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NONLINEAR DECAY AND SCATTERING OF SOLUTIONS TO A BRETHERTON EQUATION IN SEVERAL SPACE DIMENSIONS

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Abstract. We consider a Cauchy problem for the n-dimensional Bretherton equation. We establish the existence of a global solution and study its long-time behavior, with small data. This is done using the oscillatory integral techniques considered in [5].

1. Introduction

For the Bretherton equation, we consider the initial-value problem (I.V.P)

$$\begin{gathered} u_{tt} + u + \triangle u + \triangle^2 u = F(u), \quad x \in \mathbb{R}^n,\ n \geq 1,\ t > 0, \\ u(x,0) = f_1(x), \\ u_t(x,0) = f_2(x), \end{gathered} \tag{1.1}$$

where $F(u) = |u|^\alpha u$ and $\alpha \geq 1$. Problem (1.1) with $n = 1$ was introduced by Kalantorov and Ladyzhenskaya in [4], where they proved the blow-up of it's solutions in finite time for large data. After an investigation on the local existence of solutions to (1.1) with $n = 1$, Scialom [7] pointed out that the global existence result for "small data" remains an open problem.

Furthermore, using a new computational method called "RATH" (Real Automated Tangent Hyperbolic function method), Zhi-bin Li *et al.* [10] showed the existence of solitary-wave solutions of some partial differential equations. Yet, for the Bretherton equation, the "RATH" method showed the non-existence of solitary-wave solution. Our scattering result here seems to confirm the computation result of Zhi-bin Li *et al.* for the non-existence of solitary-wave solution to the Bretherton equation, at least for small data. Indeed it is well known that affirmative results on scattering are interpreted as the nonexistence of solitary-wave solution of arbitrary small amplitude, see [2, 6]. Our aim in this paper is to study the global existence, the uniform in x decay to zero and the scattering as $t \to \infty$, for solutions of (1.1) with sufficiently small data. More precisely, we show the following two theorems:

Theorem 1.1. *Let $\alpha > 5$ and $f_1, f_2 \in \mathbb{H}^s(\mathbb{R}^n) \cap \mathbb{L}^1(\mathbb{R}^n)$, $n \geq 1$, with $s \geq \frac{3}{2}n$. If $|f_1|_1 + \|f_1\|_{3n/2} + |f_2|_1 + \|f_2\|_{3n/2} < \delta$ with δ sufficiently small, then the solution u*

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of (1.1) *is unique in* $C(\mathbb{R}, \mathbb{H}^s(\mathbb{R}^n))$ *and satisfies*

$$|u(x,t)|_\infty \leq c(1+t)^{-1/4}, \quad t \geq 0, \tag{1.2}$$

where c *does not depend of* x *and* t*. Moreover, there is scattering for* $t \to \pm\infty$*, that is, there exist* u_+, u_-*, solutions of the linear problem* (2.1)*, such that* $\|u(t)-u_\pm(t)\|_2$ *tends to 0 as* $t \to \pm\infty$*.*

Theorem 1.2. *Let* $\alpha > 1+\frac{4}{\theta}$ *and* $f_1, f_2 \in \mathbb{H}^{r+\frac{5}{2}n+1}(\mathbb{R}^n) \cap \mathbb{L}^q_{r+\frac{5}{2}n}(\mathbb{R}^n)$*,* $n \geq 1$*, with* $r > \frac{n}{p}$*. If* $\|f_1\|_{r+\frac{5}{2}n,q} + \|f_2\|_{r+\frac{5}{2}n,q} + \|f_1\|_{r+\frac{5}{2}n+1} + \|f_2\|_{r+\frac{5}{2}n+1} < \delta$ *with* δ *small, then the solution* u *of the I.V.P* (1.1) *satisfies*

$$\|u(x,t)\|_{r,p} \leq c(1+t)^{-\frac{\theta}{4}}, \quad t \geq 0, \tag{1.3}$$

where $p = 2/(1-\theta), q = 1/(1+\theta)$*, and* $\theta \in]0,1[$*.*

Notation. The notation $\|\cdot\|_{r,p}$ is used to denote the norm in $\mathbb{L}^p_r$ such that for $u \in \mathbb{L}^p_r(\mathbb{R}^n)$, $\|u\|_{r,p} = \|u\|_{\mathbb{L}^p_r} = \|(1-\Delta)^{r/2}u\|_{\mathbb{L}^p} < \infty$. Also, $|\cdot|_p$ instead of $\|\cdot\|_{0,p}$ denotes the norm in $\mathbb{L}^p$, and $\mathbb{H}^s$ with norm $\|\cdot\|_s$ is used instead of $\mathbb{L}^2_s$. Throughout this paper, c represents a generic constant independent of t and x. The Fourier transform of a function f is denoted by $\widehat{f}(\xi)$ or $\mathcal{F}(f)(\xi)$ and $\mathcal{F}^{-1}(f) \equiv \check{f}$ denotes the inverse Fourier transform of f.

For $1 \leq p, q \leq \infty$ and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$,

$$\|f\|_{\mathbb{L}^q(\mathbb{R};\mathbb{L}^p(\mathbb{R}^n))} = \Big(\int_{-\infty}^{+\infty} \Big(\int_{\mathbb{R}^n} |f(x,t)|^p dx \Big)^{q/p} dt \Big)^{1/q}.$$

2. Local existence result

In this section, we write the Cauchy problem associated with (1.1) in it's integral form and we prove the local existence and uniqueness of it's solution. Our method of proof is based on linear estimates and a contraction mapping argument. Thereupon, we state a locally well-posed theorem for (1.1).

Theorem 2.1. *Let* $s > n/2$ *be a real number, and* $f_1, f_2 \in \mathbb{H}^s(\mathbb{R}^n)$*,* $n \geq 1$*. Then there exists* $T_0 > 0$ *which depends on* $\|f_1\|_s$ *and* $\|f_2\|_s$*, and a unique solution of* (1.1) *in [0,T], such that* $u \in \mathcal{C}(0, T_0; \mathbb{H}^s(\mathbb{R}^n))$*.*

Proof. Consider first the linear part of (1.1):

$$\begin{gathered} u_{tt} + u + \triangle u + \triangle^2 u = 0, \quad x \in \mathbb{R}^n,\ n \geq 1,\ t > 0, \\ u(x,0) = f_1(x), \\ u_t(x,0) = f_2(x), \end{gathered} \tag{2.1}$$

The formal solution of (2.1) is

$$u(x,t) = V_1(t)f_1(x) + V_2(t)f_2(x) \tag{2.2}$$

where

$$\begin{gathered} V_1(t)f_1(x) = [\frac{1}{2}(e^{it\phi(\xi)} + e^{-it\phi(\xi)})\hat{f}_1(\xi)]^\vee(x), \\ V_2(t)f_2(x) = [\frac{1}{2i\phi(\xi)}(e^{it\phi(\xi)} - e^{-it\phi(\xi)})\hat{f}_2(\xi)]^\vee(x) \end{gathered}$$

with $\phi(\xi) = (1 - |\xi|^2 + |\xi|^4)^{1/2}$.

We define

$$S_1(t)f_1(x) = \frac{1}{2(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\xi+it\phi(\xi)}\hat{f}_1(\xi)d\xi,$$
$$S_2(t)f_2(x) = \frac{1}{2i(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\xi+it\phi(\xi)}\frac{\hat{f}_2(\xi)}{\phi(\xi)}d\xi.$$

Then

$$V_1(t)f_1(x) = S_1(t)f_1(x) + S_1(-t)f_1(x),$$
$$V_2(t)f_2(x) = S_2(t)f_2(x) - S_2(-t)f_2(x).$$

Note that

$$\Phi(\xi) \geq \frac{1}{2\sqrt{3}}(1+|\xi|^2). \tag{2.3}$$

Indeed, $\phi(\xi)^2 = 1 - |\xi|^2 + |\xi|^4 = (1-\frac{1}{2}|\xi|^2)^2 + \frac{3}{4}|\xi|^4$ so that if $|\xi| \leq 1$ then

$$\phi(\xi)^2 \geq \frac{1}{4} + \frac{3}{4}|\xi|^4 \geq \frac{1}{4}(\frac{1}{3} + \frac{2}{3}|\xi|^2 + \frac{1}{3}|\xi|^4) = \frac{1}{12}(1+|\xi|^2)^2$$

and if $|\xi| \geq 1$ then

$$\phi(\xi)^2 \geq \frac{3}{4}|\xi|^4 \geq \frac{1}{12}(1+|\xi|^2)^2.$$

Remark 2.2. Since (1.1) will not change when t is switched to $-t$, the solution $u(t)$ in Theorem 2.1 can be extended to $u \in \mathcal{C}([-T_0, T_0]; \mathbb{H}^s(\mathbb{R}^n))$.

Remark 2.3. Note that, since the negative sign of t in $S_j(-t)$ acts only on the sign of the phase function, the estimates of $S_j(t)f(x)$ below hold also for $S_j(-t)f(x)$. Hence, to estimate $V_j(t)f(x)$ one only has to estimate $S_j(t)f(x)$, $j = 1, 2$.

