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MULTIPLICITY AND CONCENTRATION OF POSITIVE SOLUTIONS FOR FRACTIONAL NONLINEAR SCHRÖDINGER EQUATIONS WITH CRITICAL GROWTH

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Abstract. In this article we consider the multiplicity and concentration behavior of positive solutions for the fractional nonlinear Schrödinger equation

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = u^{2_s^*-1} + f(u), \quad x \in \mathbb{R}^N, u \in H^s(\mathbb{R}^N), \quad u(x) > 0,$$

where ε is a positive parameter, $s \in (0,1)$, $N > 2s$ and $2_s^* = \frac{2N}{N-2s}$ is the fractional critical exponent, and f is a $\mathcal{C}^1$ function satisfying suitable assumptions. We assume that the potential $V(x) \in \mathcal{C}(\mathbb{R}^N)$ satisfies $\inf_{\mathbb{R}^N} V(x) > 0$, and that there exits k points $x^j \in \mathbb{R}^N$ such that for each $j = 1, \dots, k$, $V(x^j)$ are strictly global minimum. By using the variational method, we show that there are at least k positive solutions for a small $\varepsilon > 0$. Moreover, we establish the concentration property of solutions as ε tends to zero.

1. Introduction

In this article we study the multiplicity and concentration phenomena of positive solutions to the following fractional nonlinear Schrödinger equation

$$\begin{gathered} \varepsilon^{2s}(-\Delta)^s u + V(x)u = u^{2_s^*-1} + f(u), \quad x \in \mathbb{R}^N, \\ u \in H^s(\mathbb{R}^N), \quad u(x) > 0, \end{gathered} \tag{1.1}$$

where $\varepsilon > 0$ is a positive parameter, $s \in (0,1)$, $N > 2s$ and $2_s^* = \frac{2N}{N-2s}$ is the fractional critical exponent. Here $(-\Delta)^s$ is the fractional Laplace operator defined, up to a normalization constant, by the Riesz potential as

$$(-\Delta)^s u(x) = -\int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} dy, \quad x \in \mathbb{R}^N;$$

see [13] for further details. This type of operator has a prevalent role in physics, biology, chemistry and finance. Recently, a great attention has been paid to the problems driven by fractional Laplacian, such as [1, 3, 7, 14] and references therein.

Solutions of (1.1) are related to the existence of standing wave solutions for the fractional nonlinear Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \hbar^{2s}(-\Delta)^s \psi + W(x)\psi - F(x, |\psi|)\psi, \tag{1.2}$$

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where $\hbar$ is the Planck's constant, the potential W is a suitable power of the density function, and $\psi(x,t)$ is the quantum mechanical probability amplitude for a given particle to have position x at time t. This equation was introduced by Laskin [20, 21], and it is based on the classical Schrödinger equation (corresponding to the case $s=1$), in which the Brownian motion of the quantum paths is related by Lévy flight.

For the classical Schrödinger equation, there a broad literature on the existence, multiplicity and concentration of positive solutions in the last decades, see for example [4, 26, 25, 33] and references listed therein. In particular, Cao and Noussair [4] considered the equation

$$-\Delta u+\mu u=Q(x)|u|^{p-2}u,\quad x\in\mathbb{R}^N$$

where $2<p<\frac{2N}{N-2}$ and $\mu>0$. They studied how the shape of the graph of $Q(x)$ affects the number of both positive and nodal solutions. Similarly, the multiplicity positive solutions of Kirchhoff type problem has been established by [35].

Nonlinear Schrödinger equations involving the fractional Laplacian have been studied extensively by many authors. See for example [9, 11, 12, 15, 17, 18, 23, 28]. Secchi [28] used the variational method to study the equation

$$\hbar^{2s}(-\Delta)^s u+V(x)u=f(u),\quad x\in\mathbb{R}^N.$$

Roughly speaking, under only a basic assumptions on subcritical nonlinearity f, the existence was obtained for small $\hbar>0$ whenever

$$\liminf_{|x|\to\infty}V(x)>\inf_{x\in\mathbb{R}^N}V(x). \tag{1.3}$$

In [17], the authors studied the fractional equation

$$\hbar^{2s}(-\Delta)^s u+V(x)u-u^p=0,\quad x\in\mathbb{R}^N, \tag{1.4}$$

under certain assumptions on the potential V. They showed that concentration points must be critical points for V. When $V(x)$ is a bounded function satisfies (1.3), and has a nondegenerate critical point, via a Lyapunov Schmidt type reduction, Chen and Zheng [9] obtained the existence and concentration phenomenon of solutions of (1.4) under further constraints in the space dimension N and the values of s. Moreover, Dávila et al. [12] studied equation (1.4) by Lyapunov Schmidt variational reduction, they recovered various existence results already known for the case $s=1$. In particular, and they constructed a single-peak solution around a minimizer of V in an open bounded set Ω whenever $\inf_{\partial\Omega}V>\inf_{\Omega}V$. Dávila et al. [11] considered the fractional Schrödinger equation in a bounded domain with zero Dirichlet datum, and built a family of solutions that concentrate at an interior point of the domain.

For the existence and multiplicity of solutions for the fractional Schrödinger equation with critical nonlinearities, we refer to [19, 29, 30, 32]. In [29], we used Ljusternik-Schnirelmann theory and Nehari manifold methods studied the equation (1.1) when V satisfying condition (1.3). We should mention a recent work of He and Zou [19] concerned the existence and concentration behavior of the fractional Schrödinger equation (1.1). Under a local condition imposed on V, they obtained the multiplicity of positive solutions concentrating around a set of local minimum of V.

Now some natural questions arise: If the potential V has k global minimum points, does the multiplicity of positive solutions of (1.1) exist? If so, what is the concentration profile of theses solutions as $\varepsilon \to 0$? These questions are the primary motivation of our paper.

In this article, we use the following assumptions:

- (H1) $V \in \mathcal{C}(\mathbb{R}^N)$ with $\inf_{x\in\mathbb{R}^N} V(x) = V_0 > 0$;
- (H2) there exits points $x^1, x^2, \dots, x^k$ in $\mathbb{R}^N$ such that $V(x^j)$ is a strict global minimum, namely satisfies $V(x^j) = V_0, j = 1, \dots, k$;
- (H3) $f : \mathbb{R} \to \mathbb{R}$ is a function of class $\mathcal{C}^1$ and $f(t) = 0$ for $t \le 0$;
- (H4) $\lim_{t\to 0} f(t)/t = 0$, $\lim_{t\to+\infty} f(t)/t^{2_s^*-1} = 0$;
- (H5) The function $t^{-1}f(t)$ is increasing for $t > 0$ and $\lim_{t\to+\infty} f(t)/t = +\infty$.

From conditions (H3)–(H5), we have

$$\frac{1}{2}f(t)t - F(t) \ge 0, \quad t^2 f'(t) - tf(t) \ge 0, \quad \forall t \in \mathbb{R}, \tag{1.5}$$

where $F(t) = \int_0^t f(u)du$. In particular, $\frac{1}{2}f(t)t - F(t)$ is increasing for $t \in \mathbb{R}$. Our main result reads as follows.

Theorem 1.1. *Suppose* (H1)–(H5) *are satisfied. Then, there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, problem* (1.1) *has at least k distinct positive solutions u_ε^j, $j = 1, 2, \dots, k$. Moreover, each u_ε^j has a maximum point $\overline{z}_\varepsilon^j \in \mathbb{R}^N$ with*

$$\lim_{\varepsilon\to 0} V(\overline{z}_\varepsilon^j) = V(x^j) = V_0,$$

and there exists $C^j > 0$ such that

$$u_\varepsilon^j(x) \le C^j \Big|\frac{x - \overline{z}_\varepsilon^j}{\varepsilon}\Big|^{-(N+2s)}.$$

Before proving our main result, some remarks are in order:

(i) As far as we know, there is no result on the multiplicity of positive solutions for problem (1.1) when V has multiple global minimum points. At present work, we prove the functional of autonomous problem has a minimizer over Pohozaev manifold. From this, we can conclude that the forthcoming Lemma 2.5. Next, inspired by [4], we use Ekeland's variational principle to get the existence of solutions of our problem. Then we also obtain the concentration of positive solutions for fractional nonlinear Schrödinger equation with critical growth.

