Electronic Journal of Differential Equations, Vol. **2002**(2002), No. 45, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

Existence and regularity of a global attractor for doubly nonlinear parabolic equations *

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Abstract

In this paper we consider a doubly nonlinear parabolic partial differential equation

$$\frac{\partial \beta(u)}{\partial t} - \Delta_p u + f(x,t,u) = 0 \quad \text{in } \Omega \times \mathbb{R}^+,$$

with Dirichlet boundary condition and initial data given. We prove the existence of a global compact attractor by using a dynamical system approach. Under additional conditions on the nonlinearities β, f, and on p, we prove more regularity for the global attractor and obtain stabilization results for the solutions.

1 Introduction

This paper is devoted to the study of a doubly nonlinear parabolic P.D.E. related to the p-Laplacian operator. More precisely, we are interested in the existence, uniqueness and long time behaviour of the solutions of problem

$$\begin{gathered} \frac{\partial \beta(u)}{\partial t} - \Delta_p u + f(x,t,u) = 0 \quad \text{in } \Omega \times (0,\infty) \\ u = 0 \quad \text{on } \partial\Omega \times (0,\infty) \\ \beta(u(.,0)) = \beta(u_0) \quad \text{in } \Omega, \end{gathered} \tag{1.1}$$

where $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$, $1 < p < +\infty$ and Ω is a regular bounded open subset of $\mathbb{R}^N$, $N \geq 1$.

These problems arise in many applications in the fields of mechanics, physics and biology (non Newtonian fluids, gas flow in porous media, spread of biological populations, etc.). There are a lot of works dedicated to the existence of solutions [1, 2, 3, 5, 15] and to the large time behaviour of these equations [4, 6, 10, 13, 16, 20].

* *Mathematics Subject Classifications:* 35K15, 35K60, 35K65.
Key words: p-Laplacian, a-priori estimate, long time behaviour, dynamical system, absorbing set, global attractor.

Submitted January 15, 2001. Published May 24, 2002.

Our work is inspired by the results of El Hachimi and de Thélin [8, 9] and of Eden, Michaux and Rakotoson [6]. The aim here is to study the long time behaviour of solutions of (1.1) via a dynamical systems approach (in the framework of Foias and Temam [11]). As is well known, the presence of a dissipative term, in many infinete dimensional systems, implies the existence of a compact set $\mathcal{A}$ which attracts all the trajectories. This set, called the global attractor, has usually finite Haussdorf and fractal dimensions, and it is studied by reducing it to a finite dimensional system.

For $p = 2$, problem (1.1) has been studied in [6, 7]. Here, we shall consider general p under the same assumptions on β and f as in these references, and extend some of the results therein.

This paper is organized as follows: After some preliminaries in Section 2, we give, in section 3, an existence result for solutions of problem (1.1). Then section 4 is devoted to the existence of the global attractor $\mathcal{A}$. Finally in section 5 we give, under restrictive conditions on β, f, p, a supplementary regularity result for $\mathcal{A}$ and a stabilization result for the solutions of (1.1).

2 Preliminaries

Notation Let β be a continuous function with $\beta(0) = 0$. For $t \in \mathbb{R}$, we define $\Psi(t) = \int_0^t \beta(\tau)d\tau$. Then the Legendre transform of Ψ is defined as $\Psi^*(\tau) = \sup_{s\in\mathbb{R}}\{\tau s - \Psi(s)\}$. Let Ω be a regular open bounded subset of $\mathbb{R}^N$ and $\partial\Omega$ its boundary. For $T > 0$, we set $Q_T = \Omega \times (0, T)$ and $S_T = \partial\Omega \times (0, T)$. The norm in a space X will be denoted by $\|\cdot\|_X$. However, $\|\cdot\|_r$ is the norm when $X = L^r(\Omega)$ with $1 \le r \le +\infty$, and $\|\cdot\|_{1,q}$ when $X = W^{1,q}(\Omega)$ with $1 \le q \le +\infty$. Let $\langle\cdot,\cdot\rangle_{X,X'}$ denote the duality product between X and its dual X'. For $l > 1$ we denote by ℓ' the conjugate of ℓ; that is the real number l' satisfying $\frac{1}{l} + \frac{1}{l'} = 1$. For $1 \le r < +\infty$, we shall denote by $W_r^{2,1}((0, T) \times \Omega)$ the set of all functions v such that

$$\int_0^T \int_\Omega \Big(|v|^r + |Dv|^r + |D^2v|^r + \big|\frac{\partial v}{\partial T}\big|^r\Big)dx\,dt < \infty.$$

We shall consider the following hypotheses.

(H1) u_0 and $\beta(u_0)$ are in $L^2(\Omega)$.

(H2) β is an increasing locally Lipschitzian function from $\mathbb{R}$ to $\mathbb{R}$, with $\beta(0) = 0$.

(H3) For each $\zeta \in \mathbb{R}$, the map $(x, t) \to f(x, t, \zeta)$ is measurable and $\zeta \to f(x, t, \zeta)$ is continuous almost everywhere in $\Omega \times \mathbb{R}^+$. Furthermore, we assume that there exist positive constants c_1, c_2, c_3 such that, for a.e $(x, t) \in \Omega \times \mathbb{R}^+$,

$$\begin{gathered}\operatorname{sign}(\xi)f(x, t, \xi) \ge c_1|\beta(\xi)|^{q-1} - c_2,\\ \lim_{t\to0^+}\sup|f(x, t, \xi)| \le c_3(|\xi|^{q-1} + 1)\end{gathered} \tag{2.1}$$

with $q > \sup(2, p)$. Also assume that $|f(x, t, \xi)| \le a(|\xi|)$ almost everywhere in $\Omega \times \mathbb{R}^+$, where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing function.

