

Solutions to a nonlinear drift-diffusion model for semiconductors *

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Abstract

A nonlinear drift-diffusion model for semiconductors is analyzed to show the existence of non-vacuum global solutions and stationary solutions. The long time behavior of the solutions is studied by establishing the existence of an absorbing set and a compact attractor of the dynamical system. Parallel results on vacuum solutions are also obtained under weaker conditions on model parameters.

1 Introduction

In this paper we analyze a nonlinear drift-diffusion model for semiconductors. Let $n = n(t, x)$ and $p = p(t, x)$ be the densities of electrons and holes, respectively, in a region $\Omega \subset \mathbb{R}^l$ ($l \leq 3$) occupied by a semiconductor. Consider the flow of the charge carriers in Ω . Conservation of mass yields

$$\frac{\partial n}{\partial t} = G_n + \operatorname{div} J_n \quad \text{and} \quad \frac{\partial p}{\partial t} = G_p - \operatorname{div} J_p \quad (1.1)$$

where J_n and J_p are the current densities, and G_n and G_p are generation-recombination rates for electrons and holes, respectively.

In the standard drift-diffusion model, temperature is assumed to be constant T_0 (the lattice temperature) and thus the current densities are given by

$$J_n = \mu_n(\nabla(k_b n T_0) - n \nabla u) \quad \text{and} \quad J_p = -\mu_p(\nabla(k_b p T_0) + p \nabla u) \quad (1.2)$$

where μ_n, μ_p are the mobilities, k_b is the Boltzmann constant, and the Einstein relations that relate the diffusion coefficients to the mobilities are already assumed. The electric potential $u = u(t, x)$ is governed by the Poisson's equation

$$\nabla \cdot (\epsilon_0 \nabla u) = n - p - N \quad (1.3)$$

* 1991 Mathematics Subject Classifications: 35K57, 35K55, 35J60, 78A35.

Key words and phrases: Drift-diffusion model, semiconductors, nonlinear diffusion, degenerated parabolic and elliptic equations, attractors.

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Submitted May 28, 1998. Published May 10, 1999.

(WF) partially supported by WVU Senate Research Grant R-97-051

and by Army Research Office grant DAAG55-98-1-0261.

(IK) partially supported by Air Force Office grant AFOSR-F49620-95-1-0447.

with electrical permittivity ϵ_0 and impurity doping profile $N(x)$. Noting that $P_n = k_b n T_0$ and $P_p = k_b p T_0$ are the pressures, if the electron and hole temperatures, T_n and T_p , are variables, then the pressures should be expressed as

$$P_n = k_b n T_n \quad \text{and} \quad P_p = k_b p T_p.$$

Using an isentropic assumption analogous to the case of gas dynamics, we further assume that P_n and P_p are functions of n and p only, i.e.,

$$k_b T_n n = P_n(n) \quad \text{and} \quad k_b T_p p = P_p(p) \quad (1.4)$$

where P_n and P_p are given functions. For example, in the case of polytropic gas, $P_n = kn^\gamma$ ($\gamma > 1$). For simplicity we will assume

$$P_n(\cdot) = P_p(\cdot) = P(\cdot).$$

Thus the current relations in (1.2) should be modified to become

$$J_n = \mu_n(\nabla P(n) - n\nabla u) \quad \text{and} \quad J_p = -\mu_p(\nabla P(p) + p\nabla u). \quad (1.5)$$

Here we obtain this model by using the analogy to gas dynamics, but such current expressions are also known as results of the so-called Fermi-Dirac statistics where $P(n)$ is related to a Fermi integral and approximately equal to $kn^{5/3}$ when n is large. The standard model (1.2) is the special case of (1.5) when $P(\cdot)$ is linear, which corresponds to the Boltzmann statistics (see, e.g., [15]).

For the generation-recombination rates G_n and G_p for the carriers, it is often assumed that

$$G_n = G_p = G$$

since electrons and holes are generated or combined in pairs. This rate G is determined by certain deviation of the density levels from the equilibrium level. The common form used in the literature for the standard model (1.2) is

$$G = g - Q(n, p)(np - 1) \quad (1.6)$$

with different models for the coefficients $Q = Q(n, p)$. For example, in the Shockley-Read-Hall model, $Q_{\text{SRH}} \sim (1 + n + p)^{-1}$, and in the Auger model, $Q_{\text{AU}} \sim n + p$ (see [15, 11]). The nonnegative function g in (1.6) represents a distributed generation rate applied to the system (see [3] for an application). For the nonlinear model (1.5), the equilibrium state is given by

$$H(n) + H(p) = h_i$$

where $H(\cdot)$ is the enthalpy function

$$H(s) = \int_1^s \frac{P'(s)}{s} ds \quad (1.7)$$

associated with the pressure $P(\cdot)$, and h_i is a material constant. This equilibrium relation is derived from the zero quasi-Fermi potential conditions (see

[16]), and includes the standard case $np = 1$ since $H(s) = \ln s$ when P is linear. Therefore we assume that the generation-recombination rate has the form

$$G = g - Q(n, p)(S(H(n) + H(p)) - 1) \quad (1.8)$$

where the scaling function S is nonnegative, nondecreasing with $S(h_i) = 1$. When $P(\cdot)$ is linear and S is exponential, the above model becomes the common model (1.6). But when P is nonlinear, (1.6) cannot be included as a special case of (1.8). We will include both models (1.6) and (1.8) in our study here. For (1.8), the two limits of $H(\cdot)$ also play an important role in our analysis:

$$h_0 = -H(0^+) = \int_0^1 \frac{P'(s)}{s} ds \quad \text{and} \quad h_\infty = H(\infty) = \int_1^\infty \frac{P'(s)}{s} ds. \quad (1.9)$$

Thus our model equations read

$$\frac{\partial n}{\partial t} - \operatorname{div} \mu_n(\nabla P(n) - n\nabla u) = G(n, p) \quad (1.10)$$

$$\frac{\partial p}{\partial t} - \operatorname{div} \mu_p(\nabla P(p) + p\nabla u) = G(n, p) \quad (1.11)$$

where u satisfies (1.3).

The boundary conditions are the same as for standard drift-diffusion model (see, e.g., [15, 11]). Let $\partial\Omega = \Sigma_N \cup \Sigma_D$, and ν the outward normal on $\partial\Omega$. Σ_N represents the insulated portion of the boundary, and Σ_D the contact portion. Thus we have the following boundary conditions:

$$\begin{aligned} n = \bar{n}(x), \quad p = \bar{p}(x), \quad u = \bar{u}(x) \quad \text{on } \Sigma_D \\ \frac{\partial n}{\partial \nu} = \frac{\partial p}{\partial \nu} = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Sigma_N. \end{aligned} \quad (1.12)$$

Initial conditions (n^0, p^0) are prescribed for (n, p) . For compatibility, we also assume that u^0 is given by (1.3) at $t = 0$.

There have been some mathematical studies devoted to the analysis of the standard drift-diffusion model since the 1970's. For example, existence of stationary solutions has been shown (e.g., [10, 3]) by using decoupling mappings and fixed point theorems, and local and global solutions for the evolutionary problem have been established (e.g., [8, 20, 5, 6, 7]). We refer the interested reader to the books [15, 11] for more complete references. As for this nonlinear model under consideration, very few studies can be found in the literature. Gajewski and Gröger [9] study the time-dependent model based on the Fermi-Dirac statistics with the assumptions that S is the exponential function and $h_0 = h_\infty = \infty$. In the common example $P(s) = s^\gamma$ with $\gamma > 1$, $h_0 = \gamma/(\gamma - 1)$ is finite. In [16] Markowich and Unterreiter study the stationary problem for the case h_0 is finite and $G = 0$, and show the existence of positive solutions under some smallness assumptions on the data, and in general the existence of nonnegative solutions (vacuum solutions). Similar results on nonnegative solutions for the time-dependent problem are obtained by Jüngel [12] for a more

general model, but it does not include the generation-recombination rates like (1.8). It is also worth noting that this model can be derived from the hydrodynamic model when the equations of momentum conservation are reduced to the current relations (1.5) by neglecting the convective terms (see [19]). In [14], it is shown rigorously in the case of one space dimension that such a model is the limit of the hydrodynamic model with the isentropic assumption as the relaxation time tends to zero. Models of this type are known in the literature as electro-diffusion systems ([18]), and appear in other applications such as electrophoresis in electro-chemistry ([4]) and ion flow through membrane channels in physiology ([2, 17]).

In this paper, we study the model (1.10)-(1.11) with (1.3) and boundary conditions (1.12) for generation-recombination rates given by either (1.6) or (1.8). In particular, we will establish the existence of global non-vacuum solutions for the time-dependent problem and non-vacuum stationary solutions. Under some general assumptions (Assumption (A) below), we show the existence of non-vacuum global solutions when G is from the common model (1.6). For G given by (1.8), we need to require a relation (see (2.19) below) among the constants h_0 , h_∞ and h_i for the positive lower bound for (n, p) ; this relation is always satisfied if $h_0 = \infty$. If we allow vacuum solutions (nonnegative solutions), then this restriction on the data can be removed. We will also study the long-time behavior of the system, showing the existence of an absorbing set and a compact global attractor. We use the following examples to illustrate some of our results in this paper.

Example 1. Consider

$$P(s) = s^\gamma \quad \text{and} \quad G(n, p) = g(x) - \bar{q} \frac{np - 1}{n + p + 1}$$

where $\gamma > 1$ and $\bar{q} > 0$ (the Shockley-Read-Hall recombination model). Then, from any positive initial data (n^0, p^0) , the system (1.10)-(1.11) with (1.3) has a global positive (non-vacuum) solution (n, p) in $H^1 \cap L^\infty$ (Theorem 2.1), and there is a positive stationary solution (Corollary 4.3). As a dynamical system, (1.10)-(1.11) has an absorbing set

$$\mathcal{B} = \{(n, p) \in L^\infty(\Omega)^2 : 0 < \underline{c} \leq n(x), p(x) \leq \bar{c}\}$$

(Theorem 3.3). In some cases (in particular, the one-dimensional Ω case) the global solution is unique (Theorem 2.3), and the dynamical system has a compact attractor \mathcal{A} that attracts all sets that are bounded in terms of the L^∞ -norm (Theorem 3.6).

Example 2. For the case

$$P(s) = s^\gamma \quad \text{and} \quad G(n, p) = g(x) - \bar{q}(\exp(n^{\gamma-1} + p^{\gamma-1}) - 1)$$

with $\gamma > 1$ and $\bar{q} > 0$, beginning with any non-negative initial data (n^0, p^0) , the system has a global solution and a stationary solution, all in L^∞ with $n, p \geq 0$,

and $n^\gamma, p^\gamma, u \in H^1(\Omega)$, and there is an absorbing set as in Example 1 above but with $\underline{c} = 0$ (Theorem 5.1).

Throughout this paper we make the following general assumptions on the model parameters:

Assumption (A):

- (i) $P(\cdot)$ is from $[0, \infty)$ to $[0, \infty)$, $P(0) = 0$, and $P'(\cdot) > 0$ is locally Lipschitz continuous in $(0, \infty)$.
- (ii) ϵ_0, μ_n and μ_p are positive constants.
- (iii) Ω is an open, connected, bounded domain in \mathbb{R}^l ($l = 1, 2$ or 3) of class $C^{0,1}$, and $\partial\Omega = \Sigma_D \cup \Sigma_N$ with Σ_D nonempty and relatively closed.
- (iv) $N, g \in L^\infty(\Omega)$ with $g \geq 0$ a.e. in Ω .
- (v) $Q = Q(n, p)$ is locally Lipschitz continuous and $0 \leq \frac{Q(n, p)}{1+n+p} \leq \bar{Q}$ (constant) for all $n, p \geq 0$.
- (vi) S is locally Lipschitz continuous, nonnegative, nondecreasing on its domain $Dom(S) \supseteq (-2h_0, 2h_\infty)$, and $S(h_i) = 1$.
- (vii) \bar{n}, \bar{p} and $\bar{u} \in H^1(\Omega) \cap L^\infty(\Sigma_D)$ with $\inf_{\Sigma_D} \bar{n} > 0, \inf_{\Sigma_D} \bar{p} > 0$.
- (viii) $n^0, p^0 \in L^\infty(\Omega)$ with $\inf_{\Omega} n^0 > 0, \inf_{\Omega} p^0 > 0$.

For the model (1.6), the condition (vi) on S is not needed. Under (v) on Q above, (1.6) includes the most commonly used Shockley-Read-Hall model $Q_{SRH} \sim (1 + n + p)^{-1}$ and the Auger model $Q_{AU} \sim n + p$. See e.g., [15, 11].

Define

$$V = \{\phi \in H^1(\Omega) : \phi = 0 \text{ on } \Sigma_D\},$$

which is the space known as $H_0^1(\Omega \cup \Sigma_N)$ (see, e.g., [22]). Denote the usual L^2 product as $\langle \phi, \psi \rangle$. V is equipped with the H^1 equivalent norm $|\nabla \cdot|_2$. By the Poincaré inequality, for $2 \leq r \leq 6$, there exists constant $\alpha_r > 0$ such that

$$|\phi|_r \leq \alpha_r |\nabla \phi|_2 \text{ for all } \phi \in V. \tag{1.13}$$

Moreover, the embedding of V into each $L^r(\Omega)$ is compact.

Given $T > 0$, a triplet (n, p, u) is said to be a weak solution to the model equations (1.10)–(1.11), (1.3) with (1.12) if

$$(n, p, u) \in (\bar{n}, \bar{p}, \bar{u}) + L^2(0, T; V)^3 \cap H^1(0, T; V^*)^3$$

with $(n, p) = (n^0, p^0)$ at $t = 0$, $n > 0$ and $p > 0$, and the triplet (n, p, u) satisfies

$$\left\langle \frac{\partial n}{\partial t}, \phi \right\rangle_{V^* \times V} + \mu_n \langle \nabla P(n) - n \nabla u, \nabla \phi \rangle = \langle G(n, p), \phi \rangle \tag{1.14}$$

$$\left\langle \frac{\partial p}{\partial t}, \psi \right\rangle_{V^* \times V} + \mu_p \langle \nabla P(p) + p \nabla u, \nabla \psi \rangle = \langle G(n, p), \psi \rangle \tag{1.15}$$

$$\epsilon_0 \langle \nabla u, \nabla \chi \rangle + \langle n - p - N, \chi \rangle = 0 \tag{1.16}$$

for all $(\phi, \psi, \chi) \in V^3$, and a.e. in $(0, T)$.

As in the case of the standard drift-diffusion model, it is more convenient to use a set of new variables, the so-called quasi-Fermi potentials, in place of (n, p) when showing existence of solutions. For positive (n, p) , define new variables (v, w) as follows:

$$v = H(n) - u \quad \text{and} \quad w = H(p) + u \quad (1.17)$$

where the enthalpy function H is as defined in (1.7). Note that $H(\cdot)$ is strictly increasing from $(0, \infty)$ onto $(-h_0, h_\infty)$. If we denote

$$\Gamma = H^{-1} : \quad s = \Gamma(\tau) \quad \text{iff} \quad \tau = H(s),$$

then $\Gamma(\cdot)$ is strictly increasing from $(-h_0, h_\infty)$ onto $(0, \infty)$ with $\Gamma(0) = 1$. The relations (1.17) can then be expressed as

$$n = \Gamma(v + u) \quad \text{and} \quad p = \Gamma(w - u), \quad (1.18)$$

and the current expressions in (1.5) can be simplified as

$$J_n = \mu_n(\nabla P(n) - n\nabla u) = \mu_n\Gamma(v + u)\nabla v,$$

$$J_p = -\mu_p(\nabla P(p) + p\nabla u) = -\mu_p\Gamma(w - u)\nabla w.$$

The boundary conditions for (v, w, u) are the same as in (1.12) with $\Gamma(\bar{v} + \bar{u}) = \bar{n}$ and $\Gamma(\bar{w} - \bar{u}) = \bar{p}$. Hence we can solve for (v, w, u) instead of (n, p, u) . When showing the existence of non-vacuum solutions, we will establish positive lower and upper bounds for (n, p) or $(\Gamma(v + u), \Gamma(w - u))$ and hence $v + u$ and $w - u$ will be always inside the domain of Γ . With such bounds established, the two sets of variables, (v, w, u) and (n, p, u) , become equivalent. In studying vacuum solutions, we will not use the (v, w) variables since (n, p) may become zero.