(ii) Obviously, in the present article the conditions on nonlinear term f are weaker than in [19]. Furthermore, by (H1) we see that $V_\infty = \liminf_{|x|\to\infty} V(x) \ge \inf_{x\in\mathbb{R}^N} V(x)$, and hence our conditions on V are weaker than the global condition (1.3). If $V_\infty = +\infty$, Cheng [8] proved that the embeddings

$$H = \Big\{u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V(x)|u|^2 dx\Big\} \hookrightarrow L^p(\mathbb{R}^N),$$

are compact for $p \in (2, 2_s^*)$. From this we obtain the existence result by variational methods. Hence, in our this work we only study the case $V_\infty < +\infty$.

This article is organize as follows. In section 2, we collect some preliminary results that will be used later. In section 3, we study the multiplicity of positive solutions for an equivalent problem to (1.1) by Ekeland's variational principle. In section 4, we study the concentration behavior of these solutions, and then prove our main result.

In this paper, we will use the following notation: C, $C_0, C_1, C_2 \dots$ are positive (possibly different) constants. $B_r(z_0)$ denotes the ball centered at z_0 with radius r.

$u^+ = \max\{u, 0\}$ and $u^- = u^+ - u$. $o_n(1)$ and $o_\varepsilon(1)$ denotes the vanishing quantities as $n \to \infty$ and $\varepsilon \to 0$.

2. Preliminary results

In this section, we recall some known results for the readers convenience and for later use. First, we will give some useful facts of the fractional order Sobolev spaces. For any $s \in (0,1)$, the Hilbert space $H^s(\mathbb{R}^N)$ is defined by

$$H^s(\mathbb{R}^N) = \Big\{ u \in L^2(\mathbb{R}^N) : \frac{|u(x) - u(y)|}{|x-y|^{\frac{N+2s}{2}}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \Big\},$$

equipped with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \Big(\int_{\mathbb{R}^N} |u|^2 dx + \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \,dx\,dy \Big)^{1/2}.$$

Note that by [13], the embeddings $H^s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ for any $p \in [2, 2_s^*]$ are continuous, and local compact for $p \in [2, 2_s^*)$. Let S be the best Sobolev constant, i.e.,

$$S = \inf_{u \in \dot{H}^s(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \,dx\,dy}{(\int_{\mathbb{R}^N} |u|^{2_s^*} dx)^{2/2_s^*}} > 0, \tag{2.1}$$

where $\dot{H}^s(\mathbb{R}^N)$ is the homogeneous fractional sobolev space, defined as the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm $\|u\|^2_{\dot{H}^s(\mathbb{R}^N)} = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \,dx\,dy$. The constant S is well defined, as can be seen in [10].

Let $v(x) = u(\varepsilon x)$. Then (1.1) becomes

$$(-\Delta)^s v + V(\varepsilon x) v = v^{2_s^*-1} + f(v), \quad x \in \mathbb{R}^N. \tag{2.2}$$

Since equations (1.1) and (2.2) are equivalent, we shall thereafter focus on (2.2). Let E_ε be the Hilbert subsequence of $H^s(\mathbb{R}^N)$ under the norm

$$\|v\|_\varepsilon = \Big(\int_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^2}{|x-y|^{N+2s}} \,dx\,dy + \int_{\mathbb{R}^N} V(\varepsilon x)|v|^2 dx \Big)^{1/2}.$$

Associated with (2.2), we have the energy functional I_ε defined by

$$I_\varepsilon(v) = \frac{1}{2}\|v\|_\varepsilon^2 - \frac{1}{2_s^*}\int_{\mathbb{R}^N} |v|^{2_s^*} dx - \int_{\mathbb{R}^N} F(v) dx.$$

It is well known that I_ε is well-defined on E_ε and belongs to $\mathcal{C}^1(E_\varepsilon, \mathbb{R})$. Furthermore, let us define the solution manifold of (2.2)

$$\mathcal{M}_\varepsilon = \big\{ v \in E_\varepsilon \backslash \{0\} : \|v\|_\varepsilon^2 = \int_{\mathbb{R}^N} f(v) v dx + \int_{\mathbb{R}^N} |v|^{2_s^*} dx \big\}.$$

The ground energy associated with (2.2) is defined as

$$c_\varepsilon = \inf_{v \in \mathcal{M}_\varepsilon} I_\varepsilon(v).$$

To show our main theorem, we will consider the autonomous problem

$$(-\Delta)^s u + \mu u = f(u) + u^{2_s^*-1}, \quad x \in \mathbb{R}^N \tag{2.3}$$

where $\mu > 0$, and the $\mathcal{C}^1$ functional in E_μ defined as

$$J_\mu(u) = \frac{1}{2}\|u\|_\mu^2 - \frac{1}{2_s^*}\int_{\mathbb{R}^N} |u|^{2_s^*} dx - \int_{\mathbb{R}^N} F(u) dx,$$

whose critical points are the solutions of (2.3). In this case $E_\mu := H^s(\mathbb{R}^N)$ is endowed with the norm

$$\|u\|_\mu^2 = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} \mu|u|^2dx.$$

The solution manifold of (2.3) is defined as follows

$$\mathcal{N}_\mu = \big\{u \in E_\mu\backslash\{0\} : \|u\|_\mu^2 = \int_{\mathbb{R}^N} f(u)udx + \int_{\mathbb{R}^N} |u|^{2_s^*}dx\big\}.$$

Denote the ground energy associated with (2.3) by

$$m_\mu = \inf_{u\in\mathcal{N}_\mu} J_\mu(u).$$

By hypotheses (H4) and (H5), we obtain

$$0 < m_\mu = \inf_{u\in E_\mu\backslash\{0\}} \sup_{t\geq 0} J_\mu(tu) = \inf_{\eta\in\Gamma} \sup_{t\in[0,1]} J_\mu(\eta(t)), \tag{2.4}$$

where $\Gamma = \{\eta \in \mathcal{C}^1([0,1], E_\mu) : \eta(0) = 0, J_\mu(\eta(1)) < 0\}$.

Furthermore, by [19, Assertion 3.1], for each given $\mu > 0$, we have

$$m_\mu < \frac{s}{N} S^{N/(2s)}. \tag{2.5}$$

Lemma 2.1 ([18, 28])**.** *Assume that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$ and satisfies*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n(x)|^2dx = 0,$$

for some $R > 0$. Then $u_n \to 0$ strongly in $L^p(\mathbb{R}^N)$ for every $p \in (2, 2_s^*)$.

Lemma 2.2 ([19])**.** *Assume that* (H3)–(H5)*. For any $\mu > 0$, problem* (2.3) *has a positive ground state solution u_μ.*

Let $H_r^s(\mathbb{R}^N)$ be the subspace of $H^s(\mathbb{R}^N)$ consisting of radial symmetric functions. Define

$$m_{r,\mu} = \inf_{u\in\mathcal{N}_{r,\mu}} J_\mu(u),$$

where $\mathcal{N}_{r,\mu} = \{u \in H_r^s(\mathbb{R}^N)\backslash\{0\} : \langle J_\mu'(u), u\rangle = 0\}$.

Lemma 2.3. *The embeddings $H_r^s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ for $p \in (2, 2_s^*)$ are compact.*

Proof. The proof follows in a similar way as ins [6, Corollary 4.7.2]. In our case we have to consider the space $H_r^s(\mathbb{R}^N)$ instead of $H_r^1(\mathbb{R}^N)$. We omit the details. □

By Lemma 2.3, we can easily show that functional J_μ has a critical point $u_{r,\mu} \in H_r^s(\mathbb{R}^N)$ with

$$J_\mu(u_{r,\mu}) = m_{r,\mu}.$$

Let $u \in H^s(\mathbb{R}^N)$ be a weak solution of (2.3), we have the following Pohozaev equality (see [15])

$$P(u) = \frac{N-2s}{2}\|u\|_{\dot{H}^s(\mathbb{R}^N)}^2 - N\int_{\mathbb{R}^N}(F(u) + \frac{1}{2_s^*}|u|^{2_s^*} - \frac{\mu}{2}|u|^2)dx = 0.$$

Define the Pohozaev manifold $\mathcal{P}_\mu = \{u \in H^s(\mathbb{R}^N)\backslash\{0\} : P(u) = 0\}$. By Lemma 2.2, we see that $\mathcal{P}_\mu$ is not empty.