To prove the existence theorem for (1.1), we need the following inequalities.

Lemma 2.4. *Let $f_1, f_2 \in \mathbb{H}^s(\mathbb{R}^n), s \geq 0$, and $V_1(t), V_2(t)$ defined in (2.2). Then*

$$\|V_1(t)f_1(x)\|_s \leq c\|f_1\|_s \tag{2.4}$$
$$\|V_2(t)f_2\|_s \leq c\|f_2\|_{s-2} \leq c\|f_2\|_s. \tag{2.5}$$

The proof of the above lemma follows directly from the definition of $V_1(t)$ and $V_2(t)$ in (2.2) and the use of the inequality (2.3). □

Thereafter, with Lemma 2.4 in hand, one can use the contraction mapping principle to prove the local well-posedness result in Theorem 2.1. Then, thanks to the Duhamela principle, the solution of (1.1) verifies the integral equation

$$u(x,t) = V_1(t)f_1(x) + V_2(t)f_2(x) + \int_0^t V_2(t-\tau)(|u|^\alpha u)(\tau)d\tau. \tag{2.6}$$

Let us define

$$\varphi(u)(t) = V_1(t)f_1(x) + V_2(t)f_2(x) + \int_0^t V_2(t-\tau)(|u|^\alpha u)(\tau)d\tau \tag{2.7}$$

and the complete metric space

$$F = \{v \in \mathcal{C}(0,T; \mathbb{H}^s(\mathbb{R}^n)),\ s > n/2, \quad \sup_{[0,T]} \|v(t)\|_s \leq a\},$$

where a is a positive real constant.

We begin by showing that $\varphi : F \to F$ is a contraction. The use of the definition of φ in (2.7), Lemma 2.4 and the fact that $\mathbb{H}^s(\mathbb{R}^n))$, $s > n/2$ is an Algebra, lead for all $0 \le t \le T$, to

$$\begin{aligned} \|\varphi(u)(t)\|_s &\le c(\|f_1\|_s + \|f_2\|_s) + c\int_0^t \|(|u|^\alpha u)(\tau)\|_s d\tau \\ &\le c(\|f_1\|_s + \|f_2\|_s) + c\int_0^t \|u(\tau)\|_s^{\alpha+1} d\tau \\ &\le c(\|f_1\|_s + \|f_2\|_s) + cT(\sup_{[0,T]} \|u\|_s)^{\alpha+1}. \end{aligned} \tag{2.8}$$

Thereby, taking μ as a positive constant such that $\|f_1\|_s + \|f_2\|_s < \mu$, we get for $u \in F$,

$$\sup_{[0,T]} \|\varphi(u)(t)\|_s \le c\{\mu + a^{\alpha+1}T\}$$

so that choosing $a = 2c\mu$, we obtain

$$\sup_{[0,T]} \|\varphi(u)(t)\|_s \le c\{\mu + 2^{\alpha+1}c^{\alpha+1}\mu^{\alpha+1}T\} = c\mu\{1 + 2^{\alpha+1}c^{\alpha+1}\mu^\alpha T\}.$$

Then, fixing T such that

$$2^{\alpha+1}c^{\alpha+1}\mu^\alpha T < 1 \tag{2.9}$$

we get

$$\sup_{[0,T]} \|\varphi(u)(t)\|_s \le 2c\mu = a.$$

This shows that φ maps F into F. The next step is to prove that φ is in fact a contraction. We consider u and v in F with the same initial values. Thus

$$(\varphi(u) - \varphi(v))(t) = \int_0^t V_2(t-\tau)(|u|^\alpha u - |v|^\alpha v)(\tau) d\tau.$$

To estimate $\sup_{[0,T]} \|(\varphi(u) - \varphi(v))(t)\|_s$ we use Lemma 2.4, Taylor formula and the fact that $\mathbb{H}^s(\mathbb{R}^n))$, $s > \frac{n}{2}$ is an Algebra; it follows that for all $0 \le t \le T$,

$$\begin{aligned} \|(\varphi(u) - \varphi(v))(t)\|_s &\le c\int_0^t \|(|u|^\alpha + |v|^\alpha)(u-v)(\tau)\|_s d\tau \\ &\le c\int_0^t \|(|u|^\alpha + |v|^\alpha)(\tau)\|_s \|(u-v)(\tau)\|_s d\tau \\ &\le cT(\sup_{[0,T]} \|u\|_s^\alpha + \sup_{[0,T]} \|v\|_s^\alpha) \sup_{[0,T]} \|u-v\|_s. \end{aligned}$$

which leads, with $a = 2c\mu$ as above, to

$$\|(\varphi(u) - \varphi(v))(t)\|_s \le 2^{\alpha+1}c^{\alpha+1}\mu^\alpha T \sup_{[0,T]} \|u-v\|_s. \tag{2.10}$$

Hence, with the choice of T as above in (2.11), we get from (2.10) that φ is a contraction map in F. Thus, the application of contraction mapping principle gives the result of local existence and uniqueness in Theorem 2.1.

For the sequel, we need the following inequalities which are obtained by obvious computations including the inequality (2.3): $\forall \xi \in \mathbb{R}^n$,

$$|\nabla\phi(\xi)| = \frac{|\xi||2|\xi|^2 - 1|}{(1 - |\xi|^2 + |\xi|^4)^{1/2}} \tag{2.11}$$

$$|D^2\phi(\xi)| \le c. \tag{2.12}$$

3. Linear Estimates

The purpose of this section is to study the linear equation associated with (1.1) and to establish linear estimates needed for the next section. The following result is concerning the decay of solutions of the linear problem (2.1).

Lemma 3.1. *Let $V_1(t)$ and $V_2(t)$ be defined as in (2.2). Let $f_1, f_2 \in \mathbb{H}^{\frac{3}{2}n}(\mathbb{R}^n) \cap \mathbb{L}^1(\mathbb{R}^n)$, $n \geq 1$. Then there exists a constant c independent of f_1, f_2, t and $x \in \mathbb{R}^n$ such that*

$$|V_j(t)f_j|_\infty \leq c(|f_j|_1 + \|f_j\|_{3n/2}(1+t)^{-1/4}, \quad j = 1, 2, \tag{3.1}$$

for all $t \geq 0$. Moreover, let $f_1, f_2 \in \mathbb{H}^{\frac{3}{2}n}(\mathbb{R}^n) \cap \mathbb{L}^1_{\frac{5}{2}n+k}(\mathbb{R}^n)$, $n \geq 1$, $k \in \mathbb{R}^+$; then we have

$$\|V_j(t)f_j\|_{k,\infty} \leq c(\|f_j\|_{\frac{5}{2}n+k,1} + \|f_j\|_{\frac{3}{2}n+k})(1+t)^{-1/4}, \quad j = 1, 2. \tag{3.2}$$

Before proving the above lemma, we prove the following lemma.

Lemma 3.2. *Given $x \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$, the phase function*

$$\Psi(\xi) = \phi(\xi) + t^{-1}(x, \xi)$$

has a finite number of stationary points. Moreover, if ξ_s is a stationary point of Ψ, then any point η_s verifying $|\eta_s| = |\xi_s|$ is also a stationary point of Ψ.

Proof. Since

$$\nabla\Psi(\xi) = \nabla\phi(\xi) + xt^{-1} = \frac{\xi(2|\xi|^2 - 1)}{(1 - |\xi|^2 + |\xi|^4)^{1/2}} + xt^{-1},$$

we have

$$\nabla\Psi(\xi) = 0 \iff \xi(2|\xi|^2 - 1) + xt^{-1}(1 - |\xi|^2 + |\xi|^4)^{1/2} = 0$$

and making the scalar product with

$$labele3.3\xi(2|\xi|^2 - 1) + xt^{-1}(1 - |\xi|^2 + |\xi|^4)^{1/2} \tag{3.3}$$

we get

$$\nabla\Psi(\xi) = 0 \Leftrightarrow |\xi|^2(2|\xi|^2 - 1)^2 + |xt^{-1}|^2(1 - |\xi|^2 + |\xi|^4) = 0 \Leftrightarrow P(|\xi|) = 0 \tag{3.4}$$

where $P(y) = 4y^6 - 4y^4 + y^2 - |xt^{-1}|^2(1 - y^2 + y^4)$, $y \in \mathbb{R}_+$. The stationary points of Ψ are such that their norms are the roots of $P(y)$. Then since $P(y)$ is polynomial of degree 6 so that it has at most 6 roots, we deduce that Ψ has a finite number of stationary points in $\mathbb{R}^n$. Furthermore, since $P(0) = -|xt^{-1}|^2 \leq 0$ and $P(y) \to +\infty$ as $y \to +\infty$, and since $P(y)$ is continuous, we deduce that there exists at least one stationary point of Ψ. Therefore, Ψ has a finite number of stationary points. Moreover, we note that if ξ_s is a stationary point of Ψ and if η_s is a point verifying $|\eta_s| = |\xi_s|$, then we have $P(|\eta_s|) = P(|\xi_s|) = 0$ and consequently from (3.4) η_s is also a stationary point of Ψ. This completes the proof of Lemma 3.2. $\square$