The rest of our presentation is organized as follows. In section 2 we analyze the time-dependent model (1.14)–(1.16), and show the existence of global positive $H^1 \cap L^\infty$ solutions under Assumption (A). We carry out the proof by first showing existence of the time-discretized systems, and then passing to the limit to obtain solutions to the continuous system. In this process, it is necessary for us to establish proper lower and upper a.e. bounds for the solutions. When the generation-recombination term G is given by (1.8), we need to assume h_0 sufficiently large to obtain a positive lower bound. With an additional regularity assumption, we can also establish the uniqueness of the global solution. We use the H^{-1} topology to deal with the nonlinear diffusion in $P(\cdot)$ for the uniqueness. In section 3, we investigate the system as an infinite-dimensional dynamical system acting on a set of admissible initial data (n^0, p^0) . We first sharpen the L^∞ bounds for the solutions, so that an absorbing set can be constructed. Again, for the lower bound, more restrictions on the model parameters are needed. We then establish the existence of a global attractor of the dynamical system. The semigroup defined by the system is continuous with respect to the H^{-1} topology, and we develop these results by using both the H^{-1} and L^∞ topologies. In section 4 we prove the existence of solutions to the corresponding stationary system. All the discussions above are for the non-vacuum solutions,

and the extra conditions (e.g. h_0 is sufficiently large) we impose to obtain the positive lower bounds for (n, p) are not needed for the upper bounds. In section 5, we will discuss parallel results about vacuum solutions by removing these conditions. In general vacuum solutions (n, p) may have less regularity than the non-vacuum solutions, and we prove that in this case the solutions (n, p, u) are in L^∞ with $P(n)$, $P(p)$ and u all in H^1 .

2 Existence of Global Solutions

In this section we establish the existence of global solutions (n, p, u) to the model equations (1.10)–(1.11) with (1.3) and boundary conditions (1.12).

2.1 The Time-Discretized System

Given an integer $m > 0$, let $h = T/m$. Consider the time-discretized version of the system (1.14)–(1.16): $(n^k, p^k, u^k) \in (\bar{n}, \bar{p}, \bar{u}) + V^3$ satisfying

$$\left\langle \frac{n^k - n^{k-1}}{h}, \phi \right\rangle + \mu_n \langle \nabla P(n^k) - n^k \nabla u^k, \nabla \phi \rangle = \langle G(n^k, p^k), \phi \rangle \tag{2.1}$$

$$\left\langle \frac{p^k - p^{k-1}}{h}, \psi \right\rangle + \mu_p \langle \nabla P(p^k) + p^k \nabla u^k, \nabla \psi \rangle = \langle G(n^k, p^k), \psi \rangle \tag{2.2}$$

$$\epsilon_0 \langle \nabla u^k, \nabla \chi \rangle + \langle n^k - p^k - N, \chi \rangle = 0 \tag{2.3}$$

for all $(\phi, \psi, \chi) \in V^3$. For the (v, w) variables, we replace (2.1)–(2.2) by

$$\left\langle \frac{n^k - n^{k-1}}{h}, \phi \right\rangle + \mu_n \langle n^k \nabla v^k, \nabla \phi \rangle = \langle G(n^k, p^k), \phi \rangle \tag{2.4}$$

$$\left\langle \frac{p^k - p^{k-1}}{h}, \psi \right\rangle + \mu_p \langle p^k \nabla w^k, \nabla \psi \rangle = \langle G(n^k, p^k), \psi \rangle \tag{2.5}$$

where $(n^k, p^k) = (\Gamma(v^k + u^k), \Gamma(w^k - u^k))$, $k = 1, 2, \dots, m$. The two sets of equations are the same if one can establish positive lower and upper bounds for $(n, p) = (\Gamma(v + u), \Gamma(w - u))$.

To show existence, we will consider the (v, w) -system. We first make certain truncations of the function Γ to ensure the properness of the nonlinearity in our setting, and then “remove” these truncations by establishing proper L^∞ bounds for the solutions and choosing the truncation constants accordingly.

For $d_k \in (0, h_\infty)$ and $c_k \in (0, h_0)$ to be determined later (see (2.17), (2.23) below), define

$$\Gamma_k(s) = \begin{cases} \Gamma(-c_k) & \text{when } s < -c_k \\ \Gamma(s) & \text{when } -c_k \leq s \leq d_k \\ \Gamma(d_k) & \text{when } s > d_k \end{cases} \tag{2.6}$$

($k = 1, 2, \dots, m$). Note that Γ_k 's are defined on $(-\infty, \infty)$. Then we consider (2.4)–(2.5) and (2.3) with the replacement of

$$n^k = \Gamma_k(v^k + u^k) \quad \text{and} \quad p^k = \Gamma_k(w^k - u^k), \quad k = 1, 2, \dots, m. \quad (2.7)$$

Given a triplet $(v^{k-1}, w^{k-1}, u^{k-1})$, we will show the existence of a solution to (2.4)–(2.5) and (2.3) with (2.7). To this end, we construct the solution map \mathcal{T} from $(\tilde{v}, \tilde{w}, \tilde{u}) \in L^2(\Omega)^3$ to the solution $(v, w, u) \in (\bar{v}, \bar{w}, \bar{u}) + V^3$ of the system

$$\left\langle \frac{\Gamma_k(v + u) - n^{k-1}}{h}, \phi \right\rangle + \mu_n \langle \tilde{n} \nabla v, \nabla \phi \rangle = \langle G(\Gamma_k(v + \tilde{u}), \tilde{p}), \phi \rangle \quad (2.8)$$

$$\left\langle \frac{\Gamma_k(w - u) - p^{k-1}}{h}, \psi \right\rangle + \mu_p \langle \tilde{p} \nabla w, \nabla \psi \rangle = \langle G(\tilde{n}, \Gamma_k(w - \tilde{u})), \psi \rangle \quad (2.9)$$

$$\epsilon_0 \langle \nabla u, \nabla \chi \rangle + \langle \Gamma_k(v + u) - \Gamma_k(w - u) - N, \chi \rangle = 0 \quad (2.10)$$

for all $(\phi, \psi, \chi) \in V^3$, where $n^{k-1} = \Gamma_{k-1}(v^{k-1} + u^{k-1})$, $p^{k-1} = \Gamma_{k-1}(w^{k-1} - u^{k-1})$, $\tilde{n} = \Gamma_k(\tilde{v} + \tilde{u})$, and $\tilde{p} = \Gamma_k(\tilde{w} - \tilde{u})$.

Well-definedness of \mathcal{T} . It is straightforward to verify that (2.8)–(2.10) is the gradient system of the following convex lower-semicontinuous functional

$$\begin{aligned} I(v, w, u) &= \frac{\mu_n}{2} \langle \tilde{n} \nabla v, \nabla v \rangle + \frac{\mu_p}{2} \langle \tilde{p} \nabla w, \nabla w \rangle + \frac{\epsilon_0}{2h} \langle \nabla u, \nabla u \rangle \\ &\quad - \int_{\Omega} \left(\int_0^{v+\tilde{u}} G(\Gamma_k(s), \tilde{p}) ds + \int_0^{w-\tilde{u}} G(\tilde{n}, \Gamma_k(s)) ds \right) dx \\ &\quad + \frac{1}{h} \int_{\Omega} \left(\int_0^{v+u} + \int_0^{w-u} \right) \Gamma_k(s) ds dx \\ &\quad - \frac{1}{h} (\langle n^{k-1}, v + u \rangle + \langle p^{k-1}, w - u \rangle + \langle N, u \rangle). \end{aligned}$$

Therefore the operator defined by (2.8)–(2.10) is maximal monotone, and hence standard result (see, e.g., [1]) yields the existence of a solution (v, w, u) to (2.8)–(2.10).

Compactness of \mathcal{T} . Choose $\phi = v - \bar{v}$, $\psi = w - \bar{w}$ and $\chi = \frac{1}{h}(u - \bar{u})$ in (2.8)–(2.10) and sum up the three to obtain

$$\begin{aligned} &\mu_n \langle \tilde{n} \nabla v, \nabla(v - \bar{v}) \rangle + \mu_p \langle \tilde{p} \nabla w, \nabla(w - \bar{w}) \rangle + \frac{\epsilon_0}{h} \langle \nabla u, \nabla(u - \bar{u}) \rangle \\ &= \langle G(\Gamma_k(v + \tilde{u}), \tilde{p}), v - \bar{v} \rangle + \langle G(\tilde{n}, \Gamma_k(w - \tilde{u})), w - \bar{w} \rangle \\ &\quad - \frac{1}{h} \langle \Gamma_k(v + u), v - \bar{v} + u - \bar{u} \rangle - \frac{1}{h} \langle \Gamma_k(w - u), w - \bar{w} - u + \bar{u} \rangle \\ &\quad + \frac{1}{h} \langle N, u - \bar{u} \rangle. \end{aligned}$$

Then using the boundedness and monotonicity of Γ_k we can show

$$\begin{aligned} & \frac{\mu_n}{2}\Gamma(-c_k)|\nabla(v - \bar{v})|_2^2 + \frac{\mu_p}{2}\Gamma(-c_k)|\nabla(w - \bar{w})|_2^2 + \frac{\epsilon_0}{2h}|\nabla(u - \bar{u})|_2^2 \\ & \leq \frac{\mu_n}{2}\Gamma(d_k)|\nabla\bar{v}|_2^2 + \frac{\mu_p}{2}\Gamma(d_k)|\nabla\bar{w}|_2^2 + \frac{\epsilon_0}{2h}|\nabla\bar{u}|_2^2 \\ & \quad + \bar{G}|\Omega|^{1/2}(|v - \bar{v}|_2 + |w - \bar{w}|_2) \\ & \quad + \frac{1}{h}|\Gamma_k(\bar{v} + \bar{u})|_2|v - \bar{v}|_2 + \frac{1}{h}|\Gamma_k(\bar{w} - \bar{u})|_2|w - \bar{w}|_2 \\ & \quad + \frac{1}{h}(|\Gamma_k(\bar{v} + \bar{u})|_2 + |\Gamma_k(\bar{w} - \bar{u})|_2 + |N|_2)|u - \bar{u}|_2 \end{aligned}$$

where $\bar{G} = \max_{0 \leq n, p \leq \Gamma(d_k)} G(n, p)$. Hence, by the Poincaré inequality, we obtain

$$|\nabla(v - \bar{v})|_2 + |\nabla(w - \bar{w})|_2 + |\nabla(u - \bar{u})|_2 \leq \text{const.} \tag{2.11}$$

where the constant depends on h, c_k and d_k . This immediately implies that the range of map \mathcal{T} is a bounded set in $H^1(\Omega)^3$, and hence \mathcal{T} is compact from $L^2(\Omega)^3$ to $L^2(\Omega)^3$ due to the compact embedding of $H^1(\Omega)$ into L^2 .

Continuity of \mathcal{T} . Suppose $(\tilde{v}_j, \tilde{w}_j, \tilde{u}_j) \rightarrow (\tilde{v}, \tilde{w}, \tilde{u})$ in $L^2(\Omega)^3$ as $j \rightarrow \infty$, and denote their images under \mathcal{T} by (v_j, w_j, u_j) and (v, w, u) , respectively. Then from (2.8)–(2.10), by using the boundedness and monotonicity of Γ_k , we can obtain

$$\begin{aligned} & \mu_n\Gamma(-c_k)|\nabla(v_j - v)|_2^2 + \mu_p\Gamma(-c_k)|\nabla(w_j - w)|_2^2 + \frac{\epsilon_0}{h}|\nabla(u_j - u)|_2^2 \\ & \leq -\mu_n\langle(\tilde{n}_j - \tilde{n})\nabla v, \nabla(v_j - v)\rangle - \mu_p\langle(\tilde{p}_j - \tilde{p})\nabla w, \nabla(w_j - w)\rangle \\ & \quad + \langle G(\Gamma_k(v_j + \tilde{u}_j), \tilde{p}_j) - G(\Gamma_k(v + \tilde{u}), \tilde{p}), v_j - v \rangle \\ & \quad + \langle G(\tilde{n}_j, \Gamma_k(w_j - \tilde{u}_j)) - G(\tilde{n}, \Gamma_k(w - \tilde{u})), w_j - w \rangle \\ & \leq \mu_n|(\tilde{n}_j - \tilde{n})\nabla v|_2|\nabla(v_j - v)|_2 + \mu_p|(\tilde{p}_j - \tilde{p})\nabla w|_2|\nabla(w_j - w)|_2 \\ & \quad + |G(\Gamma_k(v_j + \tilde{u}_j), \tilde{p}_j) - G(\Gamma_k(v + \tilde{u}), \tilde{p})|_2|v_j - v|_2 \\ & \quad + |G(\tilde{n}_j, \Gamma_k(w_j - \tilde{u}_j)) - G(\tilde{n}, \Gamma_k(w - \tilde{u}))|_2|w_j - w|_2. \end{aligned}$$

To show \mathcal{T} is continuous, we argue that every subsequence of (v_j, w_j, u_j) has a further subsequence converging to (v, w, u) . This can be shown from the above inequality, the Poincaré inequality, and using the following facts: (i) every L^2 convergent sequence has a pointwise convergent subsequence; (ii) G and Γ_k are Lipschitz continuous; (iii) the V -norm of the solution (v, w, u) is bounded by (2.11); (iv) the dominated convergence theorem. (For details, see, e.g., [10].)

Therefore, by Schauder’s fixed point theorem, \mathcal{T} has at least one fixed point (v, w, u) , which obviously is a solution to the system (2.4)–(2.5) and (2.3) with the substitution of (2.7).

By choosing proper positive constants d_k and c_k , we will show in the next subsection that such a solution (v^k, w^k, u^k) has the following L^∞ bounds:

$$-c_k \leq v^k(x) + u^k(x) \leq d_k \quad \text{and} \quad -c_k \leq w^k(x) - u^k(x) \leq d_k. \tag{2.12}$$

Thus the truncations in Γ become unnecessary, and the solution (v^k, w^k, u^k) becomes a solution to (2.4)–(2.5), (2.3) with the original Γ . Furthermore, we see that $n^k = \Gamma(v^k + u^k)$, $p^k = \Gamma(w^k - u^k)$ and u^k form a solution to (2.1)–(2.3).

2.2 The L^∞ Bounds for Solutions

Now we establish independently the L^∞ bounds (2.12) for the solution (v, w, u) to the truncated version of (2.4)–(2.5), (2.3). Hence when we choose the truncation constants d_k and c_k accordingly, we can replace Γ_k by Γ and (v, w, u) becomes a solution to (2.4)–(2.5).

The Upper Bounds. Set

$$D_0 = \max\{\sup_{\Omega} n^0, \sup_{\Omega} p^0, \sup_{\Sigma_D} \bar{n}, \sup_{\Sigma_D} \bar{p}, 1\} \quad (2.13)$$

For $B > 0$ to be determined below (in (2.15)), set $d_k \in (0, h_\infty)$ such that

$$\Gamma(d_k) = D_0(1 + Bh)^k$$

and let $\phi_d = (v + u - d_k)_+$. We also require $h < 1/B$. Clearly $\phi_d \in V$. Choose this ϕ_d as the test function in both (2.4) and (2.3) to obtain

$$\begin{aligned} & \frac{1}{h} \langle \Gamma(d_k) - \Gamma_{k-1}(v^{k-1} + u^{k-1}), \phi_d \rangle + \mu_n \Gamma(d_k) \langle \nabla v, \nabla \phi_d \rangle \\ &= \langle G(\Gamma(d_k), \Gamma_k(w - u)), \phi_d \rangle, \end{aligned}$$

and

$$\epsilon_0 \langle \nabla u, \nabla \phi_d \rangle + \langle \Gamma(d_k) - \Gamma_k(w - u) - N, \phi_d \rangle = 0.$$

Since $\phi_d \geq 0$ and $\nabla \phi_d = \nabla(v + u)$ when $v + u > d_k$, combining the two inequalities, we have

$$\begin{aligned} & \mu_n \Gamma(d_k) |\nabla \phi_d|_2^2 \\ & \leq \langle G(\Gamma(d_k), \Gamma_k(w - u)), \phi_d \rangle + \frac{1}{h} \langle \Gamma(d_{k-1}) - \Gamma(d_k), \phi_d \rangle + \frac{\mu_n}{\epsilon_0} \Gamma(d_k) \langle N, \phi_d \rangle. \end{aligned} \quad (2.14)$$

Note that

$$\frac{1}{h} \langle \Gamma(d_{k-1}) - \Gamma(d_k), \phi_d \rangle = -\Gamma(d_k) \frac{B}{1 + Bh} \leq -\Gamma(d_k) \frac{B}{2},$$

and for both models (1.6) and (1.8) for G , we have

$$G(n, p) \leq g(x) + Q(n, p) \leq |g|_\infty + \bar{Q}(1 + n + p).$$

Then (2.14) becomes

$$\mu_n |\nabla \phi_d|_2^2 \leq \left\langle \frac{|g|_\infty + \bar{Q}(1 + 2D_0)}{D_0} - \frac{B}{2} + \frac{\mu_n}{\epsilon_0} N, \phi_d \right\rangle \leq 0$$

if we choose

$$B = 2\left(\frac{|g|_\infty + \bar{Q}(1 + 2D_0)}{D_0} + \frac{\max(\mu_n, \mu_p)}{\epsilon_0}|N|_\infty\right). \tag{2.15}$$

Therefore $\phi_d = 0$ a.e. and $v + u \leq d_k$ a.e. in Ω .

