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EPIDEMIC REACTION-DIFFUSION SYSTEMS WITH TWO TYPES OF BOUNDARY CONDITIONS

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Abstract. We investigate an epidemic reaction-diffusion system with two different types of boundary conditions. For the problem with the Neumann boundary condition, the global dynamics is fully determined by the basic reproduction number $\mathcal{R}_0$. For the problem with the free boundary condition, the disease will vanish if the basic reproduction number $\mathcal{R}_0 < 1$ or the initial infected radius g_0 is sufficiently small. Furthermore, it is shown that the disease will spread to the whole domain if $\mathcal{R}_0 > 1$ and the initial infected radius g_0 is suitably large. Main results reveal that besides the basic reproduction number, the size of initial epidemic region and the diffusion rates of the disease also have an important influence to the disease transmission.

1. Introduction and model derivation

Mathematical modeling has been shown to be an effective approach to study the spread of infectious diseases as they can capture the main factors underlying the transmission mechanisms and provide feasible control strategies for health agencies. One of the simplest epidemic models is the Kermack-McKendrick model, which can be divided the population into susceptible (S), infectious (I) and recovered individuals (R) [15]. In recent years, mathematical analyses for epidemic models have received wide attentions (see, e.g., [5, 7, 18, 19, 23, 25, 28, 30, 31, 33]).

In the classical SIR models, it is assumed that recovered individuals have gotten permanent immunity. However, the acquired immunity may disappear and recovered individuals will become susceptible after a period of time [24]. Moreover, for some bacterial agent diseases, infected individuals may recover after some treatments and go back directly to the susceptible class because of transient antibody [24]. Li et al [23] proposed the following SIRS epidemic system with nonlinear response function $Sf(I)$ and transfer from the infected class to the susceptible class,

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which is governed by a set of ordinary differential equations

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda - \mu S - Sf(I) + \gamma_1 I + \delta R, \\
\frac{dI}{dt} &= Sf(I) - (\mu + \gamma_1 + \gamma_2 + \alpha)I, \\
\frac{dR}{dt} &= \gamma_2 I - (\mu + \delta)R,
\end{aligned} \tag{1.1}$$

where $\Lambda > 0$ is the recruitment rate of susceptible individuals, $\gamma_1 \geq 0$ denotes the transfer rate from the infected class to the susceptible class, $\gamma_2 \geq 0$ represents the transfer rate from the infected class to the recovered class, $\alpha \geq 0$ stands for the disease-induced death rate, $\delta \geq 0$ is the immunity loss rate, and $\mu > 0$ is the natural death rate.

Li et al [23] obtained the global dynamics of system (1.1), which is determined by the basic reproduction number

$$\mathcal{R}_0 = \frac{\Lambda\beta}{\mu(\mu + \gamma_1 + \gamma_2 + \alpha)},$$

with LaSalle's invariance principle and the Lyapunov direct method.

Most of epidemic systems are governed by a set of ordinary differential equations, which only reflect the epidemiological process as the time changes. To closely match the reality, we consider a SIRS epidemic reaction-diffusion system as follows

$$\begin{gathered}
\frac{\partial S(x,t)}{\partial t} = D\Delta S(x,t) + \Lambda - \mu S(x,t) - S(x,t)f(I(x,t)) \\
+ \gamma_1 I(x,t) + \delta R(x,t), \quad x \in \Omega,\ t > 0, \\
\frac{\partial I(x,t)}{\partial t} = D\Delta I(x,t) + S(x,t)f(I(x,t)) - (\mu + \gamma_1 + \gamma_2 + \alpha)I(x,t), \\
x \in \Omega,\ t > 0, \\
\frac{\partial R(x,t)}{\partial t} = D\Delta R(x,t) + \gamma_2 I(x,t) - (\mu + \delta)R(x,t), \quad x \in \Omega,\ t > 0, \\
\frac{\partial S}{\partial \nu} = \frac{\partial I}{\partial \nu} = \frac{\partial R}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0, \\
S(x,0) = S_0(x) > 0, \quad I(x,0) = I_0(x) > 0, \quad R(x,0) = R_0(x) > 0, \quad x \in \overline{\Omega},
\end{gathered} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. ν is the outward normal to $\partial\Omega$. $D > 0$ stands for the diffusion rate. To continue our study, we make the same hypotheses on f as in [23]. Namely, f is a real locally Lipschitz function on $\mathbb{R}_+ = [0, +\infty)$ satisfying the following assumptions

- (A1) $f(0) = 0$, $f(I) > 0$, and $f'(I) \geq 0$ for $I > 0$.
- (A2) $\frac{f(I)}{I}$ is continuous and nonincreasing for $I > 0$, and $\lim_{I \to 0^+} \frac{f(I)}{I}$ exists, denoted by $\beta > 0$.
- (A3) $f''(I) \leq 0$ for $I > 0$.

In recent years, the free boundary problems have received tremendous attentions (see, e.g., [4, 5, 7, 8, 10, 11, 12, 14, 17, 22, 26, 29, 32]). To make a better understanding for the dynamics of spatial transmission of the disease, the free boundary condition is introduced to epidemic systems. Kim et al [16] investigated a reaction-diffusion SIR epidemic system with the free boundary condition and derived some sufficient conditions for the disease vanishing or spreading. Huang and Wang [13] studied a diffusive SIR system with the free boundary condition. The dynamical behavior of the susceptible population is obtained. A SIS reaction-diffusion-advection system with the free boundary condition was proposed to discuss the persistence and eradication of infectious disease [9]. Cao et al [6] explored a free boundary problem of a diffusive SIRS system with nonlinear incidence. The estimate of the expanding speed was discussed.

Motivated by the works mentioned above, we make further investigation for a SIRS epidemic system with nonlinear incidence and the free boundary condition. For the sake of simplicity, we assume that the environment is radially symmetric. We study the behavior of the positive solution $(S(z,t), I(z,t), R(z,t); g(t))$ with $z = |x|$ and $x \in \mathbb{R}^n$ for the following problem

$$
\begin{gathered}
\begin{aligned}
\frac{\partial S(z,t)}{\partial t} &= D\Delta S(z,t) + \Lambda - \mu S(z,t) - S(z,t) f(I(z,t)) \\
&\quad + \gamma_1 I(z,t) + \delta R(z,t), \quad z > 0,\ t > 0, \\
\frac{\partial I(z,t)}{\partial t} &= D\Delta I(z,t) + S(z,t) f(I(z,t)) - (\mu + \gamma_1 \\
&\quad + \gamma_2 + \alpha) I(z,t), \quad 0 < z < g(t),\ t > 0,
\end{aligned} \\
\frac{\partial R(z,t)}{\partial t} = D\Delta R(z,t) + \gamma_2 I(z,t) - (\mu + \delta) R(z,t), \quad 0 < z < g(t),\ t > 0, \\
S_z(0,t) = I_z(0,t) = R_z(0,t) = 0, \quad t > 0, \\
I(z,t) = R(z,t) = 0, \quad z \geq g(t),\ t > 0, \\
g'(t) = -\mu_1 I_z(g(t),t), \quad g(0) = g_0 > 0, \quad t > 0, \\
S(z,0) = S_0(z), \quad I(z,0) = I_0(z), \quad R(z,0) = R_0(z), \quad z \geq 0,
\end{gathered} \tag{1.3}
$$

where g_0, D and μ_1 are positive constants. From the biological perspective, the Neumann boundary condition at $x = 0$ indicates that the left boundary is fixed, with the population confined to its right. Beyond the free boundary $z = g(t)$, there only exist susceptible individuals. The equation $g'(t) = -\mu_1 I_z(g(t),t)$ is a special case of the well-known Stefan condition, which has been proposed in [17]. $[0, g_0]$ is the initial epidemic region where infective individuals I and removed individuals R exist. The constant μ_1 denotes the ratio of expanding speed of the free boundary. The initial functions S_0, I_0 and R_0 are nonnegative and satisfy

$$
\begin{gathered}
S_0 \in C^2([0,+\infty)), \quad I_0, R_0 \in C^2([0, g_0]), \\
I_0(z) = R_0(z) = 0, \quad z \in [g_0, +\infty), \quad I_0(z) > 0, \quad z \in [0, g_0).
\end{gathered} \tag{1.4}
$$

The organization of this article is as follows. In Section 2, we study the Neumann boundary problem in a bounded domain. We first show that the solution of system (1.2) is positive and bounded, then study the global dynamics of steady states for system (1.2). Main results reveal that if $\mathcal{R}_0 < 1$, then the disease-free steady state is globally asymptotically stable; while if $\mathcal{R}_0 > 1$, the endemic steady state is globally asymptotically stable. In Section 3, we discuss the free boundary problem. We firstly investigate the existence and uniqueness of the solution to system (1.3). We derive some sufficient conditions for the disease vanishing or spreading. In Section 4, we perform some numerical simulations to illustrate theoretical results. At last, we give discussions and conclusions in Section 5.

2. Fixed domain

In this section, we aim to study system (1.2) with the Neumann boundary condition in a bounded domain. The well-posedness of the solutions for system (1.2) is discussed in Theorem 2.1. Furthermore, the global asymptotic stabilities of steady states of system (1.2) are explored in Theorems 2.2 and 2.3.

