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WIRTINGER-BEESACK INTEGRAL INEQUALITIES

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Abstract. A uniform method of obtaining various types of integral inequalities involving a function and its first or second derivative is extended to integral inequalities involving a function and its third derivative

1. Introduction

Integral inequalities of the form

$$\int_I sh^2 dt \le \int_I rh''^2 dt, \quad h \in H, \tag{1.1}$$

have appeared in publications such as [1, 2]. In the above equation I is the interval (α, β), with $-\infty \le \alpha < \beta \le \infty$, $r > 0$, $r \in AC(I)$,

$$s = (r\varphi'')''\varphi^{-1} \tag{1.2}$$

with a given function $\varphi \in AC^1(I)$ such that $\varphi > 0$ on the interval I, $r\varphi'' \in AC^1(I)$, $\omega = (r\varphi')'\varphi + 2r\varphi\varphi'' - 2r\varphi'^2 \le 0$ and H is the class of functions $h \in AC^1(I)$ satisfying some integral and limit conditions.

In this article, we assume that $r \in AC^1(I)$, $\varphi \in AC^2(I)$ and $r\varphi''' \in AC^2(I)$ are such that $r > 0$, $\varphi > 0$ on the interval I. Putting

$$s = -(r\varphi''')'''\varphi^{-1}, \tag{1.3}$$

we obtain the integral inequality

$$\int_I sh^2 dt \le \int_I rh'''^2 dt, \quad h \in H. \tag{1.4}$$

The method used here consists in determining auxiliary functions depending on the given function r and the auxiliary function φ so that these functions determine the class H for which the inequality (1.4) holds.

2. Main result

Let $I = (\alpha, \beta)$ be an arbitrary open interval with $-\infty \le \alpha < \beta \le \infty$. We denote by $AC^k(I)$ the set of functions whose k derivative is absolutely continuous on the

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interval I. Let $r \in AC^1(I)$ and $\varphi \in AC^2(I)$ be given functions such that $r > 0$, $\varphi > 0$ on the interval I and $r\varphi''' \in AC^2(I)$. Let us put

$$s = -(r\varphi''')'''\varphi^{-1},$$

Let us denote by H the set of functions $h \in AC^2(I)$ for which

$$\int_I rh'''^2 dt < \infty, \quad \int_I sh^2 dt > -\infty \tag{2.1}$$

and satisfy the limit conditions

$$\liminf_{t\to\alpha} S(t,h,h',h'') < \infty, \quad \limsup_{t\to\beta} S(t,h,h',h'') > -\infty, \tag{2.2}$$

$$\liminf_{t\to\alpha} S(t,h,h',h'') \le \limsup_{t\to\beta} S(t,h,h',h''), \tag{2.3}$$

where

$$\begin{aligned} &S(t,h,h',h'') \\ &= \nu_0(t)h^2 + \nu_1(t)h'^2 + \nu_2(t)h''^2 + 2\varepsilon_{01}(t)hh' + 2\varepsilon_{02}(t)hh'' + 2\varepsilon_{12}h'h'', \end{aligned} \tag{2.4}$$

$$\nu_0(t) = [(r\varphi''')'\varphi]'\varphi^{-2} - \frac{1}{2}r\varphi'''\varphi^{-3}(\varphi^2)'' - 3(\varphi'\varphi^{-1})^3\Big(\frac{r\varphi''}{\varphi'}\Big)' - 2r\varphi'^3\varphi^{-2}(\varphi'\varphi^{-2})', \tag{2.5}$$

$$\nu_1(t) = -6(r\varphi''\varphi^{-1})' - 2r(\varphi''\varphi^{-1})' + 4r(\varphi'\varphi^{-1})^3, \tag{2.6}$$

$$\nu_2(t) = r\varphi'\varphi^{-1}, \tag{2.7}$$

$$\varepsilon_{01}(t) = -(r\varphi'''\varphi)'\varphi^{-2} + 3(r\varphi''\varphi^{-2})'\varphi' + r[(\varphi''\varphi^{-1})^2 - 4(\varphi'\varphi^{-1})^4], \tag{2.8}$$