The upper bounds above can be improved to be independent of k and h (i.e. time-independent) if the number h_∞ is sufficiently large. In the following we give the proof of this claim. For τ such that $H(D_0) \leq \tau < h_\infty$, let $d_k = \tau$ for all k . Set $\phi_\tau = (v + u - \tau)_+$ as the test function in both (2.4) and (2.3) and then combine the two to obtain

$$\mu_n \Gamma(\tau) |\nabla \phi_\tau|_2^2 \leq \langle G(\Gamma(\tau)), \Gamma_k(w - u), \phi_\tau \rangle + \frac{\mu_n}{\epsilon_0} \Gamma(\tau) \langle N, \phi_\tau \rangle.$$

Hence

$$|\nabla \phi_\tau|_2^2 \leq \left\langle \frac{g + \bar{Q}(1 + 2D_0)}{D_0 \min(\mu_n, \mu_p)} + \frac{N}{\epsilon_0}, \phi_\tau \right\rangle$$

Then by applying Lemma A.1 in the Appendix, we obtain $v(x) + u(x) \leq d_0^*$ a.e. in Ω with

$$d_0^* = H(D_0) + 4\alpha_6^2 |\Omega|^{1/6} \left(\frac{|g|_2 + \bar{Q}(1 + 2D_0) |\Omega|^{1/2}}{D_0 \min(\mu_n, \mu_p)} + \frac{|N|_2}{\epsilon_0} \right) \tag{2.16}$$

provided that $h_\infty > d_0^*$. Thus we obtain the time-independent bound d_0^* .

The same upper bound for $w(x) - u(x)$ can be established similarly.

In summary, we have established the upper bound $\Gamma(d_k)$ for both $n = \Gamma(v + u)$ and $p = \Gamma(w - u)$ as

$$\Gamma(d_k) = \begin{cases} D_0(1 + Bh)^k & \text{in general} \\ \Gamma(d_0^*) & \text{if } h_\infty > d_0^* \end{cases} \tag{2.17}$$

where D_0 is given in (2.13), B in (2.15) and d_0^* in (2.16).

The Lower Bounds. Set

$$C_0 = \min\left\{ \inf_\Omega n^0, \inf_\Omega p^0, \inf_{\Sigma_D} \bar{n}, \inf_{\Sigma_D} \bar{p} \right\}. \tag{2.18}$$

For $0 < \tilde{C}_0 \leq C_0$ to be chosen below, set $c_k \in (0, h_0)$ such that

$$\Gamma(-c_k) = \tilde{C}_0(1 + Bh)^{-k}$$

and let $\phi_c = (v + u + c_k)_-$. Then $\phi_c \leq 0$ and $\phi_c \in V$. Choose this ϕ_c as the test function in both (2.4) and (2.3), and combine the two to obtain

$$\begin{aligned} & \mu_n \Gamma(-c_k) |\nabla \phi_c|_2^2 \\ &= \frac{1}{h} \langle \Gamma(-c_k) - \Gamma_{k-1}(v^{k-1} + u^{k-1}), -\phi_c \rangle - \langle G(\Gamma(-c_k)), \Gamma_k(w - u), -\phi_c \rangle \\ & \quad + \frac{\mu_n}{\epsilon_0} \Gamma(-c_k) \langle \Gamma(-c_k) - \Gamma_k(w - u) - N, -\phi_c \rangle \\ &\leq \left\langle \frac{1}{h} (\Gamma(-c_k) - \Gamma(-c_{k-1})) - G(\Gamma(-c_k)), \Gamma_k(w - u) \right\rangle - \frac{\mu_n}{\epsilon_0} \Gamma(-c_k) \langle N, -\phi_c \rangle \\ &\leq -\Gamma(-c_k) \langle B + \frac{\mu_n}{\epsilon_0} N, -\phi_c \rangle \quad (\text{if } G(\Gamma(-c_k)), \Gamma(w - u) \geq 0) \\ &\leq 0 \quad (\text{since } B \text{ is from (2.15)}) \end{aligned}$$

Then we have $\phi_c \equiv 0$, i.e., $v(x) + u(x) \geq -c_k$ a.e. in Ω .

Now we verify that $G(\Gamma(-c_k), \Gamma(w - u)) \geq 0$ for the two models (1.6) and (1.8) separately, by choosing different constants \tilde{C}_0 . For G given by (1.6), choose $\tilde{C}_0 = \min(C_0, 1/D_0)$. Then

$$G(\Gamma(-c_k), \Gamma(w - u)) \geq g - Q(\Gamma(-c_k)\Gamma(d_k) - 1) \geq Q(1 - \tilde{C}_0 D_0) \geq 0.$$

For G given by (1.8), we need to assume

$$\min(d_0^*, h_\infty) < h_i + h_0. \quad (2.19)$$

From (2.17) we see $H(\Gamma_k(w - u)) \leq d_k$ is no larger than either d_0^* or h_∞ . By (2.19), $h_i - \min(d_0^*, h_\infty) > -h_0$, hence we can choose $\tilde{C}_0 = \min(C_0, \Gamma(h_i - \min(d_0^*, h_\infty)))$. Then

$$H(\Gamma(-c_k)) + H(\Gamma(d_k)) \leq H(\tilde{C}_0) + \min(d_0^*, h_\infty) \leq h_i$$

and hence

$$G(\Gamma(-c_k), \Gamma(w - u)) \geq g - Q(S(H(\Gamma(-c_k)) + H(\Gamma(d_k)) - 1)) \geq 0$$

since S is nondecreasing and $S(h_i) = 1$.

Hence we have the positive lower bound $\Gamma(-c_k) = \tilde{C}_0(1 + Bh)^{-k}$ for n and p , where

$$\tilde{C}_0 = \begin{cases} \min\{C_0, 1/D_0\} & \text{for } G \text{ in (1.6)} \\ \min\{C_0, \Gamma(h_i - \min(d_0^*, h_\infty))\} & \text{for } G \text{ in (1.8) with} \\ & \min(d_0^*, h_\infty) < h_i + h_0 \end{cases} \quad (2.20)$$

In the case $h_\infty > d_0^*$, $\Gamma(d_0^*)$ is a time-independent upper bound for both n and p , then we can also establish a time-independent positive lower bound for n and p provided that $h_0 > c_0^*$ (to be given below). To prove this claim, let $c_k = \tau$ for all k with $-H(C_0) \leq \tau < h_0$. Set $\phi_\tau = (v + u + \tau)_- = -(-(v + u) - \tau)_+ \in V$. Following the same calculations as above leads to

$$\mu_n \Gamma(-\tau) |\nabla \phi_\tau|_2^2 \leq \langle G(\Gamma(-\tau), \Gamma_k(w - u)), \phi_\tau \rangle + \frac{\mu_n}{\epsilon_0} \Gamma(-\tau) \langle N, \phi_\tau \rangle.$$

As in choosing \tilde{C}_0 above, we can see that $G(\Gamma(-\tau), \Gamma_k(w - u)) \geq 0$ if we choose τ such that $\Gamma(-\tau) \leq 1/\Gamma(d_0^*)$ when G is given by (1.6), and choose $\tau \geq d_0^* - h_i$ when G is by (1.8). Thus we have

$$|\nabla \phi_\tau|_2^2 \leq \langle \frac{N}{\epsilon_0}, \phi_\tau \rangle.$$

From Lemma A.1 we conclude that $v + u \geq -c_0^*$ a.e. in Ω for

$$c_0^* = \tilde{c} + 4 \frac{\alpha_6^2}{\epsilon_0} |\Omega|^{1/6} |N|_2 \quad (2.21)$$

where

$$\tilde{c} = \begin{cases} \max\{-H(C_0), -H(1/\Gamma(d_0^*))\} & \text{for model (1.6)} \\ \max\{-H(C_0), d_0^* - h_i\} & \text{for model (1.8)} \end{cases} \quad (2.22)$$

provided that $h_0 > c_0^*$.

Similarly we can show the same lower bounds for $w(x) - u(x)$ in all situations.

In summary, we have established the positive lower bound $\Gamma(-c_k)$ for both $n = \Gamma(v + u)$ and $p = \Gamma(w - u)$ as

$$\Gamma(-c_k) = \begin{cases} \tilde{C}_0(1 + Bh)^{-k} & \text{in general} \\ \Gamma(-c_0^*) & \text{if } h_\infty > d_0^* \text{ and } h_0 > c_0^* \end{cases} \quad (2.23)$$

where \tilde{C}_0 is from (2.20) and c_0^* from (2.21).

Bounds for u . To establish the L^∞ bounds for u in (2.3), we use the same L^∞ estimate technique as in showing the time-independent upper bound for $v + u$ above. Let $\bar{\tau} = \sup_{\Sigma_D} |\bar{u}|$. For $\tau \geq \bar{\tau}$, choose $u_\tau = (u - \tau)^+$ as the test function in (2.3) to obtain

$$|\nabla u_\tau|_2^2 \leq \left\langle \frac{1}{\epsilon_0}(n - p - N), u_\tau \right\rangle.$$

By Lemma A.1 we obtain $u \leq \bar{\tau} + 4\frac{\alpha_6^2}{\epsilon_0}|\Omega|^{1/6}|n - p - N|_2$. Applying the same argument for $-u$ we can obtain the lower bound estimate. Hence we have the following L^∞ estimates for u in (2.3):

$$|u(t, x)| \leq \sup_{\Sigma_D} |\bar{u}(x)| + 4\frac{\alpha_6^2}{\epsilon_0}|\Omega|^{1/6}|n(t, \cdot) - p(t, \cdot) - N(\cdot)|_2 \quad \text{a.e. in } (0, T) \times \Omega. \quad (2.24)$$

2.3 Energy Estimates

Now we establish some energy estimates for solutions to the (n, p) -system (2.1)–(2.3), which will be needed in the next subsection for convergence argument.

First we note that

$$C_T \equiv \tilde{C}_0 e^{-BT} \leq \Gamma(-c_k) \leq n^k, p^k \leq \Gamma(d_k) \leq D_0 e^{BT} \equiv D_T \quad (2.25)$$

for all $k = 1, 2, \dots, m$. Set

$$\kappa_1 = \min P'(s) \quad \text{and} \quad \kappa_2 = \max P'(s) \quad \text{over } s \in [C_T, D_T].$$

Choose $\chi = u^k - \bar{u} \in V$ in (2.3) to obtain

$$\begin{aligned} & \frac{\epsilon_0}{2} (|\nabla u^k|_2^2 - |\nabla \bar{u}|_2^2 + |\nabla(u^k - \bar{u})|_2^2) \\ & \leq |n^k - p^k - N|_2 |u^k - \bar{u}|_2 \leq \alpha_2 |n^k - p^k - N|_2 |\nabla(u^k - \bar{u})|_2, \end{aligned}$$

and hence

$$|\nabla u^k|_2^2 \leq |\nabla \bar{u}|_2^2 + \frac{\alpha_2^2}{\epsilon_0} |n^k - p^k - N|_2^2 \leq |\nabla \bar{u}|_2^2 + \frac{\alpha_2^2}{\epsilon_0} (2D_T |\Omega|^{1/2} + |N|_2)^2 \equiv M_1.$$

Then choose $\phi = n^k - \bar{n} \in V$ in (2.1) to yield

$$\begin{aligned} & \frac{1}{2h} (|n^k - \bar{n}|_2^2 - |n^{k-1} - \bar{n}|_2^2 + |n^k - n^{k-1}|_2^2) + \mu_n \kappa_1 |\nabla(n^k - \bar{n})|_2^2 \\ & \leq \kappa_2 |\nabla \bar{n}|_2 |\nabla(n^k - \bar{n})|_2 + D_T |\nabla u^k|_2 |\nabla(n^k - \bar{n})|_2 + \overline{G}_T |\Omega|^{1/2} |n^k - \bar{n}|_2, \end{aligned}$$

where $\overline{G}_T = \max G(n, p)$ over $n, p \in [C_T, D_T]$, which leads to

$$\begin{aligned} & \frac{1}{2h} (|n^k - \bar{n}|_2^2 - |n^{k-1} - \bar{n}|_2^2 + |n^k - n^{k-1}|_2^2) + \frac{\mu_n \kappa_1}{4} |\nabla(n^k - \bar{n})|_2^2 \\ & \leq \frac{1}{\mu_n \kappa_1} \left\{ \kappa_2^2 |\nabla \bar{n}|_2^2 + D_T^2 M_1 + \alpha_2^2 \overline{G}_T^2 |\Omega| \right\} \equiv \frac{1}{\mu_n} M_2. \end{aligned}$$

Sum up the above for $1 \leq k \leq m$ to obtain

$$\sum_{k=1}^m |n^k - n^{k-1}|_2^2 + h \frac{\mu_n \kappa_1}{2} \sum_{k=1}^m |\nabla(n^k - \bar{n})|_2^2 \leq |n^0 - \bar{n}|_2^2 + \frac{2}{\mu_n} M_2 T. \quad (2.26)$$

Similarly we can obtain for p^k the estimate

$$\sum_{k=1}^m |p^k - p^{k-1}|_2^2 + h \frac{\mu_p \kappa_1}{2} \sum_{k=1}^m |\nabla(p^k - \bar{p})|_2^2 \leq |p^0 - \bar{p}|_2^2 + \frac{2}{\mu_p} M_2 T. \quad (2.27)$$

2.4 Convergence to the Continuous System

In this subsection we use the solutions $\{(n^k, p^k, u^k)\}_{k=0}^m$ to construct solutions of continuous time to (1.14)–(1.16).

Let $t_k = kh$, and define the following interpolation (in time) functions $n_t^{(1)}(t, x)$ and $n_h^{(2)}(t, x)$ as

$$n_h^{(1)}(t, x) = n^k(x) + \frac{t - t_k}{h} (n^k(x) - n^{k-1}(x))$$

and $n_h^{(2)}(t, x) = n^k(x)$ on $(t_{k-1}, t_k]$, ($k = 1, 2, \dots, m$). From this definition, we see that

$$\begin{aligned} \int_0^T |n_h^{(1)}(t) - n_h^{(2)}(t)|_2^2 dt & \leq \frac{h}{3} \sum_{k=1}^m |n^k - n^{k-1}|_2^2, \\ \int_0^T |\nabla(n_h^{(1)}(t) - \bar{n})|_2^2 dt & \leq h \sum_{k=0}^m |\nabla(n^k - \bar{n})|_2^2, \end{aligned}$$

and

$$\int_0^T |\nabla(n_h^{(2)}(t) - \bar{n})|_2^2 dt \leq h \sum_{k=1}^m |\nabla(n^k - \bar{n})|_2^2.$$

Similarly we define such functions for the other two variables p and u , and the same estimates hold. From (2.26) and (2.27) we then have

$$\int_0^T |n_h^{(1)}(t) - n_h^{(2)}(t)|_2^2 dt \leq \text{const.} h, \quad (2.28)$$

$$\int_0^T |\nabla(n_h^{(1)}(t) - \bar{n})|_2^2 dt, \quad \int_0^T |\nabla(n_h^{(2)}(t) - \bar{n})|_2^2 dt \leq \text{const.} \tag{2.29}$$

and the same hold for p .

