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POSITIVE SOLUTIONS AND EIGENVALUES OF NONLOCAL BOUNDARY-VALUE PROBLEMS

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Abstract. We study the ordinary differential equation $x'' + \lambda a(t)f(x) = 0$ with the boundary conditions $x(0) = 0$ and $x'(1) = \int_\eta^1 x'(s)dg(s)$. We characterize values of λ for which boundary-value problem has a positive solution. Also we find appropriate intervals for λ so that there are two positive solutions.

1. Introduction

This paper concerns the ordinary differential equation

$$x'' + \lambda a(t)f(x) = 0, \quad \text{a.e. } t \in [0,1] \tag{1.1}$$

with the boundary conditions

$$x(0) = 0 \tag{1.2}$$

$$x'(1) = \int_\eta^1 x'(s)dg(s), \tag{1.3}$$

where $\lambda > 0$, $\eta \in (0,1)$ and the integral in (1.3) is meant in the sense of Riemann-Stieljes. In this paper it is assumed that

(H1) The function $f : [0,\infty) \to [0,\infty)$ is continuous.
(H2) The function $a : [0,1] \to [0,\infty)$ is continuous and does not vanish identically on any subinterval.
(H3) The function $g : [0,1] \to \mathbb{R}$ is increasing and such that $g(\eta) = 0 < g(\eta^+)$ and $g(1) < 1$.

In recent years, nonlocal boundary-value problems of this form have been studied extensively in the literature [6, 7, 8, 9, 10]. This class of problems includes, as special cases, multi-point boundary-value problems considered by many authors (see [4, 12] and the references therein). In fact, condition (1.2)-(1.3) is the continuous version of the multi-point condition

$$x(0) = 0, \quad x'(1) = \sum_{i=1}^m \alpha_i x'(\xi_i) \tag{1.4}$$

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which happens when g is a piece-wise constant function that is increasing and has finitely many jumps, where $\alpha_1,\ \alpha_2,\dots\alpha_m \in \mathbb{R}$ have the same sign, $m \geq 1$ is an integer, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$.

In the sequel, in this paper we shall denote by $\mathbb{R}$ the real line and by I the interval [0,1], $C(I)$ will denote the space of all continuous functions $x : I \to \mathbb{R}$. Let

$$C_0^1(I) = \{x \in C(I) : x' \text{ is absolutely continuous on } I \text{ and } x(0) = 0\}.$$

Then $C_0^1(I)$ is a Banach space when it is furnished with the super-norm $\|x\| = \sup_{t\in I} |x(t)|$.

By a solution x of (1.1)-(1.3) we mean $x \in C_0^1(I)$ satisfying equation (1.1) for almost all $t \in I$ and condition (1.3). By a positive solution x of (1.1)-(1.3) if x is nonnegative and is not identically zero on I. If, for a particular λ, the boundary-value problem (1.1)-(1.3) has a positive solution x, then λ is called an eigenvalue and x a corresponding eigenfunction. Recently, several eigenvalue characterizations for kinds of boundary-value problems have been carried out, for this we refer to [1, 2, 3, 5, 14, 15].

In this paper, we will use the notation

$$f_0 = \lim_{x\to 0^+} \frac{f(x)}{x}, \quad f_\infty = \lim_{x\to\infty} \frac{f(x)}{x}.$$

This paper is organized as follows. In section 2, we will present some preliminary results, including a fixed point theorem due to Krasnosel'skii [11], which is the basic tool used in this paper. We shall establish the eigenvalue intervals in terms of f_0 and f_∞ in section 3. The investigation of the existence of double positive solutions is carried out in section 4.

2. Preliminaries

First, we present a fixed point theorem in cones due to Krasnosel'skii, which can be found in [11].

