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ALMOST PERIODIC SOLUTIONS FOR HIGHER-ORDER HOPFIELD NEURAL NETWORKS WITHOUT BOUNDED ACTIVATION FUNCTIONS

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Abstract. In this paper, we consider higher-order Hopfield neural networks (HHNNs) with time-varying delays. Based on the fixed point theorem, Lyapunov functional method, differential inequality techniques, and without assuming the boundedness on the activation functions, we establish sufficient conditions for the existence and local exponential stability of the almost periodic solutions. The results of this paper are new and they complement previously known results.

1. Introduction

Consider the following higher-order Hopfield neural networks (HHNNs), with time-varying delays,

$$\begin{aligned} x_i'(t) = -c_i x_i(t) &+ \sum_{j=1}^{n} a_{ij}(t) g_j(x_j(t-\tau_{ij}(t))) \\ &+ \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t) g_j(x_j(t-\sigma_{ijl}(t))) g_l(x_l(t-\nu_{ijl}(t))) + I_i(t), \end{aligned} \tag{1.1}$$

for $i = 1, 2, \dots, n$, where n corresponds to the number of units in a neural network, $x_i(t)$ corresponds to the state vector of the ith unit at the time t, $c_i > 0$ represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs, $a_{ij}(t)$ and $b_{ijl}(t)$ are the first- and second-order connection weights of the neural network, $\tau_{ij}(t) \geq 0$, $\sigma_{ijl}(t) \geq 0$ and $\upsilon_{ijl}(t) \geq 0$ correspond to the transmission delays, $I_i(t)$ denote the external inputs at time t, and g_j is the activation function of signal transmission.

Due to the fact that high-order neural networks have stronger approximation property, faster convergence rate, greater storage capacity, and higher fault tolerance than lower-order neural networks, high-order neural networks have been the object of intensive analysis by numerous authors in recent years. In particular, there have been extensive results on the problem of the existence and stability of

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equilibrium points and periodic solutions of HHNNs (1.1) in the literature. We refer readers to [1, 2, 7, 8] and the references cited therein. The assumption

- (T0) for each $j \in \{1, 2, \dots, n\}$, $g_j : \mathbb{R} \to \mathbb{R}$ is bounded, i.e., there exists a constant L_j such that
$$|g_j(u)| \le L_j, \quad \text{for all } u \in \mathbb{R} \tag{1.2}$$

has been considered as a fundamental condition for the existence and stability of equilibrium points and periodic solutions solutions of HHNNs (1.1). To the best of our knowledge, few authors have considered the problems of periodic and almost periodic solutions of HHNNs (1.1) without the assumptions (T0). Thus, it is worth while to investigate the existence and stability of almost periodic solutions of HHNNs (1.1) in this case.

In this paper we shall study the existence and stability of almost periodic solutions for (1.1). By applying the fixed point theorem, Lyapunov functional method and differential inequality techniques, we derive some new sufficient conditions ensuring the existence and local exponential stability of the almost periodic solution of (1.1). These results are new and they complement previously known results. In particular, an example is also provided to illustrate the effectiveness of the new results.

Throughout this paper, for $i, j, l = 1, 2, \dots, n$, it will be assumed that I_i, a_{ij}, b_{ijl}, τ_{ij}, σ_{ijl}, $\nu_{ijl} : \mathbb{R} \to \mathbb{R}$ are almost periodic functions, and there exist constants τ, $\overline{a_{ij}}$, $\overline{b_{ijl}}$ and $\overline{I_i}$ such that
$$\begin{gathered}
\tau = \max\Big\{ \max_{1\le i,j\le n} \sup_{t\in\mathbb{R}} \tau_{ij}(t), \max_{1\le i,j,l\le n} \sup_{t\in\mathbb{R}} \sigma_{ijl}(t), \max_{1\le i,j,l\le n} \sup_{t\in\mathbb{R}} \nu_{ijl}(t) \Big\}, \\
\sup_{t\in\mathbb{R}} |b_{ijl}(t)| = \overline{b_{ijl}}, \quad \sup_{t\in\mathbb{R}} |a_{ij}(t)| = \overline{a_{ij}}, \quad \sup_{t\in\mathbb{R}} |I_i(t)| = \overline{I_i}.
\end{gathered} \tag{1.3}$$
We also assume that the following conditions hold:

- (H1) For each $j \in \{1, 2, \dots, n\}$, there exists a nonnegative constant L_j^g such that $g_j(0) = 0$, $|g_j(u) - g_j(v)| \le L_j^g |u - v|$, for all $u,\ v \in \mathbb{R}$.
- (H2) Assume that there exist nonnegative constants L, q and δ such that
$$L = \max_{1\le i\le n} \{\frac{\overline{I_i}}{c_i}\}, \quad \delta = \max_{1\le i\le n} \{c_i^{-1}[\sum_{j=1}^n \overline{a_{ij}} L_j^g + \sum_{j=1}^n \sum_{l=1}^n \overline{b_{ijl}} L_j^g L_l^g]\} < 1,$$
$$\frac{L}{1-\delta} \le 1, \quad q = \max_{1\le i\le n} \Big\{c_i^{-1}\Big(\sum_{j=1}^n \overline{a_{ij}} L_j^g + \frac{2L}{1-\delta} \sum_{j=1}^n \sum_{l=1}^n \overline{b_{ijl}} L_j^g L_l^g \Big)\Big\} < 1.$$

For convenience, we introduce the following notation. We use $x = (x_1, x_2, \dots, x_n)^T$ in $\mathbb{R}^n$ to denote a column vector, in which the symbol $(^T)$ denotes the transpose of a vector. We let $|x|$ denote the absolute-value vector given by $|x| = (|x_1|, |x_2|, \dots, |x_n|)^T$, and define $\|x\| = \max_{1\le i\le n} |x_i|$. A vector $x \ge 0$ means that all entries of x are greater than or equal to zero. $x > 0$ is defined similarly. For vectors x and y, $x \ge y$ (resp. $x > y$) means that $x - y \ge 0$ (resp. $x - y > 0$).

For the rest of this paper, we set
$$\begin{gathered}
\{x_j(t)\} = (x_1(t), x_2(t), \dots, x_n(t))^T, \\
B = \{\varphi | \varphi = \{\varphi_j(t)\} = (\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t))^T\},
\end{gathered}$$
where φ is an almost periodic function on R. For all $\varphi \in B$, we define the induced module $\|\varphi\|_B$ by $\|\varphi\|_B = \sup_{t\in\mathbb{R}} \|\varphi(t)\|$. Therefore B is a Banach space.

The initial conditions associated with system (1.1) are of the form

$$x_i(s) = \varphi_i(s), s \in [-\tau,\ 0],\ i = 1, 2, \dots, n, \tag{1.4}$$

where $\varphi = (\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t))^T \in C([-\tau, 0]; R^n)$.

Definition. [3, 4] Let $u(t) : \mathbb{R} \to \mathbb{R}^n$ be continuous in t. $u(t)$ is said to be almost periodic on $\mathbb{R}$ if, for any $\varepsilon > 0$, the set $T(u, \varepsilon) = \{\delta : |u(t + \delta) - u(t)| < \varepsilon$, for all $t \in \mathbb{R}\}$ is relatively dense, i.e., for $\forall \varepsilon > 0$, it is possible to find a real number $l = l(\varepsilon) > 0$, for any interval with length $l(\varepsilon)$, there exists a number $\delta = \delta(\varepsilon)$ in this interval such that $|u(t + \delta) - u(t)| < \varepsilon$, for for all $t \in \mathbb{R}$.

The remaining part of this paper is organized as follows. In Section 2, we shall derive new sufficient conditions for the existence of almost periodic solutions of (1.1). In Section 3, we present some new sufficient conditions for the local exponential stability of the almost periodic solution of (1.1). In Section 4, we shall give some examples and remarks to illustrate our results obtained in the previous sections.

2. Existence of Almost Periodic Solutions

Theorem 2.1. *Let conditions (H1) and (H2) hold. Then, there exists a unique almost periodic solution to* (1.1) *in the region* $B^* = \{\varphi | \varphi \in B, \|\varphi - \varphi_0\|_B \le \frac{\delta L}{1-\delta}\}$, *where*

$$\begin{aligned}
\varphi_0(t) &= \Big\{ \int_{-\infty}^t \exp(-c_j(t-s)) I_j(s) ds \Big\} \\
&= \Big(\int_{-\infty}^t \exp(-c_1(t-s)) I_1(s) ds, \int_{-\infty}^t \exp(-c_2(t-s)) I_2(s) ds, \\
&\quad \dots, \int_{-\infty}^t \exp(-c_n(t-s)) I_n(s) ds \Big)^T.
\end{aligned}$$