For any small $\tau > 0$, it follows from (H4) that there exists a $C_\tau > 0$ such that

$$|f(t)| \le \tau|t| + C_\tau|t|^{2_s^*-1}, \quad |F(t)| \le \frac{\tau}{2}|t|^2 + \frac{C_\tau}{2_s^*}|t|^{2_s^*}. \tag{2.6}$$

For $u \in \mathcal{P}_\mu$ and $\tau < \mu$, by (2.1) and (2.6), we obtain

$$\begin{aligned} &\frac{N-2s}{2}\|u\|^2_{\dot{H}^s(\mathbb{R}^N)} + \frac{N\mu}{2}\int_{\mathbb{R}^N}|u|^2dx \\ &= N\int_{\mathbb{R}^N}(F(u) + \frac{1}{2_s^*}|u|^{2_s^*})dx \\ &\le \frac{N\tau}{2}\int_{\mathbb{R}^N}|u|^2dx + \frac{N(1+C_\tau)}{2_s^*}S^{-\frac{N}{N-2s}}\|u\|^{\frac{2N}{N-2s}}_{\dot{H}^s(\mathbb{R}^N)}. \end{aligned} \tag{2.7}$$

Then, for all $u \in \mathcal{P}_\mu$,

$$\|u\|^2_{\dot{H}^s(\mathbb{R}^N)} \ge S^{N/(2s)}(1+C_\tau)^{-\frac{N-2s}{2s}} > 0$$

Moreover,

$$J_\mu(u) = J_\mu(u) - \frac{1}{N}P(u) = \frac{s}{N}\|u\|^2_{\dot{H}^s(\mathbb{R}^N)} \ge \frac{s}{N}S^{N/(2s)}(1+C_\tau)^{-\frac{N-2s}{2s}}.$$

Thus, J_μ is bounded below on $\mathcal{P}_\mu$. Set

$$c = \inf_{u\in\mathcal{P}_\mu} J_\mu(u).$$

It follows that $c > 0$. As in [27], we shall establish the following result.

Proposition 2.4. *Assume that* (H3)–(H5) *hold. Then J_μ has a minimizer over $\mathcal{P}_\mu$. Moreover, it is a critical point of J_μ in E_μ.*

Proof. Let $\{u_n\} \subset \mathcal{P}_\mu$ be such that $J_\mu(u_n) \to c$ as $n \to \infty$, and u_n^* denotes the symmetric radial decreasing rearrangement of u_n. By using a Polya-Szegö type inequality ([24]), we have that

$$\int_{\mathbb{R}^{2N}} \frac{|u_n^*(x) - u_n^*(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \le \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy$$

and by rearrangement properties, we also have that

$$\int_{\mathbb{R}^N} G(u_n^*)dx = \int_{\mathbb{R}^N} G(u_n)dx,$$

where $G(u) = F(u) + \frac{1}{2_s^*}|u|^{2_s^*} - \frac{\mu}{2}|u|^2$. It follows that

$$P(u_n^*) \le P(u_n). \tag{2.8}$$

For any $n \in \mathbb{N}$, setting $\omega_n(x) = u_n^*(\frac{x}{\theta_n})$, $\theta_n > 0$. We can choose $0 < \theta_n \le 1$ such that $\omega_n(x) \in \mathcal{P}_\mu$. Indeed, if

$$\int_{\mathbb{R}^{2N}} \frac{|u_n^*(x) - u_n^*(y)|^2}{|x-y|^{N+2s}}\,dx\,dy = \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy,$$

we take $\theta_n = 1$. Now we consider the case

$$\int_{\mathbb{R}^{2N}} \frac{|u_n^*(x) - u_n^*(y)|^2}{|x-y|^{N+2s}}\,dx\,dy < \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy.$$

Set

$$g_n(t) = P(u_n^*(\frac{x}{t})) = \frac{N-2s}{2}t^{N-2s}\|u_n^*\|^2_{\dot{H}^s(\mathbb{R}^N)} - t^N N\int_{\mathbb{R}^N} G(u_n^*)dx$$

$$
\begin{aligned}
&= \frac{N-2s}{2} t^{N-2s} \|u_n^*\|^2_{\dot H^s(\mathbb{R}^N)} - t^N N \int_{\mathbb{R}^N} G(u_n) dx \\
&= \frac{N-2s}{2} \Big(t^{N-2s} \|u_n^*\|^2_{\dot H^s(\mathbb{R}^N)} - t^N \|u_n\|^2_{\dot H^s(\mathbb{R}^N)} \Big).
\end{aligned}
$$

It is clear that $g_n(0) = 0$, $g_n(1) < 0$ and $g_n(t) \to -\infty$ as $t \to +\infty$. Then there exists a $t \in (0,1)$ such that $g_n(t) = 0$. Hence, there exists $0 < \theta_n \le 1$ such that $\omega_n \in \mathcal{P}_\mu$ and $\omega_n \in H^s_r(\mathbb{R}^N)$, it follows that

$$
\begin{aligned}
c \le J_\mu(\omega_n) &= \frac{\theta_n^{N-2s}}{2} \|u_n^*\|^2_{\dot H^s(\mathbb{R}^N)} - \theta_n^N \int_{\mathbb{R}^N} \Big(F(u_n^*) + \frac{|u_n^*|^{2_s^*}}{2_s^*} - \frac{\mu |u_n^*|^2}{2} \Big) dx \\
&\le \frac{\theta_n^{N-2s}}{2} \|u_n\|^2_{\dot H^s(\mathbb{R}^N)} - \theta_n^N \int_{\mathbb{R}^N} \Big(F(u_n) + \frac{|u_n|^{2_s^*}}{2_s^*} - \frac{\mu |u_n|^2}{2} \Big) dx \\
&\le \frac{1}{2} \|u_n\|^2_\mu - \frac{1}{2_s^*} \int_{\mathbb{R}^N} |u_n|^{2_s^*} dx - \int_{\mathbb{R}^N} F(u_n) dx \\
&= J_\mu(u_n) = c + o_n(1).
\end{aligned}
$$

This yields

$$
J_\mu(\omega_n) = c + o_n(1). \tag{2.9}
$$

Now we show that $\{\omega_n\}$ is bounded in E_μ. By (2.6), (2.9) and $\{\omega_n\} \subset \mathcal{P}_\mu$, we have

$$
c + o_n(1) = J_\mu(\omega_n) - \frac{1}{N} P(\omega_n) = \frac{s}{N} \|\omega_n\|^2_{\dot H^s(\mathbb{R}^N)}, \tag{2.10}
$$

and

$$
\begin{aligned}
&\frac{N-2s}{2} \|\omega_n\|^2_{\dot H^s(\mathbb{R}^N)} + \frac{N\mu}{2} \int_{\mathbb{R}^N} |\omega_n|^2 dx \\
&\le \frac{N\tau}{2} \int_{\mathbb{R}^N} |\omega_n|^2 dx + \frac{N(1+C_\tau)}{2_s^*} \int_{\mathbb{R}^N} |\omega_n|^{2_s^*} dx \,.
\end{aligned} \tag{2.11}
$$

Taking $\tau = \mu/2$ in (2.11), and arguing as in (2.7) we obtain

$$
\frac{\mu}{4} \int_{\mathbb{R}^N} |\omega_n|^2 dx \le \frac{(1+C_\tau)}{2_s^*} \int_{\mathbb{R}^N} |\omega_n|^{2_s^*} dx \le \frac{(1+C_\tau)}{2_s^*} S^{-\frac{N}{N-2s}} \|\omega_n\|^{\frac{2N}{N-2s}}_{\dot H^s(\mathbb{R}^N)}.
$$

This and (2.10) lead to the boundedness of $\{\omega_n\}$ in E_μ. Then, up to a subsequence, there exists $\omega \in E_\mu$ such that $\omega_n \rightharpoonup \omega$ weakly in E_μ and $\omega_n \to \omega$ a.e. in $\mathbb{R}^N$.