Using these interpolation functions, we can write the equations (2.1)–(2.3) as follows

$$\begin{aligned} \langle n_h^{(1)}(t) - n^0, \phi \rangle & \tag{2.30} \\ + \int_0^t \left\{ \mu_n \langle P'(n_h^{(2)}(s)) \nabla n_h^{(2)}(s) - n_h^{(2)}(s) \nabla u_h^{(2)}(s), \nabla \phi \rangle - \langle G_h, \phi \rangle \right\} ds & = 0 \end{aligned}$$

$$\begin{aligned} \langle p_h^{(1)}(t) - p^0, \psi \rangle & \tag{2.31} \\ + \int_0^t \left\{ \mu_p \langle P'(p_h^{(2)}(s)) \nabla p_h^{(2)}(s) + p_h^{(2)}(s) \nabla u_h^{(2)}(s), \nabla \psi \rangle - \langle G_h, \psi \rangle \right\} ds & = 0 \end{aligned}$$

$$\epsilon_0 \langle \nabla u_h^{(2)}(t), \nabla \chi \rangle + \langle n_h^{(2)}(t) - p_h^{(2)}(t) - N, \chi \rangle = 0 \tag{2.32}$$

for all $(\phi, \psi, \chi) \in V^3$, $t \in [0, T]$. Here we use $G_h = G(n_h^{(2)}(s), p_h^{(2)}(s))$. By the L^∞ bounds (2.25), we can see easily from (2.30)–(2.31) that

$$\int_0^T \left| \frac{\partial n_h^{(1)}(t)}{\partial t} \right|_{V^*}^2 dt, \quad \int_0^T \left| \frac{\partial p_h^{(1)}(t)}{\partial t} \right|_{V^*}^2 dt \leq \text{const.} \tag{2.33}$$

Therefore (2.29) and (2.33) together yield that

$$\{n_h^{(1)}(t) - \bar{n}\}_{h>0}, \quad \{p_h^{(1)}(t) - \bar{p}\}_{h>0} \quad \text{are bounded in } L^2(0, T; V) \cap H^1(0, T; V^*) \tag{2.34}$$

(uniformly in $h > 0$), and hence Aubin’s Lemma implies that they are precompact in $L^2(0, T; L^2(\Omega))$. Therefore we can conclude that there exist a sequence of positive numbers $\{h\}$ and a pair of functions $(n(t), p(t))$ in $L^2(0, T; L^2(\Omega))$ such that, as $h \rightarrow 0^+$,

$$(n_h^{(1)}(t), p_h^{(1)}(t)) \rightarrow (n(t), p(t)) \quad \text{strongly in } L^2(0, T; L^2(\Omega))$$

and, from (2.29), by taking further subsequences if necessary,

$$(n_h^{(1)}(t), p_h^{(1)}(t)) \rightarrow (n(t), p(t)) \quad \text{weakly in } L^2(0, T; H^1(\Omega)).$$

Hence $(n(t) - \bar{n}, p(t) - \bar{p})$ are in $L^2(0, T; V)$. In view of (2.28), the above convergence results also hold for $(n_h^{(2)}(t), p_h^{(2)}(t))$. Since every L^2 -convergent sequence has an a.e. convergent subsequence, we can further assume that

$$(n_h^{(2)}(t, x), p_h^{(2)}(t, x)) \rightarrow (n(t, x), p(t, x)) \quad \text{a.e. in } (0, T) \times \Omega.$$

Hence we see from (2.25) that (n, p) are in $L^\infty((0, T) \times \Omega)$, and the expressions for d_k and c_k in (2.17) and (2.23) clearly lead to

$$\tilde{C}_0 e^{-Bt} \leq n(t, x), \quad p(t, x) \leq D_0 e^{Bt} \quad \text{a.e. in } (0, T) \times \Omega \tag{2.35}$$

(or with the upper bound replaced by $\Gamma(d_0^*)$ when $h_\infty > h^*$, and the lower bound by $\Gamma(-c_0^*)$ if additionally $h_0 > c_0^*$.) For the u -component, it is easy to see from (2.32) that there exists $u(t) \in L^2(0, T; H^1(\Omega))$ satisfying

$$\epsilon_0 \langle \nabla u(t), \nabla \chi \rangle + \langle n(t) - p(t) - N, \chi \rangle = 0$$

for all $\chi \in V$ such that

$$u(t) - \bar{u} \in V \quad \text{and} \quad u_h^{(2)}(t) \rightarrow u(t) \quad \text{strongly in } H^1(\Omega), \text{ a.e. in } (0, T).$$

Finally, we can pass to the limit $h \rightarrow 0^+$ in (2.30)–(2.31), by the dominated convergence theorem, to obtain

$$\begin{aligned} \langle n(t) - n^0, \phi \rangle + \int_0^t \{ \mu_n \langle \nabla P(n(s)) - n(s) \nabla u(s), \nabla \phi \rangle - \langle G(n(s), p(s)), \phi \rangle \} ds &= 0 \\ \langle p(t) - p^0, \psi \rangle + \int_0^t \{ \mu_p \langle \nabla P(p(s)) + p(s) \nabla u(s), \nabla \psi \rangle - \langle G(n(s), p(s)), \psi \rangle \} ds &= 0 \end{aligned}$$

a.e. in $(0, T)$, which imply that $n(t)$ and $p(t)$ are in $H^1(0, T; V^*)$, and the triplet (n, p, u) is a solution to (1.14)–(1.16).

We summarize our discussions of §2.1–§2.4 in the following theorem about the existence of global solutions. Recall the various constants D_0 defined in (2.13), \tilde{C}_0 in (2.23), B in (2.15), d_0^* in (2.16), and c_0^* in (2.21). They are all independent of $T > 0$.

Theorem 2.1 (Existence of global non-vacuum solutions.) *Suppose Assumption (A) holds, and the generation-recombination rate G is given either by (1.6) or by (1.8) under the condition $\min(d_0^*, h_\infty) < h_i + h_0$. Then for any given $T > 0$, there exists at least one solution (n, p, u) to (1.14)–(1.16) that has the following estimates:*

$$\begin{aligned} \tilde{C}_0 e^{-Bt} &\leq n(t, x), \quad p(t, x) \leq D_0 e^{Bt}, \\ |u(t, x)| &\leq \sup_{\Sigma_D} |\bar{u}| + 4 \frac{\alpha_6^2}{\epsilon_0} |\Omega|^{1/6} |n(t, \cdot) - p(t, \cdot) - N(\cdot)|_2 \end{aligned}$$

a.e. in $(0, T) \times \Omega$. In the case $h_\infty > d_0^*$, the upper bound for (n, p) can be replaced by the constant $\Gamma(d_0^*)$. If additionally $h_0 > c_0^*$, the lower bound for (n, p) can be also replaced by the constant $\Gamma(-c_0^*)$.

Remark 2.2 *The condition $\min(d_0^*, h_\infty) < h_i + h_0$ is used only to obtain the lower bound constant \tilde{C}_0 when G is given by (1.8), and it is true if $h_0 = \infty$.*

2.5 Uniqueness

In this subsection we discuss the uniqueness of the global solutions for the system (1.14)–(1.15) with (1.16). It turns out that we need to impose a regularity assumption on the Laplacian operator with the mixed boundary conditions as

in (1.12), and to use the weaker H^{-1} topology to deal with the nonlinear $P(\cdot)$. These extra cares are not necessary in the case of linear diffusion ($P(\cdot)$ linear), as shown in [6, 7].

We make the following additional regularity assumption for the uniqueness:

Assumption (A1): For $f \in L^2(\Omega)$, the weak solution $z \in V$ to

$$\langle \nabla z, \nabla \chi \rangle = \langle f, \chi \rangle \quad \text{for all } \chi \in V$$

is in $H^2(\Omega)$, and there exists a constant (depending on Ω and Σ_D, Σ_N) such that

$$|z|_{H^2(\Omega)} \leq \text{const.} |f|_2.$$

Such a regularity assumption is in fact an assumption on the domain Ω and the partition of its boundary $\partial\Omega = \Sigma_D \cup \Sigma_N$. In the case of one-dimensional domain, it is always satisfied. When Ω is a rectangle in \mathbb{R}^2 and Σ_D consists of the two opposite sides of the rectangle, we know that u is in H^2 . In general, such regularity for the solution is true if Ω is of the class $C^{1,1}$ and Σ_N is both open and closed (see [22, Theorem 2.24]), which includes cases such as $\Sigma_N = \emptyset$ (the pure Dirichlet problem) or when Σ_N and Σ_D are disjoint portions of $\partial\Omega$ (annulus domains).

We also need to use the weaker H^{-1} topology to deal with the nonlinear $P(\cdot)$. Let Δ_0 be the Laplacian operator defined on $H^2(\Omega) \cap V$. Then $-\Delta_0$ is a positive self-adjoint operator on $L^2(\Omega)$, and $-\Delta_0 \in \mathcal{L}(V, V^*)$ with

$$\langle -\Delta_0 \phi, \psi \rangle_{V^* \times V} = \langle \nabla \phi, \nabla \psi \rangle \quad \text{for } \phi, \psi \in V.$$

Let $(-\Delta_0)^{-1} \in \mathcal{L}(V^*, V)$ be the Riesz map of V^* equipped with the norm

$$|\phi|_{V^*}^2 = \langle \phi, (-\Delta_0)^{-1} \phi \rangle_{V^* \times V} \quad \text{for } \phi \in V^*.$$

That is, $|\phi|_{V^*} = |(-\Delta_0)^{-1} \phi|_V$ for $\phi \in V^*$. We also see that

$$\langle \nabla \phi, \nabla (-\Delta_0)^{-1} \xi \rangle = \langle \phi, \xi \rangle \quad \text{for } \phi \in V, \xi \in L^2(\Omega),$$

and, by Assumption (A1),

$$|(-\Delta_0)^{-1} \xi|_{H^2(\Omega)} \leq \text{const.} |\xi|_2 \quad \text{for } \xi \in L^2(\Omega).$$

Theorem 2.3 (Uniqueness.) *Under Assumptions (A) and (A1), the $H^1 \cap L^\infty$ weak solution (n, p, u) to (1.14)–(1.16) is unique.*

Proof. Let (n, p, u) and $(\hat{n}, \hat{p}, \hat{u})$ be $H^1 \cap L^\infty$ weak solutions to (1.14)–(1.16) corresponding to initial data (n^0, p^0) and (\hat{n}^0, \hat{p}^0) . From (1.14) we have the equation for $n - \hat{n}$:

$$\begin{aligned} & \left\langle \frac{\partial(n - \hat{n})}{\partial t}, \phi \right\rangle_{V^* \times V} + \mu_n \langle \nabla(P(n) - P(\hat{n})), \nabla \phi \rangle - \mu_n \langle n \nabla u - \hat{n} \nabla \hat{u}, \nabla \phi \rangle \\ & = \langle G(n, p) - G(\hat{n}, \hat{p}), \phi \rangle \end{aligned} \tag{2.36}$$

for all $\phi \in V$ and a similar equation for $p - \hat{p}$ from (1.15). For $u - \hat{u}$ there holds

$$\epsilon_0 \langle \nabla(u - \hat{u}), \nabla \chi \rangle + \langle n - \hat{n} - (p - \hat{p}), \chi \rangle = 0$$

for all $\chi \in V$, and hence

$$|\nabla(u - \hat{u})|_2 \leq \frac{\alpha_2}{\epsilon_0} (|n - \hat{n}|_2 + |p - \hat{p}|_2). \quad (2.37)$$

Now choose $\phi = (-\Delta_0)^{-1}(n - \hat{n}) \in V$ in (2.36), and we estimate the terms as follows. The first and the second terms in (2.36) become:

$$\left\langle \frac{\partial(n - \hat{n})}{\partial t}, (-\Delta_0)^{-1}(n - \hat{n}) \right\rangle_{V^* \times V} = \frac{d}{dt} \left(\frac{1}{2} |n - \hat{n}|_{V^*}^2 \right)$$

and

$$\langle \nabla(P(n) - P(\hat{n})), \nabla(-\Delta_0)^{-1}(n - \hat{n}) \rangle = \langle P(n) - P(\hat{n}), n - \hat{n} \rangle \geq \kappa_1 |n - \hat{n}|_2^2.$$

The third term is

$$\begin{aligned} & |\langle n \nabla u - \hat{n} \nabla \hat{u}, \nabla(-\Delta_0)^{-1}(n - \hat{n}) \rangle| \\ & \leq |n - \hat{n}|_2 |\nabla u|_4 |\nabla(-\Delta_0)^{-1}(n - \hat{n})|_4 + |\hat{n}|_\infty |\nabla(u - \hat{u})|_2 |\nabla(-\Delta_0)^{-1}(n - \hat{n})|_2 \\ & \leq |n - \hat{n}|_2 |\nabla u|_4 \text{const.} |n - \hat{n}|_2^{3/4} |n - \hat{n}|_{V^*}^{1/4} \quad (\text{see (2.39) below}) \\ & \quad + |\hat{n}|_\infty \frac{\alpha_2}{\epsilon_0} (|n - \hat{n}|_2 + |p - \hat{p}|_2) |n - \hat{n}|_{V^*} \quad (\text{from (2.37)}) \\ & \leq \text{const.} \left(|n - \hat{n}|_2^{7/4} |n - \hat{n}|_{V^*}^{1/4} + (|n - \hat{n}|_2 + |p - \hat{p}|_2) |n - \hat{n}|_{V^*} \right) \end{aligned} \quad (2.38)$$

where the constant depends on $|\nabla u|_4$ and $|\hat{n}|_\infty$, and we have used the following inequality:

$$\begin{aligned} |\nabla(-\Delta_0)^{-1}(n - \hat{n})|_4 & \leq |\nabla(-\Delta_0)^{-1}(n - \hat{n})|_6^{3/4} |\nabla(-\Delta_0)^{-1}(n - \hat{n})|_2^{1/4} \\ & \leq \text{const.} |\nabla(-\Delta_0)^{-1}(n - \hat{n})|_V^{3/4} |n - \hat{n}|_{V^*}^{1/4} \\ & \quad (\text{by the embedding of } V \text{ into } L^6) \\ & \leq \text{const.} |(-\Delta_0)^{-1}(n - \hat{n})|_{H^2}^{3/4} |n - \hat{n}|_{V^*}^{1/4} \\ & \leq \text{const.} |n - \hat{n}|_2^{3/4} |n - \hat{n}|_{V^*}^{1/4}. \end{aligned} \quad (2.39)$$

The right side of (2.36) becomes

$$\begin{aligned} & |\langle G(n, p) - G(\hat{n}, \hat{p}), (-\Delta_0)^{-1}(n - \hat{n}) \rangle| \\ & \leq \text{const.} (|n - \hat{n}|_2 + |p - \hat{p}|_2) |(-\Delta_0)^{-1}(n - \hat{n})|_2 \\ & = \text{const.} (|n - \hat{n}|_2 + |p - \hat{p}|_2) |n - \hat{n}|_{V^*} \end{aligned}$$

where the constant depends on the Lipschitz continuity of G and $|(n, p)|_\infty$, $|(\hat{n}, \hat{p})|_\infty$.

Hence we obtain

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} |n - \hat{n}|_{V^*}^2 \right) + \mu_n \kappa_1 |n - \hat{n}|_2^2 \\ & \leq \text{const.} \left(|n - \hat{n}|_2^{7/4} |n - \hat{n}|_{V^*}^{1/4} + (|n - \hat{n}|_2 + |p - \hat{p}|_2) |n - \hat{n}|_{V^*} \right). \end{aligned}$$

A similar inequality can be obtained for $p - \hat{p}$. Combining the two yields

$$\begin{aligned} & \frac{d}{dt} (|n - \hat{n}|_{V^*}^2 + |p - \hat{p}|_{V^*}^2) + 2\kappa_1 \min\{\mu_n, \mu_p\} (|n - \hat{n}|_2^2 + |p - \hat{p}|_2^2) \\ & \leq \text{const.} \left(|n - \hat{n}|_2^{7/4} |n - \hat{n}|_{V^*}^{1/4} + |p - \hat{p}|_2^{7/4} |p - \hat{p}|_{V^*}^{1/4} \right. \\ & \quad \left. + (|n - \hat{n}|_2 + |p - \hat{p}|_2) (|n - \hat{n}|_{V^*} + |p - \hat{p}|_{V^*}) \right) \end{aligned}$$

and hence, by Young's inequality ($ab \leq a^p/p + b^q/q$),

$$\frac{d}{dt} (|n - \hat{n}|_{V^*}^2 + |p - \hat{p}|_{V^*}^2) \leq K (|n - \hat{n}|_{V^*}^2 + |p - \hat{p}|_{V^*}^2)$$

where the constant K depends only on the L^∞ bounds of the solutions and $|\nabla u|_4$. Thus Gronwall's inequality leads to

$$|n - \hat{n}|_{V^*}^2 + |p - \hat{p}|_{V^*}^2 \leq e^{Kt} (|n^0 - \hat{n}^0|_{V^*}^2 + |p^0 - \hat{p}^0|_{V^*}^2). \tag{2.40}$$

Hence (2.40) and (2.37) imply the uniqueness of $H^1 \cap L^\infty$ solutions. \square

Remark 2.4 *The H^2 -regularity of the Laplacian operator in Assumption (A1) can be replaced by $W^{1,\infty}$ -regularity.*

Indeed, in the proof above, the only place we use (A1) is in the third term estimate (2.38), and when u is in $W^{1,\infty}$, this term can be easily estimated as

$$\begin{aligned} & |\langle n \nabla u - \hat{n} \nabla \hat{u}, \nabla (-\Delta_0)^{-1} (n - \hat{n}) \rangle| \\ & \leq (|n - \hat{n}|_2 |\nabla u|_\infty + |\hat{n}|_\infty |\nabla (u - \hat{u})|_2) |n - \hat{n}|_{V^*} \end{aligned}$$

and then proceed as above to obtain (2.40) with K depending on $|\nabla u|_\infty$.

3 Long-Time Behavior of Solutions

In this section we study the system (1.14)–(1.15) for (n, p) as an infinite-dimensional dynamical system acting on an admissible set of initial data (n^0, p^0) for its long-time behavior. In particular, we will establish sharper L^∞ bounds for any $L^\infty \cap H^1$ global solutions so that an absorbing set can be constructed, and then show the existence of a global compact attractor of the dynamical system.