2.1. **Well-posedness of solutions.**

We denote the positive cone in $\mathbb{R}^3$ by

$$\mathbb{R}^3_+ = \{\phi = (S, I, R)^T \in \mathbb{R}^3 : S \geq 0, I \geq 0,\ R \geq 0\}.$$

Take $p > 3$ so that the space $W^{1,p}(\Omega, \mathbb{R}^3)$ is continuously embedded in the continuous function space $C(\Omega, \mathbb{R}^3)$ [1]. We consider the well-posedness of the solutions in the phase space

$$\mathcal{X}_+ = \{\phi \in W^{1,p}(\Omega, \mathbb{R}^3) : \phi(\overline{\Omega}) \subset \mathbb{R}^3_+ \text{ and } \partial\phi/\partial\nu = 0 \text{ on } \partial\Omega\}.$$

We rewrite system (1.2) as

$$\begin{gathered}\phi_t + \mathcal{S}(\phi)\phi = \mathcal{F}(x, \phi), \quad x \in \phi,\ t > 0,\\ B\phi = 0, \quad x \in \partial\Omega,\ t > 0,\end{gathered}$$

where $\mathcal{S}(e)\phi = -\sum_{i,k} \partial_i(a_{i,k}(e)\partial_k\phi)$, $B\phi = \frac{\partial\phi}{\partial\nu}$, $a_{i,k} = a(e)\delta_{i,k}$, $1 \leq i, k \leq 3$, and

$$a(e) = \begin{pmatrix} D & 0 & 0\\ 0 & D & 0\\ 0 & 0 & D\end{pmatrix},$$

for $e(e_1, e_2, e_3) \in \mathbb{R}^3_+$. Here $\delta_{i,k}$ is the Kronecker delta function, and

$$\mathcal{F}(x, \phi) = \Big(\Lambda - \mu S - Sf(I) + \gamma_1 I + \delta R, Sf(I) - (\mu + \gamma_1 + \gamma_2 + \alpha)I, \gamma_2 I - (\mu + \delta)R\Big)^T,$$

for $\phi = (S, I, R)$. Clearly, $a(e) \in C^2(\mathbb{R}^3_+, L)(\mathbb{R}^3_+))$, where we identified $L(\mathbb{R}^3_+)$ with the space of 3×3 real matrices.

Theorem 2.1. *For every initial value (S_0, I_0, R_0), system (1.2) admits a unique nonnegative solution defined on $[0, +\infty) \times \overline{\Omega}$, such that*

$$(S, I, R) \in C\left([0, +\infty), \mathcal{X}_+\right) \cap C^{2,1}\left([0, +\infty) \times \overline{\Omega}, \mathbb{R}^3\right).$$

Proof. In view of [2, Theorem 1] or [3, Theorems 14.4 and 14.6], system (1.2) admits a unique nonnegative classical solution (S, I, R) defined on $[0, \varrho_0) \times \Omega$ such that

$$(S, I, R) \in C\left([0, \varrho_0), \mathcal{X}_+\right) \cap C^{2,1}\left([0, \varrho_0) \times \overline{\Omega}, \mathbb{R}^3\right),$$

where $\varrho_0 > 0$ is the maximal interval of existence of the solution for system (1.2). According to [3, Theorem 15.1], the solution of system (1.2) is nonnegative. Motivated by the idea developed in [2, Theorem 5.2], we need to show that any nonnegative solution $(S(x, t), I(x, t), R(x, t))$ of system (1.2) is bounded.

Denote $N = S + I + R$, from system (1.2), we get that

$$\frac{\partial N}{\partial t} \leq D\Delta N + \Lambda - \mu N.$$

By [21, Lemma 1], $\frac{\Lambda}{\mu}$ is the globally attractive steady state for the reaction-diffusion equations

$$\frac{\partial N(x, t)}{\partial t} = D\Delta N + \Lambda - \mu N, \quad x \in \Omega,\ t > 0,$$

$$\frac{\partial N}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0.$$

In view of the parabolic comparison theorem ([27, Theorem 7.3.4]), $S+I+R$ is bounded. Since S, I, and R are nonnegative, $S(x,t)$, $I(x,t)$, and $R(x,t)$ of system (1.2) are bounded. That is, $\varrho_0 = +\infty$. By [2, Theorem 5.2], the global existence of the solution can be obtained. The proof is complete. $\square$

2.2. **Global dynamics for system** (1.2)**.** In this subsection, we investigate the global dynamics of steady states of system (1.2) by constructing suitable Lyapunov functions. Firstly, we obtain that the state space $\mathit{\Pi}$ is positively invariant for system (1.2)

$$\mathit{\Pi} := \big\{(S,I,R)^T : S(x,\cdot) + I(x,\cdot) + R(x,\cdot) \le \frac{\Lambda}{\mu},\ \text{for } x \in \overline{\Omega}\big\}.$$

Obviously, system (1.2) always has the disease-free steady state $E_0(\frac{\Lambda}{\mu},0,0)$. If $\mathcal{R}_0 > 1$, from [23], system (1.2) has a unique endemic steady state $E^* = (S^*, I^*, R^*)$, where

$$S^* = \frac{(\mu+\gamma_1+\gamma_2+\alpha)I^*}{f(I^*)}, \quad R^* = \frac{\gamma_2 I^*}{\mu+\delta}.$$

Here I^* is a unique positive zero of $\mathcal{H}$ defined by

$$\mathcal{H}(I) = \mu(\mu+\gamma_1+\gamma_2+\alpha)\frac{I}{f(I)} + \big(\mu+\alpha+\frac{\mu\gamma_2}{\mu+\delta}\big)I - \Lambda.$$

Theorem 2.2. *If $\mathcal{R}_0 < 1$, the disease-free steady state E_0 of system* (1.2) *is globally asymptotically stable in $\mathit{\Pi}$.*

Proof. We define the Lyapunov function

$$V_0 = \int_\Omega I(x,t)dx.$$

From (A2), we obtain that $f(I) \le \beta I,$ for $I \in \mathbb{R}^+$. By the divergence theorem and the Neumann boundary condition, we obtain

$$D\int_\Omega \Delta I dx = 0.$$

The derivative of V_0 along solutions of system (1.2) is

$$\begin{aligned}
\frac{\partial V_0}{\partial t} &= D\int_\Omega \Delta I dx + \int_\Omega [S(x,t)f(I(x,t)) - (\mu+\gamma_1+\gamma_2+\alpha)I(x,t)]dx \\
&\le \int_\Omega [\beta S(x,t)I(x,t) - (\mu+\gamma_1+\gamma_2+\alpha)I(x,t)]dx \\
&\le \int_\Omega \big(\frac{\Lambda\beta}{\mu} - (\mu+\gamma_1+\gamma_2+\alpha)\big)I(x,t)dx \\
&= (\mu+\gamma_1+\gamma_2+\alpha)\int_\Omega (R_0-1)I(x,t)dx.
\end{aligned}$$

We have $\frac{\partial V_0}{\partial t} \le 0$, and the equality holds if and only if $I \equiv 0$. By LaSalle's invariance principle, the disease-free steady state E_0 is globally asymptotically stable if $\mathcal{R}_0 < 1$. The proof is complete. $\square$

Next we study the global asymptotic stability of the endemic steady state E^*. We study the following equivalent system constituted by I, R, and $N = S + I + R$,

$$\begin{gathered}
\frac{\partial I(x,t)}{\partial t} = D\Delta I + (N - I - R)f(I) - (\mu + \gamma_1 + \gamma_2 + \alpha)I, \quad x \in \Omega,\ t > 0, \\
\frac{\partial R(x,t)}{\partial t} = D\Delta R + \gamma_2 I - (\mu + \delta)R, \quad x \in \Omega,\ t > 0, \\
\frac{\partial N(x,t)}{\partial t} = D\Delta N + \Lambda - \mu N - \alpha I, \quad x \in \Omega,\ t > 0, \\
\frac{\partial I}{\partial \nu} = \frac{\partial R}{\partial \nu} = \frac{\partial N}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0, \\
I(x,0) = I_0(x) > 0, \quad R(x,0) = R_0(x) > 0, \\
N(x,0) = N_0(x) > 0, \quad x \in \overline{\Omega}.
\end{gathered} \tag{2.1}$$

If $\mathcal{R}_0 > 1$, this has a unique endemic steady state $\overline{E}^* = (I^*, R^*, N^*)$. Hence,

$$\begin{aligned}
\frac{\partial I(x,t)}{\partial t} &= D\Delta I(x,t) + f(I)\{N - I - R - (\mu + \gamma_1 + \gamma_2 + \alpha)\frac{I}{f(I)}\} \\
&\quad - f(I)\{[N^* - I^* - R^* - (\mu + \gamma_1 + \gamma_2 + \alpha)\frac{I^*}{f(I^*)}]\}.
\end{aligned}$$

We rewrite system (2.1) as

$$\begin{gathered}
\begin{aligned}
\frac{\partial I(x,t)}{\partial t} &= D\Delta I(x,t) + f(I)\left\{(N - N^*) - (I - I^*) - (R - R^*)\right\} \\
&\quad - f(I)(\mu + \gamma_1 + \gamma_2 + \alpha)[\frac{I}{f(I)} - \frac{I^*}{f(I^*)}], \quad x \in \Omega,\ t > 0,
\end{aligned} \\
\frac{\partial R(x,t)}{\partial t} = D\Delta R(x,t) + \gamma_2 I(x,t) - (\mu + \delta)R(x,t), \quad x \in \Omega,\ t > 0, \\
\frac{\partial N(x,t)}{\partial t} = D\Delta N(x,t) + \Lambda - \mu N(x,t) - \alpha I(x,t), \quad x \in \Omega,\ t > 0, \\
\frac{\partial I}{\partial \nu} = \frac{\partial R}{\partial \nu} = \frac{\partial N}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0, \\
I(x,0) = I_0(x) > 0, \quad R(x,0) = R_0(x) > 0, \\
N(x,0) = N_0(x) > 0, \quad x \in \overline{\Omega}.
\end{gathered} \tag{2.2}$$

Theorem 2.3. *If $\mathcal{R}_0 > 1$, then the endemic steady state $\overline{E}^*$ of system* (2.2) *is globally asymptotically stable in Π.*

Proof. We define the Lyapunov function

$$V_1 = \int_\Omega \int_{I^*}^{I} \frac{u - I^*}{f(u)} du dx + \frac{1}{2\gamma_2} \int_\Omega (R - R^*)^2 dx + \frac{1}{2\alpha} \int_\Omega (N - N^*)^2 dx.$$