Proof. For each $\varphi \in B$, we consider the almost periodic solution $x^\varphi(t)$ to the nonlinear almost periodic differential equations

$$\begin{aligned}
x_i'(t) &= -c_i x_i(t) + \sum_{j=1}^n a_{ij}(t) g_j(\varphi_j(t - \tau_{ij}(t))) \\
&\quad + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(t) g_j(\varphi_j(t - \sigma_{ijl}(t))) g_l(\varphi_l(t - \nu_{ijl}(t))) + I_i(t),
\end{aligned} \tag{2.1}$$

for $i = 1, 2, \dots, n$. Then $\tau_{ij}(t)$, $b_{ij}(t)$ and $I_i(t)$ are almost periodic functions. According to [3, pp. 80-112] and [4, pp. 90-100], we know that the auxiliary system (2.1) has exactly one almost periodic solution

$$\begin{aligned}
x^\varphi(t) &= (x_1^\varphi(t),\ x_2^\varphi(t), \dots, x_n^\varphi(t))^T \\
&= \Big(\int_{-\infty}^t e^{-c_1(t-s)} \Big[\sum_{j=1}^n a_{1j}(s) g_j\big(\varphi_j(s - \tau_{1j}(s))\big) \\
&\quad + \sum_{j=1}^n \sum_{l=1}^n b_{1jl}(s) g_j(\varphi_j(s - \sigma_{1jl}(s))) g_l(\varphi_l(s - \nu_{1jl}(s))) + I_1(s) \Big] ds,
\end{aligned}$$

$$\dots, \int_{-\infty}^{t} e^{-c_n(t-s)}\Big[\sum_{j=1}^{n} a_{nj}(s)g_j(\varphi_j(s-\tau_{nj}(s)))$$
$$+\sum_{j=1}^{n}\sum_{l=1}^{n} b_{njl}(s)g_j(\varphi_j(s-\sigma_{njl}(s)))g_l(\varphi_l(s-\nu_{njl}(s)))+I_n(s)\Big]ds\Big)^T.$$

Now, we define a mapping $T: B\to B$ by setting

$$T(\varphi)(t)=x^{\varphi}(t),\quad \forall \varphi\in B.$$

Since $B^*=\{\varphi\in B, \|\varphi-\varphi_0\|_B\le \frac{\delta L}{1-\delta}\}$, it is easy to see that B^* is a closed convex subset of B. According to the definition of the norm of Banach space B, we get

$$\begin{aligned}
\|\varphi_0\|_B &= \sup_{t\in\mathbb{R}}\max_{1\le i\le n}\Big\{\int_{-\infty}^{t} I_i(s)\exp(-c_i(t-s))ds\Big\}\\
&\le \sup_{t\in\mathbb{R}}\max_{1\le i\le n}\{\frac{\overline{I_i}}{c_i}\}\\
&= \max_{1\le i\le n}\{\frac{\overline{I_i}}{c_i}\}=L.
\end{aligned}$$

Therefore, for for all $\varphi\in B^*$, we have

$$\|\varphi\|_B\le\|\varphi-\varphi_0\|_B+\|\varphi_0\|_B\le\frac{\delta L}{1-\delta}+L=\frac{L}{1-\delta}\le 1. \tag{2.2}$$

In view of (H1), we have

$$|g_j(u)|\le L_j^g|u|\quad\text{for all } u\in\mathbb{R},\ j=1,2,\dots,n. \tag{2.3}$$

Now, we prove that the mapping T is a self-mapping from B^* to B^*. In fact, for all $\varphi\in B^*$, from (2.2) and (2.3), we obtain

$$\begin{aligned}
&\|T\varphi-\varphi_0\|_B\\
&=\sup_{t\in\mathbb{R}}\max_{1\le i\le n}\Big\{\Big|\int_{-\infty}^{t} e^{-c_i(t-s)}\Big[\sum_{j=1}^{n} a_{ij}(s)g_j(\varphi_j(s-\tau_{ij}(s)))\\
&\quad+\sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(s)g_j(\varphi_j(s-\sigma_{ijl}(s)))g_l(\varphi_l(s-\nu_{ijl}(s)))\Big]ds\Big|\Big\}\\
&\le\sup_{t\in\mathbb{R}}\max_{1\le i\le n}\{\int_{-\infty}^{t} e^{-c_i(t-s)}\Big[\sum_{j=1}^{n}\overline{a_{ij}}L_j^g\|\varphi\|_B+\sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}L_j^gL_l^g\|\varphi\|_B^2\Big]ds\}\\
&\le\sup_{t\in\mathbb{R}}\max_{1\le i\le n}\{\int_{-\infty}^{t} e^{-c_i(t-s)}\Big[\sum_{j=1}^{n}\overline{a_{ij}}L_j^g\frac{L}{1-\delta}+\sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}L_j^gL_l^g(\frac{L}{1-\delta})^2\Big]ds\}\\
&\le\sup_{t\in\mathbb{R}}\max_{1\le i\le n}\{\int_{-\infty}^{t} e^{-c_i(t-s)}\Big[\sum_{j=1}^{n}\overline{a_{ij}}L_j^g+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b_{ijl}}L_j^gL_l^g\Big]ds\frac{L}{1-\delta}\}\\
&\le\max_{1\le i\le n}\{c_i^{-1}\Big[\sum_{j=1}^{n}\overline{a_{ij}}L_j^g+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b_{ijl}}L_j^gL_l^g\Big]\}\frac{L}{1-\delta}\\
&=\frac{\delta L}{1-\delta},
\end{aligned}$$