Next, we use Lemma 3.2 to prove Lemma 3.1. Let us recall that, thanks to Remark 2.2, the inequality (3.1) of proposition (3.1) holds for $V_1(t)$ and $V_2(t)$

whenever it holds for $S_1(t)$ and $S_2(t)$. If $0 \le t \le 1$, we have, thanks to the Schwartz inequality,

$$\begin{aligned} |S_1(t)f_1(x)| &= \frac{1}{2(2\pi)^n}\Big|\int_{\mathbb{R}^n} e^{it\Psi(\xi)}\hat{f}_1(\xi)d\xi\Big| \\ &\le c\int_{\mathbb{R}^n} |\hat{f}_1(\xi)|d\xi \\ &\le c\Big(\int_{\mathbb{R}^n}(1+|\xi|^2)^{-n}d\xi\Big)^{1/2}\|f_1\|_n \le c\|f_1\|_n \\ &\le c(1+t)^{-1/4}\|f_1\|_{3n/2} \end{aligned} \tag{3.5}$$

If $t \ge 1$, let $\Omega = \{\xi \in \mathbb{R}^n, |\xi| \le t^{\frac{1}{4n}}\}$ and $q_t(\xi) = \chi_\Omega(\xi)e^{it\phi(\xi)}$; then thanks to the Schwartz and the Young inequality,

$$\begin{aligned} &|S_1(t)f_1(x)| \\ &= \frac{1}{2(2\pi)^n}\Big|\Big(\int_\Omega + \int_{\Omega^c}\Big)e^{it\phi(\xi)+ix\cdot\xi}\hat{f}_1(\xi)d\xi\Big| \\ &\le c|\check{q}_t(x) * f_1(x)|_\infty + c\Big(\int_{\Omega^c}(1+|\xi|^2)^{-\frac{3}{2}n}d\xi\Big)^{1/2}\Big(\int_{\Omega^c}(1+|\xi|^2)^{\frac{3}{2}n}|\hat{f}_1(\xi)|^2d\xi\Big)^{1/2} \\ &\le c|\check{q}_t(x)|_\infty|f_1(x)|_1 + ct^{-1/4}\|f_1\|_{3n/2} \end{aligned} \tag{3.6}$$

where the first factor in the second term of the right hand side of (3.6) is a bound given $\forall t \ge 1$ by

$$\begin{aligned} \Big(\int_{\{|\xi|\ge t^{\frac{1}{4n}}\}}(1+|\xi|^2)^{-\frac{3}{2}n}d\xi\Big)^{1/2} &\le \Big(\int_{\{r\ge t^{\frac{1}{4n}}\}} r^{-3n}r^{n-1}dr\Big)^{1/2} \\ &= c(t^{-1/2})^{1/2} = ct^{-1/4}. \end{aligned}$$

It remains to estimate $\check{q}_t(x)$. We need for the sequel, the following notations: We take $\Omega = \{\xi \in \mathbb{R}^n, |\xi| \le t^{\frac{1}{4n}}\}$ and let $\mathcal{E}_s = \{\xi \in \mathbb{R}^n, \nabla\Psi(\xi) = 0\}$ be the set of stationary points of Ψ. Hence from Lemma 3.2, $\mathcal{E}_s$ has a finite number of elements. Then set

$$s(t^{-1/4}) = \bigcup_{\zeta\in\mathcal{E}_s} B(\zeta, t^{-1/4})\bigcup\{\xi \in \mathbb{R}^n, |\xi| \le t^{-1/4}\}$$

where for each $\zeta \in \mathcal{E}_s$, $B(\zeta, t^{-1/4}) = \{\xi \in \mathbb{R}^n, |\xi - \zeta| \le t^{-1/4}\}$. Let

$$\mathcal{A} = s(t^{-1/4})\bigcup\Big\{\frac{1}{\sqrt{2}}(1-t^{-1/4}) \le |\xi| \le \frac{1}{\sqrt{2}}(1+t^{-1/4})\Big\}.$$

Hence

$$\check{q}_t(x) = \int_\Omega e^{it\phi(\xi)+ix\cdot\xi}d\xi = \Big(\int_{\Omega\cap\mathcal{A}} + \int_{\Omega\cap\mathcal{A}^c}\Big)e^{it\phi(\xi)+ix\cdot\xi}d\xi = I_1 + I_2. \tag{3.7}$$

Since from Lemma 3.2, $card(\mathcal{E}_s) < \infty$, we get

$$\begin{aligned}
|I_1| &\le \int_{\Omega\cap\mathcal{A}} d\xi \\
&\le \sum_{\zeta\in\mathcal{E}_s}\int_{B(\zeta,t^{-1/4})} d\xi + \int_{\{|\xi|\le t^{-1/4}\}} d\xi + \int_{\{\frac{1}{\sqrt{2}}(1-t^{-1/4})\le|\xi|\le\frac{1}{\sqrt{2}}(1+t^{-1/4})\}} d\xi \\
&\le \int_{\{0\le r\le t^{-1/4}\}} r^{n-1}dr + \int_{\{\frac{1}{\sqrt{2}}(1-t^{-1/4})\le r\le\frac{1}{\sqrt{2}}(1+t^{-1/4})\}} r^{n-1}dr \\
&\le ct^{-\frac{n}{4}} + \int_{\{\frac{1}{\sqrt{2}}(1-t^{-1/4})\le r\le\frac{1}{\sqrt{2}}(1+t^{-1/4})\}} r^{n-1}dr \\
&\le ct^{-\frac{n}{4}} + ct^{-1/4} \le ct^{-1/4}.
\end{aligned} \tag{3.8}$$

For I_2, we point out that on

$$\mathcal{A}^c = \{s(t^{-1/4})\}^c \bigcap \{\{|\xi| \le \frac{1}{\sqrt{2}}(1-t^{-1/4})\} \cup \{|\xi| \ge \frac{1}{\sqrt{2}}(1+t^{-1/4})\}\},$$

Ψ has no stationary point so that we can integrate I_2 by parts as follows:

$$\begin{aligned}
|I_2| &= |\int_{\Omega\cap\mathcal{A}^c} e^{it\Psi(\xi)}d\xi| \\
&= t^{-1}|\int_{\Omega\cap\mathcal{A}^c} \frac{1}{\nabla\Psi(\xi)}\nabla(e^{it\Psi(\xi)})d\xi| \\
&\le t^{-1}\int_{\Omega\cap\mathcal{A}^c} |\nabla(\frac{1}{\nabla\Psi(\xi)})|d\xi + t^{-1}\int_{\partial\{\Omega\cap\mathcal{A}^c\}} \frac{d\xi}{|\nabla\Psi(\xi)|} \\
&\le ct^{-1}\int_{\Omega\cap\mathcal{A}^c} \{|\nabla(\frac{1}{\nabla\Psi(\xi)})| + |\nabla(\frac{1}{|\nabla\Psi(\xi)|})|\}d\xi \\
&\le ct^{-1}\int_{\Omega\cap\mathcal{A}^c} \frac{|D^2\Psi(\xi)|}{|\nabla\Psi(\xi)|^2}d\xi.
\end{aligned} \tag{3.9}$$

For the rest of this article, we consider a point $\xi_s \in \mathcal{E}_s$; then we have

$$\begin{aligned}
\mathcal{A}^c \subset &\{\xi\in\mathbb{R}^n, |\xi-\xi_s| > t^{-1/4}\} \cap \{|\xi| > t^{-1/4}\} \cap \{\{|\xi| < \frac{1}{\sqrt{2}}(1-t^{-1/4})\} \\
&\cup \{|\xi| > \frac{1}{\sqrt{2}}(1+t^{-1/4})\}\}.
\end{aligned}$$

Hence from (3.9), we obtain

$$|I_2| \le ct^{-1}\int_{\Omega\cap\{E_1\cup E_2\}\cap\{|\xi-\xi_s|>t^{-1/4}\}\cap\{|\xi|>t^{-1/4}\}} \frac{|D^2\Psi(\xi)|}{|\nabla\Psi(\xi)|^2}d\xi \tag{3.10}$$

where $E_1 = \{\xi\in\mathbb{R}^n, |\xi| < \frac{1}{\sqrt{2}}(1-t^{-1/4})\}$ and $E_2 = \{\xi\in\mathbb{R}^n, |\xi| > \frac{1}{\sqrt{2}}(1+t^{-1/4})\}$. For the sequel, we need the following inequality: with E_1 and E_2 as defined in (3.10), we claim that on $\{E_1\cup E_2\}\cap\{|\xi-\xi_s| > t^{-1/4}\}\cap\{|\xi| > t^{-1/4}\}$,

$$|\nabla\Psi(\xi)| \ge ct^{-1/4}\frac{|\xi|(1+|\xi|)}{(1-|\xi|^2+|\xi|^4)^{1/2}}. \tag{3.11}$$

To prove this inequality, let us give the following remark.