Then we have

$$\begin{gathered}
D \int_\Omega \frac{I - I^*}{f(I)} \Delta I dx = -D \int_\Omega \frac{[f(I) - If'(I)]\|\nabla I\|^2}{f^2(I)} dx - DI^* \int_\Omega \frac{f'(I)\|\nabla I\|^2}{f^2(I)} dx, \\
\frac{D}{\gamma_2} \int_\Omega (R - R^*)\Delta R dx = -\frac{D}{\gamma_2} \int_\Omega \|\nabla R\|^2 dx, \\
\frac{D}{\alpha} \int_\Omega (N - N^*)\Delta N dx = -\frac{D}{\alpha} \int_\Omega \|\nabla N\|^2 dx.
\end{gathered}$$

The derivative of V_1 along solutions of system (2.2) is

$$\begin{aligned}
\frac{\partial V_1}{\partial t} &= \int_\Omega \frac{I-I^*}{f(I)}\frac{\partial I}{\partial t}dx + \frac{1}{\gamma_2}\int_\Omega (R-R^*)\frac{\partial R}{\partial t}dx + \frac{1}{\alpha}\int_\Omega (N-N^*)\frac{\partial N}{\partial t}dx \\
&= -D\int_\Omega \frac{[f(I)-If'(I)]\|\nabla I\|^2}{f^2(I)}dx - DI^*\int_\Omega \frac{f'(I)\|\nabla I\|^2}{f^2(I)}dx \\
&\quad - \frac{D}{\gamma_2}\int_\Omega \|\nabla R\|^2 dx - \frac{D}{\alpha}\int_\Omega \|\nabla N\|^2 dx \\
&\quad + \int_\Omega \frac{I-I^*}{f(I)} f(I)\{(N-N^*)-(I-I^*)-(R-R^*)\}dx \\
&\quad - \int_\Omega \frac{I-I^*}{f(I)} f(I)\big\{(\mu+\gamma_1+\gamma_2+\alpha)[\frac{I}{f(I)} - \frac{I^*}{f(I^*)}]\big\}dx \\
&\quad + \int_\Omega \frac{R-R^*}{\gamma_2}[\gamma_2(I-I^*)-(\mu+\delta)(R-R^*)]dx \\
&\quad + \int_\Omega \frac{N-N^*}{\alpha}[-\mu(N-N^*)-\alpha(I-I^*)]dx \\
&= -D\int_\Omega \frac{[f(I)-If'(I)]\|\nabla I\|^2}{f^2(I)}dx - DI^*\int_\Omega \frac{f'(I)\|\nabla I\|^2}{f^2(I)}dx \\
&\quad - \frac{D}{\gamma_2}\int_\Omega \|\nabla R\|^2 dx - \frac{D}{\alpha}\int_\Omega \|\nabla N\|^2 dx \\
&\quad - \int_\Omega (I-I^*)^2 dx - (\mu+\gamma_1+\gamma_2+\alpha)\int_\Omega (I-I^*)[\frac{I}{f(I)} - \frac{I^*}{f(I^*)}]dx \\
&\quad - \frac{\mu+\delta}{\gamma_2}\int_\Omega (R-R^*)^2 dx - \frac{\mu}{\alpha}\int_\Omega (N-N^*)^2 dx.
\end{aligned}$$

Thus, $\frac{\partial V_1}{\partial t} \le 0$, and the equality holds if and only if $S \equiv S^*$, $I \equiv I^*$, and $R \equiv R^*$. By LaSalle's invariance principle, the endemic steady state $\overline{E}^*$ is globally attractive if $\mathcal{R}_0 > 1$. The proof is complete. $\square$

From Theorem 2.3, we immediately obtain the following corollary.

Corollary 2.4. *If $\mathcal{R}_0 > 1$, then the endemic steady state E^* of system* (1.2) *is globally asymptotically stable in Π.*

3. Free boundary problem

In this section, we study the free boundary problem of system (1.3). Let $g_\infty := \lim_{t\to\infty} g(t)$, then $g_\infty \in (0, +\infty]$. If $g_\infty < \infty$ and $\lim_{t\to\infty} \|I(\cdot,t)\|_{C[0,g(t)]} = 0$, then the vanishing occurs. If $g_\infty = \infty$, then the spreading occurs. In this case, the moving domain $(0, g(t))$ becomes the whole domain $(0, +\infty)$.

3.1. **Existence and uniqueness of solutions.**

We use a contraction mapping theorem. The proof depends mainly on some existing arguments [5, 16, 17], with some modifications. We sketch the details here for completeness.

Theorem 3.1. *For any given (S_0, I_0, R_0) satisfying* (1.4) *and any $\iota \in (0, 1)$, there exists a $T > 0$ such that system* (1.3) *admits a unique bounded solution*

$$(S, I, R; g) \in C^{1+\iota, \frac{1+\iota}{2}}(Z_T^\infty) \times \big[C^{1+\iota, \frac{1+\iota}{2}}(Z_T)\big]^2 \times C^{1+\frac{\iota}{2}}([0, T]);$$

Furthermore,

$$\|S\|_{C^{1+\iota,\frac{1+\iota}{2}}(Z_T^\infty)}+\|I\|_{C^{1+\iota,\frac{1+\iota}{2}}(Z_T)}+\|R\|_{C^{1+\iota,\frac{1+\iota}{2}}(Z_T)}+\|g\|_{C^{1+\frac{\iota}{2}}([0,T])}\le\mathcal{K},$$

where

$$\begin{gathered}Z_T^\infty=\{(z,t)\in\mathbb{R}^2:z\in[0,+\infty),t\in[0,T]\},\\ Z_T=\{(z,t)\in\mathbb{R}^2:z\in[0,g(t)],t\in[0,T]\}.\end{gathered}$$

Here $\mathcal{K}$ *and* T *only depend on* g_0, ι, $\|S_0\|_{C^2([0,+\infty))}$, $\|I_0\|_{C^2([0,g_0])}$, *and* $\|R_0\|_{C^2([0,g_0])}$.

Proof. Let $\kappa(s)$ be a function in $C^3[0,+\infty)$ satisfying

$$\kappa(s)=\begin{cases}1, & \text{if } |s-g_0|<\frac{g_0}{8},\\ 0, & \text{if } |s-g_0|>\frac{g_0}{2},\end{cases}\quad\text{and } |\kappa'(s)|<\frac{5}{g_0}\quad\text{for all } s.$$

We considering the transformation

$$(y,t)\to(x,t),\quad\text{where } x=y+\kappa(|y|)\big(g(t)-\frac{g_0y}{|y|}\big),\quad y\in\mathbb{R}^n.$$

Then

$$(s,t)\to(z,t),\quad\text{where } z=s+\kappa(s)(g(t)-g_0),\ 0\le s<\infty.$$

By adopting the method similar to [5], the free boundary $z=g(t)$ can be changed to the line $s=g_0$. Direct calculations yield that

$$\begin{gathered}\frac{\partial s}{\partial z}=\frac{1}{1+\kappa'(s)(g(t)-g_0)}:=\mathcal{A}(g(t),s),\\ \frac{\partial^2 s}{\partial z^2}=-\frac{\kappa''(s)(g(t)-g_0)}{[1+\kappa'(s)(g(t)-g_0)]^3}:=\mathcal{B}(g(t),s),\\ -\frac{1}{g(t)}\frac{\partial s}{\partial t}=\frac{\kappa(s)}{1+\kappa'(s)(g(t)-g_0)}:=\mathcal{C}(g(t),s).\end{gathered}$$

We set

$$\begin{gathered}S(z,t)=S(s+\kappa(s)(g(t)-g_0),t):=m(s,t),\\ I(z,t)=I(s+\kappa(s)(g(t)-g_0),t):=n(s,t),\\ R(z,t)=R(s+\kappa(s)(g(t)-g_0),t):=j(s,t).\end{gathered}$$

We rewrite system (1.3) as

$$\begin{gathered}m_t-\mathcal{A}D\Delta_s m-(\mathcal{B}D+g'\mathcal{C})\,m_s=\Lambda-\mu m-mf(n)-\gamma_1 n+\delta j,\quad s>0,\ t>0,\\ n_t-\mathcal{A}D\Delta_s n-(\mathcal{B}D+g'\mathcal{C})n_s=mf(n)-(\mu+\gamma_1+\gamma_2+\alpha)n,\quad 0<s<g_0,\ t>0,\\ j_t-\mathcal{A}D\Delta_s j-(\mathcal{B}D+g'\mathcal{C})j_s=\gamma_2 n-(\mu+\delta)j,\quad 0<s<g_0,\ t>0,\\ m_s(0,t)=n_s(0,t)=j_s(0,t)=0,\quad t>0,\\ n(s,t)=j(s,t)=0,\quad s\ge g_0,\ t>0,\\ g'(t)=-\mu_1 n_s(g_0,t),\quad g(0)=g_0>0,\ t>0,\\ m(s,0)=m_0(s),\quad n(s,0)=n_0(s),\quad j(s,0)=j_0(s),\ s\ge 0,\end{gathered}$$

where $m_0=S_0$, $n_0=I_0$, and $j_0=R_0$.