3.1 Construction of an Absorbing Set

The existence of an absorbing set amounts to careful analysis of the dependence of bounds for solutions on the initial conditions. To this end, we will work on the continuous system (1.14)–(1.15) for sharper L^∞ bounds of the solutions. Suppose (n, p) is a $H^1 \cap L^\infty$ global solution to (1.14)–(1.15) with initial data (n^0, p^0) . We will construct two positive functions $C(t)$ and $D(t)$ such that

$$C(t) \leq n(t, x), \quad p(t, x) \leq D(t) \quad \text{a.e. in } (0, \infty) \times \Omega$$

where $C(\infty)$ and $D(\infty)$ are independent of the initial conditions (n^0, p^0) . With these bounds we can easily construct an absorbing set.

We make the following assumptions in addition to the general Assumption (A).

Assumption (A2): $\inf_{s \geq 1} P'(s) \geq r_0 > 0$ and $0 \leq Q(n, p) \leq \bar{Q}_0$.

The condition on P yields $h_\infty = \infty$ and thus we have the time-independent upper bound $\Gamma(d_0^*)$ from Theorem 2.1 (but it depends on initial conditions). It is satisfied for $P(s) = ks^\gamma$ when $\gamma \geq 1$. We also remark that it is possible to relax the condition on Q to the case of linear growth $Q(n, p) \leq \bar{Q}_0 + \alpha(n + p)$ with sufficiently small $\alpha > 0$. This can be easily seen in the following upper bound estimates.

Upper Bounds. Let

$$d_0 = (\max\{\sup_\Omega n^0, \sup_\Omega p^0\} - 1)^+ \quad \text{and} \quad \bar{d} = \max\{\sup_{\Sigma_D} \bar{n}, \sup_{\Sigma_D} \bar{p}, 1\}. \quad (3.1)$$

For $d \geq \bar{d}$ and $\lambda > 0$ (to be determined below), set

$$D(t) = d(1 + d_0 e^{-\lambda t}) \quad \text{and} \quad \phi_d = (n - D(t))^+, \quad \psi_d = (p - D(t))^+.$$

Clearly $\phi_d, \psi_d \in V$ and $\phi_d(0) = \psi_d(0) = 0$. Since (n, p, u) is in $L^\infty((0, T) \times \Omega)$ for any given $T > 0$, we see that ϕ_d and ψ_d are in $L^\infty((0, T) \times \Omega)$. Use this ϕ_d as a test function in (1.14). Since $n = \phi_d + D(t)$ and $\nabla n = \nabla \phi_d$ when $\phi_d \geq 0$ and $\langle \frac{\partial}{\partial t} \phi_d, \phi_d \rangle = \frac{d}{dt} |\phi_d|_2^2$, we have

$$\frac{d}{dt} \left(\frac{1}{2} |\phi_d|_2^2 \right) + \langle D'(t), \phi_d \rangle + \mu_n r_0 |\nabla \phi_d|_2^2 \leq \mu_n \langle n \nabla u, \nabla \phi_d \rangle + \langle G(n, p), \phi_d \rangle. \quad (3.2)$$

Since $n \nabla \phi_d = (\phi_d + D(t)) \nabla \phi_d = \nabla (\frac{1}{2} \phi_d^2 + D(t) \phi_d)$, we use $\frac{1}{2} \phi_d^2 + D(t) \phi_d \in V$ as a test function in (1.16) to obtain

$$\langle n \nabla u, \nabla \phi_d \rangle = \langle \nabla u, \nabla (\frac{1}{2} \phi_d^2 + D(t) \phi_d) \rangle = -\frac{1}{\epsilon_0} \langle n - p - N, \frac{1}{2} \phi_d^2 + D(t) \phi_d \rangle.$$

For G from either (1.6) or (1.8), we have $G(n, p) \leq g + Q(n, p) \leq g + \bar{Q}_0$ by Assumption (A2). Thus (3.2) becomes

$$\frac{d}{dt} \left(\frac{1}{2\mu_n} |\phi_d|_2^2 \right) + r_0 |\nabla \phi_d|_2^2 \leq -\frac{1}{\epsilon_0} \langle n - p - N, \frac{1}{2} \phi_d^2 + D(t) \phi_d \rangle + \frac{1}{\mu_n} \langle Q + g - D'(t), \phi_d \rangle. \quad (3.3)$$

For the p -component we obtain similarly

$$\frac{d}{dt} \left(\frac{1}{2\mu_p} |\psi_d|_2^2 + r_0 |\nabla \psi_d|_2^2 \right) \leq \frac{1}{\epsilon_0} \langle n - p - N, \frac{1}{2} \psi_d^2 + D(t) \psi_d \rangle + \frac{1}{\mu_p} \langle Q + g - D'(t), \psi_d \rangle. \tag{3.4}$$

It is straightforward to verify that

$$(n - p)(\phi_d - \psi_d) \geq 0, \quad (n - p - N)(\phi_d^2 - \psi_d^2) \geq -\frac{N^2}{4}(\phi_d + \psi_d).$$

Hence combining (3.3)–(3.4) yields

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2\mu_n} |\phi_d|_2^2 + \frac{1}{2\mu_p} |\psi_d|_2^2 \right) + r_0 (|\nabla \phi_d|_2^2 + |\nabla \psi_d|_2^2) \\ & \leq \frac{1}{8\epsilon_0} \langle N^2, \phi_d + \psi_d \rangle + \frac{D(t)}{\epsilon_0} \langle N, \phi_d - \psi_d \rangle + \langle Q + g - D'(t), \frac{1}{\mu_n} \phi_d + \frac{1}{\mu_p} \psi_d \rangle. \end{aligned} \tag{3.5}$$

The following is similar to the proof of Lemma A.1, but we need to use Lemma A.3 instead of Lemma A.2 because the upper bound $D(t)$ is a function of t . Denote

$$\Omega_d^{(n)}(t) = \{x \in \Omega : n(t, x) > D(t)\}, \quad \Omega_d^{(p)}(t) = \{x \in \Omega : p(t, x) > D(t)\}$$

and

$$\varphi(d) = \sup_{t \geq 0} \left(|\Omega_d^{(n)}(t)| + |\Omega_d^{(p)}(t)| \right).$$

Clearly φ is nonnegative, non-increasing in d . Note that, for $f \in L^\infty$,

$$\begin{aligned} \langle f, \phi_d \rangle & \leq |f|_\infty |\Omega_d^{(n)}|^{3/4} |\phi_d|_4 \\ & \leq |f|_\infty |\Omega_d^{(n)}|^{3/4} \alpha_4 |\nabla \phi_d|_2 \leq \frac{r_0}{2} |\nabla \phi_d|_2^2 + \frac{\alpha_4^2}{2r_0} |f|_\infty^2 |\Omega_d^{(n)}|^{3/2} \end{aligned}$$

and similarly for ψ_d . Hence by properly identifying f in (3.5) we obtain

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{\mu_n} |\phi_d|_2^2 + \frac{1}{\mu_p} |\psi_d|_2^2 \right) + r_0 (|\nabla \phi_d|_2^2 + |\nabla \psi_d|_2^2) \\ & \leq \frac{\alpha_4^2}{r_0} \left[\frac{|N|_\infty^2}{8\epsilon_0} + \frac{\overline{Q}_0 + |g|_\infty}{\min(\mu_n, \mu_p)} + \left(\frac{|N|_\infty}{\epsilon_0} + \frac{\lambda}{\min(\mu_n, \mu_p)} \right) D(t) \right]^2 \times \\ & \quad \left(|\Omega_d^{(n)}|^{3/2} + |\Omega_d^{(p)}|^{3/2} \right), \end{aligned}$$

where we have also used the fact that $-D'(t) \leq \lambda D(t)$. Using the Poincaré inequality again and the fact that $D(t) \geq 1$, we reduce the above further into

$$\frac{d}{dt} (|\phi_d|_2^2 + |\psi_d|_2^2) + \frac{r_0 \max(\mu_n, \mu_p)}{\alpha_2^2} (|\phi_d|_2^2 + |\psi_d|_2^2) \leq K_1 D^2(t) \varphi(d)^{3/2} \tag{3.6}$$

where we denote

$$K_1 = \frac{\alpha_4^2 \max(\mu_n, \mu_p)}{r_0} \left(\frac{|N|_\infty^2 + 8|N|_\infty}{8\epsilon_0} + \frac{\overline{Q}_0 + |g|_\infty + \lambda}{\min(\mu_n, \mu_p)} \right)^2. \tag{3.7}$$

Choose

$$\lambda = \frac{r_0 \max(\mu_n, \mu_p)}{3\alpha_2^2} \quad (3.8)$$

(independent of initial data), and multiply (3.6) with $e^{3\lambda t}$ and then integrate to yield

$$|\phi_d|_2^2 + |\psi_d|_2^2 \leq K_1 \varphi(d)^{3/2} e^{-3\lambda t} \int_0^t e^{3\lambda s} D^2(s) ds \leq \frac{K_1}{\lambda} \varphi(d)^{3/2} D^2(t).$$

Now since for $\hat{d} > d \geq \bar{d}$,

$$|\phi_d|_2^2 \geq \int_{\Omega_{\hat{d}}^{(n)}} |\phi_d|^2 dx \geq (\hat{d} - d)^2 (1 + d_0 e^{-\lambda t})^2 |\Omega_{\hat{d}}^{(n)}| = d^{-2} (\hat{d} - d)^2 |\Omega_{\hat{d}}^{(n)}| D^2(t)$$

and similarly for $|\psi_d|_2^2$, we obtain from the above that

$$|\Omega_{\hat{d}}^{(n)}(t)| + |\Omega_{\hat{d}}^{(p)}(t)| \leq \frac{K_1}{\lambda} d^2 (\hat{d} - d)^{-2} \varphi(d)^{3/2}.$$

Hence,

$$\varphi(\hat{d}) \leq \frac{K_1}{\lambda} d^2 (\hat{d} - d)^{-2} \varphi(d)^{3/2}$$

for $\hat{d} > d \geq \bar{d}$. By applying Lemma A.3 to $\varphi(d)$, we conclude that $\varphi(d) = 0$ for

$$d = 2\bar{d}(1 + 2^8 K_1^{3/2} \lambda^{-3/2} |\Omega|^{3/4}). \quad (3.9)$$

Hence $\phi_d = \psi_d = 0$ a.e. for $t \geq 0$ and $x \in \Omega$. That is,

$$n(t, x), \quad p(t, x) \leq d(1 + d_0 e^{-\lambda t})$$

where d in (3.9) and λ in (3.8) are independent of the initial data.

Lower Bounds. Let

$$c_0 = \min\{1/d_0, \inf_{\Omega} n^0, \inf_{\Omega} p^0\}, \quad \bar{c} = \min\{\inf_{\Sigma_D} \bar{n}, \inf_{\Sigma_D} \bar{p}, 1\}. \quad (3.10)$$

For $C(t)$ to be determined below such that $C(t) \leq \bar{c}$ and $C(0) \leq c_0$, let

$$\phi_c = (n - C(t))^{-}, \quad \psi_c = (p - C(t))^{-} \in V$$

Choose ϕ_c and ψ_c as test functions in (1.14)–(1.15) and follow the same calculations as in (3.2) to (3.4) above to obtain (note that here ϕ_c and ψ_c are non-positive)

$$\frac{d}{dt} \left(\frac{1}{2\mu_n} |\phi_c|_2^2 \right) \leq -\frac{1}{\epsilon_0} \langle n - p - N, \frac{1}{2} \phi_c^2 + C(t) \phi_c \rangle + \frac{1}{\mu_n} \langle -G(n, p) + C'(t), -\phi_c \rangle$$

and

$$\frac{d}{dt} \left(\frac{1}{2\mu_p} |\psi_c|_2^2 \right) \leq \frac{1}{\epsilon_0} \langle n - p - N, \frac{1}{2} \psi_c^2 + C(t) \psi_c \rangle + \frac{1}{\mu_p} \langle -G(n, p) + C'(t), -\psi_c \rangle.$$

Using the fact that $(n - p)(\phi_c - \psi_c) \geq 0$ and $(n - p)(\phi_c^2 - \psi_c^2) \geq 0$, we combine the two above to yield

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2\mu_n} |\phi_c|_2^2 + \frac{1}{2\mu_p} |\psi_c|_2^2 \right) &\leq \frac{1}{2\epsilon_0} \langle N, \phi_c^2 - \psi_c^2 \rangle + \left\langle \frac{C(t)}{\epsilon_0} N, \phi_c - \psi_c \right\rangle \\ &\quad + \left\langle -G(n, p) + C'(t), -\frac{1}{\mu_n} \phi_c - \frac{1}{\mu_p} \psi_c \right\rangle \end{aligned} \tag{3.11}$$

Our choice of $C(t)$ will depend on the form of $G(n, p)$, and we need to make additional assumptions. Hence we discuss the two models (1.6) and (1.8) separately below.

Case 1: G is by (1.6). In this case we assume further that

$$\inf_{n, p \geq 0} (1 + n + p)Q(n, p) \geq \bar{q} > 0. \tag{3.12}$$

This is true for the Shockley-Read-Hall model $Q_{\text{SRH}} \sim (1 + n + p)^{-1}$, and it also includes the case when Q has a positive lower bound. We choose

$$C(t) = \frac{c}{1 + e^{-\omega t/c_0}} \quad \text{with } \omega = \min\{c_0, \lambda\} \tag{3.13}$$

for $0 < c \leq \bar{c}$, where c_0 and \bar{c} are as in (3.10) and λ in (3.8). Thus $C'(t) \leq C(t)e^{-\omega t} \omega / c_0 \leq C(t)$. By (3.12) on Q ,

$$Q(n, p) \geq \frac{\bar{q}}{1 + n + p} \geq \frac{\bar{q}}{2D(t)} \geq \frac{\bar{q}}{2d(1 + e^{-\omega t/c_0})} = \frac{\bar{q}}{2cd} C(t)$$

since $d_0 \leq 1/c_0$ and $\omega \leq \lambda$. When $n < C(t)$, since $p \leq D(t)$ from the upper bound, we have

$$np \leq C(t)D(t) \leq cd \leq 1/2 \quad \text{if we choose } c \leq 1/2d.$$

Hence,

$$\langle -G(n, p), -\phi_c \rangle = \langle -g + Q(np - 1), -\phi_c \rangle \leq \left\langle -\frac{1}{2}Q, -\phi_c \right\rangle \leq \left\langle -\frac{\bar{q}}{4cd} C(t), -\phi_c \right\rangle.$$

Similarly we can obtain

$$\langle -G(n, p), -\psi_c \rangle \leq \left\langle -\frac{\bar{q}}{4cd} C(t), -\psi_c \right\rangle.$$

Substituting these estimates in to (3.11) we obtain

$$\frac{d}{dt} \left(\frac{1}{2\mu_n} |\phi_c|_2^2 + \frac{1}{2\mu_p} |\psi_c|_2^2 \right) \leq \frac{|N|_\infty}{2\epsilon_0} (|\phi_c|_2^2 + |\psi_c|_2^2) + C(t) \left\langle F, -\frac{1}{\mu_n} \phi_c - \frac{1}{\mu_p} \psi_c \right\rangle \tag{3.14}$$

where

$$F = \frac{\max(\mu_n, \mu_p)}{\epsilon_0} |N|_\infty + 1 - \frac{\bar{q}}{4cd}.$$

Therefore when we choose

$$c = \min \left\{ \bar{c}, 1/2d, \frac{\bar{q}}{4d(\max(\mu_n, \mu_p)|N|_\infty/\epsilon_0 + 1)} \right\}, \tag{3.15}$$

we have $F \leq 0$. Hence

$$\frac{d}{dt} \left(\frac{1}{2\mu_n} |\phi_c|_2^2 + \frac{1}{2\mu_p} |\psi_c|_2^2 \right) \leq \frac{\max(\mu_n, \mu_p)}{\epsilon_0} |N|_\infty \left(\frac{1}{2\mu_n} |\phi_c|_2^2 + \frac{1}{2\mu_p} |\psi_c|_2^2 \right) \tag{3.16}$$

and Gronwall's inequality leads to $\phi_c = \psi_c = 0$ a.e. for $t \geq 0$ and $x \in \Omega$. Therefore $n(t, x), p(t, x) \geq C(t)$ when $C(t)$ is given by (3.13) with c in (3.15).