where $\delta = \max_{1\le i\le n}\{c_i^{-1}[\sum_{j=1}^n \overline{a_{ij}}L_j^g + \sum_{j=1}^n\sum_{l=1}^n \overline{b_{ijl}}L_j^gL_l^g]\}$. This implies that $T(\varphi)(t)\in B^*$. Next, we prove that the mapping T is a contraction mapping on B^*. In view of (2.2) and (H1), for all $\phi,\psi\in B^*$, we have

$$
\begin{aligned}
&|T(\phi(t)) - T(\psi(t))|\\
&= \Big(|(T(\phi(t)) - T(\psi(t)))_1|,\dots,|(T(\phi(t)) - T(\psi(t)))_n|\Big)^T\\
&= \Big(|\int_{-\infty}^t e^{-c_1(t-s)}\Big[\sum_{j=1}^n a_{1j}(s)(g_j(\phi_j(s-\tau_{1j}(s))) - g_j(\psi_j(s-\tau_{1j}(s))))\\
&\quad + \sum_{j=1}^n\sum_{l=1}^n b_{1jl}(s)\big(g_j(\phi_j(s-\sigma_{1jl}(s)))g_l(\phi_l(s-\nu_{1jl}(s)))\\
&\quad - g_j(\psi_j(s-\sigma_{1jl}(s)))g_l(\psi_l(s-\nu_{1jl}(s)))\big)\Big]ds|,\dots,\\
&\quad |\int_{-\infty}^t e^{-c_n(t-s)}\Big[\sum_{j=1}^n a_{nj}(s)(g_j(\phi_j(s-\tau_{nj}(s))) - g_j(\psi_j(s-\tau_{nj}(s))))\\
&\quad + \sum_{j=1}^n\sum_{l=1}^n b_{njl}(s)\big(g_j(\phi_j(s-\sigma_{njl}(s)))g_l(\phi_l(s-\nu_{njl}(s)))\\
&\quad - g_j(\psi_j(s-\sigma_{njl}(s)))g_l(\psi_l(s-\nu_{njl}(s)))\big)\Big]ds|\Big)^T\\
&\le \Big(\int_{-\infty}^t e^{-c_1(t-s)}\Big[\sum_{j=1}^n \overline{a_{1j}}L_j^g\sup_{t\in\mathbb{R}}|\phi_j(t)-\psi_j(t)|\\
&\quad + \sum_{j=1}^n\sum_{l=1}^n \overline{b_{1jl}}(|g_j(\phi_j(s-\sigma_{1jl}(s)))g_l(\phi_l(s-\nu_{1jl}(s)))\\
&\quad - g_j(\psi_j(s-\sigma_{1jl}(s)))g_l(\phi_l(s-\nu_{1jl}(s)))|\\
&\quad + |g_j(\psi_j(s-\sigma_{1jl}(s)))g_l(\phi_l(s-\nu_{1jl}(s)))\\
&\quad - g_j(\psi_j(s-\sigma_{1jl}(s)))g_l(\psi_l(s-\nu_{1jl}(s)))|)\Big]ds,\\
&\quad \dots,\int_{-\infty}^t e^{-c_n(t-s)}\Big[\sum_{j=1}^n \overline{a_{nj}}L_j^g\sup_{t\in\mathbb{R}}|\phi_j(t)-\psi_j(t)|\\
&\quad + \sum_{j=1}^n\sum_{l=1}^n \overline{b_{njl}}(|g_j(\phi_j(s-\sigma_{njl}(s)))g_l(\phi_l(s-\nu_{njl}(s)))\\
&\quad - g_j(\psi_j(s-\sigma_{njl}(s)))g_l(\phi_l(s-\nu_{njl}(s)))|\\
&\quad + |g_j(\psi_j(s-\sigma_{njl}(s)))g_l(\phi_l(s-\nu_{njl}(s)))\\
&\quad - g_j(\psi_j(s-\sigma_{njl}(s)))g_l(\psi_l(s-\nu_{njl}(s)))|)\Big]ds\Big)^T\\
&\le \Big(\int_{-\infty}^t e^{-c_1(t-s)}[\sum_{j=1}^n \overline{a_{1j}}L_j^g\sup_{t\in\mathbb{R}}|\phi_j(t)-\psi_j(t)|\\
&\quad + \sum_{j=1}^n\sum_{l=1}^n \overline{b_{1jl}}L_j^gL_l^g(\sup_{t\in\mathbb{R}}|\phi_l(t)| + \sup_{t\in\mathbb{R}}|\psi_j(t)|)\|\phi-\psi\|_B]ds,\dots,
\end{aligned}
$$