Theorem 2.1. *Let X be a Banach space and K $(\subset X)$ be a cone. Assume that $\Omega_1,\ \Omega_2$ are open subsets of X with $0 \in \Omega_1,\ \bar{\Omega}_1 \subset \Omega_2$, and let*

$$T : K \cap (\bar{\Omega}_2 \backslash \Omega_1) \to K$$

be a continuous and compact operator such that either

(i) $\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$; *or*
(ii) $\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

We will apply Theorem 2.1 to find positive solutions to boundary-value problem (1.1)-(1.3). To do so, we need to re-formulate the problem as an operator equation of the form $x = T_\lambda x$, for an appropriate operator T_λ. In fact, following from [7], we have:

Lemma 2.2. *A function $x \in C_0^1(I)$ is a solution of the boundary-value problem* (1.1)-(1.3) *if and only if x is a solution of the operator equation $x = T_\lambda x$, where T_λ is defined by*

$$(T_\lambda x)(t) = \frac{\lambda t}{1 - g(1)} \int_\eta^1 \int_s^1 a(r) f(x(r)) dr dg(s) + \lambda \int_0^t \int_s^1 a(r) f(x(r)) dr\, ds\,. \quad (2.1)$$

In order to apply Theorem 2.1, we define

$$K = \{x \in C_0^1(I) : x(t) \geq 0,\ x'(t) \geq 0 \text{ and } x \text{ is concave}\}.$$

One may readily verify that K is a cone in $C_0^1(I)$. Moreover, we have the following elementary fact.

Lemma 2.3. *If $x \in K$, then, for any $\tau \in [0,1]$ it holds $x(t) \geq \tau\|x\|$, $t \in [\tau, 1]$.*

Theorem 2.4. *Assume that* (H1)-(H3) *hold, then $T_\lambda(K) \subseteq K$ and T_λ is continuous and completely continuous.*

3. Eigenvalue intervals

For the sake of simplicity, let

$$A = \frac{1}{1-g(1)} \int_\eta^1 \int_s^1 a(r)drdg(s) + \int_0^1 \int_s^1 a(r)\,dr\,ds \tag{3.1}$$

$$B = \frac{1}{1-g(1)} \int_\eta^1 \int_s^1 a(r)drdg(s) + \int_\eta^1 \int_s^1 a(r)\,dr\,ds. \tag{3.2}$$

Theorem 3.1. *Suppose that* (H1)-(H3) *hold, then the boundary-value problem* (1.1)-(1.3) *has at least one positive solution for each*

$$\lambda \in (1/\eta f_\infty B, 1/f_0 A). \tag{3.3}$$

Proof. We construct the sets Ω_1 and Ω_2 in order to apply Theorem 2.1. Let λ be given as in (3.3) and choose $\varepsilon > 0$ such that

$$\frac{1}{\eta(f_\infty - \varepsilon)B} \leq \lambda \leq \frac{1}{(f_0+\varepsilon)A}.$$

First, there exists $r > 0$ such that

$$f(x) \leq (f_0+\varepsilon)x, \quad 0 < x \leq r.$$

So, for any $x \in K$ with $\|x\| = r$, we have

$$\begin{aligned}
&(T_\lambda x)(t)\\
&\leq \frac{\lambda}{1-g(1)} \int_\eta^1 \int_s^1 a(r)f(x(r))dr\,dg(s) + \lambda \int_0^1 \int_s^1 a(r)f(x(r))\,dr\,ds\\
&\leq \frac{\lambda}{1-g(1)} \int_\eta^1 \int_s^1 a(r)(f_0+\varepsilon)x(r)dr\,dg(s) + \lambda \int_0^1 \int_s^1 a(r)(f_0+\varepsilon)x(r)\,dr\,ds\\
&\leq \lambda(f_0+\varepsilon)r\{\frac{1}{1-g(1)} \int_\eta^1 \int_s^1 a(r)dr\,dg(s) + \int_0^1 \int_s^1 a(r)\,dr\,ds\}\\
&\leq \lambda(f_0+\varepsilon)Ar \leq r = \|x\|.
\end{aligned}$$

Consequently, $\|T_\lambda x\| \leq \|x\|$. So, if we set $\Omega_1 = \{x \in K : \|x\| < r\}$, then

$$\|T_\lambda x\| \leq \|x\|, \quad \forall x \in K \cap \partial\Omega_1. \tag{3.4}$$