(H4) For each $M > 0$ and $|\zeta)| \leq M$, $\frac{\partial f}{\partial t}(x,t,\zeta)$ exists, there exists a positive constant C_M such that $|\frac{\partial f}{\partial t}(x,t,\zeta)| \leq C_M$ for almost every $(x,t) \in \Omega \times \mathbb{R}^+$.

(H5) There exist $c_4 > 0$ such that $\zeta \to f(x,t,\zeta) + c_4\beta(\zeta)$, is increasing for almost $(x,t) \in \Omega \times \mathbb{R}$.

Remarks (i) By hypothesis (H5) and properties of β, if the function f_0 : $(x,t) \to |f(x,t,0)|$ is bounded by a positive constant d, for a.e. $(x,t) \in \Omega \times \mathbb{R}^+$,

$$\operatorname{sign}(u) f(x,t,u) \geq c_3|\beta(u)| - d. \tag{2.2}$$

When this condition is satisfied, Condition (2.1) is also satisfied.
(ii) From (H1), it follows that $\Psi^*(\beta(u_0)) \in L^1(\Omega)$.
(iii) When β satisfies the condition $|\beta(s)| \leq d_1|s| + d_2$, for any $s \in \mathbb{R}$, with positive constants d_1 and d_2, as in [6], we have the implications:

$$u_0 \in L^2(\Omega) \Rightarrow \beta(u_0) \in L^2(\Omega) \Rightarrow \Psi^*(\beta(u_0)) \in L^2(\Omega).$$

Definition By a weak solution to (1.1), we mean a function u such that:

$$u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^q(0,T;L^q(\Omega)) \cap L^\infty(\tau,T;L^\infty(\Omega)) \quad \forall \tau > 0,$$
$$\frac{\partial \beta(u)}{\partial t} \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^{q'}(0,T;L^{q'}(\Omega)),$$

for all $\phi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega))$ it holds

$$\int_0^T \langle \frac{\partial \beta(u)}{\partial t}, \phi \rangle_{X,X'} dt + \int_0^T \int_\Omega F(\nabla u)\nabla \phi \, dx dt = -\int_0^T \int_\Omega f(x,t,u)\phi \, dx dt;$$

and if $\frac{\partial \phi}{\partial t} \in L^2(0,T;L^2(\Omega))$, with $\phi(T) = 0$, then

$$\int_0^T \langle \frac{\partial \beta(u)}{\partial t}, \phi \rangle_{X,X'} dt = -\int_0^T \int_\Omega (\beta(u) - \beta(u_0)) \frac{\partial \phi}{\partial t} dx dt,$$

where $X = L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$, $X' = L^1(\Omega) + W^{-1,p'}(\Omega)$ and $F(\xi) = |\xi|^{p-2}\xi$ for any $\xi \in \mathbb{R}^N$.

3 Existence and uniqueness

Our main result reads as follows.

Theorem 3.1 *Under Hypotheses (H1)-(H5), Problem (1.1) has a weak solution u such that $u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(\tau,T;W_0^{1,p}(\Omega) \cap L^\infty(\Omega))$, for all $\tau > 0$ and $\beta(u) \in L^q(Q_T) \cap L^\infty(0,T;L^2(\Omega))$.*

Remark For a solution u of (1.1), by the first equation in (3.1), we have

$$\frac{\partial \beta(u)}{\partial t} \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^{q'}(0,T;L^{q'}(\Omega)).$$

Since $q > \sup(2,p)$, we get $\beta(u) \in L^q(Q_T) \cap L^\infty(0,T;L^2(\Omega))$ wich is a subset of $L^{q'}(0,T;L^{q'}(\Omega) + W_0^{-1,p'}(\Omega))$. Thus, from Lion's lemma of compactness [14, p.23], we deduce that at least $\beta(u)$ is in $C(0,T;L^{q'}(\Omega))$; so that the third condition (1.1) makes sense.

Proof of the main result

a) Existence. The proof of Theorem 3.1 is based on a priori estimates. From β, we construct a sequence $\beta_\varepsilon \in C^1(\mathbb{R})$ such that: $\varepsilon \le \beta'_\varepsilon$, $\beta_\varepsilon(0) = 0$, $\beta_\varepsilon \to \beta$ in $C_{\rm loc}(\mathbb{R})$ and $|\beta_\varepsilon| \le |\beta|$.

Let $(u_{0\varepsilon})_{\varepsilon>0}$ be a sequence in $D(\Omega)$ such that $u_{0\varepsilon} \to u_0$ almost everywhere in Ω and $\|u_{0\varepsilon}\|_{L^2(\Omega)}, \|\beta_\varepsilon(u_{0\varepsilon})\|_{L^2(\Omega)} \le c$, with a constant $c > 0$. Consider the problem

$$\begin{gathered}
\frac{\partial \beta_\varepsilon(u_\epsilon)}{\partial t} - \operatorname{div} F_\varepsilon(\nabla u_\varepsilon) + f(x,t,u_\epsilon) = 0 \quad \text{in } Q_T \\
u_\epsilon = 0 \quad \text{in } S_T \\
\beta_\varepsilon(u_\epsilon)_{|t=0} = \beta_\varepsilon(u_{0\varepsilon}) \quad \text{in } \Omega,
\end{gathered} \tag{3.1}$$

where $F_\varepsilon(\xi) = (|\xi|^2 + \varepsilon)^{(p-2)/2}\xi$, for $\xi \in \mathbb{R}^N$.

Remark In this paper, we shall denote by c_i different constants, depending on p and Ω, but not on ε, or T. Sometimes we shall refer to a constant depending on specific parameters: $c(\tau)$, $c(T)$, $c(\tau,T)$, etc.