Next we are going to show that $\omega_n \to \omega$ strongly in E_μ and $\omega \in \mathcal{P}_\mu$, which implies c is attained by ω. Since $\{\omega_n\} \subset H^s_r(\mathbb{R}^N)$, then by (H3) and Lemma 2.3, we obtain

$$
\lim_{n\to\infty} \int_{\mathbb{R}^N} F(\omega_n) dx = \int_{\mathbb{R}^N} F(\omega) dx. \tag{2.12}
$$

Then, by Fatou's lemma,

$$
\begin{aligned}
&\frac{N-2s}{2} \|\omega\|^2_{\dot H^s(\mathbb{R}^N)} + \frac{N\mu}{2} \int_{\mathbb{R}^N} |\omega|^2 dx - N \int_{\mathbb{R}^N} F(\omega) dx \\
&\le \liminf_{n\to\infty} \Big(\frac{N-2s}{2} \|\omega_n\|^2_{\dot H^s(\mathbb{R}^N)} + \frac{N\mu}{2} \int_{\mathbb{R}^N} |\omega_n|^2 dx - N \int_{\mathbb{R}^N} F(\omega_n) dx \Big) \\
&= \frac{N}{2_s^*} \liminf_{n\to\infty} \int_{\mathbb{R}^N} |\omega_n|^{2_s^*} dx.
\end{aligned}
$$

This yields

$$
P(\omega) \le \frac{N}{2_s^*} \liminf_{n\to\infty} \int_{\mathbb{R}^N} (|\omega_n|^{2_s^*} - |\omega|^{2_s^*}) dx. \tag{2.13}
$$

We claim that $P(\omega) \geq 0$. Indeed, if $P(\omega) < 0$, then there exists a $0 < \theta < 1$ such that $\widetilde{\omega} = \omega(\frac{x}{\theta}) \in \mathcal{P}_\mu$. Therefore

$$\begin{aligned}
c \leq J_\mu(\widetilde{\omega}) &= J_\mu(\widetilde{\omega}) - \frac{1}{N} P(\widetilde{\omega}) = \frac{s}{N} \int_{\mathbb{R}^{2N}} \frac{|\widetilde{\omega}(x) - \widetilde{\omega}(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \\
&= \frac{s}{N} \theta^{N-2s} \int_{\mathbb{R}^{2N}} \frac{|\omega(x) - \omega(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \\
&< \frac{s}{N} \liminf_{n\to\infty} \int_{\mathbb{R}^{2N}} \frac{|\omega_n(x) - \omega_n(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \\
&= \liminf_{n\to\infty} J_\mu(\omega_n) = c.
\end{aligned}$$

This contradiction proves our claim. It follows from (2.13) that

$$0 \leq P(\omega) \leq \frac{N}{2_s^*} \liminf_{n\to\infty} \int_{\mathbb{R}^N} (|\omega_n|^{2_s^*} - |\omega|^{2_s^*}) dx. \tag{2.14}$$

Setting $\overline{\omega}_n = \omega_n - \omega$. By the Brezis-Lieb lemma [2] and (2.12), we obtain

$$P(\omega_n) - P(\omega) = \frac{N-2s}{2} \|\overline{\omega}_n\|^2_{\dot{H}^s(\mathbb{R}^N)} + \frac{N\mu}{2} \int_{\mathbb{R}^N} |\overline{\omega}_n|^2 dx - \frac{N}{2_s^*} \int_{\mathbb{R}^N} |\overline{\omega}_n|^{2_s^*} dx + o_n(1).$$

It follows from $\omega_n \in \mathcal{P}_\mu$ and (2.14) that

$$\frac{N-2s}{2} \|\overline{\omega}_n\|^2_{\dot{H}^s(\mathbb{R}^N)} + \frac{N\mu}{2} \int_{\mathbb{R}^N} |\overline{\omega}_n|^2 dx \leq \frac{N}{2_s^*} \int_{\mathbb{R}^N} |\overline{\omega}_n|^{2_s^*} dx + o_n(1). \tag{2.15}$$

Note that $\{\overline{\omega}_n\}$ is bounded in E_μ, up to a subsequence, we can assume that

$$\|\overline{\omega}_n\|^2_{\dot{H}^s(\mathbb{R}^N)} \to l \geq 0 \quad \text{and} \quad \int_{\mathbb{R}^N} |\overline{\omega}_n|^{2_s^*} dx \to L \geq 0.$$

By (2.1) and (2.15), we obtain that $l = 0$ if and only if $L = 0$. We now show that the case $l > 0$ can not occur by contradiction. From (2.1) and (2.15), we obtain

$$l \geq S^{N/(2s)}. \tag{2.16}$$

By the definition of $\overline{\omega}_n$, we have

$$\begin{aligned}
J_\mu(\omega_n) &= J_\mu(\omega_n) - \frac{1}{N} P(\omega_n) \\
&= \frac{s}{N} \|\overline{\omega}_n\|^2_{\dot{H}^s(\mathbb{R}^N)} + \frac{s}{N} \|\omega\|^2_{\dot{H}^s(\mathbb{R}^N)} + o_n(1).
\end{aligned}$$

It follows from (2.9) and (2.16) that

$$c \geq \frac{s}{N} S^{N/(2s)} + \frac{s}{N} \|\omega\|^2_{\dot{H}^s(\mathbb{R}^N)} + o_n(1). \tag{2.17}$$

On the other hand, from Lemma 2.2, we know that problem (2.3) has a ground state solution u_μ. Moreover, $u_\mu \in \mathcal{P}_\mu$. By (2.5), we obtain

$$c \leq J_\mu(u_\mu) = m_\mu < \frac{s}{N} S^{N/(2s)},$$

this leads to a contradiction to (2.17), and thus $l = 0$. Therefore, $\omega_n \to \omega$ in E_μ, and (2.14) implies $\omega \in \mathcal{P}_\mu$.

Next we verify that ω is a critical point of J_μ in E_μ. By the Lagrange multiplier, there exists a real number λ such that

$$J'_\mu(\omega) = \lambda P'(\omega). \tag{2.18}$$

First we claim that $P'(\omega) \neq 0$. Indeed, if not, in a weak sense the equation $P'(\omega) = 0$ can be written as

$$(N-2s)(-\Delta)^s\omega + N\mu\omega = Nf(\omega) + N|\omega|^{2_s^*-2}\omega. \tag{2.19}$$

So ω solves the equation (2.19). Then the Pohozaev equality applied to (2.19), we obtain

$$\frac{(N-2s)^2}{2}\|\omega\|^2_{\dot{H}^s(\mathbb{R}^N)} - N^2\int_{\mathbb{R}^N}(F(\omega) + \frac{1}{2_s^*}|\omega|^{2_s^*} - \frac{\mu}{2}|\omega|^2)dx = 0.$$

It follows from $P(\omega) = 0$ that $2s\|\omega\|^2_{\dot{H}^s(\mathbb{R}^N)} = 0$, this is contradict with $\omega \neq 0$. Thus $P'_\mu(\omega) \neq 0$. We now show that $\lambda = 0$. As above in the weak sense, we write (2.18) as

$$\begin{aligned}
&(-\Delta)^s\omega + \mu\omega - f(\omega) - |\omega|^{2_s^*-2}\omega \\
&= \lambda[(N-2s)(-\Delta)^s\omega + N(\mu\omega - f(\omega) - |\omega|^{2_s^*-2}\omega].
\end{aligned}$$

So ω solves the equation

$$(\lambda(N-2s)-1)(-\Delta)^s\omega + (\lambda N - 1)[\mu\omega - f(\omega) - |\omega|^{2_s^*-2}\omega] = 0.$$

By the Pohozaev equality and $\omega \in \mathcal{P}_\mu$, we deduce that

$$\begin{gathered}
\frac{(\lambda(N-2s)-1)(N-2s)}{2}\|\omega\|^2_{\dot{H}^s(\mathbb{R}^N)} + (\lambda N - 1)N\int_{\mathbb{R}^N} G(\omega)dx = 0, \\
\frac{N-2s}{2}\|\omega\|^2_{\dot{H}^s(\mathbb{R}^N)} + N\int_{\mathbb{R}^N} G(\omega)dx = 0.
\end{gathered}$$

It can be checked that $2\lambda s\|\omega\|^2_{\dot{H}^s(\mathbb{R}^N)} = 0$. Thus $\lambda = 0$. Hence, by (2.18) we have $J'_\mu(\omega) = 0$. The proof is complete. □

By the same arguments as in Proposition 2.4, we conclude that

$$c_r = \inf_{u\in\mathcal{P}_{r,\mu}} J_\mu(u)$$

is attained by $u \in \mathcal{P}_{r,\mu}$, where $\mathcal{P}_{r,\mu} = \{u \in H^s_r(\mathbb{R}^N)\backslash\{0\} : P(u) = 0\}$.