$$
\begin{aligned}
&\int_{-\infty}^{t} e^{-c_n(t-s)}[\sum_{j=1}^{n}\overline{a_{nj}}L_j^g \sup_{t\in\mathbb{R}}|\phi_j(t)-\psi_j(t)| \\
&+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b_{njl}}L_j^gL_l^g(\sup_{t\in\mathbb{R}}|\phi_l(t)|+\sup_{t\in\mathbb{R}}|\psi_j(t)|)\|\phi-\psi\|_B]ds\Big)^T \\
\le{}& \Big(c_1^{-1}(\sum_{j=1}^{n}\overline{a_{1j}}L_j^g+\frac{2L}{1-\delta}\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b_{1jl}}L_j^gL_l^g)\|\phi-\psi\|_B, \\
&\dots, c_n^{-1}(\sum_{j=1}^{n}\overline{a_{nj}}L_j^g+\frac{2L}{1-\delta}\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b_{njl}}L_j^gL_l^g)\|\phi-\psi\|_B\Big)^T,
\end{aligned}
$$

which implies

$$
\begin{aligned}
\|T(\phi)-T(\psi)\|_B &\le \max_{1\le i\le n}\{c_i^{-1}(\sum_{j=1}^{n}\overline{a_{ij}}L_j^g+\frac{2L}{1-\delta}\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b_{ijl}}L_j^gL_l^g)\}\|\phi-\psi\|_B \\
&= q\|\phi-\psi\|_B.
\end{aligned}
$$

Note that $q=\max_{1\le i\le n}\{c_i^{-1}(\sum_{j=1}^{n}\overline{a_{ij}}L_j^g+\frac{2L}{1-\delta}\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b_{ijl}}L_j^gL_l^g)\}<1$; it is clear that the mapping T is a contraction. Therefore the mapping T possesses a unique fixed point $Z^*\in B^*$, $TZ^*=Z^*$. By (2.1), Z^* satisfies (1.1). So Z^* is an almost periodic solution of (1.1) in B^*. The proof is complete. $\square$

3. Stability of the almost periodic solution

In this section, we establish some results for the stability of the almost periodic solution of (1.1).

Theorem 3.1. *Let*

$$
\max_{1\le i\le n}\{c_i^{-1}[\sum_{j=1}^{n}\overline{a_{ij}}L_j^g+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b_{ijl}}L_j^gL_l^g(1+2\frac{L}{1-\delta})]\}<1.
$$

Suppose that all the conditions of Theorem 2.1 are satisfied. Then (1.1) *has exactly one almost periodic solution $Z^*(t)=(x_1^*(t),x_2^*(t),\dots,x_n^*(t))^T\in B^*$. Moreover, $Z^*(t)$ is locally exponentially stable, the domain of the attraction of $Z^*(t)$ is the set*

$$
G_1(Z^*)=\{\varphi|\varphi=(\varphi_1(t),\varphi_2(t),\dots,\varphi_n(t))^T\in C([-\tau,\ 0];\ R^n),\ \|\varphi-\varphi^*\|_1<1\},
$$

where $\varphi^*=\{\varphi_j^*(t)\}$, $\varphi_j^*(t)=x_j^*(t)$, $j=1,2,\dots,n$, $t\in[-\tau,0]$, and $\|\varphi-\varphi^*\|_1=\sup_{-\tau\le s\le 0}\max_{1\le j\le n}|\varphi_j(s)-\varphi_j^*(s)|$. Namely, there exist constants $\lambda>0$ and $M>1$ such that for every solution $Z(t)=\{x_j(t)\}$ to system (1.1) *with initial value $\varphi=\{\varphi_j(t)\}\in G_1(Z^*)$, we have*

$$
|x_i(t)-x_i^*(t)|\le M\|\varphi-\varphi^*\|_1e^{-\lambda t},\quad \forall t>0,\ i=1,2,\dots,n.
$$

Proof. From Theorem 2.1, system (1.1) has exactly one almost periodic solution $Z^*(t)=\{x_j^*(t)\}\in B^*$. Let $Z(t)=\{x_j(t)\}$ be an arbitrary solution of system (1.1) with initial value $\varphi=\{\varphi_j(t)\}\in G_1(Z^*)$, let $y(t)=\{y_j(t)\}=\{x_j(t)-x_j^*(t)\}=$

$Z(t) - Z^*(t)$. Then

$$\begin{aligned} y_i'(t) &= -c_i y_i(t) + \sum_{j=1}^{n} a_{ij}(t)(g_j(x_j(t-\tau_{ij}(t))) - g_j(x_j^*(t-\tau_{ij}(t)))) \\ &\quad + \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t)(g_j(x_j(t-\sigma_{ijl}(t)))g_l(x_l(t-\nu_{ijl}(t))) \\ &\quad - g_j(x_j^*(t-\sigma_{ijl}(t)))g_l(x_l^*(t-\nu_{ijl}(t)))), \quad i = 1,2,\dots,n. \end{aligned} \tag{3.1}$$