We denote $g^*=-\mu_1 n_0'(g_0)$, and for $0<T\le\frac{g_0}{8(1+g^*)}$, we set

$$\begin{gathered}H_T=\big\{g\in C^1[0,T]:g(0)=g_0,\ g'(0)=g^*,\ \|g'-g^*\|_{C([0,T])}\le 1\big\},\\ M_T=\big\{m\in C([0,+\infty)\times[0,T]):m(s,0)=m_0(s),\|m-m_0\|_{L^\infty([0,+\infty)\times[0,T])}\le 1\big\},\end{gathered}$$

$$
\begin{aligned}
N_T = \Big\{ & n \in C([0,+\infty) \times [0,T]) : n(s,0) \equiv 0 \text{ for } s \geq g_0,\ 0 \leq t \leq T, \\
& n(s,0) = n_0(s),\ \text{for } 0 \leq s \leq g_0,\ \|n - n_0\|_{L^\infty([0,+\infty)\times[0,T])} \leq 1 \Big\}, \\
J_T = \Big\{ & j \in C([0,+\infty) \times [0,T]) : j(s,0) \equiv 0 \text{ for } s \geq g_0,\ 0 \leq t \leq T, \\
& j(s,0) = j_0(s),\ \text{for } 0 \leq s \leq g_0,\ \|j - j_0\|_{L^\infty([0,+\infty)\times[0,T])} \leq 1 \Big\}.
\end{aligned}
$$

Since $g_1, g_2 \in H_T$ and $g_1(0) = g_2(0) = g_0$, one gets

$$
\|g_1 - g_2\|_{C([0,T])} \leq T \|g_1' - g_2'\|_{C([0,T])}.
$$

$\Gamma_T := M_T \times N_T \times J_T \times H_T$ is a complete metric space with the metric

$$
\begin{aligned}
&\mathfrak{D}\big((m_1, n_1, j_1; g_1), (m_2, n_2, j_2; g_2)\big) \\
&= \|m_1 - m_2\|_{L^\infty([0,+\infty)\times[0,T])} + \|n_1 - n_2\|_{L^\infty([0,+\infty)\times[0,T])} \\
&\quad + \|j_1 - j_2\|_{L^\infty([0,+\infty)\times[0,T])} + \|g_1' - g_2'\|_{C([0,T])}.
\end{aligned}
$$

By adopting standard L^p theory and the Sobolev embedding theorem [20], for $(m, n, j; g) \in \Gamma_T$, the initial boundary value problem

$$
\begin{gathered}
\widetilde{m}_t - \mathcal{A}D\Delta_s \widetilde{m} - (\mathcal{B}D + g'\mathcal{C})\widetilde{m}_s = \Lambda - \mu m - m f(n) - \gamma_1 n + \delta j, \quad s > 0,\ t > 0, \\
\widetilde{n}_t - \mathcal{A}D\Delta_s \widetilde{n} - (\mathcal{B}D + g'\mathcal{C})\widetilde{n}_s = m f(n) - (\mu + \gamma_1 + \gamma_2 + \alpha) n, \quad 0 < s < g_0,\ t > 0, \\
\widetilde{j}_t - \mathcal{A}D\Delta_s \widetilde{j} - (\mathcal{B}D + g'\mathcal{C})\widetilde{j}_s = \gamma_2 n - (\mu + \delta) j, \quad 0 < s < g_0,\ t > 0, \\
\widetilde{m}_s(0,t) = \widetilde{n}_s(0,t) = \widetilde{j}_s(0,t) = 0, \quad t > 0, \\
\widetilde{n}(s,t) = \widetilde{j}(s,t) = 0, \quad s \geq g_0,\ t > 0, \\
\widetilde{m}(s,0) = m_0(s), \quad \widetilde{n}(s,0) = n_0(s), \quad \widetilde{j}(s,0) = j_0(s), \quad s \geq 0,
\end{gathered}
$$

has a unique solution

$$
(\widetilde{m}, \widetilde{n}, \widetilde{j}) \in \Big[C^{1+\iota, \frac{1+\iota}{2}}([0,+\infty) \times [0,T]) \Big]^3,
$$

and it satisfies

$$
\begin{aligned}
\|\widetilde{m}\|_{C^{1+\iota, \frac{1+\iota}{2}}([0,+\infty)\times[0,T])} &\leq \mathcal{K}_1, \\
\|\widetilde{n}\|_{C^{1+\iota, \frac{1+\iota}{2}}([0,g_0]\times[0,T])} &\leq \mathcal{K}_1, \\
\|\widetilde{j}\|_{C^{1+\iota, \frac{1+\iota}{2}}([0,g_0]\times[0,T])} &\leq \mathcal{K}_1,
\end{aligned}
$$

where $\mathcal{K}_1$ is a constant depending on ι, g_0, $\|S_0\|_{C^2([0,+\infty))}$, $\|I_0\|_{C^2([0,g_0])}$, and $\|R_0\|_{C^2([0,g_0])}$.

We define

$$
\widetilde{g}(t) = g_0 - \mu_1 \int_0^t \widetilde{n}_s(g_0, \tau) d\tau.
$$

Then it follows that $\widetilde{g}'(t) = -\mu_1 \widetilde{n}_s(g_0, t)$, $\widetilde{g}(0) = g_0$, and $\widetilde{g}'(0) = -\mu_1 n_0'(g_0) = g^*$. Thus, $\widetilde{g}'(t) \in C^{\iota/2}([0,T])$ and

$$
\|\widetilde{g}'(t)\|_{C^{\iota/2}([0,T])} \leq \mathcal{K}_2 := \mu_1 \mathcal{K}_1.
$$

Next, we define a map $\mathfrak{F} : \Gamma_T \to [C([0,+\infty) \times [0,T])]^3 \times C^1([0,T])$ by

$$
\mathfrak{F}(m(s,t), n(s,t), j(s,t); g(t)) = (\widetilde{m}(s,t), \widetilde{n}(s,t), \widetilde{j}(s,t); \widetilde{g}(t)).
$$

Then $(m(s,t), n(s,t), j(s,t); g(t)) \in \Gamma_T$ is a fixed point of $\mathfrak{F}$.

From [7], there is a $T > 0$ such that $\mathfrak{F}$ is a contraction mapping in Γ_T. In view of the contraction mapping theorem, there exists a $(m(s,t), n(s,t), j(s,t); g(t))$ in Γ_T such that

$$\mathfrak{F}(m(s,t), n(s,t), j(s,t); g(t)) = (m(s,t), n(s,t), j(s,t); g(t)).$$

Thus, $(S(z,t), I(z,t), R(z,t); g(t))$ is the solution of system (1.3). Further, by employing the Schauder estimates, $h(t) \in C^{1+\frac{\iota}{2}}([0,T])$, $S \in C^{2+\iota,1+\frac{\iota}{2}}((0,+\infty)\times[0,T])$ and $I, R \in C^{2+\iota,1+\frac{\iota}{2}}((0,g(t)) \times [0,T])$. Hence, $(S(z,t), I(z,t), R(z,t); g(t))$ is the classical solution of system (1.3). The proof is complete. $\square$

To show the existence of solution for $t > 0$, we need to show the following lemma. For mathematical considerations, we assume that $\gamma_1 = \delta = 0$.

Lemma 3.2. *Let $(S, I, R; g)$ be a bounded solution to system (1.3) defined on $t \in (0, T_0)$ for some $T_0 \in (0, +\infty]$. Then there exist positive constants $\mathcal{C}_1$ and $\mathcal{C}_2$ independent of T_0 such that*

$$\begin{gathered} 0 < S(z,t) \le \mathcal{C}_1, \quad \text{for } \ 0 \le z < +\infty,\ t \in (0,T_0), \\ 0 < I(z,t), \quad R(z,t) \le \mathcal{C}_2,\ \text{for } \ 0 \le z < g(t),\ t \in (0,T_0). \end{gathered}$$

Proof. By employing the strong maximum principle to system (1.3) in $[0, g(t)] \times [0, T_0)$, $S(z,t), I(z,t), R(z,t) > 0$ for $0 \le z < g(t)$, $0 < t < T_0$. Note that $S(z,t)$ satisfies

$$\begin{gathered} S_t - D\Delta S = \Lambda - \mu S - Sf(I), \quad z > 0,\ t > 0, \\ S(z,0) = S_0(z) \ge 0, \quad z \ge 0. \end{gathered}$$

Thus, $S(z,t) \le \mathcal{C}_1 := \max\{\|S_0(z)\|_{L^\infty(0,+\infty)}, \frac{\Lambda}{\mu}\}$. Let $H(z,t) = S(z,t) + I(z,t) + R(z,t)$. Then

$$\begin{gathered} H_t - D\Delta H = \Lambda - \mu H - \alpha I, \quad 0 < z < g(t),\ t > 0, \\ H = S \le C_1, \quad z = g(t),\ t > 0, \\ H(z,0) = S_0(z) + I_0(z) + R_0(z), \quad 0 \le z \le g_0. \end{gathered}$$

Hence, there exists a constant $\mathcal{C}_2 > 0$ such that

$$S + I + R \le \mathcal{C}_2, \quad \text{for } (z,t) \in [0, g(t)] \times [0, T_0).$$

The proof is complete. $\square$

Similar to the proof of [22, Lemma 3.2], we have the following result.

Lemma 3.3. *There exists a positive constant $\mathcal{C}_3$ independent of T_0 such that $0 < g'(t) \le \mathcal{C}_3$ for $t \in (0, T_0)$.*

By adopting the similar arguments to [22, Theorem 3.3], combined with Lemmas 3.2 and 3.3, we obtain the following result.