Lemma 2.5. $m_\mu = m_{r,\mu}$.

Proof. We first show that $m_\mu = c$. By Lemma 2.2, we have that $u_\mu \in \mathcal{P}_\mu$. Then

$$m_\mu = J_\mu(u_\mu) \geq c.$$

On the other hand, by Proposition 2.4, J_μ has a minimizer $\omega \in \mathcal{P}_\mu$ and $J'_\mu(\omega) = 0$. It is easy to see that there exists a sufficiently large $R > 0$ such that $J_\mu(R\omega) < 0$. Define a path $\eta(t) = \{tR\omega : t \in [0,1]\}$, which is an element of Γ. By (2.4), (H5) and $\omega \in \mathcal{N}_\mu$, we obtain

$$m_\mu \leq \max_{t\in[0,1]} J_\mu(\eta(t)) = \max_{t\in[0,1]} J_\mu(tR\omega) = J_\mu(\omega) = c.$$

Then we have shown that $m_\mu = c$. Similarly, we also obtain that

$$m_{r,\mu} = c_r. \tag{2.20}$$

Let u^*_μ denote the symmetric radial rearrangement of u_μ. Arguing as in the proof of Proposition 2.4, there exists $0 < \theta \leq 1$ such that $u^*_\mu(\theta^{-1}x) \in \mathcal{P}_{r,\mu}$. Then, it follows from $u_\mu \in \mathcal{P}_\mu$ that

$$m_{r,\mu} = \inf_{u\in\mathcal{P}_{r,\mu}} J_\mu(u)$$

$$
\begin{aligned}
&\le J_\mu(u_\mu^*(\theta^{-1}x)) = J_\mu(u_\mu^*(\theta^{-1}x)) - \frac{1}{N}P(u_\mu^*(\theta^{-1}x))\\
&= \frac{s}{N}\theta^{N-2s}\int_{\mathbb{R}^{2N}} \frac{|u_\mu^*(x)-u_\mu^*(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\\
&\le \frac{s}{N}\int_{\mathbb{R}^{2N}} \frac{|u_\mu^*(x)-u_\mu^*(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\\
&\le \frac{s}{N}\int_{\mathbb{R}^{2N}} \frac{|u_\mu(x)-u_\mu(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\\
&= J_\mu(u_\mu) = m_\mu.
\end{aligned}
$$

In addition, from (2.20), we have

$$
m_\mu = c = \inf_{u\in\mathcal{P}_\mu} J_\mu(u) \le \inf_{u\in\mathcal{P}_{r,\mu}} J_\mu(u) = c_r = m_{r,\mu}.
$$

Therefore, we have the desired result. □

3. Multiplicity of solutions

In this section, we investigate the effect of the shape of the graph of the potential V of problem (2.2) on the number of positive solutions. We introduce the map $\varphi : \dot{H}^s(\mathbb{R}^N) \to \dot{H}^s(\mathbb{R}^N)$

$$
\varphi(u) = \frac{1}{|B_1(x)|}\int_{B_1(x)} |u(z)|dz, \quad \forall x \in \mathbb{R}^N,
$$

where $|B_1(x)|$ is the Lebesgue measure of $B_1(x)$. Let

$$
h(u) = \frac{1}{2}\max_{x\in\mathbb{R}^N} \varphi(u), \quad \widehat{u}(x) = (\varphi(u) - h(u))^+.
$$

Define the function $\Phi : \dot{H}^s(\mathbb{R}^N)\backslash\{0\} \to \mathbb{R}^N$ by

$$
\Phi(u) = \frac{1}{\|\widehat{u}\|_{L^1(\mathbb{R}^N)}}\int_{\mathbb{R}^N} x\widehat{u}(x)dx.
$$

From [5], we know the map Φ is continuous in $\dot{H}^s(\mathbb{R}^N)\backslash\{0\}$, $\Phi(u) = 0$ if u is a radial function, and $\Phi(u(x-z)) = \Phi(u) + z$ for $z \in \mathbb{R}^N$.

For $a > 0$, let $U_a(x^j)$ be the hypercube $\prod_{i=1}^N (x_i^j - a, x_i^j + a)$ centered at $x^j = (x_1^j, \dots, x_N^j)$, $j = 1, 2, \dots, k$. $\overline{U_a(x^j)}$ and $\partial U_a(x^j)$ are the closure and the boundary of $U_a(x^j)$ respectively. By assumptions (H1) and (H2), we choose numbers $K, a > 0$ such that $\overline{U_a(x^j)}$ are disjoint, $V(x) > V(x^j)$ for $x \in \partial U_a(x^j)$, and $\cup_{j=1}^k U_a(x^j) \subset \prod_{i=1}^N (-K, K)$.

Let $U_{a/\varepsilon}^j \equiv U_{a/\varepsilon}(\frac{x^j}{\varepsilon})$, and for $j = 1, 2, \dots, k$, let

$$
\begin{aligned}
\mathcal{M}_\varepsilon^j &= \{u \in E_\varepsilon : u \in \mathcal{M}_\varepsilon \text{ and } \Phi(u) \in U_{a/\varepsilon}^j\},\\
\mathcal{O}_\varepsilon^j &= \{u \in E_\varepsilon : u \in \mathcal{M}_\varepsilon \text{ and } \Phi(u) \in \partial U_{a/\varepsilon}^j\}.
\end{aligned}
$$

It is easy to show that $\mathcal{M}_\varepsilon^j$ and $\mathcal{O}_\varepsilon^j$ are non-empty sets for $j = 1, 2, \dots, k$. Define for $j = 1, 2, \dots, k$

$$
b_\varepsilon^j = \inf_{u\in\mathcal{M}_\varepsilon^j} I_\varepsilon(u), \quad \widetilde{b}_\varepsilon^j = \inf_{u\in\mathcal{O}_\varepsilon^j} I_\varepsilon(u). \tag{3.1}
$$

First we derive basic properties of these quantities.

Lemma 3.1. *Suppose that* (H1)–(H5) *are satisfied. There exists* $\varepsilon_\sigma > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_\sigma)$,

$$b_\varepsilon^j < m_{V_0} + \sigma, \quad \textit{for } j = 1, 2, \dots, k.$$

Proof. Let j be fixed. From Lemma 2.5, we see that there exists $u_{r,V_0} \in \mathcal{N}_{r,V_0}$ such that $m_{V_0} = J_{V_0}(u_{r,V_0})$. For small $\varepsilon > 0$, we take $\psi_\varepsilon(x) \in C_0^\infty(\mathbb{R}^N, [0,1])$ such that $\psi_\varepsilon(x) = 1$ if $|x| < \frac{1}{\sqrt{\varepsilon}} - 1$; $\psi_\varepsilon \equiv 0$ if $|x| > \frac{1}{\sqrt{\varepsilon}}$, and $0 \le \psi_\varepsilon(x) \le 1$. Set $Q_\varepsilon^j = u_{r,V_0}(x - \varepsilon^{-1}x^j)\psi_\varepsilon(x - \varepsilon^{-1}x^j)$. By (H5), it is easy to see that there exists $\theta_\varepsilon^j > 0$ such that $\theta_\varepsilon^j Q_\varepsilon^j \in \mathcal{M}_\varepsilon$. by the properties of Φ, we have

$$\Phi(\theta_\varepsilon^j Q_\varepsilon^j) = \Phi(u_{r,V_0}\psi_\varepsilon) + \varepsilon^{-1}x^j. \tag{3.2}$$

By Lebesgue's theorem, we obtain

$$\begin{gathered} \|Q_\varepsilon^j\|_{\dot H^s(\mathbb{R}^N)} = \|u_{r,V_0}\|_{\dot H^s(\mathbb{R}^N)} + o_\varepsilon(1), \\ \int_{\mathbb{R}^N} V(\varepsilon x)|Q_\varepsilon^j|^2 dx = V_0 \int_{\mathbb{R}^N} |u_{r,V_0}|^2 dx + o_\varepsilon(1). \end{gathered} \tag{3.3}$$

Then the continuity of Φ and $u_{r,V_0} \in H_r^s(\mathbb{R}^N)$ implies

$$\lim_{\varepsilon \to 0} \Phi(u_{r,V_0}\psi_\varepsilon) = \Phi(u_{r,V_0}) = 0.$$

It follows from (3.2) that $\Phi(\theta_\varepsilon^j Q_\varepsilon^j) = \varepsilon^{-1}x^j + o_\varepsilon(1)$, we then deduce that $\Phi(\theta_\varepsilon^j Q_\varepsilon^j) \in U_{a/\varepsilon}^j$ for small enough $\varepsilon > 0$. Thus $\theta_\varepsilon^j Q_\varepsilon^j \in \mathcal{M}_\varepsilon^j$ for small ε.