Since $\max_{1\le i\le n}\{c_i^{-1}[\sum_{j=1}^n \overline{a_{ij}} L_j^g + \sum_{j=1}^n \sum_{l=1}^n \overline{b_{ijl}} L_j^g L_l^g (1 + 2\frac{L}{1-\delta})]\} < 1$, we can easily get

$$-c_i + \sum_{j=1}^{n} \overline{a_{ij}} L_j^g + \sum_{j=1}^{n}\sum_{l=1}^{n} \overline{b_{ijl}} L_j^g L_l^g (1 + 2\frac{L}{1-\delta}) < 0, \quad i = 1,2,\dots,n, \tag{3.2}$$

which implies that we can choose a positive constant λ such that

$$(\lambda - c_i) + \sum_{j=1}^{n} \overline{a_{ij}} L_j^g e^{\lambda\tau} + \sum_{j=1}^{n}\sum_{l=1}^{n} \overline{b_{ijl}} L_j^g L_l^g (e^{2\lambda\tau} + 2e^{\lambda\tau}\frac{L}{1-\delta}) < 0, \tag{3.3}$$

for $i = 1,2,\dots,n$. We consider the Lyapunov functional

$$V_i(t) = |y_i(t)|e^{\lambda t}, \quad i = 1,2,\dots,n. \tag{3.4}$$

Calculating the upper right derivative of $V_i(t)$ along the solution $y(t) = \{y_j(t)\}$ of system (3.1) with the initial value $\bar{\varphi} = \varphi - \varphi^*$, we have from (2.2), (2.3), (3.1) and (H1) that

$$\begin{aligned} &D^+(V_i(t)) \\ &\le -c_i|y_i(t)|e^{\lambda t} + \Big[\sum_{j=1}^{n} |a_{ij}(t)||g_j(x_j(t-\tau_{ij}(t))) - g_j(x_j^*(t-\tau_{ij}(t)))| \\ &\quad + \sum_{j=1}^{n}\sum_{l=1}^{n} |b_{ijl}(t)||g_j(x_j(t-\sigma_{ijl}(t)))g_l(x_l(t-\nu_{ijl}(t))) \\ &\quad - g_j(x_j^*(t-\sigma_{ijl}(t)))g_l(x_l^*(t-\nu_{ijl}(t)))|\Big]e^{\lambda t} + \lambda|y_i(t)|e^{\lambda t} \\ &\le (\lambda - c_i)|y_i(t)|e^{\lambda t} + \sum_{j=1}^{n} |a_{ij}(t)|L_j^g|y_j(t-\tau_{ij}(t))| \\ &\quad + \Big[\sum_{j=1}^{n}\sum_{l=1}^{n} |b_{ijl}(t)|(|g_j(x_j(t-\sigma_{ijl}(t)))g_l(x_l(t-\nu_{ijl}(t))) \\ &\quad - g_j(x_j^*(t-\sigma_{ijl}(t)))g_l(x_l(t-\nu_{ijl}(t)))| + |g_j(x_j^*(t-\sigma_{ijl}(t)))g_l(x_l(t-\nu_{ijl}(t))) \\ &\quad - g_j(x_j^*(t-\sigma_{ijl}(t)))g_l(x_l^*(t-\nu_{ijl}(t)))|)\Big]e^{\lambda t} \\ &\le (\lambda - c_i)|y_i(t)|e^{\lambda t} + [\sum_{j=1}^{n} \overline{a_{ij}} L_j^g|y_j(t-\tau_{ij}(t))| \\ &\quad + \sum_{j=1}^{n}\sum_{l=1}^{n} \overline{b_{ijl}} L_j^g L_l^g (|y_j(t-\sigma_{ijl}(t))||y_l(t-\nu_{ijl}(t)) \\ &\quad + x_l^*(t-\nu_{ijl}(t))| + |x_j^*(t-\sigma_{ijl}(t))||y_l(t-\nu_{ijl}(t))|)]e^{\lambda t} \end{aligned}$$

$$\begin{aligned}
&\leq (\lambda - c_i)|y_i(t)|e^{\lambda t} + \Big[\sum_{j=1}^{n} \overline{a_{ij}} L_j^g |y_j(t-\tau_{ij}(t))| \\
&\quad + \sum_{j=1}^{n}\sum_{l=1}^{n} \overline{b_{ijl}} L_j^g L_l^g (|y_j(t-\sigma_{ijl}(t))||y_l(t-\nu_{ijl}(t))| \\
&\quad + |y_j(t-\sigma_{ijl}(t))|\frac{L}{1-\delta} + \frac{L}{1-\delta}|y_l(t-\nu_{ijl}(t))|)\Big]e^{\lambda t},
\end{aligned} \tag{3.5}$$

where $i = 1, 2, \dots, n$. Set

$$\|\varphi - \varphi^*\|_1 = \sup_{-\tau \leq s \leq 0} \max_{1 \leq j \leq n} |\varphi_j(s) - \varphi_j^*(s)| > 0.$$