Remark 3.3. Let $\xi_s \in \mathcal{E}_s$. Then for any $\xi \in \mathbb{R}^n$, there exists an index set J empty or not, with $J \in \{1, \dots, n\}$ such that

$$\operatorname{sgn}(\xi_i) = \begin{cases} -\operatorname{sgn}(\xi_{si}) & \text{if } i \in J \\ \operatorname{sgn}(\xi_{si}) & \text{if } i \in J^c. \end{cases}$$

Let us prove now inequality (3.11). In view of Remark 3.3, let $\xi_s \in \mathcal{E}_s$ and let J be an index set such that

$$\operatorname{sgn}(\xi_i) = \begin{cases} -\operatorname{sgn}(\xi_{si}) & \text{if } i \in J \\ \operatorname{sgn}(\xi_{si}) & \text{if } i \in J^c. \end{cases}$$

Moreover, define a point η_s by

$$\eta_{si} = \begin{cases} \xi_{si} & \text{if } i \in J \\ -\xi_{si} & \text{if } i \in J^c \end{cases}$$

where J is the same index set as above. Hence from Lemma 3.2, η_s is also a stationary point and then thanks to Remark 3.3 and the definition of η_s, we have on $E_2 = \{\xi \in \mathbb{R}^n, |\xi| > \frac{1}{\sqrt{2}}(1 + t^{-1/4})\}$ and for $|\xi_s| \geq \frac{1}{\sqrt{2}}$,

$$\begin{aligned}
|\nabla\Psi(\xi)| &= |\nabla\Psi(\xi) - \nabla\Psi(\eta_s)| = |\nabla\phi(\xi) - \nabla\phi(\eta_s)| \\
&= |\xi \frac{(2|\xi|^2 - 1)}{(1 - |\xi|^2 + |\xi|^4)^{1/2}} - \eta_s \frac{(2|\eta_s|^2 - 1)}{(1 - |\eta_s|^2 + |\eta_s|^4)^{1/2}}| \\
&= (\sum_{i\in J} + \sum_{i \in J^c})|\xi_i \frac{(2|\xi|^2 - 1)}{(1 - |\xi|^2 + |\xi|^4)^{1/2}} - \eta_{si} \frac{(2|\eta_s|^2 - 1)}{(1 - |\eta_s|^2 + |\eta_s|^4)^{1/2}}| \\
&= \sum_{i\in J} |\xi_i \frac{(2|\xi|^2 - 1)}{(1 - |\xi|^2 + |\xi|^4)^{1/2}} - \xi_{si} \frac{(2|\xi_s|^2 - 1)}{(1 - |\xi_s|^2 + |\xi_s|^4)^{1/2}}| \\
&\quad + \sum_{i\in J^c} |\xi_i \frac{(2|\xi|^2 - 1)}{(1 - |\xi|^2 + |\xi|^4)^{1/2}} + \xi_{si} \frac{(2|\xi_s|^2 - 1)}{(1 - |\xi_s|^2 + |\xi_s|^4)^{1/2}}| \\
&= \sum_{i\in J} |\operatorname{sgn}(\xi_i)|\xi_i| \frac{(2|\xi|^2 - 1)}{(1 - |\xi|^2 + |\xi|^4)^{1/2}} - \operatorname{sgn}(\xi_{si})|\xi_{si}| \frac{(2|\xi_s|^2 - 1)}{(1 - |\xi_s|^2 + |\xi_s|^4)^{1/2}}| \\
&\quad + \sum_{i\in J^c} |\operatorname{sgn}(\xi_i)|\xi_i| \frac{(2|\xi|^2 - 1)}{(1 - |\xi|^2 + |\xi|^4)^{1/2}} + \operatorname{sgn}(\xi_{si})|\xi_{si}| \frac{(2|\xi_s|^2 - 1)}{(1 - |\xi_s|^2 + |\xi_s|^4)^{1/2}}| \\
&= \sum_{i\in J} |\operatorname{sgn}(\xi_i)|\xi_i| \frac{(2|\xi|^2 - 1)}{(1 - |\xi|^2 + |\xi|^4)^{1/2}} + \operatorname{sgn}(\xi_i)|\xi_{si}| \frac{(2|\xi_s|^2 - 1)}{(1 - |\xi_s|^2 + |\xi_s|^4)^{1/2}}| \\
&\quad + \sum_{i\in J^c} |\operatorname{sgn}(\xi_i)|\xi_i| \frac{(2|\xi|^2 - 1)}{(1 - |\xi|^2 + |\xi|^4)^{1/2}} + \operatorname{sgn}(\xi_i)|\xi_{si}| \frac{(2|\xi_s|^2 - 1)}{(1 - |\xi_s|^2 + |\xi_s|^4)^{1/2}}| \\
&= (\sum_{i\in J} + \sum_{i\in J^c})(\frac{|\xi_i|(2|\xi|^2 - 1)}{(1 - |\xi|^2 + |\xi|^4)^{1/2}} + \frac{|\xi_{si}|(2|\xi_s|^2 - 1)}{(1 - |\xi_s|^2 + |\xi_s|^4)^{1/2}}) \\
&\geq \frac{|\xi|(\sqrt{2}|\xi| - 1)(\sqrt{2}|\xi| + 1)}{(1 - |\xi|^2 + |\xi|^4)^{1/2}} \geq t^{-1/4} \frac{|\xi|(|\xi| + 1)}{(1 - |\xi|^2 + |\xi|^4)^{1/2}}.
\end{aligned}$$

Again on $E_2 = \{\xi \in \mathbb{R}^n, |\xi| > \frac{1}{\sqrt{2}}(1 + t^{-1/4})\}$ but now for $|\xi_s| \le \frac{1}{\sqrt{2}}$, we write thanks to the definition of η_s,

$$\begin{aligned}|\nabla\Psi(\xi)| &= |\nabla\phi(\xi) - \nabla\phi(-\eta_s)|\\ &= |\xi\frac{(2|\xi|^2-1)}{(1-|\xi|^2+|\xi|^4)^{1/2}} + \eta_s\frac{(2|\eta_s|^2-1)}{(1-|\eta_s|^2+|\eta_s|^4)^{1/2}}|\\ &= |\xi\frac{(2|\xi|^2-1)}{(1-|\xi|^2+|\xi|^4)^{1/2}} - \eta_s\frac{(1-2|\eta_s|^2)}{(1-|\eta_s|^2+|\eta_s|^4)^{1/2}}|\end{aligned}$$

so that thanks to Remark 3.3 and the definition of η_s, we follow the same lines as above to obtain

$$\begin{aligned}|\nabla\Psi(\xi)| &= |\xi|\frac{(2|\xi|^2-1)}{(1-|\xi|^2+|\xi|^4)^{1/2}} + |\xi_s|\frac{(1-2|\xi_s|^2)}{(1-|\xi_s|^2+|\xi_s|^4)^{1/2}}\\ &\ge ct^{-1/4}\frac{|\xi|(|\xi|+1)}{(1-|\xi|^2+|\xi|^4)^{1/2}}.\end{aligned}$$

For this time, on $E_1 = \{\xi \in \mathbb{R}^n, |\xi| < \frac{1}{\sqrt{2}}(1 - t^{-1/4})\}$ and if $|\xi_s| \ge \frac{1}{\sqrt{2}}$, we write thanks to the definition of η_s above,

$$\begin{aligned}|\nabla\Psi(\xi)| &= |\nabla\phi(-\eta_s) - \nabla\phi(\xi)|\\ &= |-\eta_s\frac{(2|\eta_s|^2-1)}{(1-|\eta_s|^2+|\eta_s|^4)^{1/2}} - \xi\frac{(2|\xi|^2-1)}{(1-|\xi|^2+|\xi|^4)^{1/2}}|\\ &= |-\eta_s\frac{(2|\eta_s|^2-1)}{(1-|\eta_s|^2+|\eta_s|^4)^{1/2}} + \xi\frac{(1-2|\xi|^2)}{(1-|\xi|^2+|\xi|^4)^{1/2}}|\end{aligned}$$

so that thanks to Remark 3.3 and the definition of η_s, we follow the same lines as above to get

$$\begin{aligned}|\nabla\Psi(\xi)| &= |\xi_s|\frac{(2|\xi_s|^2-1)}{(1-|\xi_s|^2+|\xi_s|^4)^{1/2}} + |\xi|\frac{(1-2|\xi|^2)}{(1-|\xi|^2+|\xi|^4)^{1/2}}\\ &\ge ct^{-1/4}\frac{|\xi|(|\xi|+1)}{(1-|\xi|^2+|\xi|^4)^{1/2}}.\end{aligned}$$

Finally, still on $E_1 = \{\xi \in \mathbb{R}^n, |\xi| < \frac{1}{\sqrt{2}}(1 - t^{-1/4})\}$ but now for $|\xi_s| \le \frac{1}{\sqrt{2}}$, we write with the definition of η_s,

$$\begin{aligned}|\nabla\Psi(\xi)| &= |\nabla\phi(\eta_s) - \nabla\phi(\xi)|\\ &= |\eta_s\frac{(2|\eta_s|^2-1)}{(1-|\eta_s|^2+|\eta_s|^4)^{1/2}} - \xi\frac{(2|\xi|^2-1)}{(1-|\xi|^2+|\xi|^4)^{1/2}}|\\ &= |-\eta_s\frac{(1-2|\eta_s|^2)}{(1-|\eta_s|^2+|\eta_s|^4)^{1/2}} + \xi\frac{(1-2|\xi|^2)}{(1-|\xi|^2+|\xi|^4)^{1/2}}|\end{aligned}$$

so that proceeding as above, we find thanks to Remark 3.3 and the definition of η_s,

$$|\nabla\Psi(\xi)| \ge ct^{-1/4}\frac{|\xi|(|\xi|+1)}{(1-|\xi|^2+|\xi|^4)^{1/2}}.$$

This completes the proof of (3.11).