Lemma 3.2 *There exists a unique solution of (3.1), such that $u_\varepsilon \in L^\infty(Q_T) \cap L^\infty(0,T;W_0^{1,p}(\Omega))$. Moreover, $u_\varepsilon \in W_r^{2,1}((0,T)\times\Omega)$ for $1 \le r < \infty$,*

Proof. The proof is similar to that in [6, lemma 5] and we shall give here only a sketch. For a fixed positive integer m, consider the function

$$f_m(x,t,u) = \begin{cases} f(x,t,u) & \text{if } |\beta(u)| \le m \\ c_1(|\beta(u)|^{q-1} - m^{q-1})\operatorname{sign}(u) & \\ +f(x,t,\beta^{-1}(u)\operatorname{sign}(u))) & \text{otherwise.} \end{cases}$$

Then

$$\operatorname{sign}(u) f_m(x,t,u) \ge c_1|\beta_\varepsilon(u)|^{q-1} - c_2.$$

Indeed, if $|\beta(u)| \le m$, by properties of β_ε, we get

$$\operatorname{sign}(u) f_m(x,t,u) = \operatorname{sign}(u) f(x,t,u) \ge c_1|\beta(u)|^{q-1} - c_2 \ge c_1|\beta_\varepsilon(u)|^{q-1} - c_2,$$

and if $|\beta(u)| \geq m$ then, as $\operatorname{sign}(u)/\operatorname{sign}(\beta^{-1}(m\operatorname{sign}(u))) = 1$, we deduce by properties of β_ε that

$$\begin{aligned}\operatorname{sign}(u) f_m(x,t,u) &\geq c_1(|\beta(u)|^{q-1} - m^{q-1}) + c_1|\beta(\beta^{-1}(m\operatorname{sign}(u)))|^{q-1} - c_2 \\ &\geq c_1|\beta(u)|^{q-1} - c_2 \geq c_1|\beta_\varepsilon(u)|^{q-1} - c_2.\end{aligned}$$

For $\sigma \in [0,1]$, define the map $K(\sigma,.)$ by $K(\sigma,v) = u_{\varepsilon,\sigma}$ which is the solution to

$$\begin{gathered}\frac{\partial \beta_\varepsilon(u_{\epsilon,\sigma})}{\partial t} - divF_\varepsilon(\nabla u_{\varepsilon,\sigma}) + \sigma f_m(x,t,v) = 0 \quad \text{in } Q_T, \\ u_{\epsilon,\sigma} = 0 \quad \text{in } S_T, \\ \beta_\varepsilon(\,u_{\epsilon,\sigma})_{|t=0} = \beta_\varepsilon(\sigma u_{0\varepsilon}) \quad \text{in } \Omega,\end{gathered} \tag{3.2}$$

For each $\sigma \in [0,1]$, the operator $K(\sigma,.)$ is compact from $L^p(0,T;W_0^{1,p}(\Omega))$ into itself. Indeed, for a fixed $v \in L^p(0,T;W_0^{1,p}(\Omega))$, one has a unique solution $u_{\varepsilon,\sigma} \in L^p(0,T;W_0^{1,p}(\Omega)) \cap W_r^{2,1}((0,T)\times\Omega)$ by using the theory of Ladyzenskaya et al [12, chap. V]. Therefore, arguing exactly as in [6, Lemma5], we deduce that, for each $\sigma \in [0,1]$, $K(\sigma,.)$ is a compact operator from $L^p(0,T;W_0^{1,p}(\Omega))$ into itself and that the map $\sigma \to K(\sigma,.)$ is continuous and $K(0,v) = u_{\varepsilon,0} = 0$. Thus, from Leray-Schauder fixed-point theorem, there exists a fixed point $u_\varepsilon \equiv u_{\varepsilon,1} = K(1,v)$. Moreover, arguing also as in [6, Lemma 5] and using (3.6), we obtain $|\beta_\varepsilon(u_\varepsilon)|_{L^\infty(0,T;L^\infty(\Omega))} \leq c(u_{0\varepsilon})$, where $c(u_{0\varepsilon})$ is a positive constant depending only on $u_{0\varepsilon}$. Thus, $f_m(x,t,u_\varepsilon) = f(x,t,u_\varepsilon)$ for $m \geq c(u_{0\varepsilon})$ and then u_ε is a solution of (3.1).

The uniqueness property of a solutions can be derived from [4, Theorem 3, p. 1095]. If we show that $\frac{\partial\beta_\varepsilon(u_\varepsilon)}{\partial t} \in L^2(0,T;L^2(\Omega))$. To avoid repetition, we claim that it is a consequence of Lemma 3.4 below.

Now we give the a priori estimates needed for the remainder of the proof.

Lemma 3.3 *Under the hypothesis (H1)-(H3), there exists constants c_i such that for any $\varepsilon \in]0,1[$ and any $\tau > 0$, the following estimates hold*

$$\|u_\epsilon\|_{L^\infty(\tau,T;L^\infty(\Omega))} \leq c_4(\tau,T), \tag{3.3}$$

$$\|\beta_\varepsilon(u_\epsilon)\|_{L^\infty(0,T;L^2(\Omega))\cap L^q(Q_T)} \leq c_5(T) \tag{3.4}$$

$$|u|_{L^p(0,T;W_0^{1,p}(\Omega))} \leq c_6(T). \tag{3.5}$$

Proof (i) Multiplying the first equation in (3.1) by $|\beta_\varepsilon(u_\epsilon)|^k\beta_\varepsilon(u_\epsilon)$ and using the growth condition on f and the properties of β_ε, we deduce that

$$\frac{1}{k+2}\frac{d}{dt}\int_\Omega |\beta_\varepsilon(u_\epsilon)|^{k+2}dx + c_{14}\int_\Omega |\beta_\varepsilon(u_\epsilon)|^{k+q}dx \leq c_{15}\int_\Omega |\beta_\varepsilon(u_\epsilon)|^{k+1}dx \tag{3.6}$$