Theorem 3.4. *The solution of system (1.3) exists and is unique for $t \in (0, \infty)$.*

3.2. **Spreading and vanishing.**

Theorem 3.5. *If $\mathcal{R}_0 < 1$, then $\lim_{t\to\infty} S(z,t) = \frac{\Lambda}{\mu}$, $\lim_{t\to\infty} \|I(\cdot,t)\|_{C[0,g(t)]} = 0$, and $\lim_{t\to\infty} \|R(\cdot,t)\|_{C[0,g(t)]} = 0$ uniformly in any bounded subset of $[0,+\infty)$. Moreover, $g_\infty < \infty$.*

Proof. From the comparison principle, $S(z,t) \le \overline{S}(t)$ for $z \ge 0$ and $t \in (0,+\infty)$, where

$$\overline{S}(t) := \frac{\Lambda}{\mu} + \big(\|S_0\|_\infty - \frac{\Lambda}{\mu}\big)e^{-\mu t}.$$

$\overline{S}(t)$ is the solution of the problem

$$\frac{d\overline{S}}{dt} = \Lambda - \mu\overline{S}, \quad t > 0;\ \overline{S}(0) = \|S_0\|_\infty.$$

Since $\lim_{t\to\infty} \overline{S}(t) = \frac{\Lambda}{\mu}$, it follows that $\limsup_{t\to\infty} S(z,t) \le \frac{\Lambda}{\mu}$ uniformly for $z \in [0,+\infty)$. From $\mathcal{R}_0 < 1$, there exists T_0 such that $S(z,t) \le \frac{\Lambda}{\mu}\frac{1+\mathcal{R}_0}{2\mathcal{R}_0}$ in $[0,+\infty) \times [T_0,+\infty)$. We find that $I(z,t)$ satisfies

$$\begin{gathered}
I_t - D\Delta I \le \big[\frac{\beta\Lambda}{\mu}\frac{1+\mathcal{R}_0}{2\mathcal{R}_0} - (\mu+\gamma_2+\alpha)\big]I(z,t), \quad 0 < z < g(t),\ t > T_0,\\
I(z,t) = 0, \quad I_z(0,t) = 0, \quad z = g(t),\ t > 0,\\
I(z,T_0) > 0, \quad 0 \le z \le g(T_0).
\end{gathered}$$

Because of

$$\frac{\beta\Lambda}{\mu(\mu+\gamma_2+\alpha)}\frac{1+\mathcal{R}_0}{2\mathcal{R}_0} < 1,$$

we have $\lim_{t\to\infty} \|I(\cdot,t)\|_{C[0,g(t)]} = 0$. From (1.3), we have $\lim_{t\to\infty} \|R(\cdot,t)\|_{C[0,g(t)]} = 0$. Next, we show that $g_\infty < +\infty$. In fact,

$$\begin{aligned}
&\frac{d}{dt}\int_0^{g(t)} z^{n-1}I(z,t)dt\\
&= \int_0^{g(t)} z^{n-1}I_t(z,t)dz + g'(t)g^{n-1}(t)I(g(t),t)\\
&= \int_0^{g(t)} Dz^{n-1}\Delta I dz + \int_0^{g(t)} z^{n-1}I(z,t)\big[\frac{Sf(I)}{I} - (\mu+\gamma_2+\alpha)\big]dz\\
&= \int_0^{g(t)} D(z^{n-1}I_z(z,t))_z dz + \int_0^{g(t)} z^{n-1}I(z,t)\big[\frac{Sf(I)}{I} - (\mu+\gamma_2+\alpha)\big]dz\\
&= -\frac{D}{\mu_1}g^{n-1}g'(t) + \int_0^{g(t)} z^{n-1}I(z,t)\big[\frac{Sf(I)}{I} - (\mu+\gamma_2+\alpha)\big]dz.
\end{aligned}$$

Integrating from T_0 to t ($t > T_0$) gives

$$\begin{aligned}
\int_0^{g(t)} z^{n-1}I(z,t) &= \int_0^{g(T_0)} z^{n-1}I_t(z,T_0)dz + \frac{D}{n\mu_1}g^n(T_0) - \frac{D}{n\mu_1}g^n(t)\\
&\quad + \int_{T_0}^t\int_0^{g(s)} z^{n-1}I(z,s)\big[\frac{Sf(I)}{I} - (\mu+\gamma_2+\alpha)\big]dz\,ds.
\end{aligned}$$

Since $0 < S(z,t) \le \frac{\Lambda}{\mu}\frac{1+\mathcal{R}_0}{2\mathcal{R}_0}$ for $z \in [0,g(t))$ and $t \ge T_0$, it follows that

$$\big[\frac{Sf(I)}{I} - (\mu+\gamma_2+\alpha)\big] \le \beta S - (\mu+\gamma_2+\alpha) \le \frac{\beta\Lambda}{\mu}\frac{1+\mathcal{R}_0}{2\mathcal{R}_0} - (\mu+\gamma_2+\alpha) \le 0.$$

For $t \geq T_0$, it follows that

$$\int_0^{g(t)} z^{n-1} I(z,t)dz \leq \int_0^{g(T_0)} z^{n-1} I(z,T_0)dz + \frac{D}{n\mu_1} g^n(T_0) - \frac{D}{n\mu_1} g^n(t), \quad \text{for } t \geq T_0.$$

Hence, $g_\infty < \infty$. From system (1.3), $\lim_{t\to\infty} S(z,t) = \frac{\Lambda}{\mu}$ uniformly in any bounded subset of $[0,+\infty)$. The proof is complete. □

By using an argument analogous to [7] with some minor modifications, we can obtain the following result.

Lemma 3.6. *Suppose that* $T \in (0,+\infty)$, $\overline{g} \in C^1([0,T])$, $\overline{S} \in C([0,+\infty)\times[0,T]) \cap C^{2,1}([0,+\infty)\times[0,T])$, $\overline{I}, \overline{R} \in C(\overline{Z}_T^*) \cap C^{2,1}(Z_T^*)$ *with* $Z_T^* = \{(z,t) \in \mathbb{R}^2 : 0 < z < \overline{g}(t), 0 < t \leq T\}$, *and*

$$\begin{gathered}
\overline{S}_t - D\Delta \overline{S} \geq \Lambda - \mu \overline{S}, \quad z > 0,\ 0 < t \leq \overline{T}, \\
\overline{I}_t - D\Delta \overline{I} \geq (\beta \overline{S} - (\mu + \gamma_2 + \alpha))\overline{I}, \quad 0 < z < \overline{g}(t),\ 0 < t \leq \overline{T}, \\
\overline{R}_t - D\Delta \overline{R} \geq \gamma_2 \overline{I} - \mu \overline{R}, \quad 0 < z < \overline{g}(t),\ 0 < t \leq \overline{T}, \\
\overline{S}_z(0,t) \geq 0, \quad \overline{I}_z(0,t) \geq 0, \quad \overline{R}_z(0,t) \geq 0, \quad 0 < t \leq \overline{T}, \\
\overline{I}(z,t) = \overline{R}(z,t) = 0, \quad z \geq \overline{g}(t), \quad 0 < t \leq \overline{T}, \\
\overline{g}'(t) \geq -\mu_1 \overline{I}_z(\overline{g}(t),t), \quad \overline{g}(0) = g_0 > 0, \quad 0 < t \leq \overline{T}, \\
\overline{S}(z,0) = S_0(z), \quad \overline{I}(z,0) = I_0(z), \quad \overline{R}(z,0) = R_0(z), \quad z \geq 0.
\end{gathered}$$

Then the solution $(S,I,R;g)$ of system (1.3) satisfies

$$\begin{gathered}
S(z,t) \leq \overline{S}(z,t), \quad g(t) \leq \overline{g}(t), \quad \text{for } z \in (0,+\infty) \text{ and } t \in (0,T], \\
I(z,t) \leq \overline{I}(z,t), \quad R(z,t) \leq \overline{R}(z,t), \quad \text{for } z \in (0,g(t)) \text{ and } t \in (0,T].
\end{gathered}$$

Theorem 3.7. *If* $g_\infty < \infty$, *then* $\lim_{t\to\infty} S(z,t) = \frac{\Lambda}{\mu}$, $\lim_{t\to\infty} \|I(\cdot,t)\|_{C[0,g(t)]} = 0$, *and* $\lim_{t\to\infty} \|R(\cdot,t)\|_{C[0,g(t)]} = 0$ *uniformly in any bounded subset of* $[0,+\infty)$.

Proof. By contradiction, we assume that $\limsup_{t\to\infty} \|I(\cdot,t)\|_{C[0,g(t)]} = \delta_1 > 0$. There exists a sequence (z_q,t_q) in $[0,g(t))\times(0,+\infty)$ such that $I(z_q,t_q) \geq \frac{\delta_1}{2}$ for $q \in N$, and $t_q \to +\infty$. Since $0 \leq z_q < g(t) < g_\infty < \infty$, there exists a subsequence of $\{z_n\}$ converging to $z_0 \in [0,g_\infty)$. We assume $z_q \to z_0$ as $q \to \infty$.

Define

$$\begin{gathered}
S_q(z,t) = S(z,t_q+t), \quad I_q(z,t) = I(z,t_q+t), \\
R_q(z,t) = R(z,t_q+t), \quad \text{for } (z,t) \in (0,g(t_q+t))\times(-t_q,+\infty).
\end{gathered}$$

From the parabolic regularity, $\{(S_q,I_q,R_q)\}$ has a subsequence $\{(S_{q_i},I_{q_i},R_{q_i})\}$ such that $(S_{q_i},I_{q_i},R_{q_i}) \to (\widetilde{S},\widetilde{I},\widetilde{R})$ satisfies

$$\begin{gathered}
\widetilde{S}_t - D\Delta \widetilde{S} = \Lambda - \mu \widetilde{S} - \widetilde{S} f\left(\widetilde{I}\right), \quad 0 < z < g_\infty,\ t \in (-\infty,+\infty), \\
\widetilde{I}_t - D\Delta \widetilde{I} = \widetilde{S} f(\widetilde{I}) - (\mu + \gamma_2 + \alpha)\widetilde{I}, \quad 0 < z < g_\infty,\ t \in (-\infty,+\infty), \\
\widetilde{R}_t - D\Delta \widetilde{R} = \gamma_2 \widetilde{I} - \mu \widetilde{R}, \quad 0 < z < g_\infty,\ t \in (-\infty,+\infty).
\end{gathered}$$

Because $\widetilde{I}(z_0,0) \geq \delta_1/2$, we obtain $\widetilde{I} > 0$ in $[0,g_\infty)\times(-\infty,+\infty)$. Noting that

$$\widetilde{S} f(\widetilde{I}) - (\mu + \gamma_2 + \alpha)\widetilde{I} = \Big(\widetilde{S}\frac{f(\widetilde{I})}{\widetilde{I}} - (\mu + \gamma_2 + \alpha)\Big)\widetilde{I}$$

is bounded by $Q_1 := \beta \max\left\{\|S_0\|_{L^\infty}, \frac{\Lambda}{\mu}\right\} + \mu + \gamma_2 + \alpha$.