Since $\|\varphi - \varphi^*\|_1 < 1$, we can choose a positive constant $M > 1$ such that

$$M\|\varphi - \varphi^*\|_1 < 1, \quad (M\|\varphi - \varphi^*\|_1)^2 < M\|\varphi - \varphi^*\|_1. \tag{3.6}$$

It follows from (3.4) that

$$V_i(t) = |y_i(t)|e^{\lambda t} < M\|\varphi - \varphi^*\|_1, \quad \text{for all } t \in [-\tau, 0], \quad i = 1, 2, \dots, n.$$

Now we claim that

$$V_i(t) = |y_i(t)|e^{\lambda t} < M\|\varphi - \varphi^*\|_1, \quad \text{for all } t > 0,\ i = 1, 2, \dots, n. \tag{3.7}$$

Contrarily, there must exist an $i \in \{1, 2, \dots, n\}$ and $t_i > 0$ such that

$$V_i(t_i) = M\|\varphi - \varphi^*\|_1 \quad \text{and} \quad V_j(t) < M\|\varphi - \varphi^*\|_1,\ \forall t \in [-\tau,\ t_i),$$

for $j = 1, 2, \dots, n$. It follows that

$$V_i(t_i) - M\|\varphi - \varphi^*\|_1 = 0 \quad \text{and} \quad V_j(t) - M\|\varphi - \varphi^*\|_1 < 0,\ \forall t \in [-\tau, t_i),$$

for $j = 1, 2, \dots, n$. This together with (3.5), yields

$$\begin{aligned}
0 &\leq D^+(V_i(t_i) - M\|\varphi - \varphi^*\|_1) \\
&= D^+(V_i(t_i)) \\
&\leq (\lambda - c_i)|y_i(t_i)|e^{\lambda t_i} + \Big[\sum_{j=1}^{n} \overline{a_{ij}} L_j^g |y_j(t_i-\tau_{ij}(t_i))| \\
&\quad + \sum_{j=1}^{n}\sum_{l=1}^{n} \overline{b_{ijl}} L_j^g L_l^g (|y_j(t_i-\sigma_{ijl}(t_i))||y_l(t_i-\nu_{ijl}(t_i))| \\
&\quad + |y_j(t_i-\sigma_{ijl}(t_i))|\frac{L}{1-\delta} + \frac{L}{1-\delta}|y_l(t_i-\nu_{ijl}(t_i))|)\Big]e^{\lambda t_i} \\
&= (\lambda - c_i)|y_i(t_i)|e^{\lambda t_i} + \sum_{j=1}^{n} \overline{a_{ij}} L_j^g |y_j(t_i-\tau_{ij}(t_i))|e^{\lambda(t_i-\tau_{ij}(t_i))}e^{\lambda\tau_{ij}(t_i)} \\
&\quad + \sum_{j=1}^{n}\sum_{l=1}^{n} \overline{b_{ijl}} L_j^g L_l^g \Big(|y_j(t_i-\sigma_{ijl}(t_i))|e^{\lambda(t_i-\sigma_{ijl}(t_i))}|y_l(t_i-\nu_{ijl}(t_i))| \\
&\quad \times e^{\lambda(t_i-\nu_{ijl}(t_i))}e^{\lambda\sigma_{ijl}(t_i)}e^{\lambda\nu_{ijl}(t_i)}e^{-\lambda t_i} \\
&\quad + |y_j(t_i-\sigma_{ijl}(t_i))|e^{\lambda(t_i-\sigma_{ijl}(t_i))}e^{\lambda\sigma_{ijl}(t_i)}\frac{L}{1-\delta} \\
&\quad + \frac{L}{1-\delta}|y_l(t_i-\nu_{ijl}(t_i))|e^{\lambda(t_i-\nu_{ijl}(t_i))}e^{\lambda\nu_{ijl}(t_i)}\Big)
\end{aligned}$$