Setting $y_{\varepsilon,k}(t) = \|\beta_\varepsilon(u_\epsilon)\|_{L^{k+2}(\Omega)}$ and using Hölder's inequality on both sides of (3.6), there exist two constants $\alpha_0 > 0$ and $\lambda_0 > 0$ such that

$$\frac{dy_{\varepsilon,k}(t)}{dt} + \lambda_0 y_{\varepsilon,k}^{q-1}(t) \leq \alpha_0;$$

which implies from Ghidaglia's lemma [19] that

$$y_{\varepsilon,k}(t) \leq \Big(\frac{\alpha_0}{\lambda_0}\Big)^{\frac{1}{q-1}} + \frac{1}{[\lambda_0(q-2)t]^{\frac{1}{q-2}}} = c_7(t)\,, \forall t > 0. \tag{3.7}$$

As $k \to +\infty$, and for all $t \geq \tau > 0$, we have

$$|\beta_\varepsilon(u_\epsilon)(t)|_{L^\infty(\Omega)} \leq c_7(\tau); \tag{3.8}$$

which implies

$$|u_\epsilon(t)|_{L^\infty(\Omega)} \leq \max(\beta_\varepsilon^{-1}(c_7(\tau)), |\beta_\varepsilon^{-1}(-c_7(\tau))|) = \delta_\varepsilon\,. \tag{3.9}$$

Since β_ε converges to β in $C_{\rm loc}(\mathbb{R})$, then the sequence δ_ε is bounded in $\mathbb{R}$ as $\varepsilon \to +\infty$. Thus δ_ε is bounded by $\max(\beta^{-1}(c_7(\tau)), |\beta^{-1}(-c_7(\tau))|)$, which is finite. Whence (3.3) is satisfied. On the other hand, taking $k = 0$ in (3.6), using Hölder inequality and integrating on $[0, T]$ yields (3.4).

(ii) Multiplying the first equation in (3.1) by u_ϵ, integrating on Ω and using (2.1) and the properties of β_ε, gives

$$\frac{d}{dt}\Big(\int_\Omega \Psi_\varepsilon^*(\beta_\varepsilon(u_\epsilon))dx\Big) + \int_\Omega (|\nabla u_\epsilon|^2 + \epsilon)^{(p-2)/2}|\nabla u_\epsilon|^2 dx + c_1 \int_\Omega |\beta_\varepsilon(u_\epsilon)|^{q-1}dx \leq c_2, \tag{3.10}$$

where Ψ_ε^* is the Legendre transform of Ψ_ε and $\Psi_\varepsilon(t) = \int_0^t \beta_\varepsilon(s)ds$. By hypotheses (H1) and (H2), and the remark (ii) in Chapter 2, we can assume that $\int_\Omega \Psi_\varepsilon^*(\beta_\varepsilon(u_{0\epsilon}))dx$ converges to $\int_\Omega \Psi^*(\beta(u_\epsilon))dx \leq c$, where c is some positive constant. So, integrating (3.9) from 0 to T yields

$$\int_\Omega \Psi_\varepsilon^*\,(\beta_\varepsilon(u_\epsilon))dx + c_8 \int_0^T \int_\Omega |\,u_\epsilon|^p dxds \leq c_8(T). \tag{3.11}$$

Hence (3.5) follows. □

Lemma 3.4 *Assume (H1)-(H4). Then there exist constants $c_{11}(\tau)$ and $c_i(\tau, T)$ $(i = 9, 10)$ such that for $\varepsilon \in]0, 1[$ the following estimates hold*

$$\|u_\varepsilon\|_{L^\infty(\tau,T;W_0^{1,p}(\Omega))} \leq c_9(\tau, T), \tag{3.12}$$

$$\int_\tau^T \int_\Omega \beta_\varepsilon'(u_\varepsilon)\Big(\frac{\partial u_\varepsilon}{\partial t}\Big)^2 dxds \leq c_{10}(\tau, T) \tag{3.13}$$

$$\int_t^{t+\tau} \int_\Omega \beta_\varepsilon'(u_\varepsilon)\Big(\frac{\partial u_\varepsilon}{\partial t}\Big)^2 dxds \leq c_{11}(\tau), \text{ for any } t \geq \tau > 0. \tag{3.14}$$

Proof. Multiplying the first equation in (3.1) by $\frac{\partial u_\epsilon}{\partial t}$, integrating on Ω and using (3.9) and (H4), it follows that for any $t \geq \tau > 0$,

$$\int_\Omega \beta_\varepsilon'(u_\epsilon)\Big(\frac{\partial u_\epsilon}{\partial t}\Big)^2 dx + \frac{d}{dt}\Big[\frac{1}{p}\int_\Omega (|\nabla u_\epsilon|^2 + \epsilon)^{\frac{p}{2}} dx + \int_\Omega \int_0^{u_\epsilon} f(x, t, y)dy\, dx\Big] \leq |\int_\Omega \int_0^{u_\epsilon} \frac{\partial f}{\partial t}(x, t, y)dydx| \leq c_{12}(\tau)\,, \tag{3.15}$$

where $c_{12}(\tau)$ is some positive constant. Now integrating (3.10) on $[t, t+\frac{\tau}{2}]$ and observing that $\varepsilon \in]0,1[$, yields

$$\int_t^{t+\frac{\tau}{2}} \int_\Omega (|\nabla u_\epsilon|^2 + \epsilon)^{\frac{p}{2}} dxdt \leq c_{13}(\tau) \quad \forall t \geq \frac{\tau}{2}.$$

Furthermore, by (3.9) we have: $|\int_\Omega \int_0^{u_\varepsilon(x,t)} f(x,t,y)dy\,dx| \leq c_{13}(\tau)$. Then, applying the uniform Gronwall's lemma [19, p.89] with $a_1 = c_{13}(\tau)$, $a_2 = c_{14}(\tau)$, $h = c_{12}(\tau)$ and

$$y(t) = \int_\Omega (|\nabla u_\epsilon|^2 + \epsilon)^{p/2} dx + \int_\Omega \int_0^{u_\varepsilon(x,t)} f(x,t,y)dydx,$$

gives

$$\int_\Omega |\nabla u_\epsilon|^p dx + \int_\Omega \int_0^{u_\varepsilon(x,t)} f(x,t,y)dydx \leq \frac{a_1 + a_2}{\tau} + c_{15}(\tau) \quad \forall t \geq \tau > 0. \tag{3.16}$$

