mathbb{R})}$ by

$$A \leq B \quad \text{iff} \quad a \leq b, \quad \forall a \in A \quad \text{and} \quad \forall b \in B. \tag{4.3}$$

In particular, $a \leq B$ implies that $a \leq b, \quad \forall b \in B$ and if $A \leq A$, then it follows that $A$ is a singleton set.

**Definition 4.1.** A multi-map $T : C(J, \mathbb{R}) \to 2^{C(J,\mathbb{R})}$ is said to be isotone increasing if for any $x, y \in C(J, \mathbb{R})$ with $x < y$ we have that $Tx \leq Ty$.

We need the following fixed point theorem of Dhage [6] in the sequel.

**Theorem 4.2.** *Let $[\alpha,\beta]$ be an order interval in a Banach space $X$ and let $T : [\alpha,\beta] \to 2^{[\alpha,\beta]}$ be a completely continuous and isotone increasing multi-map. Further if the cone $K$ in $X$ is normal, then $T$ has a least $x_*$ and a greatest fixed point $y^*$ in $[\alpha,\beta]$. Moreover, the sequences $\{x_n\}$ and $\{y_n\}$ defined by $x_{n+1} \in Tx_n, x_0 = \alpha$ and $y_{n+1} \in Ty_n, y_0 = \beta$, converge to $x_*$ and $y^*$ respectively.*

We consider the following assumptions in the sequel.

(H4) The multi-map $F(t,x)$ is Carathéodory.

(H5) $F(t,x)$ is nondecreasing in $x$ almost everywhere for $t \in J$, i.e. if $x < y$, then $F(t,x) \le F(t,y)$ almost everywhere for $t \in J$.

**Remark 4.3.** Suppose that hypotheses (H3)–(H5) hold. Then the function $h : J \to \mathbb{R}$ defined by

$$h(t) = \|F(t,\alpha(t))\| + \|F(t,\beta(t))\|, \quad \text{for } t \in J,$$

is Lebesque integrable and that

$$|F(t,x)| \le h(t), \quad \forall t \in J,\ \forall x \in [\alpha,\beta].$$

**Definition 4.4.** A solution $x_M$ of (1.1) is called maximal if for any other solution of (1.1) we have that $x(t) \le x_M(t), \forall t \in J$. Similarly a minimal solution $x_m$ of (1.1) is defined.

**Theorem 4.5.** *Assume that hypotheses (H1), (H3), (H4) and (H5) hold. Then IVP (1.1) has a minimal and a maximal solution on $J$.*

*Proof.* Clearly (1.1) is equivalent to the operator inclusion

$$x(t) \in Tx(t),\ t \in J \tag{4.4}$$

where the multi-map $T : C(J,\mathbb{R}) \to 2^{C(J,\mathbb{R})}$ is defined by

$$Tx = \Big\{u \in C(J,\mathbb{R}) : u(t) = \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v(s)ds, \quad v \in S_F^1(x)\Big\}.$$

We show that the multi-map $T$ satisfies all the conditions of Theorem 4.2. First we show that $T$ is isotone increasing on $C(J,\mathbb{R})$. Let $x, y \in C(J,\mathbb{R})$ be such that $x < y$. Let $\alpha \in Tx$ be arbitrary. Then there is a $v_1 \in S_F^1(x)$ such that

$$\alpha(t) = \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v_1(s)ds.$$

Since $F(t,x)$ is nondecreasing in $x$ we have that $S_F^1(x) \le S_F^1(y)$. As a result for any $v_2 \in S_F^1(y)$ one has

$$\alpha(t) \le \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v_2(s)ds = \beta(t)$$

for all $t \in J$ and any $\beta \in Ty$. This shows that the multi-map $T$ is isotone increasing on $C(J,\mathbb{R})$ and in particular on $[\alpha,\beta]$. Since $\alpha$ and $\beta$ are lower and upper solutions of IVP (1.1) on $J$, we have

$$\alpha(t) \le \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v(s)ds, \quad t \in J$$

for all $v \in S^1_F(\alpha)$, and so $\alpha \leq T\alpha$. Similarly $T\beta \leq \beta$. Now let $x \in [\alpha, \beta]$ be arbitrary. Then by the isotonicity of $T$

$$\alpha \leq T\alpha \leq T\beta \leq \beta.$$

Therefore, $T$ defines a multi-map $T : [\alpha, \beta] \to 2^{[\alpha,\beta]}$. Finally proceeding as in Theorem 3.1, is proved that $T$ is a completely continuous multi-operator on $[\alpha, \beta]$. Since $T$ satisfies all the conditions of Theorem 4.2 and the cone $K$ in $C(J, \mathbb{R})$ is normal, an application of Theorem 4.2 yields that $T$ has a least and a greatest fixed point in $[\alpha, \beta]$. This further implies that the IVP (1.1) has a minimal and a maximal solution on $J$. This completes the proof. □

**Conclusion.** We remark that when $n = 2$ in (1.1) we obtain the existence of solution of the second order differential inclusions studied in Benchohra [2]. Again IVP (1.1) and its special cases have been discussed in Dhage and Kang [4], Dhage *et al.* [3], [5] for the existence of extremal solutions via a different approach and under the weaker continuity condition of the multifunction involved in the differential inclusions.

**Acknowledgment.** The authors are thankful to the anonymous referee for his/her helpful suggestions for the improvement of this paper.

## References

[1] Bernfield S. and Lakshmikantham V., *An Introduction to Boundary Value Problems*, Academic Press, New York, 1974.
[2] Benchohra M., *Upper and lower solutions method for second order differential inclusions*, Dynam. Systems Appl. **11** (2002), 13-20.
[3] Agarwal, R., Dhage B. C., and O'Regan, D., *The upper and lower solution method for differential inclusions via a lattice fixed point theorem*, Dynamic Systems Appl. **12** (2003), 1-7.
[4] Dhage B. C. and Kang, S.M., *Upper and lower solutions method for first order discontinuous differential inclusions*, Math. Sci. Res. J. **6** (2002), 527-533.
[5] Dhage B. C., Holambe, T. L. and Ntouyas S. K., *Upper and lower solutions method for second order discontinuous differential inclusions*, Math. Sci. Res. J. **7** (2003), 2o6-212.
[6] Dhage B. C., *A fixed point theorem for multi-valued mappings in Banach spaces with applications*, Nonlinear Anal. (to appear).
[7] Halidias N. and Papageorgiou N., *Second order multi-valued boundary value problems*, Arch. Math. (Brno) **34** (1998), 267-284.
[8] Heikila S. and Lakshmikantham V., *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*, Marcel Dekker Inc., New York, 1994.
[9] Lasota , A. and Opial,Z., *An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations*, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astronom. Phys. **13** (1965), 781-786.
[10] Martelli,M., *A Rothe's type theorem for non compact acyclic-valued maps*, Boll. Un. Mat. Ital. **4** (Suppl. Fasc.) (1975), 70-76.

Bupurao C. Dhage
Kasubai, Gurukul Colony, Ahmedpur-413 515, Dist: Latur, Maharashtra, India
*E-mail address*: bcd20012001@yahoo.co.in

Tarachand L. Holambe
GMCT's ACS College, Shankarnagar -431 505, Dist: Nanded, Maharashtra, India

Sotiris K. Ntouyas
Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece
*E-mail address*: sntouyas@cc.uoi.gr