Next, we choose R_1 such that

$$f(x) \geq (f_\infty - \varepsilon)x, \quad x \geq R_1.$$

Let $R = \max\{2r, \eta^{-1}R_1\}$ and set

$$\Omega_2 = \{x \in K : \|x\| < R\}.$$

If $x \in K$ with $\|x\| = R$, then

$$\min_{t\in[\eta,1]} x(t) \geq \eta\|x\| \geq R_1.$$

Thus, we have

$$\begin{aligned}
&(T_\lambda x)(1)\\
&= \frac{\lambda}{1-g(1)} \int_\eta^1 \int_s^1 a(r)f(x(r))dr\, dg(s) + \lambda \int_0^1 \int_s^1 a(r)f(x(r))\, dr\, ds\\
&\geq \frac{\lambda}{1-g(1)} \int_\eta^1 \int_s^1 a(r)f(x(r))dr\, dg(s) + \lambda \int_\eta^1 \int_s^1 a(r)f(x(r))\, dr\, ds\\
&\geq \frac{\lambda}{1-g(1)} \int_\eta^1 \int_s^1 a(r)(f_\infty - \varepsilon)x(r)dr\, dg(s) + \lambda \int_\eta^1 \int_s^1 a(r)(f_\infty - \varepsilon)x(r)\, dr\, ds\\
&\geq \lambda(f_\infty - \varepsilon)\eta\|x\|\{\frac{1}{1-g(1)} \int_\eta^1 \int_s^1 a(r)dr\, dg(s) + \int_\eta^1 \int_s^1 a(r)\, dr\, ds\}\\
&= \lambda(f_\infty - \varepsilon)B\eta R \geq R = \|x\|.
\end{aligned}$$

Hence,

$$\|T_\lambda x\| \geq \|x\|, \quad \forall x \in K \cap \partial\Omega_2.$$

From this inequality, (3.4), and Theorem 2.1 it follows that T_λ has a fixed point $x \in K \cap (\bar{\Omega}_2 \backslash \Omega_1)$ with $r \leq \|x\| \leq R$. Clearly, this x is a positive solution of (1.1)-(1.3). □

Theorem 3.2. *Suppose that* (H1)-(H3) *hold, then the boundary-value problem* (1.1)-(1.3) *has at least one positive solution for each*

$$\lambda \in (1/\eta f_0 B, 1/f_\infty A). \tag{3.5}$$

Proof. We construct the sets Ω_1 and Ω_2 in order to apply Theorem 2.1. Let λ be given as in (3.5) and choose $\varepsilon > 0$ such that

$$\frac{1}{\eta(f_0 - \varepsilon)B} \leq \lambda \leq \frac{1}{(f_\infty + \varepsilon)A}.$$

First, there exists $r > 0$ such that

$$f(x) \geq (f_0 - \varepsilon)x, \quad 0 < x \leq r.$$

So, for any $x \in K$ with $\|x\| = r$, we have

$$\begin{aligned}
&(T_\lambda x)(1)\\
&\geq \frac{\lambda}{1-g(1)} \int_\eta^1 \int_s^1 a(r)f(x(r))drdg(s) + \lambda \int_\eta^1 \int_s^1 a(r)f(x(r))\, dr\, ds\\
&\geq \frac{\lambda}{1-g(1)} \int_\eta^1 \int_s^1 a(r)(f_0 - \varepsilon)x(r)drdg(s) + \lambda \int_\eta^1 \int_s^1 a(r)(f_0 - \varepsilon)x(r)\, dr\, ds\\
&\geq \lambda(f_0 - \varepsilon)\eta r\{\frac{1}{1-g(1)} \int_\eta^1 \int_s^1 a(r)drdg(s) + \int_\eta^1 \int_s^1 a(r)\, dr\, ds\}\\
&\geq \lambda(f_0 - \varepsilon)B\eta r \geq r = \|x\|.
\end{aligned}$$

Consequently, $\|T_\lambda x\| \geq \|x\|$. So, if we set $\Omega_1 = \{x \in K : \|x\| < r\}$, then

$$\|T_\lambda x\| \geq \|x\|, \quad \forall x \in K \cap \partial\Omega_1. \tag{3.6}$$

Next, we can choose R_1 such that

$$f(x) \le (f_\infty + \varepsilon)x,\ \ x \ge R_1.$$

Here are two cases to be considered, namely, where f is bounded and where f is unbounded.