By using (3.9) and hypothesis (H4), (3.16) leads to

$$\int_\Omega |\nabla u_\epsilon|^p dx \leq c_{16}(\tau) \forall t \geq \tau > 0. \tag{3.17}$$

Hence (3.12) is satisfied. On the other hand, by the mean value theorem and (3.5), we conclude that for any $\tau > 0$, there exists $\tau_\varepsilon \in]\frac{\tau}{4}, \frac{\tau}{2}[$ such that

$$\int_\Omega |\nabla u_\epsilon(\tau_\varepsilon)|^p dx = \frac{2}{\tau} \int_{\frac{\tau}{4}}^{\frac{\tau}{2}} \int_\Omega |\nabla u_\epsilon|^p dxdt \leq c_{17}(\tau).$$

Now, integrating (3.15) on $[\tau_\varepsilon, T]$ and using (3.9), (3.17) and (H4), we easily deduce (3.13). To conclude (3.14), it suffices to integrate (3.15) on $[t, t+\tau]$ and to use once again (3.9), (3.17) and hypothesis (H4). Whence the lemma is proved. □

As a consequence of Lemma 3.4, we get the following lemma.

Lemma 3.5 *(i) The following estimates hold:*

$$\int_\tau^T \int_\Omega (\frac{\partial \beta_\varepsilon(u_\varepsilon)}{\partial t})^2 dx\,ds \leq c_{18}(\tau, T), \quad \textit{for } T \geq \tau > 0,$$
$$\int_t^{t+\tau} \int_\Omega (\frac{\partial \beta_\varepsilon(u_\varepsilon)}{\partial t})^2 dx\,ds \leq c_{19}(\tau), \quad \textit{for } \tau > 0.$$

(ii) When f does not depend on t,

$$\int_\tau^T \int_\Omega \beta'_\varepsilon(u_\varepsilon)(\frac{\partial u_\varepsilon}{\partial t})^2 dxds \leq c_{22}(\tau), \quad \textit{for } T \geq \tau > 0.$$

Proof. (i) Let L be the Lipschitz constant of β on $[-\delta, \delta]$, where δ is the bound in the proof of lemma 3.3 (i). It is possible to choose β_ε so that $\beta'_\varepsilon \leq L$ on $[-\delta, \delta]$. Then (3.11) implies

$$\frac{1}{L}\int_\tau^T \int_\Omega \big(\frac{\partial \beta_\varepsilon(u_\varepsilon)}{\partial t}\big)^2 dxds \leq c_{23}(\tau, T), \text{ for any } T \geq \tau > 0.$$

(ii) From (3.14), and using the notation on the equation preceding (3.16) now we have

$$\int_\Omega \beta'_\varepsilon(u_\epsilon)((u_\epsilon)_t)^2 dx + \frac{d}{dt}\Big[\int_\Omega \frac{1-p}{p}\left(|\nabla u_\epsilon|^2 + \epsilon\right)^{\frac{p}{2}} dx + y(t)\Big] \leq 0.$$

Integrating this expression on $[\tau_\varepsilon, T]$ and using (3.17), it follows (3.13). $\square$

Passage to the limit in (3.1) as $\varepsilon \to +\infty$. By estimates (3.5) and (3.12), $F_\epsilon(\nabla u_\epsilon)$ is bounded in $L^{p'}(0, T; L^{p'}(\Omega))$. Hence

$$F_\epsilon(\nabla u_\epsilon) \text{ is bounded in } L^{p'}(\tau, T; W^{-1,p'}(\Omega)), \tag{3.18}$$

By Lemma 3.5 (i),

$$\frac{\partial \beta_\varepsilon(u_\epsilon)}{\partial t} \text{ is bounded in } L^2(\tau, T; L^2(\Omega)), \forall \tau > 0. \tag{3.19}$$

Therefore, by estimates (3.3), (3.4), (3.5), (3.8), (3.12) and (3.18), there exists a subsequence (denoted again by u_ε) such that as $\varepsilon \to 0$, we have

$$u_\epsilon \to u \quad \text{weak in } L^p(0, T; W_0^{1,p}(\Omega)), \tag{3.20}$$

$$u_\epsilon \to u \quad \text{weak star in } L^\infty(\tau, T; W_0^{1,p}(\Omega)), \quad \forall \tau > 0, \tag{3.21}$$

$$\operatorname{div} F_\epsilon(\nabla u_\epsilon) \to \chi \quad \text{weak in } L^{p'}(0, T; W^{-1,p'}(\Omega)), \tag{3.22}$$

$$\beta_\varepsilon(u_\varepsilon) \to \xi \quad \text{weak in } L^q(Q_T), \tag{3.23}$$

$$\beta_\varepsilon(u_\varepsilon) \to \xi \quad \text{weak star in } L^\infty(\tau, T; L^\infty(\Omega)). \tag{3.24}$$

Now according to (3.9), (3.19), (3.23), (3.24), and Aubin's lemma [17, Corol. 4], we derive that $\beta_\varepsilon(u_\varepsilon) \to \xi$ strongly in $C([0, T], L^2(\Omega))$ and by a similar way as that in ([3], page 1048), we consequently obtain $\beta(u) = \xi$. Moreover standard monotonicity argument [3, 14] gives $\chi = \operatorname{div} F(\nabla u)$.