$$\varepsilon_{02}(t) = r[(\varphi'\varphi^{-1})'' + \varphi'\varphi''\varphi^{-2}], \tag{2.9}$$

$$\varepsilon_{12}(t) = r\varphi(\varphi'\varphi^{-2})'. \tag{2.10}$$

These assumptions apply that $\nu_0 \in AC(I)$, $\nu_1, \varepsilon_{01} \in AC^1(I)$ and $\nu_2, \varepsilon_{02}, \varepsilon_{12} \in AC^2(I)$.

The following theorem is the main result of this paper.

Theorem 2.1. *Let*

$$\omega_0(t) = [(r\varphi''' + (r\varphi'')'\varphi^{-1}]\varphi^2 + r\varphi''^2 \ge 0, \tag{2.11}$$

$$\omega_1(t) = 2r\varphi'^2 - 2r\varphi''\varphi - (r\varphi')'\varphi \ge 0 \tag{2.12}$$

almost everywhere on the interval I. Then for every function $h \in H$ the inequality (1.4) *holds.*

If $\omega_0 \ne 0$, $\omega_1 \ne 0$ and $h \ne 0$ then (1.4) *becomes an equality if and only if $h = c\varphi$ with c a non-zero constant, $\varphi \in H$, and and*

$$\lim_{t\to\alpha} S(t,h,h',h'') = \lim_{t\to\beta} S(t,h,h',h''). \tag{2.13}$$

Proof. For this proof, we use a standard method for obtaining various types of integral inequalities involving a function and its third derivative. See, for example, [1, 2] and the references cited there in.

Let $h \in AC^2(I)$. From (2.4)–(2.10) and the assumptions, we have $\varphi^{-1}h \in AC^2(I)$ and $S(t,h,h',h'') \in AC(I)$. If we substitute $h = \varphi f$, where $f \in AC^2(I)$, in the expression rh'''^2, then, after simple calculations, we obtain

$$\begin{aligned}
rh'''^2 &= r\left(\varphi''' f + 3\varphi'' f' + 3\varphi' f'' + \varphi f'''\right)^2 \\
&= rh'''[\varphi''' f^2 + 3\varphi''(f^2)' + 3\varphi'(f^2)'' + \varphi(f^2)'''] + r(3\varphi'' f' + 3\varphi' f'' + \varphi f''')^2 \\
&\quad - 6r\varphi'''(\varphi' f'^2 + \varphi f' f'') \\
&= r\varphi'''(\varphi f^2)''' - 3(r\varphi'''\varphi f'^2)' + 3[(r\varphi''')'\varphi - r\varphi'''\varphi']f'^2 \\
&\quad + r\big(3\varphi'' f' + 3\varphi' f'' + \varphi f'''\big)^2.
\end{aligned}$$

Then, using the obvious identity

$$r\varphi'''(\varphi f^2)''' + (r\varphi''')'''\varphi f^2 = [r\varphi'''(\varphi f^2)'' - (r\varphi''')'(\varphi f^2)' + (r\varphi''')''\varphi f^2]',$$

and

$$\begin{aligned}
& r\big(3\varphi'' f' + 3\varphi' f'' + \varphi f'''\big)^2 \\
&= 3[r\varphi''^2 + (r\varphi'')''\varphi - (r\varphi'')'\varphi']f'^2 + 3[2r\varphi'^2 - 2r\varphi''\varphi - (r\varphi')'\varphi]f''^2 + r\varphi^2 f'''^2 \\
&\quad + 3[2r\varphi''\varphi' f'^2 + r\varphi'\varphi f''^2 + 2r\varphi''\varphi f' f'' - (r\varphi'')'\varphi f'^2]',
\end{aligned}$$

we obtain

$$\begin{aligned}
rh'''^2 &= sh^2 + 3\omega_0 f'^2 + 3\omega_1 f''^2 + r\varphi^2 f'''^2 \\
&\quad + \Big\{[r\varphi'''(\varphi f^2)'' - (r\varphi''')' \cdot (\varphi f^2)' + (r\varphi''')''\varphi f^2] \\
&\quad + 3[2r\varphi''\varphi' - (r\varphi'')'\varphi - r\varphi'''\varphi]f'^2 + 6r\varphi''\varphi f' f'' + 3r\varphi'\varphi f''^2\Big\}'.
\end{aligned}$$