$$
\begin{aligned}
&\le (\lambda - c_i) M \|\varphi - \varphi^*\|_1 + \sum_{j=1}^n \overline{a_{ij}} L_j^g e^{\lambda\tau} M \|\varphi - \varphi^*\|_1 \\
&\quad + \sum_{j=1}^n \sum_{l=1}^n \overline{b_{ijl}} L_j^g L_l^g ((M\|\varphi - \varphi^*\|_1)^2 e^{2\lambda\tau} e^{-\lambda t_i} \\
&\quad + M\|\varphi - \varphi^*\|_1 e^{\lambda\tau} \frac{L}{1-\delta} + \frac{L}{1-\delta} M \|\varphi - \varphi^*\|_1 e^{\lambda\tau}) \\
&\le \Big[(\lambda - c_i) + \sum_{j=1}^n \overline{a_{ij}} L_j^g e^{\lambda\tau} + \sum_{j=1}^n \sum_{l=1}^n \overline{b_{ijl}} L_j^g L_l^g (e^{2\lambda\tau} + 2e^{\lambda\tau} \frac{L}{1-\delta}) \Big] M \|\varphi - \varphi^*\|_1 .
\end{aligned}
$$

Thus, we have

$$
0 \le (\lambda - c_i) + \sum_{j=1}^n \overline{a_{ij}} L_j^g e^{\lambda\tau} + \sum_{j=1}^n \sum_{l=1}^n \overline{b_{ijl}} L_j^g L_l^g (e^{2\lambda\tau} + 2e^{\lambda\tau} \frac{L}{1-\delta})
$$

which contradicts (3.3). Hence, (3.7) holds. It follows that

$$
|y_i(t)| < M \|\varphi - \varphi^*\|_1 e^{-\lambda t}, \quad t > 0, \ i = 1, 2, \dots, n.
$$

This completes the proof. □

4. An Example

In this section, we give an example to demonstrate the results obtained in previous sections.

Consider the following HHNNs with delays:

$$
\begin{aligned}
x_1'(t) &= -x_1(t) + \frac{1}{16}(\sin t) g_1(x_1(t - \sin^2 t)) + \frac{1}{16}(\cos 3t) g_2(x_2(t - 7\sin^2 t)) \\
&\quad + \frac{1}{8}(\cos t) g_1(x_1(t - 5\sin^2 t)) g_2(x_2(t - 2\sin^2 t)) + \frac{3}{4}\sin(\sqrt{2}t), \\
x_2'(t) &= -x_2(t) + \frac{1}{16}(\sin 2t) g_1(x_1(t - \cos^2 t)) + \frac{1}{16}(\cos 8t) g_2(x_2(t - 5\sin^2 t)) \\
&\quad + \frac{1}{8}(\cos 4t) g_1(x_1(t - \sin^2 t)) g_2(x_2(t - 4\sin^2 t)) + \frac{3}{4}\cos(\sqrt{2}t),
\end{aligned} \tag{4.1}
$$

where $g_1(x) = g_2(x) = |x|$. Observe that $c_1 = c_2 = L_1^g = L_2^g = 1$, $\overline{a_{ij}} = \frac{1}{16}$, $i, j = 1, 2$, $\overline{b_{112}} = \overline{b_{212}} = \frac{1}{8}$, $\overline{b_{ijl}} = 0$, $i, j, l = 1, 2$, $ijl \ne 112$, $ijl \ne 212$. Then

$$
L = \frac{3}{4}, \quad \delta = \max_{1\le i\le 2} \{ c_i^{-1} [\sum_{j=1}^2 \overline{a_{ij}} L_j^g + \sum_{j=1}^2 \sum_{l=1}^2 \overline{b_{ijl}} L_j^g L_l^g] \} = \frac{1}{4} < 1,
$$

$$
q = \max_{1\le i\le 2} \{ c_i^{-1} (\sum_{j=1}^2 \overline{a_{ij}} L_j^g + \frac{2L}{1-\delta} \sum_{j=1}^2 \sum_{l=1}^2 \overline{b_{ijl}} L_j^g L_l^g) \} = \frac{3}{8} < 1,
$$

$$
\max_{1\le i\le 2} \{ c_i^{-1} [\sum_{j=1}^2 \overline{a_{ij}} L_j^g + \sum_{j=1}^2 \sum_{l=1}^2 \overline{b_{ijl}} L_j^g L_l^g (1 + 2\frac{L}{1-\delta})] \} = \frac{1}{2} < 1.
$$

Therefore, By Theorem 3.1, system (4.1) has a unique almost periodic solution $Z^*(t)$ in the region $\|\varphi - \varphi_0\|_B \le 0.25$. Moreover, $Z^*(t)$ is locally exponentially stable, the domain of the attraction of $Z^*(t)$ is the set $G_1(Z^*)$.

We remark that (4.1) is a very simple form of HHNNs. Since $g_1(x) = g_2(x) = |x|$, one can observe that the condition (T0) is not satisfied. Therefore, all the results in [1, 2, 5, 6, 7, 8, 9] and the references cited therein can not be applicable to system (4.1). This implies that the results of this paper are essentially new.

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