Now we claim that

$$\theta_\varepsilon^j = 1 + o_\varepsilon(1).$$

We first show that $\{\theta_\varepsilon^j\}$ is bounded. In fact, if $\theta_\varepsilon^j \to \infty$ as $\varepsilon \to 0$, by $\theta_\varepsilon^j Q_\varepsilon^j \in \mathcal{M}_\varepsilon$ and $f(t)t \ge 0$, we obtain

$$\int_{\mathbb{R}^{2N}} \frac{|Q_\varepsilon^j(x) - Q_\varepsilon^j(y)|^2}{|x-y|^{N+2s}}\, dx\, dy + \int_{\mathbb{R}^N} V(\varepsilon x)|Q_\varepsilon^j|^2 dx \ge (\theta_\varepsilon^j)^{2_s^*-2} \int_{\mathbb{R}^N} |Q_\varepsilon^j|^{2_s^*} dx. \tag{3.4}$$

Note that

$$\int_{\mathbb{R}^N} |Q_\varepsilon^j|^{2_s^*} dx = \int_{\mathbb{R}^N} |u_{r,V_0}|^{2_s^*} dx + o_\varepsilon(1), \tag{3.5}$$

and

$$\begin{gathered} \int_{\mathbb{R}^N} F(u_{r,V_0}\psi_\varepsilon) dx = \int_{\mathbb{R}^N} F(u_{r,V_0}) dx + o_\varepsilon(1), \\ \int_{\mathbb{R}^N} f(u_{r,V_0}\psi_\varepsilon) u_{r,V_0}\psi_\varepsilon dx = \int_{\mathbb{R}^N} f(u_{r,V_0}) u_{r,V_0} dx + o_\varepsilon(1). \end{gathered} \tag{3.6}$$

Then by (3.3), (3.4) and (3.5), we obtain a contradiction. Thus, up to a subsequence, we may assume that $\theta_\varepsilon^j \to \theta_0^j$ with $\theta_0^j \ge 0$. Furthermore, by $\theta_\varepsilon^j Q_\varepsilon^j \in \mathcal{M}_\varepsilon$ and (2.6), we obtain

$$\|Q_\varepsilon^j\|_{V_0}^2 \le 2(C_{\frac{V_0}{2}} + 1)(\theta_\varepsilon^j)^{2_s^*-2} \int_{\mathbb{R}^N} |Q_\varepsilon^j|^{2_s^*} dx.$$

This combined with (3.3) and (3.5) gives

$$(\theta_0^j)^{2_s^*-2} \ge \frac{\|u_{r,V_0}\|_{V_0}^2}{2(C_{\frac{V_0}{2}} + 1)\int_{\mathbb{R}^N} |u_{r,V_0}|^{2_s^*} dx} > 0.$$

Letting $\varepsilon \to 0$ in $\langle I_\varepsilon'(\theta_\varepsilon^j Q_\varepsilon^j), \theta_\varepsilon^j Q_\varepsilon^j \rangle = 0$, we obtain $\theta_0^j u_{r,V_0} \in \mathcal{N}_{V_0}$. It follows from (H5) and $u_{r,V_0} \in \mathcal{N}_{V_0}$ that $\theta_0^j = 1$, which gives the desired assertion.

Consequently, by Lemma 2.5, (3.3), (3.5) and (3.6), we have

$$b_\varepsilon^j \le I_\varepsilon(\theta_\varepsilon^j Q_\varepsilon^j) = J_{V_0}(u_{r,V_0}) + o_\varepsilon(1) = m_{V_0} + o_\varepsilon(1),$$

which concludes the proof. □

Lemma 3.2. *Suppose* (H1)–(H5) *are satisfied. There exist* $\delta, \varepsilon_\delta > 0$ *such that* $\widetilde{b}_\varepsilon^j > m_{V_0} + \delta$, *for all* $\varepsilon \in (0, \varepsilon_\delta)$, $j = 1, 2, \dots, k$.

Proof. For $j = 1, 2, \dots, k$, arguing indirectly we assume that there exists a sequence $\varepsilon_n \to 0$ such that $\widetilde{b}_\varepsilon^j \to d \le m_{V_0}$. Then there exits a sequence $\{u_n\} \subset \mathcal{O}_{\varepsilon_n}^j$ such that $I_{\varepsilon_n}(u_n) \to d \le m_{V_0}$.

Noting $\{u_n\} \subset \mathcal{M}_{\varepsilon_n}$, then by (H1) we obtain

$$\|u_n\|_{V_0}^2 \le \int_{\mathbb{R}^N} f(u_n)u_n dx + \int_{\mathbb{R}^N} |u_n|^{2_s^*} dx. \tag{3.7}$$

For any n, (H5) implies there exists unique $t_n > 0$ such that $t_n u_n \in \mathcal{N}_{V_0}$; that is,

$$t_n^2 \|u_n\|_{V_0}^2 = \int_{\mathbb{R}^N} f(t_n u_n) t_n u_n dx + t_n^{2_s^*} \int_{\mathbb{R}^N} |u_n|^{2_s^*} dx.$$

It follows from (3.7) that

$$\int_{\mathbb{R}^N} \Big(\frac{f(t_n u_n)}{t_n u_n} - \frac{f(u_n)}{u_n} \Big) u_n^2 dx + (t_n^{2_s^*-2} - 1) \int_{\mathbb{R}^N} |u_n|^{2_s^*} dx \le 0.$$

This combined with (H5) yields $t_n \le 1$. Then, by (1.5) we obtain

$$\begin{aligned}
m_{V_0} &\ge \lim_{n\to\infty} I_{\varepsilon_n}(u_n) = \lim_{n\to\infty} \Big(I_{\varepsilon_n}(u_n) - \frac{1}{2}\langle I'_{\varepsilon_n}(u_n), u_n\rangle \Big) \\
&= \lim_{n\to\infty} \Big(\frac{s}{N} \int_{\mathbb{R}^N} |u_n|^{2_s^*} dx + \int_{\mathbb{R}^N} (\frac{1}{2} f(u_n)u_n - F(u_n)) dx \Big) \\
&\ge \lim_{n\to\infty} \Big(\frac{s}{N} t_n^{2_s^*} \int_{\mathbb{R}^N} |u_n|^{2_s^*} dx + \int_{\mathbb{R}^N} (\frac{1}{2} f(t_n u_n) t_n u_n - F(t_n u_n)) dx \Big) \\
&= \lim_{n\to\infty} \Big(J_{V_0}(t_n u_n) - \frac{1}{2}\langle J'_{V_0}(t_n u_n), t_n u_n\rangle \Big) \\
&= \lim_{n\to\infty} J_{V_0}(t_n u_n) \ge m_{V_0},
\end{aligned} \tag{3.8}$$

which implies that $t_n = 1 + o_n(1)$ and

$$\int_{\mathbb{R}^N} V(\varepsilon_n x)|u_n|^2 dx = \int_{\mathbb{R}^N} V_0 |u_n|^2 dx + o_n(1). \tag{3.9}$$