Now substituting $f = \varphi^{-1}h$ on the right hand side of the above identity, and using

$$\begin{gathered}
\varphi f^2 = \varphi^{-1}h^2, \\
(\varphi f^2)' = (\varphi^{-1})'h^2 + 2\varphi^{-1}hh', \\
(\varphi f^2)'' = (\varphi^{-1})''h^2 + 4(\varphi^{-1})'hh' + 2\varphi^{-1}h'^2 + 2\varphi^{-1}hh'', \\
f' = (\varphi^{-1})'h + \varphi^{-1}h', \\
f'' = (\varphi^{-1})''h + 2(\varphi^{-1})'h' + \varphi^{-1}h'',
\end{gathered}$$

we obtain the identity

$$rh'''^2 - sh^2 = [S(t,h,h',h'')]' + 3\omega_0(\varphi^{-1}h)'^2 + 3\omega_1(\varphi^{-1}h)''^2 + r\varphi^2(\varphi^{-1}h)'''^2. \tag{2.14}$$

Now let $h \in H$. Condition (1.3) implies that the function rh'''^2 is summable on I since $rh'''^2 \ge 0$ on I. It follows from assumptions that the function sh^2 and $[S(t,h,h',h'')]'$ are summable on each compact interval $[a,b] \subset I$. Thus by (2.14) we get the summability of the function

$$3\omega_0(\varphi^{-1}h)'^2 + 3\omega_1(\varphi^{-1}h)''^2 + r\varphi^2(\varphi^{-1}h)'''^2 \tag{2.15}$$

on each compact interval $[a,b] \subset I$ and we obtain the equality

$$\int_a^b rh'''^2 dt = \int_a^b sh^2 dt + S(t,h,h',h'')\Big|_a^b + \int_a^b g(t)dt. \tag{2.16}$$

for arbitrary $\alpha < a_n < b_n < \beta$, $a_n \to \alpha$, $b_n \to \beta$ and

$$\lim_{n\to\infty} S(t,h,h',h'')\Big|_{a_n} < \infty, \quad \lim_{n\to\infty} S(t,h,h',h'')\Big|_{b_n} > -\infty.$$

Thus, there is a constant C such that

$$-S(t,h,h',h'')\Big|_{a_n}^{b_n} \le C < \infty.$$

By condition (2.15), $g \ge 0$ a.e. on I. From (2.16), we infer that

$$\int_{a_n}^{b_n} sh^2 dt \le \int_{a_n}^{b_n} rh'''^2 t + C \le \int_{I_n} rh'''^2 dt + C,$$

and from this, letting $n \to \infty$, we obtain

$$\int_I sh^2 dt \le \int_I rh'''^2 dt + C < \infty.$$

From this estimate and by the second condition of (2.1), we conclude that sh^2 is summable on I. Next, in a similar way, using (2.16) and the sum ability of the function sh^2 on I, we prove that the function g is sum able on I. Thus all the integrals in (2.16) have finite limits as $a \to \alpha$ or $b \to \beta$, and hence both of the limits in (2.2) are proper and finite. Therefore the conditions (2.2) and (2.3) may be written in the equivalent form

$$-\infty < \lim_{t\to\alpha} S(t,h,h',h'') \le \lim_{t\to\beta} S(t,h,h',h'') < \infty.$$

Now by (2.16) as $a \to \alpha$ and $b \to \beta$, we obtain the equality

$$\int_I rh'''^2 dt - \int_I sh^2 dt = \lim_{t\to\beta} S(t,h,h',h'') - \lim_{t\to\alpha} S(t,h,h',h'') + \int_I g dt, \tag{2.17}$$

hence, in view of (2.15), the inequality (1.4) follows, since $g \ge 0$ a.e. on I.