For the sequel, we need the following inequality which, thanks to (2.12) and (3.11), is obviously shown: That is: On

$$\Omega \cap \{E_1 \cup E_2\} \cap \{|\xi - \xi_s| > t^{-1/4}\} \cap \{|\xi| > t^{-1/4}\}$$

where Ω is defined in (3.5) and E_1 and E_2 are defined in (3.10), we have

$$\frac{|D^2\Psi(\xi)|}{|\nabla\Psi(\xi)|^2} \leq ct^{1/2}\frac{(1-|\xi|^2+|\xi|^4)}{|\xi|^2(|\xi|+1)^2} \leq ct^{1/2}\frac{(1+|\xi|)^2}{|\xi|^2}. \tag{3.12}$$

Therefore, from (3.10), (3.11), (3.12), we get

$$\begin{aligned}
|I_2| &\leq ct^{-1}\int_{\Omega\cap\{E_1\cup E_2\}\cap\{|\xi-\xi_s|>t^{-1/4}\}\cap\{|\xi|>t^{-1/4}\}} \frac{|D^2\Psi(\xi)|}{|\nabla\Psi(\xi)|^2}d\xi \\
&\leq ct^{-1}t^{1/2}\int_{\Omega\cap\{E_1\cup E_2\}\cap\{|\xi-\xi_s|>t^{-1/4}\}\cap\{|\xi|>t^{-1/4}\}} |\xi|^{-2}(1+|\xi|)^2 d\xi \\
&\leq ct^{-1/2}\int_{\Omega\cap E_1\cap\{|\xi-\xi_s|>t^{-1/4}\}\cap\{|\xi|>t^{-1/4}\}} |\xi|^{-2}(1+|\xi|)^2 d\xi \\
&\quad + ct^{-1/2}\int_{\Omega\cap E_2\cap\{|\xi-\xi_s|>t^{-1/4}\}\cap\{|\xi|>t^{-1/4}\}} |\xi|^{-2}(1+|\xi|)^2 d\xi \\
&\leq ct^{-1/2}\{\int_{\{t^{-1/4}<|\xi|<\frac{1}{\sqrt{2}}(1-t^{-1/4})\}} |\xi|^{-2}d\xi + \int_{\{\frac{1}{\sqrt{2}}(1+t^{-1/4})<|\xi|<t^{\frac{1}{4n}}\}} d\xi\} \\
&\leq ct^{-1/2}\{\int_{\{t^{-1/4}<r<\frac{1}{\sqrt{2}}(1-t^{-1/4})\}} r^{-2}r^{n-1}dr \\
&\quad + \int_{\{\frac{1}{\sqrt{2}}(1+t^{-1/4})<r<t^{\frac{1}{4n}}\}} r^{n-1}dr\} \\
&\leq ct^{-1/2}\{\int_{\{t^{-1/4}<r<1\}} r^{-2}dr + t^{\frac{n-1}{4n}}\int_{\{\frac{1}{\sqrt{2}}(1+t^{-1/4})<r<t^{\frac{1}{4n}}\}} dr\} \leq ct^{-1/4},
\end{aligned} \tag{3.13}$$

where $E_1 = \{\xi\in\mathbb{R}^n, |\xi| < \frac{1}{\sqrt{2}}(1-t^{-1/4})\}$ and $E_2 = \{\xi\in\mathbb{R}^n, |\xi| > \frac{1}{\sqrt{2}}(1+t^{-1/4})\}$. Hence, with the estimates on I_1 and I_2 in (3.8) and (3.13) above, and thanks to (3.7), we are led to

$$|\check{q}_t(x)|_\infty \leq ct^{-1/4} \quad \forall t\geq 1.$$

Combining this inequality, (3.6) and (3.5), we find

$$|S_1(t)f_1(x)| \leq c(1+t)^{-1/4}(|f_1|_1 + \|f_1\|_{3n/2} \quad \forall t\geq 0,$$

which with Remark 2.2 leads to (3.1) for the case $j=1$. Let us prove now the inequality (3.1) for the case $j=2$ If $0\leq t\leq 1$, we have thanks to the Schwartz inequality and the inequality (2.3) on $\phi(\xi)$,

$$\begin{aligned}
|S_2(t)f_2(x)| &= \frac{1}{2(2\pi)^n}|\int_{\mathbb{R}^n} e^{it\Psi(\xi)}\frac{\hat{f}_2(\xi)}{\phi(\xi)}d\xi| \\
&\leq c\int_{\mathbb{R}^n}\frac{|\hat{f}_2(\xi)|}{|\phi(\xi)|}d\xi \\
&\leq c\int_{\mathbb{R}^n}\frac{|\hat{f}_2(\xi)|}{(1+|\xi|^2)}d\xi \\
&\leq c(\int_{\mathbb{R}^n}(1+|\xi|^2)^{-n}d\xi)^{1/2}\|f_2\|_{n-2} \\
&\leq c\|f_2\|_{n-2} \leq c(1+t)^{-1/4}\|f_2\|_{3n/2}.
\end{aligned} \tag{3.14}$$

If $t \geq 1$, then we have with the notation $\Omega = \{\xi \in \mathbb{R}^n, |\xi| \leq t^{\frac{1}{4n}}\}$ given above and thanks to (2.3), the Schwartz and the Young inequalities,

$$\begin{aligned} |S_2(t)f_2(x)| &= \frac{1}{2(2\pi)^n}\Big|\Big(\int_\Omega + \int_{\Omega^c}\Big)e^{it\phi(\xi)+ix\cdot\xi}\frac{\hat{f}_2}{\phi(\xi)}(\xi)d\xi\Big| \\ &\leq c|\check{k}_t(x) * f_2(x)|_\infty + c\Big(\int_{\Omega^c}(1+|\xi|^2)^{-\frac{3}{2}n}d\xi\Big)^{1/2}\|f_2\|_{3n/2} \\ &\leq c|\check{k}_t(x)|_\infty|f_2(x)|_1 + ct^{-1/4}\|f_2\|_{3n/2} \end{aligned} \tag{3.15}$$

where the function $k_t(\xi) = \chi_\Omega(\xi)e^{it\phi(\xi)}/\phi(\xi)$. On the other hand, with the same notations of Ω and $\mathcal{A}$ given in (3.5), (3.7), (3.9), we write

$$\check{k}_t(x) = \frac{1}{(2\pi)^n}\Big(\int_{\Omega\cap\mathcal{A}} + \int_{\Omega\cap\mathcal{A}^c}\Big)e^{it\phi(\xi)+ix\cdot\xi}\frac{1}{\phi(\xi)}d\xi = J_1 + J_2. \tag{3.16}$$

Then, with the use of the inequality (2.3), we follow the same lines as the estimation of I_1 in (3.8) to get

$$|J_1| \leq ct^{-1/4}. \tag{3.17}$$

For the estimation of J_2, we need the following inequality which with the use of (2.3), (2.11), (2.12), (3.11), (3.12), is obviously proved. That is: On

$$\begin{gathered} \Omega\cap\{E_1\cup E_2\}\cap\{|\xi-\xi_s| > t^{-1/4}\}\cap\{|\xi| > t^{-1/4}\}, \\ \frac{|D^2\Psi(\xi)|}{|\nabla\Psi(\xi)|^2|\phi(\xi)|} + \frac{|\nabla\phi(\xi)|}{|\nabla\Psi(\xi)||\phi(\xi)|^2} \leq ct^{1/2}|\xi|^{-2}(1+|\xi|^2). \end{gathered} \tag{3.18}$$