Moreover,

$$J_{V_0}(t_n u_n) = m_{V_0} + o_n(1). \tag{3.10}$$

Applying the Ekeland variational principle [34], we deduce that there exists a sequence $\{w_n\} \subset \mathcal{N}_{V_0}$ such that

$$J_{V_0}(w_n) = m_{V_0} + o_n(1), \quad \|w_n - t_n u_n\|_{V_0} = o_n(1), \quad J'_{V_0}(w_n) - \lambda_n H'_{V_0}(w_n) = o_n(1),$$

where $\lambda_n \in \mathbb{R}$ and $H_{V_0}(u) = \langle J'_{V_0}(u), u\rangle$. Then, by (1.5) we obtain

$$m_{V_0} + o_n(1) = J_{V_0}(w_n) - \frac{1}{2}\langle J'_{V_0}(w_n), w_n\rangle \ge \frac{s}{N} \int_{\mathbb{R}^N} |w_n|^{2_s^*} dx. \tag{3.11}$$

By (2.6) and $\{w_n\} \subset \mathcal{N}_{V_0}$, we conclude

$$\|w_n\|_{V_0}^2 \le 2(C_{\frac{V_0}{2}} + 1) \int_{\mathbb{R}^N} |w_n|^{2_s^*} dx. \tag{3.12}$$

This and (3.11) implies that $\{w_n\}$ is bounded in E_{V_0}. Hence, we have

$$o_n(1) = \lambda_n \langle H'_{V_0}(w_n), w_n \rangle. \tag{3.13}$$

We claim that there exists a $\alpha > 0$ such that $|\langle H'_{V_0}(w_n), w_n \rangle| \geq \alpha$. Indeed, by the definition of H_{V_0}, we have from (1.5) that

$$\begin{aligned} -\langle H'_{V_0}(w_n), w_n \rangle &= -\langle H'_{V_0}(w_n), w_n \rangle + 2\langle J'_{V_0}(w_n), w_n \rangle \\ &= (2_s^* - 2) \int_{\mathbb{R}^N} |w_n|^{2_s^*} dx + \int_{\mathbb{R}^N} (f'(w_n)w_n^2 - f(w_n)w_n) dx \\ &\geq (2_s^* - 2) \int_{\mathbb{R}^N} |w_n|^{2_s^*} dx. \end{aligned}$$

Arguing by contradiction, that $\langle H'_{V_0}(w_n), w_n \rangle = o_n(1)$, we have $\int_{\mathbb{R}^N} |w_n|^{2_s^*} dx = o_n(1)$. It follows from (3.12) that $\|w_n\|_{V_0} = o_n(1)$, which yields $J_{V_0}(w_n) = o_n(1)$. This contradicts $J_{V_0}(w_n) \to m_{V_0} > 0$, and proves our claim. Therefore, from (3.13) we have $\lambda_n = o_n(1)$. Thus

$$J_{V_0}(w_n) = m_{V_0} + o_n(1), \quad J'_{V_0}(w_n) = o_n(1). \tag{3.14}$$

Note $\{w_n\}$ is bounded in E_{V_0}. Letting $\rho_n = |w_n|^2$ and using the concentration-compactness lemma [22], we have one of cases: (i) Vanishing or (ii) Nonvanishing. If (i) Vanishing occurs, by Lemma 2.1, we have $w_n \to 0$ in $L^p(\mathbb{R}^N)$ for $p \in (2, 2_s^*)$. It follows from (H4) that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(w_n) dx = 0, \quad \lim_{n\to\infty} \int_{\mathbb{R}^N} f(w_n) w_n dx = 0.$$

This together with (3.14) implies

$$\frac{1}{2}\|w_n\|_{V_0}^2 - \frac{1}{2_s^*} \int_{\mathbb{R}^N} |w_n|^{2_s^*} dx = m_{V_0} + o_n(1), \tag{3.15}$$

$$\|w_n\|_{V_0}^2 - \int_{\mathbb{R}^N} |w_n|^{2_s^*} dx = o_n(1). \tag{3.16}$$

By the boundedness of $\{w_n\}$, up to a subsequence, we can assume that

$$\|w_n\|_{V_0}^2 \to L \geq 0 \quad \text{and} \quad \int_{\mathbb{R}^N} |w_n|^{2_s^*} dx \to L \geq 0.$$

If $L = 0$, from (3.15) we find $m_{V_0} = 0$, this contradicts $m_{V_0} > 0$. In addition $L > 0$, by (2.1) and (3.16) we obtain

$$\int_{\mathbb{R}^{2N}} \frac{|w_n(x) - w_n(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \geq S^{N/(2s)}.$$

This, (3.15) and (3.16) yields $m_{V_0} > \frac{s}{N} S^{N/(2s)}$, which contradicts (2.5). Hence, (ii) Nonvanishing occurs. Then there exits $\{z_n\} \subset \mathbb{R}^N$, $R > 0$ and $\beta > 0$, such that

$$\int_{B_R(z_n)} |w_n|^2 dx \geq \beta > 0.$$

It follows that $w_n(x + z_n) \rightharpoonup w \neq 0$ weakly in E_{V_0}. By (3.14), one gets

$$J_{V_0}(\overline{w}_n) = m_{V_0} + o_n(1), \quad J'_{V_0}(\overline{w}_n) = o_n(1),$$

where $\overline{w}_n = w_n(x+z_n)$. Then, the weak convergence of $\{\overline{w}_n\}$ implies that $J'_{V_0}(w) = 0$, and $w \in \mathcal{N}_{V_0}$. By Fatou's lemma

$$\begin{aligned} m_{V_0} &= \lim_{n\to\infty} J_{V_0}(\overline{w}_n) = \lim_{n\to\infty} \Big(J_{V_0}(\overline{w}_n) - \frac{1}{2}\langle J'_{V_0}(\overline{w}_n), \overline{w}_n\rangle \Big) \\ &= \lim_{n\to\infty} \Big(\frac{s}{N} \int_{\mathbb{R}^N} |\overline{w}_n|^{2_s^*} dx + \int_{\mathbb{R}^N} (\frac{1}{2} f(\overline{w}_n)\overline{w}_n - F(\overline{w}_n))dx \Big) \\ &\geq \frac{s}{N} \int_{\mathbb{R}^N} |w|^{2_s^*} dx + \int_{\mathbb{R}^N} (\frac{1}{2} f(w)w - F(w))dx \\ &= J_{V_0}(w) \geq m_{V_0}. \end{aligned}$$

This means that $\overline{w}_n \to w$ in E_{V_0}. Hence, $\Phi(w_n) - z_n = \Phi(\overline{w}_n) = \Phi(w) + o_n(1)$. By $\|w_n - t_n u_n\|_{V_0} = o_n(1)$, we obtain $\Phi(t_n u_n) - z_n = \Phi(w) + o_n(1)$. So $\operatorname{dist}(\varepsilon_n z_n, \partial U_a^j) = o_n(1)$. We assume that $\varepsilon_n z_n \to z_0 \in \partial U_a^j$. Thus, $V(z_0) > V_0$.

By (3.9) and $t_n = 1 + o_n(1)$, we obtain

$$\int_{\mathbb{R}^N} V(\varepsilon_n x)|t_n u_n|^2 dx = \int_{\mathbb{R}^N} V_0 |t_n u_n|^2 dx + o_n(1). \tag{3.17}$$

Moreover,

$$\int_{\mathbb{R}^N} V_0 |t_n u_n|^2 dx = \int_{\mathbb{R}^N} V_0 |w|^2 dx + o_n(1), \tag{3.18}$$

and $t_n u_n \to w$ in E_{V_0}. By Fatou's lemma,

$$\begin{aligned} \lim_{n\to\infty} \int_{\mathbb{R}^N} V(\varepsilon_n x)|t_n u_n|^2 dx &= \lim_{n\to\infty} \int_{\mathbb{R}^N} V(\varepsilon_n x + \varepsilon_n z_n)|t_n u_n(x+z_n)|^2 dx \\ &\geq \int_{\mathbb{R}^N} V(z_0)|w|^2 dx. \end{aligned}$$

It follows from (3.17) and (3.18) that

$$\int_{\mathbb{R}^N} V_0 |w|^2 dx \geq \int_{\mathbb{R}^N} V(z_0)|w|^2 dx.$$

This contradicts $V(z_0) > V_0$, which completes the proof. □

Lemma 3.3. *For any $u \in \mathcal{M}_\varepsilon^j$, there exits $\delta > 0$ and a differentiable function $t(v) > 0$ defined for $v \in E_\varepsilon$, and $\|v\|_\varepsilon < \delta$ such that $t(0) = 1$, $t(v)(u - v) \in \mathcal{M}_\varepsilon^j$ and*

$$\langle t'(0), \varphi\rangle = \frac{\langle H'_\varepsilon(u), \varphi\rangle}{\langle H'_\varepsilon(u), u\rangle} \tag{3.19}$$

for all $\varphi \in E_\varepsilon$, where $H_\varepsilon(u) = \langle I'_\varepsilon(u), u\rangle$.