If (1.4) becomes an equality for a non-vanishing function $h \in H$, then by (2.15) and (2.17), we have

$$\int_I g dt = 0, \quad \lim_{t\to\alpha} S(t,h,h',h'') = \lim_{t\to\beta} S(t,h,h',h''). \tag{2.18}$$

Since $g \ge 0$ a.e. on I, we obtain $g = 0$ a.e. on I. In view of g it follows from assumptions that it $g = 0$ a.e. on I, then $(\varphi^{-1}h)'(t_0) = 0$ for some $t_0 \in I$, and we get that $(\varphi^{-1}h)' = 0$ on I, since $(\varphi^{-1}h)' \in AC^2(I)$.

This implies that $h = C\varphi$, where $C = \mathit{const} \neq 0$, since $\varphi^{-1}h \in AC^2(I)$. Thus $\varphi \in H$, so that we obtain from the condition (2.18) we get the condition (17).

Now let (2.17) be satisfied and let $h = C\varphi$, where $C = \mathit{const} \neq 0$. This implies $g = 0$ a.e. on I, so that $\int_I g dt = 0$. In view of (2.15)-(2.18), (1.4) becomes equality. The theorem is proved. □

3. Example

Let $I = (-1,1)$, $r = (1-t^2)^a$ and $\varphi = (1-t^2)^{3-a}$ on I, where a is an arbitrary constant such that the case $a \in (-\infty;1]$ is considered. Then by (1.3), (2.11) and (2.12), we have

$$s = -(r\varphi''')'''\varphi^{-1} = 24(3-a)(2-a)(5-2a)(1-t^2)^{a-3} > 0,$$
$$\omega_0 = 4 - (3-a)(1-t^2)^{2-a}[(15-6a) + (12a-30)t^2 + (15-29a)t^4] > 0,$$
$$\omega_1 = 2(3-a)(1-t^2)^{5-a} > 0 \text{ on } I.$$

From Theorem 2.1. we obtain that the inequality (1.4) holds for every function $h \in H$, where H is the class of function $h \in AC^2((-1,1))$ satisfying the integral condition

$$\int_{-1}^{1} (1-t^2)^a h'''^2 dt < \infty \tag{3.1}$$

and the limit condition

$$-\infty < \lim_{t\to -1} S(t,h,h',h'') \le \lim_{t\to 1} S(t,h,h',h'') < \infty, \tag{3.2}$$

where by (2.3)-(2.10)

$$\begin{aligned} S(t,h,h',h'') &= \nu_0(t)(1-t^2)^{a-5}h^2 + \nu_1(t)(1-t^2)^{a-3}h'^2 \\ &\quad + \nu_2(t)(1-t^2)^{a-1}h''^2 + 2\varepsilon_{01}(t)(1-t^2)^{a-4}hh' \\ &\quad + 2\varepsilon_{02}(t)(1-t^2)^{a-3}hh'' + 2\varepsilon_{12}(t)(1-t^2)^{a-2}hh'', \end{aligned} \tag{3.3}$$

$$\begin{aligned} \nu_0(t) =& 8(3-a)t[-3(a^2-3a+1) + (12a^3-90a^2+238a-222)]t^2 \\ &+ (-4a^4+60a^3-319a^2+811a-528)t^4], \end{aligned}$$

$$\begin{gathered} \nu_1(t) = -8(3-a)t[6-a+2(7-2a)t^2], \\ \nu_2(t) = -2(3-a)t, \\ \varepsilon_{01}(t) = 4(3-a)[2a-3+(-10a^2+52a-66)t^2 + (28a^3-238a^2+728a-803)t^4], \\ \varepsilon_{02}(t) = -4(3-a)[a+(2a^2-11a+16)t^2], \\ \varepsilon_{12}(t) = -4(3-a)[1+(7-2a)t^2]. \end{gathered}$$

Since the second condition of (2.1) is satisfied trivially. Now we show that a function $h \in AC^2((-1,1))$ that satisfies the integral condition (3.1) and limit conditions $h(\pm 1) = h'(\pm 1) = h''(\pm 1) = 0$ belongs to the class H.