Therefore, following the same lines as in the proofs of (3.9) and (3.13), we find thanks to (3.18) and integration by parts (as for I_2),

$$\begin{aligned} |J_2| &= |\int_{\Omega\cap\mathcal{A}^c} e^{it\Psi(\xi)}\frac{1}{\phi(\xi)}d\xi| = t^{-1}|\int_{\Omega\cap\mathcal{A}^c}\frac{1}{\nabla\Psi(\xi)\phi(\xi)}\nabla(e^{it\Psi(\xi)})d\xi| \\ &\leq t^{-1}\{\int_{\Omega\cap\mathcal{A}^c}|\nabla(\frac{1}{\nabla\Psi(\xi)\phi(\xi)})|d\xi + \int_{\partial\{\Omega\cap\mathcal{A}^c\}}\frac{d\xi}{|\nabla\Psi(\xi)||\phi(\xi)|}\} \\ &\leq ct^{-1}\int_{\Omega\cap\mathcal{A}^c}|\nabla(\frac{1}{\nabla\Psi(\xi)\phi(\xi)})|d\xi \\ &\leq ct^{-1}\int_{\Omega\cap\mathcal{A}^c}\{\frac{|D^2\Psi(\xi)|}{|\nabla\Psi(\xi)|^2|\phi(\xi)|} + \frac{|\nabla\phi(\xi)|}{|\nabla\Psi(\xi)||\phi(\xi)|^2}\}d\xi \\ &\leq ct^{-1/2}\int_{\Omega\cap E_1\cap\{|\xi-\xi_s|>t^{-1/4}\}\cap\{|\xi|>t^{-1/4}\}}|\xi|^{-2}(1+|\xi|^2)d\xi \\ &\quad + ct^{-1/2}\int_{\Omega\cap E_2\cap\{|\xi-\xi_s|>t^{-1/4}\}\cap\{|\xi|>t^{-1/4}\}}|\xi|^{-2}(1+|\xi|^2)d\xi \\ &\leq ct^{-1/4}. \end{aligned} \tag{3.19}$$

Hence (3.16) and the estimates (3.17) and (3.19) on J_1 and J_2 above, give

$$|\check{k}_t(x)| \leq ct^{-1/4}\ \forall t \geq 1.$$

Then, this with (3.15) give

$$|S_2(t)f_2(x)| \leq c(1+t)^{-1/4}(|f_2|_1 + \|f_2\|_{3n/2}\ \forall t \geq 1,$$

Combining this inequality and (3.14), we get with Remark 2.2 the desired inequality (3.1) for the case $j = 2$. This finishes up the proof of inequality (3.4). In order to prove the inequality (3.2) of Lemma 3.1, we set $J^k = (1-\triangle)^{k/2}$ with $k \in \mathbb{R}$, and we note that

$$\begin{aligned} J^k S_1(t) f_1(x) &= J^k \mathcal{F}^{-1}(\frac{1}{2} e^{it\phi(\xi)} \hat{f}_1(\xi))(x) \\ &= \mathcal{F}^{-1}(\frac{1}{2}(1+|\xi|^2)^{k/2} e^{it\phi(\xi)} \hat{f}_1(\xi))(x) \\ &= \frac{1}{2(2\pi)^n} \int_{\mathbb{R}^n} (1+|\xi|^2)^{k/2} e^{it\phi(\xi)+ix\cdot\xi} \hat{f}_1(\xi) d\xi \end{aligned}$$

and

$$J^k S_2(t) f_2(x) = \frac{1}{2(2\pi)^n} \int_{\mathbb{R}^n} (1+|\xi|^2)^{k/2} e^{it\phi(\xi)+ix\cdot\xi} \frac{\hat{f}_2}{\phi(\xi)}(\xi) d\xi.$$

Henceforth, we can prove the inequality (3.2) of Lemma 3.1. We begin with the case $j = 1$: For $0 \le t \le 1$, we follow the same lines as in (3.5) and we get

$$|J^k S_1(t) f_1(x)| \le c(1+t)^{-1/4} \|f_1\|_{\frac{3}{2}n+k}. \tag{3.20}$$

If $t \ge 1$, let $p_t(\xi) = (1+|\xi|^2)^{-\frac{5}{4}n} \chi_\Omega(\xi) e^{it\phi(\xi)}$ where $\Omega = \{\xi \in \mathbb{R}^n, |\xi| \le t^{\frac{1}{4n}}\}$ is defined above in (3.6). Then thanks to the Schwartz and the Young inequalities, we have as in (3.6),

$$\begin{aligned} |J^k S_1(t) f_1(x)| &= \frac{1}{2(2\pi)^n} \Big| \Big(\int_\Omega + \int_{\Omega^c} \Big) e^{it\phi(\xi)+ix\cdot\xi} (1+|\xi|^2\Big)^{-\frac{k}{2}} \hat{f}_1(\xi) d\xi \Big| \\ &\le c|\check{p}_t(x) * (1-\triangle)^{\frac{(\frac{5}{2}n+k)}{2}} f_1(x)|_\infty \\ &\quad + c\Big(\int_{\Omega^c} (1+|\xi|^2)^{-\frac{3}{2}n} d\xi \Big)^{1/2} \Big(\int_{\Omega^c} (1+|\xi|^2)^{\frac{3}{2}n+k} |\hat{f}_1(\xi)|^2 d\xi \Big)^{1/2} \\ &\le c|\check{p}_t(x)|_\infty \|f_1(x)\|_{\frac{5}{2}n+k,1} + ct^{-1/4} \|f_1\|_{\frac{3}{2}n+k}. \end{aligned} \tag{3.21}$$

Then, with the same notation of $\mathcal{A}$ and Ω given in (3.7) above, we write:

$$\check{p}_t(x) = \frac{1}{(2\pi)^n} \Big(\int_{\Omega\cap\mathcal{A}} + \int_{\Omega\cap\mathcal{A}^c} \Big) (1+|\xi|^2)^{-\frac{5}{4}n} e^{it\phi(\xi)+ix\cdot\xi} d\xi = I_1' + I_2' \tag{3.22}$$

and following the same lines as in (3.8) we get

$$|I_1'| \le c \int_{\Omega\cap\mathcal{A}} (1+|\xi|^2)^{-\frac{5}{4}n} d\xi \le c \int_{\Omega\cap\mathcal{A}} d\xi \le ct^{-1/4}. \tag{3.23}$$

For the sequel, we need the following inequality which with the help of the inequalities (2.3), (2.10), (2.11), (3.11), (3.12), is easily proved. That is, for all $\xi \in \mathbb{R}^n$ and for any given $\gamma \ge 0$,

$$|\nabla(\frac{1}{\nabla\Psi(\xi)(1+|\xi|^2)^{\frac{\gamma}{2}}})| + |\nabla(\frac{1}{\phi(\xi)\nabla\Psi(\xi)(1+|\xi|^2)^{\frac{\gamma}{2}}})| \le ct^{1/2}|\xi|^{-2}(1+|\xi|^2). \tag{3.24}$$

Henceforth, thanks to the above inequality, we follow the same lines as in the proof of (3.19), and using integration by parts, we get

$$
\begin{aligned}
|I_2'| &= \frac{1}{2(2\pi)^n}\Big|\int_{\Omega\cap\mathcal{A}^c}\frac{e^{it\Psi(\xi)}}{(1+|\xi|^2)^{\frac{\gamma}{2}}}d\xi\Big| \\
&= \frac{1}{2(2\pi)^n}t^{-1}\Big|\int_{\Omega\cap\mathcal{A}^c}\frac{\nabla(e^{it\Psi(\xi)})}{\nabla\Psi(\xi)(1+|\xi|^2)^{\frac{\gamma}{2}}}d\xi\Big| \\
&\le ct^{-1}\int_{\Omega\cap\mathcal{A}^c}\Big|\nabla\Big(\frac{1}{\nabla\Psi(\xi)(1+|\xi|^2)^{\frac{\gamma}{2}}}\Big)\Big|d\xi \\
&\le ct^{-1/2}\int_{\Omega\cap E_1\cap\{|\xi-\xi_s|>t^{-1/4}\}\cap\{|\xi|>t^{-1/4}\}}|\xi|^{-2}(1+|\xi|^2)d\xi \\
&\quad + ct^{-1/2}\int_{\Omega\cap E_2\cap\{|\xi-\xi_s|>t^{-1/4}\}\cap\{|\xi|>t^{-1/4}\}}|\xi|^{-2}(1+|\xi|^2)d\xi \\
&\le ct^{-1/4}.
\end{aligned}
\tag{3.25}
$$

Therefore, (3.22) and the estimates in (3.23) and (3.25) of I_1' and I_2' above, give

$$|\check{p}_t(x)| \le ct^{-1/4} \quad t\ge 1.$$

so that thanks to (3.21) we get for all $t\ge 1$,

$$|J^kS_1(t)f_1(x)| \le c(1+t)^{-1/4}(\|f_1\|_{\frac{5}{2}n+k,1}+\|f_1\|_{\frac{3}{2}n+k}), \quad k\ge 0. \tag{3.26}$$

Finally, combining (3.20) and (3.26), and thanks to Remark 2.2, we find the case $j=1$ of the inequality (3.2) of Lemma 3.1. Likewise, following the same lines as in the proof of the case $j=1$ of (3.2), and with the use of the inequalities (2.3) and (3.24), we prove the case $j=2$ of the inequality (3.2). This, with Remark 2.2 puts an end of the proof of the inequality (3.2) and consequently of Lemma 3.1.