Case 2: G is by (1.8). Here we need to assume that

$$\inf_{0 < s \leq 1} P'(s) = s_0 > 0, \quad \inf_{n, p \geq 0} Q(n, p) \geq \bar{q}_0 > 0 \quad \text{and} \quad S(-\infty) = 0. \tag{3.17}$$

The first condition implies that $h_0 = \infty$, and when S is an exponential function, $S(-\infty) = 0$. We need to choose a different form of $C(t)$:

$$C(t) = \Gamma(-\theta - \theta_0 e^{-\hat{\omega}t}) \tag{3.18}$$

where θ_0 and $\hat{\omega}$ are given by

$$\theta_0 = \max\{-H(c_0), K_0 dd_0\}, \quad \hat{\omega} = \min\left\{\frac{s_0}{\theta_0}, \lambda\right\} \quad \text{with} \quad K_0 = \max_{d \leq \xi \leq d(1+d_0)} H'(\xi), \tag{3.19}$$

and for $\theta \geq -H(\bar{c})$. Then $C(t) \leq \Gamma(-\theta) \leq \bar{c}$, $C(0) \leq \Gamma(-\theta_0) \leq c_0$, and

$$C'(t) = \Gamma'(-\theta - \theta_0 e^{-\hat{\omega}t}) e^{-\hat{\omega}t} \theta_0 \hat{\omega} \leq C(t) \frac{\theta_0}{s_0} \hat{\omega} \leq C(t)$$

since Γ is the inverse of H and hence $\Gamma'(\cdot) = \Gamma(\cdot)/P'(\Gamma(\cdot)) \leq \Gamma(\cdot)/s_0$ when $\Gamma \leq 1$ by (3.17). We also have

$$\begin{aligned} H(D(t)) &= H(d + dd_0 e^{-\lambda t}) = H(d) + H'(\xi) dd_0 e^{-\lambda t} \\ &\leq H(d) + K_0 dd_0 e^{-\lambda t} \leq H(d) + \theta_0 e^{-\hat{\omega}t}, \end{aligned}$$

and hence $H(C(t)) + H(D(t)) \leq -\theta + H(d)$. From (3.17), there exists $\bar{k} > 0$ such that

$$S(-k) \leq 1/2 \quad \text{for all } k \geq \bar{k}.$$

Hence, when $n < C(t)$, since $p \leq D(t)$,

$$S(H(n) + H(p)) \leq S(-\theta + H(d)) \leq 1/2 \quad \text{if we choose } \theta \geq \bar{k} + H(d).$$

Therefore,

$$\begin{aligned} \langle -G(n, p), -\phi_c \rangle &= \langle -g + Q(S(H(n) + H(p)) - 1), -\phi_c \rangle \\ &\leq \langle -Q/2, -\phi_c \rangle \leq \langle -\bar{q}_0/2, -\phi_c \rangle \end{aligned}$$

and similarly for $\langle -G, -\psi_c \rangle$. Substituting these estimates in (3.11) and following a similar calculation as in the previous case from (3.14) to (3.16), we can obtain $n(t, x), p(t, x) \geq C(t)$ for $C(t)$ given in (3.18) with

$$\theta = \max\{-H(c), \bar{k} + H(d), -H(\frac{\bar{q}_0}{2(\max(\mu_n, \mu_p)|N|_\infty/\epsilon_0 + 1)})\}. \tag{3.20}$$

In summary, we have established for global solutions (n, p) the upper bound $D(t)$ given by

$$D(t) = d(1 + d_0 e^{-\lambda t}) \tag{3.21}$$

where d is independent of the initial data and given in (3.9), d_0 in (3.1), and λ in (3.8), and the lower bound $C(t)$ given by

$$C(t) = \begin{cases} \frac{c}{1 + e^{-\omega t}/c_0} & \text{when } G \text{ is by (1.6) with (3.12)} \\ \Gamma(-\theta - \theta_0 e^{-\hat{\omega} t}) & \text{when } G \text{ is by (1.8) with (3.17)} \end{cases} \tag{3.22}$$

where c and θ are independent of the initial data and given in (3.15) and (3.20) respectively, and c_0 in (3.10), ω in (3.13), and θ_0 and $\hat{\omega}$ in (3.19).

Theorem 3.1 (Absorbing L^∞ bounds.) *Suppose Assumptions (A) and (A2) hold, and G is given either by (1.6) with the condition (3.12) or by (1.8) with (3.17). Then for any global $L^\infty \cap H^\infty$ solutions (n, p, u) to (1.14)–(1.16) from initial data (n^0, p^0) , there hold the following bounds:*

$$C(t) \leq n(t, x), p(t, x) \leq D(t) \tag{3.23}$$

a.e. for $t \geq 0$ and $x \in \Omega$, where $D(t)$ and $C(t)$ are given in (3.21) and (3.22), respectively. In particular the positive constants $C(\infty)$ and $D(\infty)$ are independent of the initial data (n^0, p^0) .

Remark 3.2 *The extra conditions (3.12) or (3.17) are only needed for establishing the lower bound $C(t)$.*

Now we study the system (1.14)–(1.15) as a dynamical system acting on the set of admissible initial data

$$H_{\text{ad}} = \{(n^0, p^0) \in L^\infty(\Omega)^2 : \inf_\Omega n^0 > 0, \inf_\Omega p^0 > 0\}.$$

We will follow the framework of analyzing infinite dimensional dynamical systems given in [21]. Let $\mathcal{S}(t)$ ($t \geq 0$) denote the semigroup of solution maps defined on H_{ad} , i.e., $\mathcal{S}(t)(n^0, p^0) = (n(t), p(t))$ where $(n(t), p(t))$ is a solution to (1.14)–(1.15) with initial data (n^0, p^0) . From Theorem 2.1, we see for each $t > 0$, $\mathcal{S}(t)$ is well-defined from H_{ad} into itself. In general it may be set-valued, and under Assumption (A1), it becomes single-valued.

We first introduce a few definitions. A set $B \subset H_{\text{ad}}$ is said to be L^∞ -bounded in H_{ad} if there exist positive constants M_1 and M_2 such that

$$M_1 \leq n^0(x), p^0(x) \leq M_2 \quad \text{for all } (n^0, p^0) \in B.$$

A set $\mathcal{B} \subset H_{\text{ad}}$ is said to be an absorbing set for $\mathcal{S}(t)$ if it “absorbs” all L^∞ -bounded sets in H_{ad} ; that is, for any L^∞ -bounded set $B \subset H_{\text{ad}}$, there exists $t_B > 0$ (depending on B) such that $\mathcal{S}(t)B \subset \mathcal{B}$ for all $t \geq t_B$.

Given any L^∞ -bounded set $B \subset H_{\text{ad}}$, suppose M_1 and M_2 are the two bounds as defined above. From the L^∞ estimates established for the solutions (n, p) in Theorem 3.1, and the way d_0 and c_0 or θ_0 are chosen in (3.1) and (3.10) or (3.19), we see that, for each $t > 0$,

$$\frac{c}{1 + e^{-\omega t}/m_1} \text{ or } \Gamma(-\theta - \hat{m}_1 e^{-\hat{\omega}t}) \leq \mathcal{S}(t)(n^0, p^0) \leq d(1 + m_2 e^{-\lambda t}) \quad \text{a.e. in } \Omega \quad (3.24)$$

for all $(n^0, p^0) \in B$, where

$$m_2 = (M_2 - 1)^+, \quad m_1 = \min\{1/m_2, M_1\} \text{ or } \hat{m}_1 = \max\{-H(m_1), KdM_2\}$$

(with $K = \max_{d \leq \xi \leq dM_2} H'(\xi)$.) Set

$$t_B = \max\{\ln m_2/\lambda, -\ln m_1/\omega\} \text{ or } \max\{\ln m_2/\lambda, \ln \hat{m}_1/\hat{\omega}\},$$

then for all $t \geq t_B$, there holds

$$\frac{c}{2} \text{ or } \Gamma(-\theta - 1) \leq \mathcal{S}(t)(n^0, p^0) \leq 2d.$$

Thus we have an absorbing set as stated in the following theorem.

Theorem 3.3 (Existence of an absorbing set.) *Under the assumptions of Theorem 3.1, the dynamical system $\{\mathcal{S}(t)\}_{t \geq 0}$ given by (1.14)–(1.15) has an absorbing set \mathcal{B} defined by*

$$\mathcal{B} = \{(n, p) \in H_{\text{ad}} : c_* \leq n(x), p(x) \leq 2d\} \quad (3.25)$$

where $c_* = c/2$ if G is by (1.6) and $c_* = \Gamma(-\theta - 1)$ if G is by (1.8), and d, c , or θ are as given in (3.9), (3.15) or (3.20).

3.2 Existence of a Global Compact Attractor

In this subsection we study the existence of an attractor for the dynamical system under the assumptions in Theorem 3.1 and (A1). With these assumptions the semigroup $\{\mathcal{S}(t)\}$ is of single-valued, and there is an absorbing set as constructed in (3.25).

We also equip H_{ad} with the V^* -topology; i.e., with respect to the norm

$$\|(n^0, p^0)\|_*^2 = \langle n^0, (-\Delta_0)^{-1}n^0 \rangle_{V^* \times V} + \langle p^0, (-\Delta_0)^{-1}p^0 \rangle_{V^* \times V}.$$

Then from (2.40) in the proof of Theorem 2.3, we have the Lipschitz continuity of $\mathcal{S}(t)$ on H_{ad} with respect to this V^* -norm.

Lemma 3.4 *For each $t \geq 0$, $\mathcal{S}(t)$ is Lipschitz continuous from H_{ad} to H_{ad} with respect to the V^* -norm.*

Lemma 3.5 *For each $t > 0$ and each L^∞ -bounded set $B \in H_{\text{ad}}$, $\overline{\mathcal{S}(t)B}$ is an L^∞ -bounded set in H_{ad} , where the closure is taken with the V^* -topology in H_{ad} . Moreover $\overline{\mathcal{S}(t)B}$ is compact in H_{ad} .*

Proof. Suppose B is an L^∞ -bounded set in H_{ad} . Then from (3.24) we see that $\mathcal{S}(t)B$ is L^∞ -bounded in H_{ad} . To show that its closure in the V^* -topology is also L^∞ -bounded, we assume $z = \lim_{i \rightarrow \infty} z_i$ where each $z_i \in \mathcal{S}(t)B$. Since $\{z_i\}$ is bounded in L^∞ , there exists $\tilde{z} \in L^\infty$ and a subsequence $\{z_{i_k}\}$ such that z_{i_k} converges weakly- $*$ to \tilde{z} . Clearly the same bounds as in (3.24) hold for \tilde{z} . For any $\phi \in V^2$, $\langle z_{i_k}, \phi \rangle \rightarrow \langle \tilde{z}, \phi \rangle$. Hence $\langle \tilde{z} - z, \phi \rangle_{V^* \times V} = 0$ for all $\phi \in V^2$. Thus $z = \tilde{z}$. Therefore $z \in L^\infty$ and the same bounds as in (3.24) hold for z . That is, $\overline{\mathcal{S}(t)B}$ is also an L^∞ -bounded set. Since H_{ad} is equipped with the V^* -topology, and L^2 is compactly embedded in V^* , from the bounds in (3.24) we see immediately that $\mathcal{S}(t)B$ is precompact in H_{ad} . \square

Define the ω -limit set of the absorbing set \mathcal{B} :

$$\mathcal{A} = \omega(\mathcal{B}) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} \mathcal{S}(t)\mathcal{B}} \tag{3.26}$$

where the closure is taken with the V^* -topology in H_{ad} . Then from Lemma 3.5, we see that \mathcal{A} is L^∞ -bounded and compact in H_{ad} . As an ω -limit set of a nonempty set, \mathcal{A} is nonempty and invariant: $\mathcal{S}(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$ (see e.g. Lemma 1.1 in [21]).

We claim that \mathcal{A} attracts every L^∞ -bounded set in H_{ad} ; that is, for any L^∞ -bounded set $B \subset H_{\text{ad}}$,

$$\text{dist}(\mathcal{S}(t)B, \mathcal{A}) \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

where the distance is by the V^* -norm:

$$\text{dist}(X, Y) = \sup_{x \in X} \inf_{y \in Y} \|x - y\|_*$$

We argue by contradiction. If otherwise, there is an L^∞ -bounded set $B_0 \subset H_{\text{ad}}$ such that $\text{dist}(\mathcal{S}(t)B_0, \mathcal{A})$ does not tend to 0 as $t \rightarrow \infty$. That is, there exist $\delta > 0$, $t_i \rightarrow \infty$ and $z_i \in B_0$ such that

$$\text{dist}(\mathcal{S}(t_i)z_i, \mathcal{A}) \geq \delta > 0 \tag{3.27}$$

for all $i = 1, 2, \dots$. By (3.24), there is t_0 such that $\mathcal{S}(t)B_0 \subset \mathcal{B}$ when $t \geq t_0$. Without loss of generality, we assume $t_i \geq 2t_0$ for all $i = 1, 2, \dots$. Then the sequence $\mathcal{S}(t_i - t_0)z_i$ is contained in \mathcal{B} , and $\mathcal{S}(t_i)z_i = \mathcal{S}(t_0)\mathcal{S}(t_i - t_0)z_i$ is precompact in H_{ad} by Lemma 3.5. Thus $\mathcal{S}(t_i)z_i$ has a convergent subsequence $\mathcal{S}(t_{i_k})z_{i_k} \rightarrow z_* \in H_{\text{ad}}$ as $k \rightarrow \infty$. Note that $z_* = \lim_{k \rightarrow \infty} \mathcal{S}(t_{i_k} - t_0)y_k$ with $y_k = \mathcal{S}(t_0)z_{i_k} \in \mathcal{B}$. Hence $z_* \in \mathcal{A}$ by the definition of ω -limit set. This contradicts (3.27). Therefore \mathcal{A} attracts sets that are L^∞ -bounded in H_{ad} .

\mathcal{A} is also maximal among such attractors. Suppose there is another L^∞ -bounded attractor $\mathcal{A}_1 \supset \mathcal{A}$. Since \mathcal{B} absorbs L^∞ -bounded set, $\mathcal{S}(t)\mathcal{A}_1 \subset \mathcal{B}$

for t large enough. Hence $\mathcal{A}_1 = \mathcal{S}(t)\mathcal{A}_1 \subset \mathcal{B}$, and $\mathcal{A}_1 = \omega(\mathcal{A}_1) \subset \omega(\mathcal{B}) = \mathcal{A}$. Therefore, $\mathcal{A}_1 = \mathcal{A}$.

Finally, we show \mathcal{A} is connected. Suppose otherwise. Then there are open sets O_1 and O_2 in H_{ad} such that $O_1 \cap O_2 = \emptyset$ and $\mathcal{A} \subset O_1 \cup O_2$. Let $K = \text{co}\mathcal{A}$ be the convex hull of \mathcal{A} . Then K is connected. $K \subset H_{\text{ad}}$ because H_{ad} is convex, and K is L^∞ -bounded since \mathcal{A} is. Note that $\mathcal{A} = \mathcal{S}(t)\mathcal{A} \subset \mathcal{S}(t)K$ and $\mathcal{S}(t)K$ is also connected since $\mathcal{S}(t)$ is continuous for each $t > 0$ (Lemma 3.4). Therefore $O_1 \cup O_2$ does not cover $\mathcal{S}(t)K$ for each $t > 0$. That is, there are $z_j \in K$ such that $\mathcal{S}(j)z_j \notin O_1 \cup O_2$ for all $j = 1, 2, \dots$. Since it is easy to see that $\mathcal{S}(j)z_j$ is precompact in H_{ad} for j large enough (by Lemma 3.5), there exists a subsequence of $\mathcal{S}(j)z_j$ that converges to some $y \in H_{\text{ad}}$. Clearly $y \notin O_1 \cup O_2$. On the other hand, since \mathcal{A} attracts K , y must be in \mathcal{A} . This is a contradiction. Hence \mathcal{A} is connected.

Note that for $(n^0, p^0) \in H_{\text{ad}}$ and $t > 0$, $\mathcal{S}(t)(n^0, p^0) \in V^2$ from our existence results in Theorem 2.1. Hence $\mathcal{A} = \mathcal{S}(t)\mathcal{A}$ is contained in V^2 .

Thus we have proved the following theorem:

Theorem 3.6 (Existence of an attractor.) *In addition to the assumptions of Theorem 3.1, suppose Assumption (A1) holds. Then the dynamical system $\{\mathcal{S}(t)\}$ defined by (1.14)–(1.15) has a global attractor \mathcal{A} given in (3.26) that attracts all L^∞ -bounded sets in H_{ad} . Moreover, \mathcal{A} is maximal among such attractors, and \mathcal{A} is L^∞ -bounded, compact, connected, and contained in V^2 .*

We make two remarks on our results above. (1) Our proof here is similar to that of [21, Theorem 1.1], which cannot be applied directly because the set \mathcal{B} absorbs only sets that are L^∞ -bounded, not in terms of the V^* -topology. (2) Using the L^∞ estimates we established for the solutions and the continuity of the solution map $\mathcal{S}(t)$ with respect to the V^* -topology, we can improve the continuity of $\mathcal{S}(t)$ to an interpolation space between H^{-1} and L^2 , and the discussion above can be easily carried over to that setting.

4 Existence of Stationary Solutions

In this section we study the stationary version of the model equations (1.10)–(1.11) and (1.3) with boundary conditions (1.12). We are looking for a triplet $(n, p, u) \in (\bar{n}, \bar{p}, \bar{u}) + V^3$ with $n > 0$, $p > 0$ satisfying

$$\mu_n \langle \nabla P(n) - n \nabla u, \nabla \phi \rangle = \langle G(n, p), \phi \rangle \quad (4.1)$$

$$\mu_p \langle \nabla P(p) + p \nabla u, \nabla \psi \rangle = \langle G(n, p), \psi \rangle \quad (4.2)$$

$$\epsilon_0 \langle \nabla u, \nabla \chi \rangle + \langle n - p - N, \chi \rangle = 0 \quad (4.3)$$

for all $(\phi, \psi, \chi) \in V^3$, where $G(n, p)$ is given by either (1.6) or (1.8).