To conclude that u is a weak solution of (1.1) it suffices to observe, as in [6, p. 108], that $f(x, t, u_\varepsilon) \to f(x, t, u)$ strongly in $L^1(Q_T)$ and in $L^s(\tau, T; L^s(\Omega))$ for all $\tau > 0$ and for all $s \geq 1$, as $\varepsilon \to 0$. (One should use the growth condition on f_ε and Vitali's theorem).

b) Uniqueness. By Lemma 3.4, the solutions of (1.1) satisfy

$$\frac{\partial \beta(u)}{\partial t} \in L^2(\tau, T; L^2(\Omega)) \quad \forall \tau > 0.$$

Therefore, by [4, Theorem 3, p. 1095], we deduce that the solution is unique. $\square$

Corollary 3.6 *Under the hypotheses of Theorem 3.1 with f independent of time, Problem (1.1) generates a continuous semi-group $S(t : L^2(\Omega) \to L^2(\Omega)$ defined by $S(t)u_0 = \beta(u(t,.))$. Moreover the solution of problem (1.1) satisfies $\frac{\partial\beta(u)}{\partial t} \in L^2(\tau,+\infty; L^2(\Omega))$ for all $\tau > 0$.*

4 Existence and regularity of the attractor

For the concepts of absorbing sets and global attractors used here, we refer the reader to [19]. Using estimates in Lemma 3.3, we deduce the following statement.

Proposition 4.1 *Under hypotheses (H1)-(H5), the semi-group S(t) associated with problem (1.1) is such that*

(i) There exist absorbing sets in $L^\sigma(\Omega)$, for $1 \le \sigma \le +\infty$.

(ii) There exist absorbing sets in $W_0^{1,p}(\Omega)$.

Proof. Let u be solution of (1.1) and u_ε solution of (3.1) approximating u, then for fixed $t \ge \tau > 0$, (3.9) and Sobolev's injection theorem imply

$$\|u_\varepsilon(t)\|_{L^\sigma(\Omega)} \le c_\delta, \quad \text{for any } \sigma : 1 \le \sigma < \infty, \tag{4.1}$$

where c_σ is some positive constant depending on meas(Ω) and δ, with $\delta = \max(\beta^{-1}(c(\tau)), |\beta^{-1}(-c(\tau))|)$ as in the proof of Lemma 3.3 (i). From (4.1), we then obtain

$$\|u(t)\|_{L^\sigma(\Omega)} \le c_\delta \text{ for any } \sigma : 1 \le \sigma < \infty. \tag{4.2}$$

By letting σ tends to $+\infty$ in (4.2), we obtain

$$\|u(t)\|_{L^\infty(\Omega)} \le c_\delta. \tag{4.3}$$

Thus, by (4.2) and (4.3), the open ball $B(0, c_\delta)$ centered at 0 and with radius c_δ is an absorbing set in $L^\sigma(\Omega)$, $1 \le \sigma \le +\infty$. On the other hand, by (3.16), (3.20) and the lower semi-continuity of the norm, we get

$$\int_\Omega |\nabla u|^p(t)dx \le c_{16}(\tau), \text{ for any } t \ge \tau.$$

Therefore the open ball $B(0, c_{16}(\tau))$ is an absorbing set in $W_0^{1,p}(\Omega)$. Whence part (ii) is verified. *Box*

Assuming that the nonlinear function f does not depend on time, Proposition 4.1 then gives assumptions (1.1), (1.4) and (1.12) of [19, Theorem 1.1, p. 23], with $U = L^2(\Omega)$. So, by means of the uniform compactness lemma in [6, p. 111], we get the following result.

Theorem 4.2 *Assume that (H1)-(H5) are satisfied and that f does not depend on time. Then the semi-group $S(t)$ associated with the boundary value problem (1.1) possesses a maximal attractor A which is bounded in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, compact and connected in $L^2(\Omega)$. Its domain of attraction is the whole space $L^2(\Omega)$.*

5 More regularity for the attractor

In this section we shall show supplementary regularity estimates on the solution of problem (1.1) and by use of them, we shall obtain more regularity on the attractor obtained in Section 4. To this end, we consider the following hypotheses on the data.

(H6) $f(x,t,u) = g(u) - h(x)$, where $h \in L^\infty(\Omega)$ and $g \in C^1(\mathbb{R})$ are such that f satisfies the conditions already prescribed in (H3), (H4) and (H5).

(H7) $\beta \in C^2(\mathbb{R})$ is such that there exist $\sigma_1, \sigma_2 > 0$ with $\sigma_1 \le \beta'(s) \le \sigma_2$ for all $s \in \mathbb{R}$.

Let u_ε be solution of (3.1) with $f = g - h$. For simplicity, we shall denote

$$w := u_\varepsilon, \quad w' = \frac{\partial u_\varepsilon}{\partial t}, \quad w'' = \frac{\partial^2 u_\varepsilon}{\partial t^2}, \quad (E(\nabla w))' = \frac{\partial}{\partial t}(E(\nabla w)),$$

with $E(\xi) = |\xi|^{(p-2)/2}\xi$, for all $\xi \in \mathbb{R}^N$ and $(F_\varepsilon(\nabla w))' = \frac{\partial}{\partial t}(F_\varepsilon(\nabla w))$.

The following two lemmas are used in the proof of the main results of this section.