Let us give now the following lemma which will be useful for the $\mathbb{L}^p$–$\mathbb{L}^q$ estimates.

Lemma 3.4. *Let $f_1,f_2\in\mathbb{L}^1_{\frac{5}{2}n+k}(\mathbb{R}^n)$, $k\ge 0$, $n\ge 1$. Then*

$$\|V_j(t)f_j(x)\|_{k,\infty} \le c(1+t)^{-1/4}\|f_j\|_{\frac{5}{2}n+k,1}, \quad j=1,2. \tag{3.27}$$

Proof. Thanks to inequality (3.2) of Lemma 3.1, it suffices to use the Sobolev embedding $W^{n,1}(\mathbb{R}^n)\subset\mathbb{L}^2(\mathbb{R}^n)$ and we get

$$
\begin{aligned}
\|V_j(t)f_j(x)\|_{k,\infty} &\le c(1+t)^{-1/4}(\|f_j\|_{\frac{5}{2}n+k,1}+\|f_j\|_{\frac{3}{2}n+k}) \\
&= c(1+t)^{-1/4}(\|f_j\|_{\frac{5}{2}n+k,1}+|J^{\frac{3}{2}n+k}f_j|_2) \\
&\le c(1+t)^{-1/4}(\|f_j\|_{\frac{5}{2}n+k,1}+\|J^{\frac{3}{2}n+k}f_j\|_{n,1}) \\
&= 2c(1+t)^{-1/4}\|f_j\|_{\frac{5}{2}n+k,1}, \quad j=1,2.
\end{aligned}
$$

□

To end with this section, we give the following lemma.

Lemma 3.5. *Let $f_1,f_2\in\mathbb{H}^{k+\frac{5}{2}n+1}(\mathbb{R}^n)\cap\mathbb{L}^q_{k+\frac{5}{2}n}(\mathbb{R}^n)$, $k\ge 0$, $n\ge 1$. Then*

$$\|V_j(t)f_j(x)\|_{k,p} \le c(1+t)^{-\frac{\theta}{4}}\|f_j\|_{\frac{5}{2}n+k,q}, \quad j=1,2 \tag{3.28}$$

where $p=2/(1-\theta)$, $q=2/(1+\theta)$, $\theta\in]0,1[$.

Proof. Thanks to (2.4) and (2.5), we get for any $k \in \mathbb{R}_+$,

$$\|V_j(t)f_j(x)\|_k \le c\|f_j(x)\|_k \le c\|f_j\|_{\frac{5}{2}n+k}, \quad j = 1, 2; \tag{3.29}$$

that is

$$|J^k V_j(t)f_j(x)|_2 \le c|J^{\frac{5}{2}n+k} f_j(x)|_2, \quad j = 1, 2. \tag{3.30}$$

Moreover, from (3.27) in Lemma 3.4 we have

$$|J^k V_j(t)f_j(x)|_\infty \le c(1+t)^{-1/4}|J^{\frac{5}{2}n+k} f_j(x)|_1, \quad j = 1, 2. \tag{3.31}$$

We know that (see above),

$$J^k V_j(t)(f_j(x)) = V_j(t)(J^k f_j(x)) = J^{-\frac{5}{2}n} V_j(t)(J^{\frac{5}{2}n+k} f_j(x)).$$

Therefore, thanks to (3.30) and (3.31), we apply the interpolation theorem (see [1]) for the evolution operator $J^{-\frac{5}{2}n}V_j(t)$, $j = 1, 2$, and we find the inequality (3.28) of Lemma 3.5. This finishes up the proof of Lemma 3.5. □

4. Decay and Scattering results of Solutions to the Nonlinear Equation

Proof of Theorem Theorem 1.1. We write (1.1) in its integral form as given in (2.6):

$$u(x,t) = V_1(t)f_1(x) + V_2(t)f_2(x) + \int_0^t V_2(t-\tau)(|u|^\alpha u)(\tau)d\tau. \tag{4.1}$$

where $V_1(t)$ and $V_2(t)$ are defined in (2.3), (2.4). Then, taking the $\mathbb{L}^\infty$ norm of the both sides of (4.1) we get thanks to Lemma 3.1,

$$\begin{aligned} |u(t)|_\infty &\le c(1+t)^{-1/4}(|f_1|_1 + \|f_1\|_{3n/2} + |f_2|_1 + \|f_2\|_{3n/2} \\ &\quad + c\int_0^t (1+(t-\tau))^{-1/4}(||u|^\alpha u|_1 + \||u|^\alpha u\|_{3n/2}(\tau)d\tau \\ &\le c(1+t)^{-1/4}(|f_1|_1 + \|f_1\|_{3n/2} + |f_2|_1 + \|f_2\|_{3n/2} \\ &\quad + c\int_0^t (1+(t-\tau))^{-1/4}(|u|_\infty^{\alpha-1}|u|_2^2 + |u|_\infty^\alpha \|u\|_{3n/2}(\tau)d\tau. \end{aligned} \tag{4.2}$$

Then, we define the quantity

$$Q(t) = \sup_{0\le\tau\le t}\{(1+\tau)^{\frac{1}{4}}|u(\tau)|_\infty + \|u(\tau)\|_{3n/2}\}.$$

From (4.2)

$$\begin{aligned} |u(t)|_\infty &\le c(1+t)^{-1/4}(|f_1|_1 + \|f_1\|_{3n/2} + |f_2|_1 + \|f_2\|_{3n/2} \\ &\quad + cQ(t)^{\alpha+1}\int_0^t (1+(t-\tau))^{-1/4}(1+\tau)^{-\frac{1}{4}(\alpha-1)}d\tau. \end{aligned} \tag{4.3}$$

But since for $\alpha > 5$,

$$\begin{aligned} &\int_0^t (1+(t-\tau))^{-1/4}(1+\tau)^{-\frac{1}{4}(\alpha-1)}d\tau \\ &= (\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t)(1+(t-\tau))^{-1/4}(1+\tau)^{-\frac{1}{4}(\alpha-1)}d\tau \le c(1+t)^{-1/4} \end{aligned}$$

we deduce from (4.3) that for $\alpha > 5$,

$$(1+t)^{\frac{1}{4}}|u(t)|_\infty \le c\{|f_1|_1 + \|f_1\|_{3n/2} + |f_2|_1 + \|f_2\|_{3n/2} + Q(t)^{\alpha+1}\} \tag{4.4}$$

Furthermore, we get for $\alpha > 5$, thanks to the inequalities (2.4) and (2.5) of Lemma 2.4 and with (4.1),

$$\begin{aligned} \|u(t)\|_{3n/2} &\le c\{\|f_1\|_{3n/2} + \|f_2\|_{3n/2} + \int_0^t \||u|^\alpha u\|_{3n/2}(\tau)d\tau\} \\ &\le c\{\|f_1\|_{3n/2} + \|f_2\|_{3n/2} + \int_0^t |u|_\infty^\alpha \|u\|_{3n/2}(\tau)d\tau\} \\ &\le c\{\|f_1\|_{3n/2} + \|f_2\|_{3n/2} + Q(t)^{\alpha+1} \int_0^t (1+\tau)^{-\frac{\alpha}{4}} d\tau\} \\ &\le c\{\|f_1\|_{3n/2} + \|f_2\|_{3n/2} + Q(t)^{\alpha+1}\}. \end{aligned} \tag{4.5}$$

Therefore, (4.4) and (4.5) give

$$Q(t) \le c\{|f_1|_1 + \|f_1\|_{3n/2} + |f_2|_1 + \|f_2\|_{3n/2} + Q(t)^{\alpha+1}\}. \tag{4.6}$$

Henceforth, thanks to the inequality (4.6), if $|f_1|_1 + \|f_1\|_{3n/2} + |f_2|_1 + \|f_2\|_{3n/2} < \delta$ with $\delta > 0$ small enough, we find that $Q(t)$ is bounded. Indeed, it is well known that inequality (4.6) is satisfied if $Q(t) \in [0, \beta_1] \cup [\beta_2, \infty[$ with $0 < \beta_1 < \beta_2 < \infty$ since δ is small. Thereby, since $Q(0) \le 2\|f_1\|_{3n/2} < 2\delta$ (because $\mathbb{H}^{\frac{3}{2}n}(\mathbb{R}^n) \subset \mathbb{L}^\infty(\mathbb{R}^n)$), the continuity of $Q(t)$ and the inequality (4.6) allow us to conclude that $Q(t)$ remains bounded for all $t \ge 0$. Thus, we have obtained a bound of $Q(t)$ and consequently an a-priori estimate of the local solution which permit us to extend globally the local solution of Theorem 2.1. Moreover, this a-priori estimate provides the inequality (1.2) of Theorem 1.1. For the proof of the scattering result in the Theorem 1.1, we define

$$u_+(x,t) = u(x,t) + \int_t^{+\infty} V_2(t-\tau)(|u|^\alpha u)(\tau)d\tau \tag{4.7}$$

where $u(x,t)$ is the solution of (1.1) given by Theorem 1.1. We only consider the case of u_+ $(t \to +\infty)$ since the proof for the case of u_- $(t \to -\infty)$ is similar. Then, thanks to (4.7) and with the use of the inequalities (2.5) of Lemma 2.4 and (1.2) of Theorem 1.1, we have,

$$\begin{aligned} \|u(t) - u_+(t)\|_{2,2} &\le c \int_t^{+\infty} |(|u|^\alpha u)(\tau)|_2 d\tau \\ &\le c \int_0^t |u(\tau)|_\infty^\alpha |u(\tau)|_2 d\tau \\ &\le c \int_t^{+\infty} (1+\tau)^{-\frac{\alpha}{4}} d\tau \end{aligned}$$

and the integral on the right-hand side approaches to zero as $t \to +\infty$, since by hypothesis of Theorem 1.1, $\alpha > 5$.