At first we show that, if $h(1) = h'(1) = h''(1) = 0$ and (3.1) hold, then

$$\lim_{t\to 1} S(t,h,h',h'') = 0\,.$$

Let us consider the right-hand neighborhood U of the point 1. In [1], it has been shown that

$$|h'(t)| \le k(t)(1-t)^{\frac{1-a}{2}} \tag{3.4}$$

for $t \in U$, where

$$k(t) = \Big\{\frac{A}{1-a}\int_t^1 (1-\tau^2)^a h''^2(\tau)d\tau\Big\}^{1/2} > 0, \quad t \in U\,.$$

This function is a continuous function on I, $\lim_{t\to 1} k(t) \equiv k(1) = 0$, and

$$|h(t)| \le \frac{k(\theta)}{\sqrt{2-a}}(1-t)^{\frac{3-a}{2}}, \tag{3.5}$$

for $t \in U$, where $t < \theta < 1$ and $\lim_{t\to 1} k(t) \equiv k(1) = 0$.

It is easy to see that if we write h''' instead of h'', h'' instead of h', and h' instead of h in (3.4) and (3.5) then we obtain

$$|h''(t)| \le k(t)(1-t)^{\frac{1-a}{2}} \tag{3.6}$$

for $t \in U$, with k as above and

$$|h'(t)| \le \frac{k(\theta)}{\sqrt{2-a}}(1-t)^{\frac{3-a}{2}}, \tag{3.7}$$

for $t \in U$, where $t < \theta < 1$ and $\lim_{t\to 1} k(t) \equiv k(1) = 0$. From (3.5) we have

$$|h(t)| \le \frac{2k(\theta)}{(5-a)\sqrt{2-a}}(1-t)^{\frac{5-a}{2}}\,. \tag{3.8}$$

Based on the estimates (3.6), (3.7) and (3.8), from (3.3), we obtain

$$\begin{aligned} |S(t,h,h',h'')| &\le \frac{4k^2(\theta)}{(2-a)(5-a)^2}|\nu_0(t)| + \frac{k^2(\theta)}{2-a}|\nu_1(t)| \\ &\quad + k^2(\theta)|\nu_2(t)| + \frac{2k^2(\theta)}{(2-a)(5-a)}|\varepsilon_{01}(t)| \\ &\quad + \frac{2k^2(\theta)}{(5-a)\sqrt{2-a}}|\varepsilon_{02}(t)| + \frac{k^2(\theta)}{\sqrt{2-a}}|\varepsilon_{12}(t)| = m(t) \end{aligned}$$

Whence it follows that $\lim_{t\to 1} S(t,h,h',h'') = 0$. In an analogous way we show that if $h(-1) = h'(-1) = h''(-1) = 0$ and (3.1) hold then $\lim_{t\to -1} S(t,h,h',h'') = 0$. Therefore we get the following result.

Theorem 3.1. *If $a < 1$ and the function $h \in AC^2((-1,1))$ satisfies the integral condition*

$$\int_{-1}^{1}(1-t^2)^a h'''^2 dt < \infty$$

and the limit condition $h(\pm 1) = h'(\pm 1) = h''(\pm 1) = 0$, then

$$\int_{-1}^{1}(1-t^2)^a h'''^2 dt \ge 24(3-a)(2-a)(5-2a)\int_{-1}^{1}\frac{h^2 dt}{(1-t^2)^{3-a}}\,.$$

holds. The inequality (3.4) *becomes on equality if and only if $h = C(1-t^2)^{3-a}$, where C is a constant.*

In the particular case for $a = 0$ we obtain

$$\int_{-1}^{1} h'''^2 dt \ge 720\int_{-1}^{1}\frac{h^2 dt}{(1-t^2)^3}$$

as deduced in [3].

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