Case 1: f is bounded. Then, there exists some constant $M > 0$ such that $f(x) \le M,\ \ x \in (0, \infty)$. Let $R = \max\{2r, \lambda MA\}$ and set

$$\Omega_2 = \{x \in K : \|x\| < R\}.$$

Then, for any $x \in K$ with $\|x\| = R$, we have

$$\begin{aligned}
(T_\lambda x)(t) &\le \frac{\lambda}{1-g(1)} \int_\eta^1 \int_s^1 a(r) f(x(r)) dr dg(s) + \lambda \int_0^1 \int_s^1 a(r) f(x(r))\, dr\, ds \\
&\le \lambda M \{\frac{1}{1-g(1)} \int_\eta^1 \int_s^1 a(r) dr dg(s) + \int_0^1 \int_s^1 a(r)\, dr\, ds\} \\
&\le \lambda MA \le R = \|x\|.
\end{aligned}$$

Hence,

$$\|T_\lambda x\| \le \|x\|,\ \ \forall x \in K \cap \partial\Omega_2. \tag{3.7}$$

Case 2: f is unbounded. Then, there exists $R > \max\{2r, R_1\}$ such that

$$f(x) \le f(R),\ \ 0 < x \le R.$$

For $x \in K$ with $\|x\| = R$, we have

$$\begin{aligned}
&(T_\lambda x)(t) \\
&\le \frac{\lambda}{1-g(1)} \int_\eta^1 \int_s^1 a(r) f(x(r)) dr dg(s) + \lambda \int_0^1 \int_s^1 a(r) f(x(r))\, dr\, ds \\
&\le \frac{\lambda}{1-g(1)} \int_\eta^1 \int_s^1 a(r) f(R) dr dg(s) + \lambda \int_0^1 \int_s^1 a(r) f(R)\, dr\, ds \\
&\le \frac{\lambda}{1-g(1)} \int_\eta^1 \int_s^1 a(r)(f_\infty + \varepsilon) R dr\, dg(s) + \lambda \int_0^1 \int_s^1 a(r)(f_\infty + \varepsilon) R\, dr\, ds \\
&= \lambda(f_\infty + \varepsilon) RA \le R = \|x\|.
\end{aligned}$$

Then (3.7) is also true in this case.

Now (3.6), (3.7), and Theorem 2.1 guarantee that T_λ has a fixed point $x \in K \cap (\bar{\Omega}_2 \backslash \Omega_1)$ with $r \le \|x\| \le R$. Clearly, this x is a positive solution of (1.1)-(1.3). □

Example. Let the function $f(x)$ in (1.1) be

$$f(x) = x^\alpha + x^\beta, \tag{3.8}$$

then problem (1.1)-(1.3) has at least one positive solution for all $\lambda \in (0, \infty)$ if $0 < \alpha < 1, 0 < \beta < 1$ or $\alpha > 1, \beta > 1$.

Proof. It is easy to see that $f_0 = \infty,\ f_\infty = 0$ if $0 < \alpha < 1,\ 0 < \beta < 1$ and $f_0 = 0, f_\infty = \infty$ if $\alpha > 1, \beta > 1$. Then the results can be easily obtained by using Theorem 3.1 or Theorem 3.2 directly. □

4. Twin positive solutions

In this section, we establish the existence of two positive solutions to problem (1.1)-(1.3).