Thereafter, set $g_+(x) = f_2(x) + \int_0^{+\infty} V_2(-\tau)(|u|^\alpha u)(\tau)d\tau$. Then thanks to (4.7) and (4.1), we may write u_+ as

$$u_+(x,t) = V_1(t)f(x) + V_2(t)g_+(x). \tag{4.8}$$

Therefore, we can see that $u_+(t)$ is a solution of the linearized equation (2.1). This completes the proof of Theorem 1.1. $\square$

Proof of Theorem 1.2. To prove Theorem 1.2, we need the following inequality of Gagliardo-Nirenberg type.

Lemma 4.1. *Let u belong to $\mathbb{L}^{p_2}(\mathbb{R}^n)$ and its derivatives of order m, $D^m u$ belong to $\mathbb{L}^{r_1}(\mathbb{R}^n)$, $1 \le p_2,\ r_1 \le \infty$. For the derivatives $D^j u$ $0 \le j < m$, the following inequalities hold:*

$$|D^j u|_{p_1} \le c|D^m u|_{r_1}^a |u|_{p_2}^{1-a},$$

where

$$\frac{1}{p_1} = \frac{j}{n} + a(\frac{1}{r_1} - \frac{m}{n}) + (1-a)\frac{1}{p_2},$$

for all a in the interval $\frac{j}{m} \le a \le 1$.

The proof of the above lemma can be found in [7].

Now, we prove Theorem 1.2. We recall the notation $p = 2/(1-\theta)$, $q = 1/(1+\theta)$, $\theta \in]0,1[$. Let $r > \frac{n}{p}$ and apply the norm $\mathbb{L}_r^p(\mathbb{R}^n)$ to the two sides of (4.1). Then, thanks to Lemma 3.5, the Gagliardo-Nirenberg inequality in Lemma 4.1 and Sobolev imbeddings theorems,

$$\begin{aligned}
&\|u(x,t)\|_{r,p} \\
&\le \|V_1(t)f(x)\|_{r,p} + \|V_2(t)g(x)\|_{r,p} + \int_0^t \|V_2(t-\tau)(|u|^{\alpha-1}u)(\tau)\|_{r,p} d\tau \\
&\le c(1+t)^{-\frac{\theta}{4}}(\|f_1\|_{\frac{5}{2}n+r,q} + \|f_2\|_{\frac{5}{2}n+r,q}) + c\int_0^t (1+(t-\tau))^{-\frac{\theta}{4}} \||u|^\alpha u(\tau)\|_{\frac{5}{2}n+r,q} d\tau \\
&\le c(1+t)^{-\frac{\theta}{4}}(\|f_1\|_{\frac{5}{2}n+r,q} + \|f_2\|_{\frac{5}{2}n+r,q}) \\
&\quad + c\int_0^t (1+(t-\tau))^{-\frac{\theta}{4}} \||u|^\alpha u(\tau)\|^a_{\frac{5}{2}n+r+1,2} ||u|^\alpha u(\tau)|_1^{1-a} d\tau \\
&\le c(1+t)^{-\frac{\theta}{4}}(\|f_1\|_{\frac{5}{2}n+r,q} + \|f_2\|_{\frac{5}{2}n+r,q}) \\
&\quad + c\int_0^t (1+(t-\tau))^{-\frac{\theta}{4}} (|u|_\infty^\alpha \|u\|_{\frac{5}{2}n+r+1})^a (|u|_\infty^{\alpha-1}|u|_2^2)^{1-a} d\tau
\end{aligned} \tag{4.9}$$

where

$$a = \frac{(\frac{5}{2}n+r)/n + (1-\theta)/2}{(\frac{5}{2}n+r+1)/n + 1/2} = 1 - \frac{1/n + \theta/2}{(\frac{5}{2}n+r+1)/n+1/2}.$$

Set

$$K(t) = \sup_{0\le\tau\le t} \{(1+\tau)^{\frac{\theta}{4}} \|u(\tau)\|_{r,p} + \|u(\tau)\|_{\frac{5}{2}n+r+1}\}.$$

Hence, since (by hypothesis above) $r > n/p$, then thanks to the Sobolev imbedding theorem $\mathbb{L}_r^p(\mathbb{R}^n) \subset \mathbb{L}^\infty(\mathbb{R}^n)$ and with (4.9), we get for $\alpha > 1 + 4/\theta$,

$$\begin{aligned}
&\|u(x,t)\|_{r,p} \\
&\le c(1+t)^{-\frac{\theta}{4}}(\|f_1\|_{\frac{5}{2}n+r,q} + \|f_2\|_{\frac{5}{2}n+r,q}) \\
&\quad + c\int_0^t (1+(t-\tau))^{-\frac{\theta}{4}} (\|u\|_{r,p}^\alpha \|u\|_{\frac{5}{2}n+r+1})^a (\|u\|_{r,p}^{\alpha-1}\|u\|^2_{\frac{5}{2}n+r+1})^{1-a} d\tau. \\
&\le c(1+t)^{-\frac{\theta}{4}}(\|f_1\|_{\frac{5}{2}n+r,q} + \|f_2\|_{\frac{5}{2}n+r,q}) \\
&\quad + c\int_0^t (1+(t-\tau))^{-\frac{\theta}{4}} (K(t)^{\alpha+1}(1+\tau)^{-\alpha\frac{\theta}{4}})^a (K(t)^{\alpha+1}(1+\tau)^{-(\alpha-1)\frac{\theta}{4}})^{1-a} d\tau. \\
&\le c(1+t)^{-\frac{\theta}{4}}(\|f_1\|_{\frac{5}{2}n+r,q} + \|f_2\|_{\frac{5}{2}n+r,q})
\end{aligned}$$

$$
\begin{aligned}
&+ K(t)^{\alpha+1} \int_0^t (1+(t-\tau))^{-\frac{\theta}{4}} (1+\tau)^{-(\alpha-1)\frac{\theta}{4}} d\tau \\
&\le c(1+t)^{-\frac{\theta}{4}} \{\|f_1\|_{\frac{5}{2}n+r,q} + \|f_2\|_{\frac{5}{2}n+r,q} + K(t)^{\alpha+1}\}.
\end{aligned}
$$

We deduce from the above inequality that for $\alpha > 1 + 4/\theta$,

$$
(1+t)^{\frac{\theta}{4}} \|u(x,t)\|_{r,p} \le c\{\|f_1\|_{\frac{5}{2}n+r,q} + \|f_2\|_{\frac{5}{2}n+r,q} + K(t)^{\alpha+1}\}. \tag{4.10}
$$

Furthermore, thanks to the inequalities (2.4), (2.5) of Lemma 2.4 and following the same lines as in the proof of the inequality (4.5), we find with (4.1) and for $\alpha > 1 + 4/\theta$

$$
\|u(x,t)\|_{\frac{5}{2}n+r+1} \le c\{\|f_1\|_{\frac{5}{2}n+r+1} + \|f_2\|_{\frac{5}{2}n+r+1} + K(t)^{\alpha+1}\}. \tag{4.11}
$$

Then the combination of (4.10) and (4.11) leads to the inequality

$$
K(t) \le c\{\|f_1\|_{\frac{5}{2}n+r,q} + \|f_2\|_{\frac{5}{2}n+r,q} + \|f_1\|_{\frac{5}{2}n+r+1} + \|f_2\|_{\frac{5}{2}n+r+1} + K(t)^{\alpha+1}\}. \tag{4.12}
$$

Therefore, as above, we find that if

$$
\|f_1\|_{\frac{5}{2}n+r,q} + \|f_2\|_{\frac{5}{2}n+r,q} + \|f_1\|_{\frac{5}{2}n+r+1} + \|f_2\|_{\frac{5}{2}n+r+1}
$$

is sufficiently small, then the inequality (4.12) gives $K(t) \le c$ for all $t \ge 0$. This implies that $\|u(x,t)\|_{r,p} \le c(1+t)^{-\frac{\theta}{4}}$ for all $t \ge 0$, and Theorem 1.2 is proven. $\square$

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