Further, $\widetilde{I}_z(g_\infty, 0) \le -\sigma_0$ for some $\sigma_0 > 0$. For any $0 < \varpi < 1$, there exists a constant $\widetilde{C}$, which depends on ϖ, g_0, $\|I_0\|_{C^{1+\varpi}[0,g_0]}$, and g_∞, such that

$$\|I_0\|_{C^{1+\varpi, \frac{1+\varpi}{2}}([0,g(t))\times[0,+\infty))} + \|g\|_{C^{1+\frac{\varpi}{2}}([0,+\infty))} \le \widetilde{C}.$$

Define

$$S = \frac{g_0 z}{g(t)}, \quad m(s,t) = S(z,t), \quad n(s,t) = I(z,t), \quad j(s,t) = R(z,t).$$

It then follows that

$$I_t = n_t - \frac{g'(t)}{g(t)} s n_s, \quad I_z = \frac{g_0}{g(t)} n_s, \quad \Delta_z I = \frac{g_0^2}{g^2(t)} \Delta_s n.$$

Thus, $n(s,t)$ satisfies

$$\begin{gathered}
n_t - D\frac{g_0^2}{g^2(t)}\Delta_s n - \frac{g'(t)}{g(t)} s n_s = n\Big(\frac{f(n)m}{n} - (\mu + \gamma_2 + \alpha)\Big), \quad 0 < s < g_0,\ t > 0,\\
n_s(0,t) = n(g_0,t) = 0, \quad t > 0,\\
n(s,0) = I_0(s) \ge 0, \quad 0 \le s \le g_0.
\end{gathered}$$

Lemmas 3.2 and 3.3 yield

$$\|n\big(\frac{f(n)m}{n} - (\mu + \gamma_2 + \alpha)\big)\|_{L^\infty} \le Q_2.$$

By employing standard L^P theory and Sobolev embedding theorem [20], one gets

$$\|n\|_{C^{1+\varpi, \frac{1+\varpi}{2}}([0,g_0)\times[0,+\infty))} \le Q_3,$$

where Q_3 is a positive constant depending on ϖ, g_0, Q_1, Q_2, and $\|I_0\|_{C^2[0,g_0]}$.

Since $\|g\|_{C^{1+\frac{\varpi}{2}}([0,+\infty))} \le \widetilde{C}$, it follows that $g'(t) \to 0$ as $t \to \infty$, namely, $I_z(g(t_q), t_q) \to 0$ as $t_q \to +\infty$. Furthermore, from $\|g\|_{C^{1+\varpi, \frac{1+\varpi}{2}}([0,g(t))\times[0,+\infty))} \le \widetilde{C}$, it follows that $I_z(g(t_q), t_q + 0) = (I_q)_z(g(t_q), 0) \to \widetilde{I}_z(g_\infty, 0)$ as $q \to \infty$, which is a contradiction. Thus, $\lim_{t\to\infty} \|I(\cdot,t)\|_{C[0,g(t)]} = 0$. Then $\lim_{t\to\infty} \|R(\cdot,t)\|_{C[0,g(t)]} = 0$ and $\lim_{t\to\infty} S(z,t) = \frac{\Lambda}{\mu}$ uniformly in any bounded subset of $[0,+\infty)$. The proof is complete. □

Theorem 3.8. *If $\mathcal{R}_0 > 1$, $g_0 \le \min\big\{\sqrt{\frac{D}{16 q_0}}, \sqrt{\frac{D}{16\gamma_2}}\big\}$, and $\mu_1 \le \frac{D}{8\mathcal{M}}$, then $g_\infty < \infty$, where $q_0 = \beta\mathcal{C}_1 - \mu - \gamma_2 - \alpha > 0$ and $\mathcal{M} = \frac{4}{3}\max\{\|I_0\|_\infty, \|R_0\|_\infty\}$.*

Proof. We construct suitable upper solutions for system (1.3). As in [7], we define upper solutions as follows:

$$\begin{gathered}
\overline{S}(z,t) = \mathcal{C}_1,\\
\overline{I} = \overline{R} = \begin{cases} \mathcal{M}e^{-\gamma t}\mathcal{V}\big(\frac{z}{\overline{g}(t)}\big), & 0 \le z \le \overline{g}(t),\\ 0, & z > \overline{g}(t), \end{cases}\\
\overline{g}(t) = 2g_0\left(2 - e^{-\gamma t}\right), \quad t \ge 0, \quad \mathcal{V}(x) = 1 - x^2, \quad 0 \le x \le 1,
\end{gathered}$$

where γ and $\mathcal{M}$ are positive constants to be determined. From $\mathcal{R}_0 > 1$, we get $k_0 = \beta\mathcal{C}_1 - \mu - \gamma_2 - \alpha > 0$. A simple calculation yields

$$\overline{S}_t - D\Delta\overline{S} = 0 \ge \Lambda - \mu\overline{S},$$

$$\overline{I}_t - D\Delta\overline{I} - \big(\beta\overline{S} - (\mu+\gamma_2+\alpha)\big)\overline{I} \ge \mathcal{M}e^{-\gamma t}\big[\frac{D}{8g_0^2} - \gamma - k_0\big],$$
$$\overline{R}_t - D\Delta\overline{R} - \big(\gamma_2\overline{I} - \mu\overline{R}\big) \ge \mathcal{M}e^{-\gamma t}\big[\frac{D}{8g_0^2} - \gamma - \gamma_2\big],$$

for $0 < z < \overline{g}(t)$ and $t > 0$.

Direct calculations yield $\overline{g}'(t) = 2g_0\gamma e^{-\gamma t}$ and $-\mu_1\overline{I}_z(\overline{g}(t), t) = 2\mathcal{M}\mu_1\overline{g}^{-1}(t)e^{-\gamma t}$. Hence, $\overline{S}(z,0) \ge S_0(z)$, $\overline{I}(z,0) = \mathcal{M}\big(1 - \frac{z^2}{4g_0^2}\big) \ge \frac{3}{4}\mathcal{M}$, and $\overline{R}(z,0) = \mathcal{M}\big(1 - \frac{z^2}{4g_0^2}\big) \ge \frac{3}{4}\mathcal{M}$ for $z \in [0, g_0]$. By choosing $\mathcal{M} = \frac{4}{3}\max\{\|I_0\|_\infty, \|R_0\|_\infty\}$, $\gamma = \frac{D}{16g_0^2}$, $\mu_1 \le \frac{D}{8\mathcal{M}}$, and $g_0 \le \min\{\frac{D}{16q_0}, \frac{D}{16\gamma_2}\}$, one gets

$$\begin{gathered}
\overline{S}_t - D\Delta\overline{S} \ge \Lambda - \mu\overline{S} - \overline{S}f(\underline{I}), \quad z > 0,\ t > 0,\\
\overline{I}_t - D\Delta\overline{I} \ge \overline{S}f(\overline{I}) - (\mu+\gamma_2+\alpha)\overline{I}, \quad 0 < z < \overline{g}(t),\ t > 0,\\
\overline{R}_t - D\Delta\overline{R} \ge \gamma_2\overline{I} - \mu\overline{R}, \quad 0 < z < \overline{g}(t),\ t > 0,\\
\overline{S}_z(0,t) = \overline{I}_z(0,t) = \overline{R}_z(0,t) = 0, \quad t > 0,\\
\overline{I}(z,t) = \overline{R}(z,t) = 0, \quad z \ge g(t),\ t > 0,\\
\overline{g}'(t) = -\mu_1\overline{I}_z(\overline{g}(t), t), \quad \overline{g}(0) = 2g_0 > g_0 > 0,\ t > 0,\\
\overline{S}(z,0) \ge S_0(z), \quad \overline{I}(z,0) \ge I_0(z), \quad \overline{R}(z,0) \ge R_0(z),\ ; z \ge 0.
\end{gathered}$$

From Lemma 3.6, $g(t) \le \overline{g}(t)$ for $t > 0$. Hence, $g_\infty \le \lim_{t\to\infty}\overline{g}(t) = 4g_0 < \infty$. The proof is complete. □

Let λ_1 represent the principle eigenvalue of the operator $-\Delta$ with respect to the homogeneous Dirichlet boundary condition. We then have the following result.

Theorem 3.9. *If $\mathcal{R}_0 > 1$, then $g_\infty = \infty$ provided that $g_0 > g_0^*$, where $\lambda_1(g_0^*) = \frac{\mu+\gamma_2+\alpha}{D}(\mathcal{R}_0 - 1)$.*

Proof. By a way of contradiction, we assume that $g_\infty < \infty$. From Theorem 3.7, $\lim_{t\to\infty}\|I(\cdot,t)\|_{C[0,g(t)]} = 0$. Further, $\lim_{t\to\infty} S(z,t) = \frac{\Lambda}{\mu}$ uniformly in the bounded subset. Consequently, for $\varepsilon > 0$, there exists $T^* > 0$ such that $S(z,t) \ge \frac{\Lambda}{\mu} - \varepsilon$ for $r \in [0, g(t))$, $t \ge T^*$. $I(z,t)$ satisfies

$$\begin{gathered}
I_t - D\Delta I \ge I\Big(f'(\varepsilon)\big(\frac{\Lambda}{\mu} - \varepsilon\big) - (\mu+\gamma_2+\alpha)\Big), \quad 0 < s < g_0,\ t > T^*,\\
I_z(0,t) = I(g_0,t) = 0, \quad t > T^*,\\
I(z,T^*) > 0, \quad 0 \le z < g_0.
\end{gathered}$$