The proof for existence consists of similar arguments as in §2.1 and §2.2. Consider the solution map \mathcal{T}_0 defined from $(\bar{v}, \bar{w}, \bar{u}) \in L^2(\Omega)^3$ to $(v, w, u) \in (\bar{v}, \bar{w}, \bar{u}) + V^3$ in

$$\mu_n \langle \bar{n} \nabla v, \nabla \phi \rangle = \langle G(\Gamma_0(v + \bar{u}), \bar{p}), \phi \rangle$$

$$\begin{aligned} \mu_p \langle \tilde{p} \nabla w, \nabla \psi \rangle &= \langle G(\tilde{n}, \Gamma_0(w - \tilde{u})), \psi \rangle \\ \epsilon_0 \langle \nabla u, \nabla \chi \rangle + \langle \Gamma_0(\tilde{v} + u) - \Gamma_0(\tilde{w} - u) - N, \chi \rangle &= 0 \end{aligned}$$

for all $(\phi, \psi, \chi) \in V^3$, where

$$\Gamma_0(s) = \begin{cases} \Gamma(-c_0) & \text{when } s < -c_0 \\ \Gamma(s) & \text{when } -c_0 \leq s \leq d_0 \\ \Gamma(d_0) & \text{when } s > d_0 \end{cases}$$

($c_0 \in (0, h_0)$ and $d_0 \in (0, h_\infty)$ to be determined) and $\tilde{n} = \Gamma_0(\tilde{v} + \tilde{u})$, $\tilde{p} = \Gamma_0(\tilde{w} - \tilde{u})$.

Since the equations above are all decoupled, and each defines a maximum monotone operator, the well-definedness of \mathcal{T}_0 is obvious. The compactness and continuity of \mathcal{T}_0 can be shown similarly as for \mathcal{T} in §2.1. Therefore \mathcal{T}_0 has a fixed point $(v, w, u) \in (\bar{v}, \bar{w}, \bar{u}) + V^3$ satisfying

$$\mu_n \langle \Gamma_0(v + u) \nabla v, \nabla \phi \rangle = \langle G(\Gamma_0(v + u), \Gamma_0(w - u)), \phi \rangle \tag{4.4}$$

$$\mu_p \langle \Gamma_0(w - u) \nabla w, \nabla \psi \rangle = \langle G(\Gamma_0(v + u), \Gamma_0(w - u)), \psi \rangle \tag{4.5}$$

$$\epsilon_0 \langle \nabla u, \nabla \chi \rangle + \langle \Gamma_0(v + u) - \Gamma_0(w - u) - N, \chi \rangle = 0 \tag{4.6}$$

for all $(\phi, \psi, \chi) \in V^3$.

Now we establish the L^∞ bounds for $v + u$ and $w - u$, and hence choose the truncation constants d_0 and c_0 accordingly.

The Upper Bounds. Here we need to assume h_∞ is sufficiently large ($h_\infty > d_0$ given in (4.7) below.) Set $\hat{d} \in (0, h_\infty)$ such that

$$\Gamma(\hat{d}) = \max\{\sup_{\Sigma_D} \bar{n}, \sup_{\Sigma_D} \bar{p}, 1\}$$

and for $d \in (\hat{d}, h_\infty)$ set $\phi_d = (u + v - d)^+ \in V$ as the test function in both (4.4) and (4.6) to obtain

$$\mu_n \Gamma(d) |\nabla \phi_d|_2^2 \leq (Q + g + \frac{\mu_n}{\epsilon_0} \Gamma(d) N, \phi_d).$$

Following the same argument as for the bounds of $v + u$ in §2.2, we can conclude that $u(x) + v(x) \leq d_0$ a.e. in Ω where

$$d_0 = \hat{d} + 4\alpha_6^2 |\Omega|^{1/6} \left(\frac{\overline{Q}(1 + 2\Gamma(\hat{d})) |\Omega|^{1/2} + |g|_2}{\Gamma(\hat{d}) \min(\mu_n, \mu_p)} + 2 \frac{|N|_2}{\epsilon_0} \right) \tag{4.7}$$

provided that $d_0 < h_\infty$. The same bound can be shown for $w(x) - u(x)$.

The Lower Bounds. Here we also need to assume that $h_0 > c_0$ where c_0 is to be given below in (4.8). Set $\hat{c} \in (0, h_0)$ such that

$$\Gamma(-\hat{c}) = \min\{\inf_{\Sigma_D} \bar{n}, \inf_{\Sigma_D} \bar{p}\}.$$

Following the same argument as in §2.2 for the lower bound, we can obtain $v + u \geq -c_0$ when

$$c_0 = \hat{c}_0 + 4 \frac{\alpha_6^2}{\epsilon_0} |\Omega|^{1/6} |N|_2 \quad (4.8)$$

provided that $h_0 > c_0$, where

$$\hat{c}_0 = \begin{cases} \max\{\hat{c}, H(1/\Gamma(d_0))\} & \text{for } G \text{ in (1.6)} \\ \max\{\hat{c}, d_0 - h_i\} & \text{for } G \text{ in (1.8)} \end{cases}$$

The same lower bound can be shown for $w(x) - u(x)$.

The L^∞ bound estimates for u can be obtained using the same argument as in §2.2.

Hence when we choose the d_0 and c_0 as the truncation constants in Γ_0 , the fixed point (v, w, u) of \mathcal{T}_0 is also a solution to (4.4)–(4.6) with $\Gamma_0(\cdot)$ replaced by $\Gamma(\cdot)$, and hence $n = \Gamma(v + u)$, $p = \Gamma(w - u)$ and u form a solution to the stationary system (4.1)–(4.3).

The above discussions can be summarized in the following theorem.

Theorem 4.1 (Existence of non-vacuum stationary solutions.) *In addition to Assumption (A), suppose $h_\infty > d_0$ and $h_0 > c_0$. Then the stationary system (4.1)–(4.3) has at least one solution $(n, p, u) \in (\bar{n}, \bar{p}, \bar{u}) + V^3$ satisfying*

$$\begin{aligned} \Gamma(-c_0) &\leq n(x), \quad p(x) \leq \Gamma(d_0), \\ |u(x)| &\leq \sup_{\Sigma_D} \bar{u} + 4 \frac{\alpha_6^2}{\epsilon_0} (2\Gamma(d_0)|\Omega|^{1/2} + |N|_2)|\Omega|^{1/6} \end{aligned}$$

a.e. in Ω , where the positive constants d_0 and c_0 are as given in (4.7)–(4.8).

Remark 4.2 *The condition $h_0 > c_0$ is needed only for the lower bounds of (n, p) .*

In the case G is given by (1.6), it is also possible to replace the condition $h_0 > c_0$ by a condition on the recombination coefficient Q : $Q(n, p) > 0$. For example, the Shockley-Read-Hall model $Q_{\text{SRH}} \sim (1 + n + p)^{-1}$ satisfies this condition. Then

$$q_0 = \min Q(n, p) \quad \text{over } n, p \in [0, \Gamma(d_0)]$$

is positive, where d_0 is the upper bound in (4.7). For $c'_0 \in (0, h_0)$ such that

$$\Gamma(-c'_0) = \min\left\{\Gamma(-\hat{c}), \frac{\epsilon_0 q_0}{\epsilon_0 q_0 \Gamma(d_0) + |N|_\infty \max(\mu_n, \mu_p)}\right\}, \quad (4.9)$$

set $\chi_c = (v + u + c'_0)_- \in V$ as the test function in both (4.4) and (4.6) to obtain

$$\begin{aligned} \mu_n \Gamma(-c'_0) |\nabla \chi_c|_2^2 &= \langle Q(\Gamma(-c'_0)) \Gamma_0(w - u) - 1, -\chi_c \rangle \\ &\quad + \frac{\mu_n}{\epsilon_0} \Gamma_0(-c'_0) \langle \Gamma(-c'_0) - \Gamma_0(w - u) - N, -\chi_c \rangle \end{aligned}$$

$$\begin{aligned} &\leq \langle Q(\Gamma(-c'_0)\Gamma(d_0) - 1) - \frac{\mu_n}{\epsilon_0}\Gamma(-c'_0)N, -\chi_c \rangle \\ &\leq \langle -\frac{q_0|N|_\infty \max(\mu_n, \mu_p)}{\epsilon_0 q_0 \Gamma(d_0) + |N|_\infty \max(\mu_n, \mu_p)} \\ &\quad + \frac{\mu_n}{\epsilon_0}\Gamma(-c'_0)|N|_\infty, -\chi_c \rangle \leq 0 \end{aligned}$$

Therefore, $\chi_c \equiv 0$ and hence $v(x) + u(x) \geq -c'_0$ a.e. in Ω .

Corollary 4.3 *Under Assumption (A), suppose $h_\infty > d_0$ and G is given by (1.6) with $Q(n, p) > 0$. Then the stationary system (4.1)–(4.3) has at least one solution $(n, p, u) \in (\bar{n}, \bar{p}, \bar{u}) + V^3$ satisfying*

$$\Gamma(-c'_0) \leq n(x), \quad p(x) \leq \Gamma(d_0)$$

a.e. in Ω , where d_0 is from (4.7) and c'_0 from (4.9).

Although there is no rigorous proof, the standard stationary drift-diffusion model is believed to have multiple solutions (see, e.g., [15, 11]). Hence we do not expect the uniqueness of solutions to be true for this nonlinear model.

5 Vacuum Solutions

So far we have considered only non-vacuum solutions (n, p) (i.e. $n, p > 0$) for the system (1.14)–(1.16). Due to the nonlinear diffusion $P(\cdot)$ (e.g., $P(n) = n^\gamma$), such a system in general is degenerated if there are vacuum solutions (i.e. nonnegative solutions). We have shown in the previous sections that, under the conditions stated, the system (1.14)–(1.16) does not allow vacuum solutions to develop. In this section, we discuss how to remove some of the conditions we imposed so to obtain parallel results for the case of vacuum solutions.

In the section we use the general Assumption (A) but with nonnegative boundary and initial data in (A)(vii)–(viii).

Suppose P is as given by Assumption (A) and H is the associated enthalpy defined in (1.7). For $0 < \epsilon < 1$, define

$$P_\epsilon(s) = \begin{cases} \frac{P(\epsilon)}{\epsilon}s & \text{if } 0 < s \leq \epsilon, \\ P(s) & \text{if } s > \epsilon \end{cases} \tag{5.1}$$

if $P'(0^+) = 0$ (degeneracy). When $P'(0^+) > 0$, we set $P_\epsilon = P$ for all ϵ . Let H_ϵ be the enthalpy associated with P_ϵ :

$$H_\epsilon(s) = \int_1^s \frac{P'_\epsilon(s)}{s} ds = \begin{cases} \frac{P(\epsilon)}{\epsilon} \ln \frac{s}{\epsilon} + H(\epsilon) & \text{if } 0 < s \leq \epsilon, \\ H(s) & \text{if } s > \epsilon. \end{cases} \tag{5.2}$$

Clearly $\inf_{0 < s \leq 1} P'_\epsilon(s) > 0$ and hence $H_\epsilon(0^+) = \infty$ for each ϵ . We will also perturb the generation-recombination rate G to G_ϵ by replacing the coefficient

Q in either (1.6) or (1.8) with $Q_\varepsilon = Q + \varepsilon$ so that $\inf Q_\varepsilon \geq \varepsilon > 0$. For given nonnegative (n^0, p^0) and (\bar{n}, \bar{p}) , we use positive initial data $(n_\varepsilon^0, p_\varepsilon^0) = (n^0 + \varepsilon, p^0 + \varepsilon)$ and positive boundary data $(\bar{n}_\varepsilon, \bar{p}_\varepsilon) = (\bar{n} + \varepsilon, \bar{p} + \varepsilon)$ for the perturbed system

$$\left\langle \frac{\partial n_\varepsilon}{\partial t}, \phi \right\rangle_{V^* \times V} + \mu_n \langle \nabla P_\varepsilon(n_\varepsilon) - n_\varepsilon \nabla u_\varepsilon, \nabla \phi \rangle = \langle G_\varepsilon(n_\varepsilon, p_\varepsilon), \phi \rangle \quad (5.3)$$

$$\left\langle \frac{\partial p_\varepsilon}{\partial t}, \psi \right\rangle_{V^* \times V} + \mu_p \langle \nabla P_\varepsilon(p_\varepsilon) + p_\varepsilon \nabla u_\varepsilon, \nabla \psi \rangle = \langle G_\varepsilon(n_\varepsilon, p_\varepsilon), \psi \rangle \quad (5.4)$$

$$\varepsilon_0 \langle \nabla u_\varepsilon, \nabla \chi \rangle + \langle n_\varepsilon - p_\varepsilon - N, \chi \rangle = 0 \quad (5.5)$$

for all $(\phi, \psi, \chi) \in V^3$. By Theorem 2.1, there exists a solution $(n_\varepsilon, p_\varepsilon, u_\varepsilon) \in H^1 \cap L^\infty$ such that $(n_\varepsilon - \bar{n}_\varepsilon, p_\varepsilon - \bar{p}_\varepsilon, u_\varepsilon - \bar{u}) \in V^3$ and

$$0 < n_\varepsilon(t, x), p_\varepsilon(t, x) \leq (D_0 + 1)e^{Bt} \quad (5.6)$$

a.e. in Ω and $t > 0$, where D_0 is as in (2.13) and B in (2.15) with \bar{Q} replaced by $\bar{Q} + 1$. Note also that (5.3)–(5.4) can be equivalently written as

$$\begin{aligned} \langle n_\varepsilon(t) - n_\varepsilon^0, \phi \rangle & \quad (5.7) \\ + \int_0^t \{ \mu_n \langle \nabla P_\varepsilon(n_\varepsilon(s)) - n_\varepsilon(s) \nabla u_\varepsilon(s), \nabla \phi \rangle - \langle G_\varepsilon(n_\varepsilon(s), p_\varepsilon(s)), \phi \rangle \} ds & = 0 \end{aligned}$$

$$\begin{aligned} \langle p_\varepsilon(t) - p_\varepsilon^0, \psi \rangle & \quad (5.8) \\ + \int_0^t \{ \mu_p \langle \nabla P_\varepsilon(p_\varepsilon(s)) + p_\varepsilon(s) \nabla u_\varepsilon(s), \nabla \psi \rangle - \langle G_\varepsilon(n_\varepsilon(s), p_\varepsilon(s)), \psi \rangle \} ds & = 0 \end{aligned}$$

for all $\phi, \psi \in V$.

From (5.6), for each $T > 0$, there exist $n(t, x)$ and $p(t, x) \in L^\infty((0, T) \times \Omega)$ such that

$$(n_\varepsilon, p_\varepsilon) \rightarrow (n, p) \quad \text{as } \varepsilon \rightarrow 0 \text{ in } L^\infty \text{ weakly-*} \quad (5.9)$$

(for a sequence of $\{\varepsilon\}$, but still denoted as the same.) Let u be such that $u - \bar{u} \in L^2(0, T; V)$ and solve

$$\varepsilon_0 \langle \nabla u, \nabla \chi \rangle + \langle n - p - N, \chi \rangle = 0 \quad (5.10)$$

for all $\chi \in V$. We show below that this triplet (n, p, u) is a non-negative solution to the original system (1.14)–(1.16).