Lemma 5.1 *For $1 < p < 2$, there exists a positive constant c_{24} such that*

$$\int_\Omega |\nabla w'|^p dx \le c_{24} \int_\Omega |\nabla w|^p dx + \frac{2(p-1)}{p^2} \int_\Omega |\,(E(\nabla w))'\,|^2 dx, \tag{5.1}$$

Proof. Straightforward calculations, [8], give

$$\int_\Omega (F_\varepsilon(\nabla w))' . \nabla w' dx \ge \frac{4(p-1)}{p^2} \int_\Omega |\,(E(\nabla w))'\,|^2 dx. \tag{5.2}$$

Since $\nabla w = |E(\nabla w)|^{\frac{2-p}{p}} E(\nabla w)$, it follows that $\nabla w' = \frac{2}{p}|E(\nabla w)|^{\frac{2-p}{p}}(E(\nabla w))'$. So, as $1 < p < 2$, the Hölder and Young inequalities lead to

$$\begin{aligned} \int_\Omega |\nabla w'|^p dx &= c_{25} \int_\Omega |E(\nabla w)|^{2-p} |(E(\nabla w))'|^p dx \\ &\le \frac{c_{26}}{2} \int_\Omega |\,(E(\nabla w))\,|^2 dx \; + \frac{2(p-1)}{p^2} \int_\Omega |\,(E(\nabla w))'\,|^2 dx, \end{aligned}$$

where $c_{25} = (2/p)^p$ and c_{26} is a positive constant. Hence estimate 5.1 follows. □

Lemma 5.2 *Assuming (H1)-(H8), the sequence $(u_\varepsilon)_{\varepsilon>0}$ converges strongly to the solution u of (1.1) in $L^p(0,T;W^{1,p}(\Omega))$.*

The proof of this lemma is similar to that of [9, Lemma 2] and is omitted here. For stating the next theorem we introduce the hypothesis

(H8) $N = 1$ and $1 < p < 2$ or $N \ge 2$ and $\frac{3N}{N+2} \le p < 2$.

Theorem 5.3 *Let f and β satisfy hypotheses (H1)-(H7), and (H8) be satisfied. Let $y(t) = \int_\Omega \beta'(w)(w')^2 dx$. Then*

$$y(t) \le c_{27}(\tau), \quad \forall t, \tau, \varepsilon \textit{ with } t \ge \tau > 0 \textit{ and } 0 < \varepsilon < 1.$$

Proof. Differentiating equation (3.14) (with $f = g - h$) with respect to t (the justification can be done by passing to finite dimension as in [9]), we get

$$\beta'(w)w'' + \beta''(w)(w')^2 - \operatorname{div}((F_\epsilon(\nabla w))') + g'(w)w' = 0. \tag{5.3}$$

Now multiplying (5.3) by w', integrating over Ω and using (5.2), gives

$$\frac{1}{2}y'(t) + \frac{1}{2}\int_\Omega \big[\beta''(w)(w')^3 + \frac{4(p-1)}{p^2}|(E(\nabla w))'|^2 + g'(w)(w')^2\big]dx \le 0. \tag{5.4}$$

On the other hand, by using hypotheses (H7) and (H8) and relation (3.3) and applying successively Gagliardo-Nirenberg's inequality (see for example [12]), Young's inequality and Lemma 5.1, it follows that

$$\begin{aligned} &-\frac{1}{2}\int_\Omega \beta''(w)(w')^3dx \\ &\le c_{31}||w'||_2^{3(1+\alpha)}c_{32}||\nabla w||_p^p + \frac{4(p-1)}{p^2}\int_\Omega |(E(\nabla w))'|^2dx, \end{aligned} \tag{5.5}$$

where $\theta = \frac{1}{3}(\frac{Np}{Np+2p-2N})$ and $\alpha = \frac{N(3-p)}{3Np+6p-9N}$. Estimate (3.3) and hypothesis (H6) and (H7) imply

$$\int_\Omega g'(w)(w')^2dx \le \|g'(w)\|_{L^\infty(\Omega)}\int_\Omega (w')^2dx \le M_1\|w'\|_2^2, \tag{5.6}$$

$$\sigma_1\|w'\|_2^2 \le y(t), \tag{5.7}$$

where M_1 is a positive constant. Therefore, using (5.5) and (5.6), (5.4) becomes

$$\begin{aligned} &\frac{1}{2}y'(t) + \frac{2(p-1)}{p^2}\int_\Omega |\,(E(\nabla w))'\,|^2dx \\ &\le c_{31}\|w'\|_2^{3(1+\alpha)} + c_{32}\|\nabla w\|_p^p + M_1\|w'\|_2^2. \end{aligned} \tag{5.8}$$

Now (5.7) and estimate (3.4) give

$$\frac{1}{2}y'(t)+\frac{2(p-1)}{p^2}\int_\Omega |\,(E(\nabla w))'\,|^2dx \le c_{33}(y(t)^{\frac{3(1+\alpha)}{2}}+y(t)+1) \le c_{34}(y(t))^2+c_{35} \tag{5.9}$$

for all $t \ge \tau > 0$. By assumption (H6), equation (3.15) can be written as

$$\beta'(w)w' - \operatorname{div}(F_\varepsilon(\nabla w)) = h - g(w). \tag{5.10}$$

Taking the scalar product of (5.12) with w', we obtain

$$\begin{aligned} &\int_\Omega \beta'(w)\,(w')^2\,dx + \frac{d}{dt}\Big[\frac{1}{p}\int_\Omega \left(|\nabla w|^2+\varepsilon\right)^{\frac{p}{2}}dx\Big] \\ &= \int_\Omega (g(w)-h)w'dx \\ &\le \int_\Omega \frac{(g(w)-h)}{\sqrt{\beta'(w)}}.\sqrt{\beta'(w)}w'dx \\ &\le \frac{1}{2\sigma_2}\|g(w)-h\|_2^2 + \frac{1}{2}\int_\Omega \beta'(w)\,(w')^2\,dx. \end{aligned} \tag{5.11}$$

Hence

$$\frac{1}{2}\int_\Omega \beta'(w)\,(w')^2\,dx+\frac{d}{dt}\Big[\frac{1}{p}\int_\Omega \big(|\nabla w|^2+\varepsilon\big)^{\frac{p}{2}}\,dx\Big]\le c_{36}\|g(w)-h\|^2_{L^\infty(\Omega)}, \quad (5.12)$$

where c_{36} depends on σ_2 and meas(Ω). Estimate (3.12) of Lemma 3.4 gives

$$\frac{1}{p}\int_\Omega \big(|\nabla w|^2+\varepsilon\big)^{\frac{p}{2}}(t)dx\le c_{37}(\tau),\quad \forall t\ge\frac{\tau}{2}>0.$$