$I(z,\,t)$ has a lower solution $\underline{I}(z,t)$ satisfying

$$\begin{gathered}
\underline{I}_t - D\Delta\underline{I} = \underline{I}\Big(f'(\varepsilon)\big(\frac{\Lambda}{\mu} - \varepsilon\big) - (\mu+\gamma_2+\alpha)\Big), \quad 0 < s < g_0,\ t > T^*,\\
\underline{I}_z(0,t) = \underline{I}(g_0,t) = 0, \quad t > T^*,\\
\underline{I}(z,T^*) = I(z,T^*), \quad 0 \le z < g_0.
\end{gathered}$$

From $g_0 > g_0^*$, we can choose sufficiently small ε satisfying

$$f'(\varepsilon)\big(\frac{\Lambda}{\mu} - \varepsilon\big) - (\mu+\gamma_2+\alpha) > D\lambda_1(g_0).$$

Hence, I is unbounded in $(0, g_0) \times [T^*, +\infty)$, which leads to a contradiction. The proof is complete. $\square$

4. Numerical simulations

In this section, we perform some numerical simulations to illustrate the theoretical results. For the homogeneous system, we choose parameters

$$\Lambda = 0.1, \quad d_U = 0.01, \quad \beta = 0.3, \quad \gamma_1 = 1, \quad \delta = 1, \quad \gamma_2 = 2, \quad \alpha = 2. \tag{4.1}$$

We choose the initial conditions as follows

$$S_0(x) = 5\Big(1 + 0.5\cos\big(\frac{9}{10}\pi\big)\Big), \quad I_0(x) = 5\Big(1 + 0.8\sin\big(\frac{9}{10}\pi\big)\Big),$$
$$R_0(x) = 5\Big(1 + 0.6\sin\big(\frac{9}{10}\pi\big)\Big), \quad x \in [0, 10],$$

and the Neumann boundary condition

$$\frac{\partial S(x,t)}{\partial \nu} = \frac{\partial I(x,t)}{\partial \nu} = \frac{\partial R(x,t)}{\partial \nu} = 0, \quad t > 0,\ x \in \partial\Omega.$$

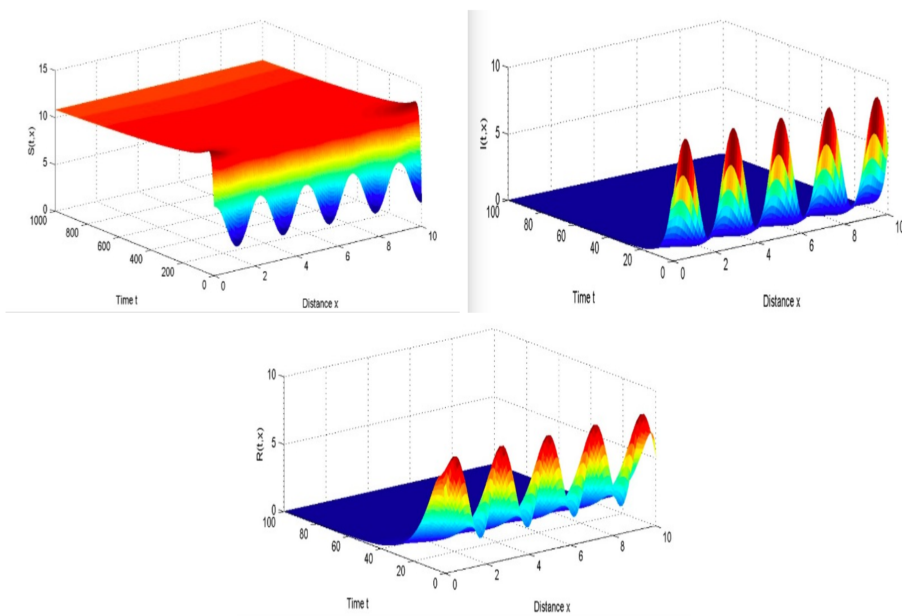

Figure 1. E_0 is globally asymptotically stable when $D = 0.1$.

For the case $D = 0.1$, by a simple computation, we get $\mathcal{R}_0 < 1$. In view of Theorem 2.1, the disease-free steady state E_0 of system (1.2) is globally asymptotically stable (see, Figure 1). Further, if $\Lambda = 50$ and the other parameters are the same as (4.1), system (1.2) exists a unique endemic steady state. By Theorem 2.1, the endemic steady state of system (1.2) is globally asymptotically stable (see, Figure 2). Similarly, for the case $D = 10000$, from Figures 3 and 4, the disease-free steady state E_0 of system (1.2) is globally asymptotically stable if $\mathcal{R}_0 < 1$, while the endemic steady state of system (1.2) is globally asymptotically stable if $\mathcal{R}_0 > 1$.

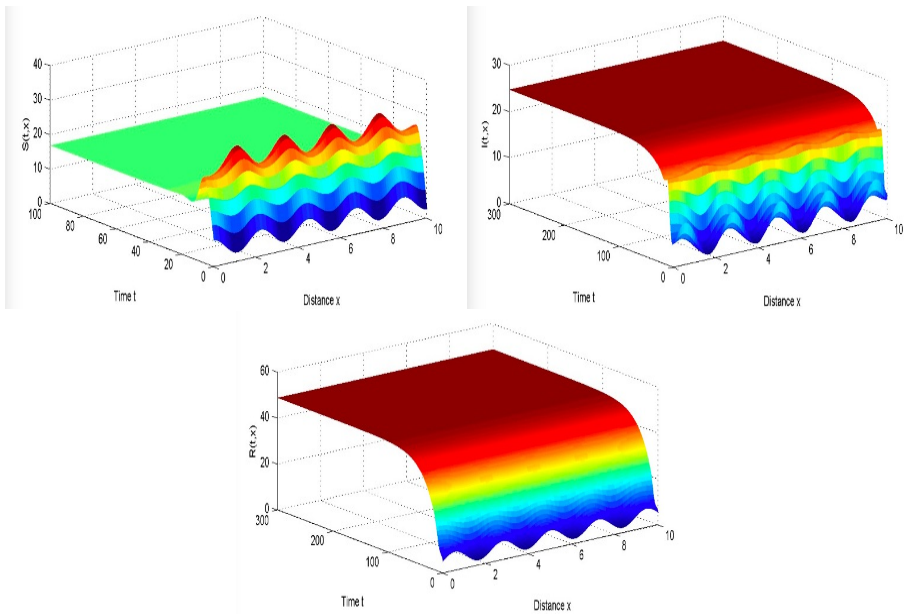

FIGURE 2. E^* is globally asymptotically stable when $D = 0.1$.

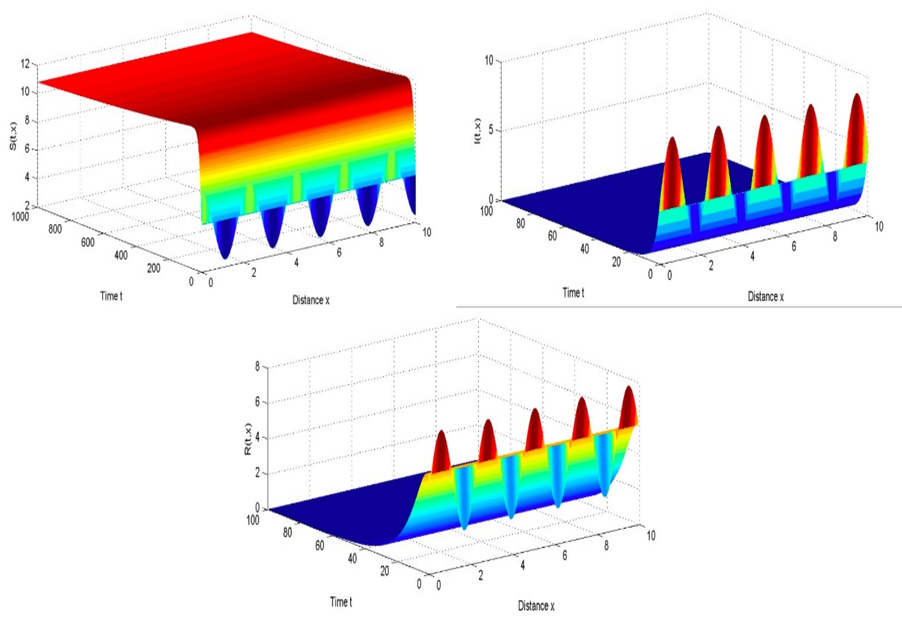

FIGURE 3. E_0 is globally asymptotically stable when $D = 10000$.

Next, we fix parameters as (4.1) and vary $\beta(x)$ with the following form

$$\beta(x) = \overline{\beta}(1 + 0.8 \cos \pi x),$$

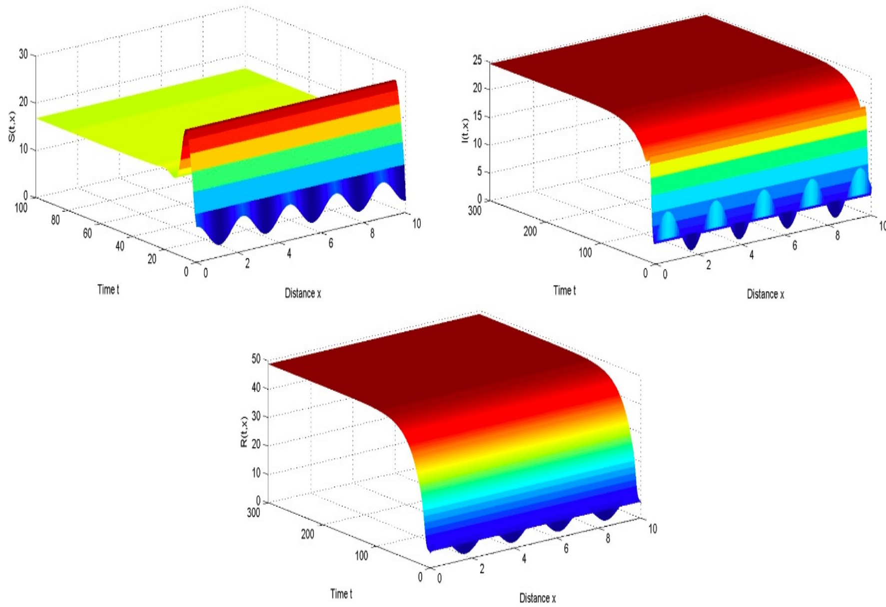

FIGURE 4. E^* is globally asymptotically stable when $D = 10000$.