From (5.5) above, we see that ∇u_ε is uniformly bounded in $L^2((0, T) \times \Omega)$, and hence, from (5.7)–(5.8) and (5.6) we have

$$\int_0^T |\nabla P_\varepsilon(n_\varepsilon)|_2^2 dt, \quad \int_0^T |\nabla P_\varepsilon(p_\varepsilon)|_2^2 dt \leq \text{const.} \quad (5.11)$$

Consequently,

$$\int_0^T \left| \frac{\partial n_\varepsilon}{\partial t} \right|_{V^*}^2 dt, \quad \int_0^T \left| \frac{\partial p_\varepsilon}{\partial t} \right|_{V^*}^2 dt \leq \text{const.} \quad (5.12)$$

From (5.12) and (5.6), we see that $\{n_\varepsilon, p_\varepsilon\}$ is uniformly bounded in $L^2(0, T; L^2(\Omega)) \cap H^1(0, T; V^*)$, hence compact in $L^2(0, T; V^*)$. Therefore there exist a subsequence of $\{\varepsilon\}$ (still denoted by $\{\varepsilon\}$) and a set $E \subset [0, T]$ with $mE = 0$ such that

$$(n_\varepsilon(t), p_\varepsilon(t)) \rightarrow (n(t), p(t)) \quad \text{in } V^* \quad \text{for all } t \in [0, T] \setminus E$$

as $\varepsilon \rightarrow 0$. From (5.5),

$$u_\varepsilon = \tilde{u} + \frac{1}{\epsilon_0}(-\Delta_0)^{-1}(n_\varepsilon - p_\varepsilon - N)$$

where $\tilde{u} \in \bar{u} + V$ satisfies $\Delta \tilde{u} = 0$. Since $(-\Delta_0)^{-1}$ is an isometry from V^* to V , we have

$$u_\varepsilon(t) \rightarrow u(t) \quad \text{in } H^1(\Omega) \quad \text{a.e. } t \in (0, T)$$

and hence, together with (5.9),

$$n_\varepsilon \nabla u_\varepsilon \rightarrow n \nabla u \quad \text{and} \quad p_\varepsilon \nabla u_\varepsilon \rightarrow p \nabla u \quad \text{weakly in } L^2(\Omega) \quad \text{a.e. } t \in (0, T).$$

From (5.11), there exists $\eta \in L^2(0, T; V)$ such that

$$P_\varepsilon(n_\varepsilon) - P_\varepsilon(\bar{n}) \rightarrow \eta \quad \text{weakly in } L^2(0, T; V)$$

(a further subsequence if necessary.) Since $\frac{\partial P_\varepsilon(n_\varepsilon)}{\partial t} = P'_\varepsilon(n_\varepsilon) \frac{\partial n_\varepsilon}{\partial t}$ and $P'_\varepsilon(s) \leq \sup P'(s)$ on bounded intervals, we see from (5.11)-(5.12) that

$$\{P_\varepsilon(n_\varepsilon) - P_\varepsilon(\bar{n})\} \text{ is uniformly bounded in } L^2(0, T; V) \cap H^1(0, T; V^*).$$

Hence by Aubin's Lemma, we can also assume

$$P_\varepsilon(n_\varepsilon) - P_\varepsilon(\bar{n}) \rightarrow \eta \quad \text{in } L^2((0, T) \times \Omega)$$

(a further subsequence if necessary.) Note that

$$\|P_\varepsilon(\phi) - P(\phi)\|_2^2 = \int_0^T \int_{\phi < \varepsilon} (P_\varepsilon(\phi) - P(\phi))^2 dx dt \leq 2P(\varepsilon)^2 T |\Omega| \rightarrow 0$$

for any $\phi \in L^\infty((0, T) \times \Omega)$. Hence we have

$$P(n_\varepsilon) - P(\bar{n}) \rightarrow \eta \quad \text{in } L^2((0, T) \times \Omega).$$

Thus

$$P(n_\varepsilon) \rightarrow P(\bar{n}) + \eta \quad \text{or} \quad n_\varepsilon \rightarrow P^{-1}(P(\bar{n}) + \eta) \quad \text{a.e. in } (0, T) \times \Omega.$$

Together with (5.9) we see $n = P^{-1}(P(\bar{n}) + \eta)$ and $n_\varepsilon \rightarrow n$ a.e. in $(0, T) \times \Omega$. The same can be shown for p_ε :

$$P_\varepsilon(p_\varepsilon) \rightarrow P(p) \quad \text{weakly in } L^2(0, T; H^1(\Omega)) \quad \text{and} \quad p_\varepsilon \rightarrow p \quad \text{a.e. in } (0, T) \times \Omega.$$

Therefore, $G_\varepsilon(n_\varepsilon, p_\varepsilon) \rightarrow G(n, p)$ a.e. in $(0, T) \times \Omega$. Passing to the limit of $\varepsilon \rightarrow 0$ in (5.7)–(5.8) we obtain

$$\begin{aligned} & \langle n(t) - n^0, \phi \rangle \\ & + \int_0^t \{ \mu_n \langle \nabla P(n(s)) - n(s) \nabla u(s), \nabla \phi \rangle - \langle G(n(s), p(s)), \phi \rangle \} ds = 0, \end{aligned}$$

$$\begin{aligned} & \langle p(t) - p^0, \psi \rangle \\ & + \int_0^t \{ \mu_p \langle \nabla P(p(s)) + p(s) \nabla u(s), \nabla \psi \rangle - \langle G(n(s), p(s)), \psi \rangle \} ds = 0 \end{aligned}$$

and u as in (5.10). That is, (n, p, u) is a solution that satisfies (1.14)–(1.16) with (1.3) in the sense that

$$\begin{aligned} (n, p, u) & \in L^\infty((0, T) \times \Omega)^3 \cap H^1(0, T; V^*)^3 \quad \text{with } n, p \geq 0, \\ (P(n), P(p), u) & \in (P(\bar{n}), P(\bar{p}), \bar{u}) + L^2(0, T; V)^3. \end{aligned} \tag{5.13}$$

Although we cannot obtain H^1 -regularity for (n, p) in general, it can be shown that

$$\text{for every } \delta > 0: \quad (n - \delta)^+, (p - \delta)^+ \in H^1(\Omega). \tag{5.14}$$

Indeed, since $P(n) \in H^1(\Omega)$, so is $(P(n) - P(\delta))^+$, and

$$|\nabla(P(n) - P(\delta))^+|_2^2 = \int_{n>\delta} |P'(n) \nabla n|^2 dx \geq \kappa_0 \int_{n>\delta} |\nabla n|^2 dx = \kappa_0 |\nabla(n - \delta)^+|_2^2$$

where $\kappa_0 = \inf_{\delta \leq s \leq |n|_\infty} P'(s) > 0$. Thus, under Assumption (A2), the argument in §3.1 for the upper bound can be applied to obtain the absorbing L^∞ upper bound $D(t)$ for n and p :

$$0 \leq n(t, x), p(t, x) \leq d(1 + d_0 e^{-\lambda t}) \quad \text{a.e. } t > 0, x \in \Omega$$

where the constants d, d_0 and λ are as in §3.1, and d is independent of the initial data. The set of admissible initial data is now extended to

$$\tilde{H}_{\text{ad}} = \{(n, p) \in L^\infty(\Omega)^2: n(x) \geq 0, p(x) \geq 0 \text{ a.e.}\}.$$

Our uniqueness argument in §2.5 requires positive lower bounds of solutions, and this restriction cannot be removed by the above perturbation technique. Hence we do not have uniqueness result for vacuum solutions, and thus there is no result on global attractors.

As for the stationary system, the existence of positive solutions $\{n_\varepsilon, p_\varepsilon\}$ to the perturbed systems is by Theorem 4.1 if $h_\infty > d_0$ (d_0 given by (4.7)). The convergence argument of $\{n_\varepsilon, p_\varepsilon\}$ to a non-negative solution (n, p) is similar (but simpler), and we skip the details.

These parallel results can be summarized in the following theorem.

Theorem 5.1 (Vacuum solutions.) *Suppose Assumption (A) holds and G is given by either (1.6) or (1.8).*

- (i) (Global solutions.) *For each $T > 0$, there exists at least one solution (n, p, u) to (1.14)–(1.16) with the regularities stated in (5.13).*
- (ii) (Absorbing set.) *If Assumption (A2) also holds, then any global solution (n, p, u) from the initial data $(n^0, p^0) \in \tilde{H}_{ad}$ satisfies*

$$0 \leq n(t, x), p(t, x) \leq d(1 + d_0 e^{-\lambda t})$$

where d, d_0 and λ are as in §3.1, and d is independent of the initial data (n^0, p^0) . Therefore, the dynamical system has an absorbing set

$$\mathcal{B} = \{(n, p) \in \tilde{H}_{ad} : 0 \leq n(x), p(x) \leq 2d\}.$$

- (iii) (Stationary solutions.) *If $h_\infty > d_0$ (d_0 as in (4.7)), then the stationary system (4.1)–(4.3) has at least a solution (n, p, u) such that $(n, p, u) \in L^\infty(\Omega)^3$ with $n(x), p(x) \geq 0$ and $(P(n), P(p), u) \in (P(\bar{n}), P(\bar{p}), \bar{u}) + V^3$.*

A Appendix

In this appendix we give the proofs of the lemmas we have used in this paper for establishing the L^∞ estimates for solutions of partial differential equations. This technique is an extension of the classical maximum principle for parabolic and elliptic equations.

Lemma A.1 *Suppose $\phi \in H^1(\Omega)$, and $\phi_\tau = (\phi - \tau)^+ \in V$ for $\tau \geq \bar{\tau}$. If ϕ_τ satisfies*

$$\langle \nabla \phi_\tau, \nabla \phi_\tau \rangle \leq \langle f, \phi_\tau \rangle$$

for some $f \in L^2(\Omega)$. Then

$$\phi(x) \leq \bar{\tau} + 4\alpha_6 |\Omega|^{1/6} |f|_2.$$

Proof. Let $\Omega_\tau = \{x \in \Omega : \phi(x) > \tau\}$. By the Poincaré inequality,

$$\alpha_6^{-2} |\phi_\tau|_6^2 \leq |\nabla \phi_\tau|_2^2 \leq \langle f, \phi_\tau \rangle \leq |f|_2 |\phi_\tau|_6 |\Omega_\tau|^{1/3},$$

that is,

$$|\phi_\tau|_6 \leq \alpha_6^2 |f|_2 |\Omega_\tau|^{1/3}.$$

For $\hat{\tau} > \tau \geq \bar{\tau}$,

$$|\phi_\tau|_6^6 \geq \int_{\Omega_{\hat{\tau}}} |\phi_\tau|^6 dx \geq (\hat{\tau} - \tau)^6 |\Omega_{\hat{\tau}}|.$$

Therefore

$$|\Omega_{\hat{\tau}}| \leq (\alpha_6^2 |f|_2)^6 (\hat{\tau} - \tau)^{-6} |\Omega_\tau|^2.$$

Then by applying Lemma A.2 below to $|\Omega_\tau|$, a non-increasing function in τ , we obtain

$$|\Omega_\tau| = 0 \quad \text{for } \tau \geq \tau^*$$

where

$$\tau^* = \bar{\tau} + 4\alpha_6^2 |\Omega|^{1/6} |f|_2.$$

Therefore we have $\phi_{\tau^*} = 0$ a.e. in Ω . Hence $\phi \leq \tau^*$ as stated. \square

For the sake of completeness, we now state and prove the elementary lemma used above. See [22, Lemma 2.9].

Lemma A.2 *Suppose $\varphi(\tau)$ is a nonnegative, non-increasing function on $[\tau_0, \tau_1]$, and there are positive constants M , γ and δ such that*

$$\varphi(\hat{\tau}) \leq M(\hat{\tau} - \tau)^{-\gamma} \varphi(\tau)^{1+\delta} \quad \text{for all } \hat{\tau}, \tau \in [\tau_0, \tau_1] \quad \text{with } \hat{\tau} > \tau. \quad (\text{A.1})$$

Then

$$\varphi(\tau^*) = 0 \quad \text{provided that } \tau^* \equiv \tau_0 + M^{1/\gamma} 2^{(1+\delta)/\delta} \varphi(\tau_0)^{\delta/\gamma} \leq \tau_1.$$

Proof. Let $\sigma = M^{1/\gamma} 2^{(1+\delta)/\delta} \varphi(\tau_0)^{\delta/\gamma}$ and $x_k = \tau_0 + \sigma(1 - 2^{-k})$ for $k = 0, 1, \dots$. By induction and (A.1), it is straightforward to show that

$$\varphi(x_k) \leq \varphi(x_0) 2^{-k\gamma/\delta} \quad k = 0, 1, \dots$$

Therefore, when $\tau^* = \tau_0 + \sigma \leq \tau_1$, $\varphi(\tau^*) \leq \varphi(x_k) \leq \varphi(x_0) 2^{-k\gamma/\delta} \rightarrow 0$ as $k \rightarrow \infty$. \square

We have also used the following extension to the above lemma in §3.1 (see (3.9)). Other variations can be found in [13, 6].

Lemma A.3 *Suppose $\varphi(\tau)$ is a nonnegative, non-increasing function on $[\tau_0, \infty)$, and there are positive constants M , γ and δ such that*

$$\varphi(\hat{\tau}) \leq M\tau^\gamma (\hat{\tau} - \tau)^{-\gamma} \varphi(\tau)^{1+\delta} \quad \text{for all } \hat{\tau} > \tau \geq \tau_0 \quad (\text{A.2})$$

Then

$$\varphi(\tau^*) = 0 \quad \text{for } \tau^* \equiv 2\tau_0(1 + 2^{(1+2\delta)/\delta}) M^{(1+\delta)/\delta\gamma} \varphi(\tau_0)^{(1+\delta)/\gamma}.$$

Proof. For any $m > 1$, from (A.2) with $\tau = \tau_0$ and $\hat{\tau} = m\tau_0$ we have

$$\varphi(m\tau_0) \leq M(m-1)^{-\gamma} \varphi(\tau_0)^{1+\delta}.$$

Now apply Lemma A.2 to φ on the interval $[m\tau_0, 2m\tau_0]$ to obtain

$$\varphi(\tau^*) = 0 \quad \text{for } \tau^* \geq \tau_0^* \equiv m\tau_0 + M^{1/\gamma} 2m\tau_0 2^{(1+\delta)/\delta} \varphi(m\tau_0)^{\delta/\gamma}$$

provided that $\tau^* \leq 2m\tau_0$. From the above estimate for $\varphi(m\tau_0)$,

$$\frac{\tau_0^*}{m\tau_0} \leq 1 + M^{(1+\delta)/\gamma} 2^{(1+2\delta)/\delta} \varphi(\tau_0)^{(1+\delta)\delta/\gamma} (m-1)^{-\delta} = 2$$

if we choose

$$m = 1 + M^{(1+\delta)/\delta\gamma} 2^{(1+2\delta)/\delta} \varphi(\tau_0)^{(1+\delta)/\gamma}.$$

Hence for this chosen m we set $\tau^* = 2m\tau_0 \geq \tau_0^*$ and thus $\varphi(\tau^*) = 0$. \square

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Addendum: June 21, 1999.

On page 33, there is an error in the argument of the strong convergence of $P(n_\varepsilon) \rightarrow P(\bar{n}) + \eta$ in $L^2((0, T) \times \Omega)$. A correct proof of the same result is given below.

First we note that, on the top of page 33, the convergence of $(n_\varepsilon, p_\varepsilon) \rightarrow (n, p)$ is strong in $L^2(0, T; V^*)$ and weak in $L^2(0, T; H)$. Then the paragraph from line 13 to line 26 on page 33 should be replaced by the following:

From (5.11), there exists $\eta \in L^2(0, T; H^1(\Omega))$ such that

$$P_\varepsilon(n_\varepsilon) \rightarrow \eta \quad \text{weakly in } L^2(0, T; H^1(\Omega))$$

(a further subsequence if necessary.) Since $n_\varepsilon, p_\varepsilon$ are in $L^\infty((0, T) \times \Omega)$, there exists G_0 such that

$$G_\varepsilon(n_\varepsilon, p_\varepsilon) \rightarrow G_0 \quad \text{weakly in } L^2((0, T) \times \Omega).$$

Hence, passing to the limit in (5.7) we obtain

$$\langle n(t) - n^0, \phi \rangle + \int_0^t \mu_n \langle \nabla \eta, \nabla \phi \rangle ds = \int_0^t \{ \mu_n \langle n \nabla u, \nabla \phi \rangle + \langle G_0, \phi \rangle \} ds.$$

By subtracting this equation from (5.7) and setting $\phi = (-\Delta_0)^{-1}(n_\varepsilon - n)$, we can obtain

$$\frac{1}{2} |n_\varepsilon(t) - n(t)|_{V^*}^2 + \mu_n \int_0^t \langle P_\varepsilon(n_\varepsilon) - \eta, n_\varepsilon - n \rangle ds = O(|n_\varepsilon - n|_{L^2(0, T; V^*)}).$$

Note that $n_\varepsilon \rightarrow n$ strongly in $L^2(0, T; V^*)$ and, for any $\phi \in L^\infty((0, T) \times \Omega)$, there holds

$$|P_\varepsilon(\phi) - P(\phi)|_2^2 = \int_0^T \int_{\phi < \varepsilon} (P_\varepsilon(\phi) - P(\phi))^2 dx dt \leq 2P(\varepsilon)^2 T |\Omega| \rightarrow 0.$$

Hence we have

$$\int_0^T \langle P(n_\varepsilon) - \eta, n_\varepsilon - n \rangle ds \rightarrow 0 \tag{B.1}$$

which is equivalent to

$$\int_0^T \langle P(n_\varepsilon), n_\varepsilon \rangle \rightarrow \int_0^T \langle \eta, n \rangle ds.$$

Since P is maximum monotone on $L^2(0, T; H)$, we can see that the above gives $P(n) = \eta$. Therefore the limit in (B.1) leads to $P(n_\varepsilon) \rightarrow P(n)$ and $n_\varepsilon \rightarrow n$ strongly in $L^2((0, T) \times \Omega)$, and thus $n_\varepsilon \rightarrow n$ a.e. in $(0, T) \times \Omega$. \diamond