Integrating (5.12) on $\big[t,t+\frac{\tau}{2}\big]$ yields

$$\int_t^{t+\frac{\tau}{2}} y(s)ds\le c_{38}(\tau),\quad \forall t\ge\frac{\tau}{2}>0. \quad (5.13)$$

Going back to (5.9) and using the uniform Gronwall lemma [19, p. 89] with $r=\tau/2$, $g(t)=c_{34}y(t)$ and $h=c_{35}$ and estimate (5.13) leads to

$$y(t+\frac{\tau}{2})\le c_{39}(\tau)\quad \forall t\ge\frac{\tau}{2}>0.$$

Hence $y(t)\le c_{39}(\tau)$, for any $t\ge\tau>0$. The proof of the theorem is now complete. □

Using Theorem 5.3, we state the main result of this section.

Theorem 5.4 *Let f,β,p satisfies hypotheses (H1)-(H8). Then, for $\tau>0$, the solution of problem (1.1) satisfies:*

$$\frac{\partial\beta(u)}{\partial t}\in L^\infty(\tau,+\infty;L^2(\Omega)), \quad (5.14)$$

$$u\in L^\infty(\tau,+\infty;B^{1+\sigma,p}_\infty(\Omega)), \quad (5.15)$$

where $B^{1+\sigma,p}_\infty(\Omega)$ is a Besov space defined by the real interpolation method [18]. Moreover, there exists a constant $c(\tau)>0$, depending on τ such that

$$\lim_{t\to+\infty}\||\nabla u|^{(p-2)/2}\frac{\partial\nabla u}{\partial t}\|_{L^2(t,t+1;L^2(\Omega))}\ \le c(\tau). \quad (5.16)$$

Proof. By Theorem 5.3 and hypothesis (H7), :$\int_\Omega(\frac{\partial\,\beta(u_\varepsilon)}{\partial t})^2dx\le\sigma_2 y(t)\le c(\tau)$ for $t\ge\tau>0$. Passing to the limit as ε goes to 0 then yields (5.14). Now integrating (5.9) on $[t,t+1]$, for any $t\ge\tau>0$, and using Theorem 5.4, yields

$$\int_t^{t+1}\int_\Omega |\,(E(\nabla u_\varepsilon))'\,|^2dx\,ds\le c(\tau),\quad \forall\tau>0. \quad (5.17)$$

Furthermore, from Lemma 5.2,

$$\nabla u_\varepsilon\to\nabla u \text{ a.e on } Q_T. \quad (5.18)$$

By (5.17) and (5.18) we derive the estimate (5.16). On the other hand, by (H8) there is some σ', $0 < \sigma' < 1$, such that :$L^2(\Omega) \subset W^{-\sigma',p'}(\Omega)$. Now Simon's regularity results [18], concerning the equation

$$-\Delta_p u = h(x) - g(u) - \beta(u)_t \in L^\infty(\tau, +\infty; B_\infty^{-\sigma',p'}(\Omega)),$$

implies that for any $t \geq \tau$,

$$\|u(.,t)\|_{B_\infty^{1+(1-\sigma')(1-p)^2,p}(\Omega)} \leq c_{41}(\tau)\|g(u) - h(.)\|_{B_\infty^{-\sigma',p'}(\Omega)} + c_{42}(\tau).$$

Hence estimate (5.15) follows. □

Remark Integrating (5.9) on $[t, t+h]$ and letting h tends to 0 leads to the estimate

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} \int_\Omega |\nabla u|^{p-2} |\frac{\partial}{\partial t} \nabla u|^2 dx\, ds \leq c(\tau), \quad \forall t \geq \tau > 0\,.$$

Let

$$\omega(u_0) = \big\{ w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) : \exists t_n \to +\infty : u(.,t_n) \to w \text{ in } W_0^{1,p}(\Omega) \big\}.$$

Corollary 5.5 *Under the hypotheses of Theorem 5.3, $\omega(u_0)$ is not empty and $\omega(u_0) \subset E$, where E is the set of solutions of the associated elliptic problem*

$$\begin{gathered} -\Delta_p w = g(w) - h(x) \quad \text{in } \Omega, \\ w = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

Proof. Note that $\omega(u_0)$ is not empty because $B_\infty^{1+r,p}(\Omega)$ is compactly imbedded in $W^{1,p}(\Omega)$. Let $w = \lim_{n\to\to+\infty} u(.,t_n) \in \omega(u_0)$. By the regularity estimate $\frac{\partial u}{\partial t} \in L^2(\tau, +\infty; L^2(\Omega))$, we can conclude as in [9] that $w \in \mathcal{E}$. □

Concluding remarks. 1) In the case $\beta(u) = u$, a regularity property stronger than (5.16) is obtained in [9]; namely,

$$|\nabla u|^{(p-2)/2} \frac{\partial \nabla u}{\partial t} \in L^2(\tau, +\infty; L^2(\Omega)) \quad \forall \tau > 0.$$

2) In [6], the authors obtained that the attractor $\mathcal{A}$ satisfies $\mathcal{A} \subset W^{2,6}(\Omega)$ if $p = 2$, and $N \leq 3$. In fact, their result still holds for $N = 4$ and the proof follows the same lines as in Theorem 5.3 with $p = 2$.
3) In [8] and [9], it is obtained that $A \subset B_\infty^{1+\frac{1}{(p-2)^2},p}(\Omega)$ if $p > 2$ and $\beta(u) = u$. Unfortunately for general β and $p > 2$, Lemma 5.1 no longer applies.
4) In a forthcoming paper, we shall study a time semi-discretization scheme associated to problem (1.1) and related questions.

Acknowledgement The authors would like to thank professor F. de Thélin for reading a preliminary version of this work.

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