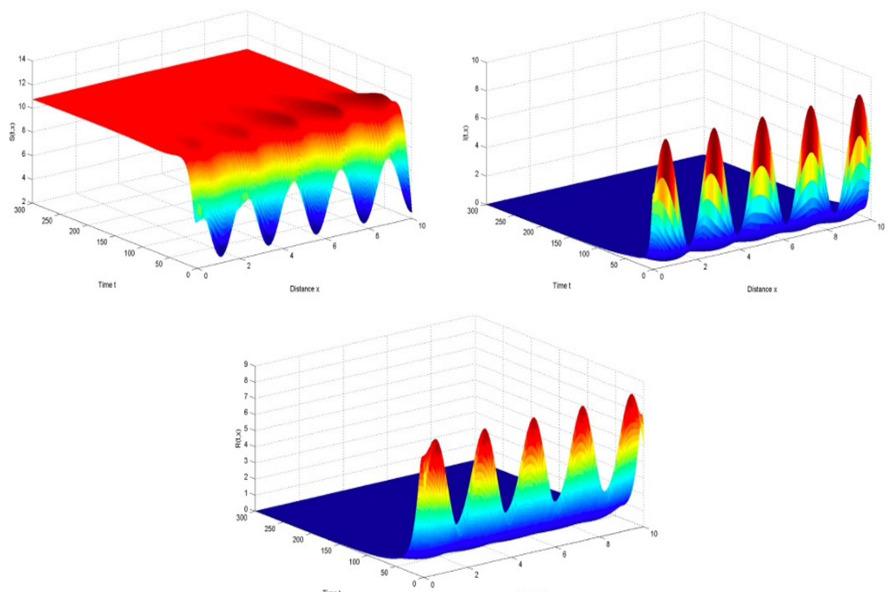

FIGURE 5. E_0 is globally asymptotically stable when $D = 0.1$.

where $\overline{\beta}$ is a positive constant. We choose the function β to explore the difference for the dynamical behavior between the homogeneous system and the heterogeneous system.

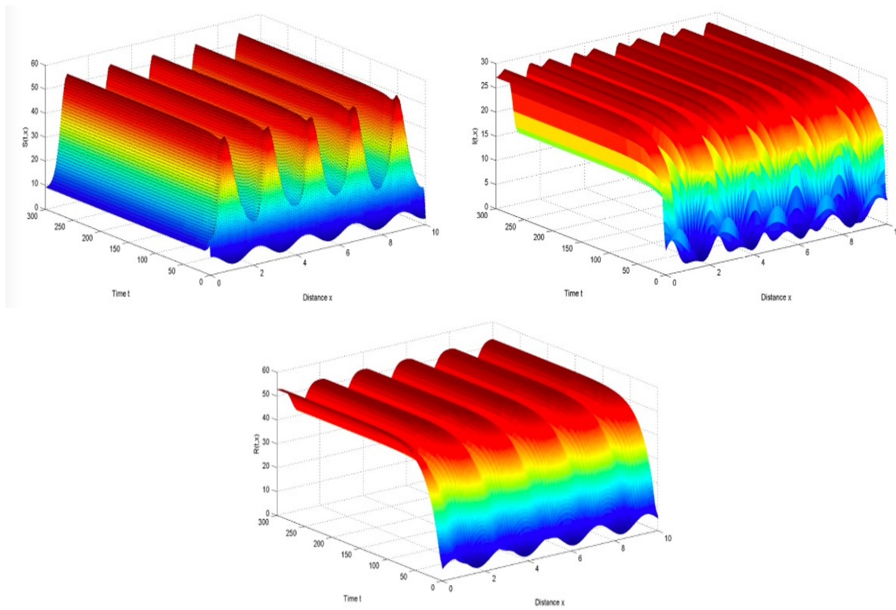

Figure 6. The endemic steady state of system (1.2) converges to a positive distribution which is not a constant when $D = 0.1$.

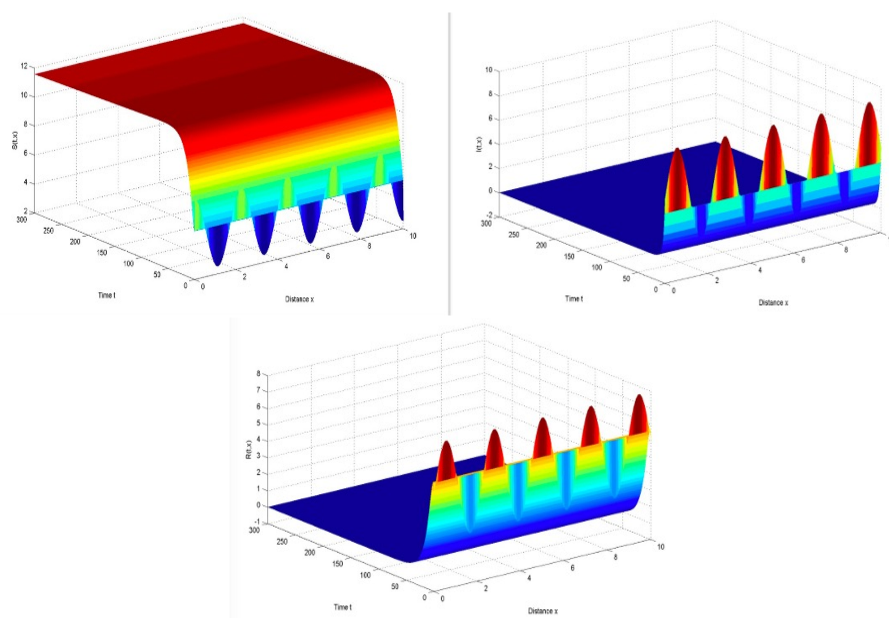

Figure 7. E_0 is globally asymptotically stable when $D = 10000$.

Let $D = 0.1$. For $\overline{\beta} = 0.3$ and the other parameters as (4.1), we have $\mathcal{R}_0 < 1$. In Figure 5, we observe that the disease-free steady state E_0 of system (1.2) is globally

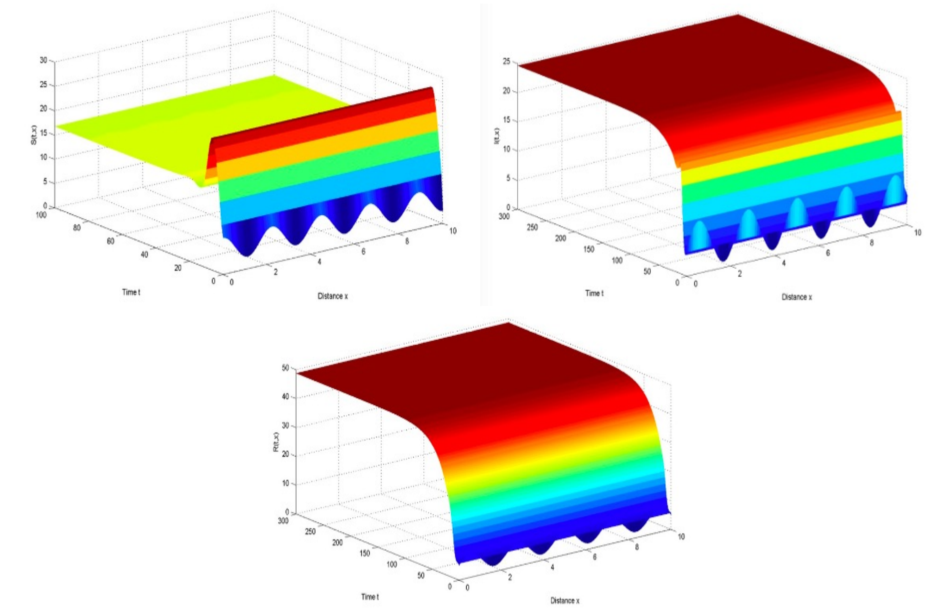

FIGURE 8. The endemic steady state of system (1.2) converges to a positive constant distribution which is the homogeneous constant steady state when $D = 10000$.

asymptotically stable. On the other hand, for $\Lambda = 50$ and the other parameters are the same as (4.1), we get $\mathcal{R}_0 > 1$. Thus, from Theorem 2.1, the endemic steady state of system (1.2) converges to a positive distribution which is not a constant (see, Figure 6).

Let $D = 10000$. For $\overline{\beta} = 0.3$ and the other parameters as (4.1), we get $\mathcal{R}_0 < 1$. In fact, in Figure 7, the disease-free steady state E_0 of system (1.2) is globally asymptotically stable. On the other hand, for $\Lambda = 50$ and the other parameters are the same as (4.1), it then follows that $\mathcal{R}_0 > 1$. From the numerical simulations, the endemic steady state of system (1.2) converges to a positive constant distribution which is the homogeneous constant steady state (see, Figure 8).

In biology, for the homogeneous system, we observe that the final state of the infectious disease is independent on its dispersal rate, while for the heterogeneous system, the final state of the infectious disease is dependent on its dispersal rate.

Discussions and conclusions

In this paper, we have proposed a SIRS epidemic reaction-diffusion system with two different kinds of boundary conditions. For the problem with the Neumann boundary condition, we have obtained the global dynamics, which are fully determined by the basic reproduction number $\mathcal{R}_0$. To make a better understanding for the transmissions dynamics for the disease, we further consider a free boundary problem of system (1.3). Main results reveal that besides the basic reproduction number, the size of initial epidemic region and the diffusion rate of the disease also play a crucial role in the disease transmission.

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