Theorem 4.1. *Suppose that* (H1)-(H3) *hold. In addition, assume there exist two constants $R > r > 0$ such that*

$$\max_{0\le x\le r} f(x) \le r/\lambda A, \quad \min_{\eta R\le x\le R} f(x) \ge R/\lambda B. \tag{4.1}$$

Then the boundary-value problem (1.1)-(1.3) *has at least one positive solution $x \in K$ with $r \le \|x\| \le R$.*

Proof. For $x \in \partial K_r = \{x \in K : \|x\| = r\}$, we have $f(x(t)) \le r/\lambda A$ for $t \in [0,1]$. Then we have

$$\begin{aligned}
(T_\lambda x)(t) &\le \frac{\lambda}{1-g(1)} \int_\eta^1 \int_s^1 a(r) f(x(r)) dr\, dg(s) + \lambda \int_0^1 \int_s^1 a(r) f(x(r))\, dr\, ds \\
&\le \frac{\lambda}{1-g(1)} \frac{r}{\lambda A} \int_\eta^1 \int_s^1 a(r) dr\, dg(s) + \lambda \frac{r}{\lambda A} \int_0^1 \int_s^1 a(r)\, dr\, ds = r\,.
\end{aligned}$$

As a result, $\|T_\lambda x\| \le \|x\|, \forall x \in \partial K_r$. For $x \in \partial K_R$, we have $f(x(t)) \ge R/\lambda B$ for $t \in [\eta, 1]$. Then we have

$$\begin{aligned}
(T_\lambda x)(1) &\ge \frac{\lambda}{1-g(1)} \int_\eta^1 \int_s^1 a(r) f(x(r)) dr dg(s) + \lambda \int_\eta^1 \int_s^1 a(r) f(x(r))\, dr\, ds \\
&\ge \frac{\lambda}{1-g(1)} \frac{R}{\lambda B} \int_\eta^1 \int_s^1 a(r) dr dg(s) + \lambda \frac{R}{\lambda B} \int_\eta^1 \int_s^1 a(r)\, dr\, ds = R.
\end{aligned}$$

As a result, $\|T_\lambda x\| \ge \|x\|$, for all $x \in \partial K_R$. Then we can obtain the result by using Theorem 2.1. □

Remark 4.2. In Theorem 4.1, if condition (4.1) is replaced by

$$\max_{0\le x\le R} f(x) \le R/\lambda A, \quad \min_{\eta r\le x\le r} f(x) \ge r/\lambda B.$$

Then (1.1) has also a solution $x \in K$ with $r \le \|x\| \le R$.

For the remainder of this section, we need the following condition:

(H4) $\sup_{r>0} \min_{\eta r\le x\le r} f(x) > 0$.

Let

$$\lambda^* = \sup_{r>0} \frac{r}{A \max_{0\le x\le r} f(x)}, \quad \lambda^{**} = \inf_{r>0} \frac{r}{B \min_{\eta r\le x\le r} f(x)}.$$

We can easily obtain that $0 < \lambda^* \le \infty$ and $0 \le \lambda^{**} < \infty$ by using (H1) and (H4).

Theorem 4.3. *Suppose that* (H1)-(H4) *hold. In addition, assume that $f_0 = \infty$ and $f_\infty = \infty$. Then the boundary-value problem* (1.1)-(1.3) *has at least two positive solutions for any $\lambda \in (0, \lambda^*)$.*

Proof. Define

$$h(r) = \frac{r}{A \max_{0\le x\le r} f(x)}.$$

Using the condition (H1), $f_0 = \infty$ and $f_\infty = \infty$, we can easily obtain that $h : (0,\infty) \to (0,\infty)$ is continuous and

$$\lim_{r\to 0} h(r) = \lim_{r\to\infty} h(r) = 0.$$

So there exists $r_0 \in (0,\infty)$ such that $h(r_0) = \sup_{r>0} h(r) = \lambda^*$. For $\lambda \in (0,\lambda^*)$, there exist two constants $r_1, r_2 (0 < r_1 < r_0 < r_2 < \infty)$ with $h(r_1) = h(r_2) = \lambda$. Thus

$$f(x) \le r_1/\lambda A, \quad 0 \le x \le r_1, \tag{4.2}$$

$$f(x) \le r_2/\lambda A, \quad 0 \le x \le r_2. \tag{4.3}$$

On the other hand, by using the condition $f_0 = \infty$ and $f_\infty = \infty$, there exist two constants $r_3, r_4 (0 < r_3 < r_1 < r_2 < \eta r_4 < \infty)$ with

$$\frac{f(x)}{x} \ge \frac{1}{\lambda \eta B}, \quad x \in (0, r_3) \cup (\eta r_4, \infty).$$

Therefore,

$$\min_{\eta r_3 \le x \le r_3} f(x) \ge r_3/\lambda B \tag{4.4}$$

$$\min_{\eta r_4 \le x \le r_4} f(x) \ge r_4/\lambda B. \tag{4.5}$$

It follows from Remark 4.2 and (4.2), (4.4) that problem (1.1)-(1.3) has a solution $x_1 \in K$ with $r_3 \le \|x_1\| \le r_1$. Also, it follows from Theorem 4.1 and (4.3), (4.5) that problem (1.1)-(1.3) has a solution $x_2 \in K$ with $r_2 \le \|x_2\| \le r_4$. As a results, problem (1.1)-(1.3) has at least two positive solutions

$$r_3 \le \|x_1\| \le r_1 < r_2 \le \|x_2\| \le r_4.$$

□

Theorem 4.4. *Suppose that* (H1)-(H4) hold. In addition, assume that $f_0 = 0$ and $f_\infty = 0$. Then, the boundary-value problem (1.1)-(1.3) has at least two positive solutions for all $\lambda \in (\lambda^{**}, \infty)$.

Proof. Define

$$g(r) = \frac{r}{B \min_{\eta r \le x \le r} f(x)}.$$

Using the conditions (H1), $f_0 = 0$ and $f_\infty = 0$, we can easily obtain that $g : (0,\infty) \to (0,\infty)$ is continuous and

$$\lim_{r\to 0} g(r) = \lim_{r\to\infty} g(r) = +\infty.$$

So there exists $r_0 \in (0,\infty)$ such that $g(r_0) = \inf_{r>0} g(r) = \lambda^{**}$. For $\lambda \in (\lambda^{**},\infty)$, there exist two constants $r_1, r_2 (0 < r_1 < r_0 < r_2 < \infty)$ with $g(r_1) = g(r_2) = \lambda$. Thus

$$f(x) \ge r_1/\lambda B, \quad \eta r_1 \le x \le r_1, \tag{4.6}$$

$$f(x) \ge r_2/\lambda B, \quad \eta r_2 \le x \le r_2. \tag{4.7}$$

On the other hand, since $f_0 = 0$, there exists a constant $r_3 (0 < r_3 < r_1)$ with

$$\frac{f(x)}{x} \le \frac{1}{\lambda A}, \quad x \in (0, r_3).$$

Therefore,

$$\max_{0 \le x \le r_3} f(x) \le r_3/\lambda A. \tag{4.8}$$

Further, using the condition $f_\infty = 0$, there exists a constant $r (r_2 < r < +\infty)$ with

$$\frac{f(x)}{x} \le \frac{1}{\lambda A}, x \in (r, \infty).$$

Let $M = \sup_{0\le x\le r} f(x)$ and $r_4 \ge A\lambda M$. It is easily seen that

$$\max_{0\le x\le r_4} f(x) \le r_4/\lambda A. \tag{4.9}$$

It follows from Theorem 4.1, (4.6) and (4.8) that (1.1)-(1.3) has a solution $x_1 \in K$ with $r_3 \le \|x_1\| \le r_1$. Also, it follows from Remark 4.2 and (4.7), (4.9) that problem (1.1)-(1.3) has a solution $x_2 \in K$ with $r_2 \le \|x_2\| \le r_4$. Therefore, problem (1.1)-(1.3) has two positive solutions

$$r_3 \le \|x_1\| \le r_1 < r_2 \le \|x_2\| \le r_4.$$

□

Example. Assume in (3.8) that $0 < \alpha < 1 < \beta$, then problem (1.1)-(1.3) has at least two positive solution for each $\lambda \in (0, \lambda^*)$, where λ^* is some positive constant.

Proof. It is easy to see that $f_0 = \infty$, $f_\infty = \infty$ since $0 < \alpha < 1 < \beta$. Then the result can be easily obtained using Theorem 4.3. □

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