Proof. The proof follows along the lines of [31, Lemma 2.4]. Define $\Psi : \mathbb{R} \times E_\varepsilon \to \mathbb{R}$ by

$$\Psi(t, v) = t^2 \|u - v\|_\varepsilon^2 - t^{2_s^*} \int_{\mathbb{R}^N} |u - v|^{2_s^*} dx - \int_{\mathbb{R}^N} f(tu - tv)(tu - tv)dx.$$

Since $u \in \mathcal{M}_\varepsilon^j$, by (1.5) we see that $\Psi(1, 0) = 0$ and

$$\begin{aligned} \Psi_t(1, 0) &= 2\|u\|_\varepsilon^2 - 2_s^* \int_{\mathbb{R}^N} |u|^{2_s^*} dx - \int_{\mathbb{R}^N} (f'(u)u^2 + f(u)u)dx \\ &\leq 2\|u\|_\varepsilon^2 - 2_s^* \int_{\mathbb{R}^N} |u|^{2_s^*} dx - 2\int_{\mathbb{R}^N} f(u)u dx \end{aligned}$$

$$= (2-2_s^*)\int_{\mathbb{R}^N}|u|^{2_s^*}dx < 0.$$

Hence, applying the implicit function theorem at point $(1,0)$, there exits $\delta>0$ and a differential function $t(v)$ defined for $\|v\|_\varepsilon<\delta$ such that $t(0)=1$, (3.19) holds and $\Psi(t(v),v)=0$, which implies $t(v)(u-v)\in\mathcal{M}_\varepsilon$. Furthermore, by the continuity of functions Φ and t, we have $t(v)(u-v)\in\mathcal{M}_\varepsilon^j$. This completes our proof. □

Lemma 3.4. *Suppose that* (H1)–(H5) *are satisfied. For a fixed j, the value b_ε^j has a minimizing sequence $\{v_n^j\}\subset\mathcal{M}_\varepsilon^j$ such that $I_\varepsilon(v_n^j)\to b_\varepsilon^j$ and $I'_\varepsilon(v_n^j)\to 0$, as $n\to\infty$.*

Proof. Applying Ekeland variational principle [16] to (3.1), we have a minimizing sequence $\{v_n^j\}\subset\mathcal{M}_\varepsilon^j$ such that

$$I_\varepsilon(v_n^j)<b_\varepsilon^j+\frac{1}{n},\quad I_\varepsilon(v_n^j)<I_\varepsilon(w)+\frac{1}{n}\|v_n^j-w\|_\varepsilon \tag{3.20}$$

for any $w\in\mathcal{M}_\varepsilon^j$.

We now apply Lemma 3.3 to v_n^j, we obtain $\delta_n>0$, function $t_n(v)$ defined for $v\in E_\varepsilon$, $\|v\|_\varepsilon<\delta_n$, such that $t_n(v)(v_n^j-v)\in\mathcal{M}_\varepsilon^j$. Choose $0<\theta<\delta_n$. Let $\varphi\in E_\varepsilon\setminus\{0\}$, and $v_\theta=\frac{\theta\varphi}{\|\varphi\|_\varepsilon}$. By (3.20), we obtain

$$I_\varepsilon(t_n(v_\theta)(v_n^j-v_\theta))-I_\varepsilon(v_n^j)>-\frac{1}{n}\|v_n^j-t_n(v_\theta)(v_n^j-v_\theta)\|_\varepsilon.$$

It follows from the mean value theorem that

$$\begin{aligned}&\langle I'_\varepsilon(v_n^j),t_n(v_\theta)(v_n^j-v_\theta)-v_n^j\rangle+o(\|v_n^j-t_n(v_\theta)(v_n^j-v_\theta)\|_\varepsilon)\\&\geq-\frac{\|v_n^j-t_n(v_\theta)(v_n^j-v_\theta)\|_\varepsilon}{n}.\end{aligned}$$

Therefore,

$$\begin{aligned}\langle I'_\varepsilon(v_n^j),\frac{\varphi}{\|\varphi\|_\varepsilon}\rangle&\leq\frac{1}{n}\frac{\|v_n^j-t_n(v_\theta)(v_n^j-v_\theta)\|_\varepsilon}{\theta}+\frac{o(\|v_n^j-t_n(v_\theta)(v_n^j-v_\theta)\|_\varepsilon)}{\theta}\\&\quad+\frac{(t_n(v_\theta)-1)}{\theta}\langle I'_\varepsilon(v_n^j)-I'_\varepsilon(t_n(v_\theta)(v_n^j-v_\theta),v_n^j-v_\theta\rangle.\end{aligned} \tag{3.21}$$

On the other hand,

$$\frac{\|v_n^j-t_n(v_\theta)(v_n^j-v_\theta)\|_\varepsilon}{\theta}\leq\frac{|t_n(v_\theta)-1|}{\theta}\|v_n^j\|_\varepsilon+t_n(v_\theta).$$

By (3.19) and the boundedness of $\{v_n^j\}$, we see that there exists $M>0$ such that

$$\lim_{\theta\to 0}\frac{\|v_n^j-t_n(v_\theta)(v_n^j-v_\theta)\|_\varepsilon}{\theta}\leq M.$$

Note that $t_n(v_\theta)(v_n^j-v_\theta)\to v_n^j$ and $I'_\varepsilon(t_n(v_\theta)(v_n^j-v_\theta))\to I'_\varepsilon(v_n^j)$ as $\theta\to 0$. Let $\theta\to 0$ in (3.21), we conclude that

$$\langle I'_\varepsilon(v_n^j),\frac{\varphi}{\|\varphi\|_\varepsilon}\rangle\leq\frac{M}{n},$$

which implies $\lim_{n\to\infty}\|I'_\varepsilon(v_n^j)\|=0$. This completes the the proof. □

Proposition 3.5. *For $j=1,2,\dots,k$, let $\{v_n^j\}\subset\mathcal{M}_\varepsilon^j$ be a sequence satisfying $I_\varepsilon(v_n^j)\to b_\varepsilon^j$ and $I'_\varepsilon(v_n^j)\to 0$. Then $\{v_n^j\}$ has a convergence subsequence in E_ε.*

Proof. Note that the sequence $\{v_n^j\}$ is bounded in E_ε, we may assume that there exists $v^j \in E_\varepsilon$ such that $v_n^j \rightharpoonup v^j$ weakly in E_ε, $v_n^j \to v^j$ strongly in $L^p_{loc}(\mathbb{R}^N)$ for every $p \in (2, 2_s^*)$, and $v_n^j \to v^j$ a.e. on $\mathbb{R}^N$.

We will show that $v^j \neq 0$. Assume to the contrary that $v^j \equiv 0$. Then by the definition of V_∞, and $v_n^j \rightharpoonup 0$ weakly in E_ε, we obtain

$$\int_{\mathbb{R}^N} V(\varepsilon x)|v_n^j|^2 dx \geq \int_{\mathbb{R}^N} V_\infty |v_n^j|^2 dx + o_n(1). \tag{3.22}$$

By (H5), there exists $t_n^j > 0$ such that $t_n^j v_n^j \in \mathcal{N}_{V_\infty}$, that is

$$(t_n^j)^2 \|v_n^j\|^2_{V_\infty} = \int_{\mathbb{R}^N} f(t_n^j v_n^j) t_n^j v_n^j dx + \int_{\mathbb{R}^N} |t_n^j v_n^j|^{2_s^*} dx. \tag{3.23}$$

From (2.6) and $v_n^j \in \mathcal{M}_\varepsilon^j$, we obtain

$$\|v_n^j\|^2_\varepsilon \leq 2(C_{\frac{V_0}{2}} + 1) \int_{\mathbb{R}^N} |v_n^j|^{2_s^*} dx.$$

Arguing by contradiction we obtain $\int_{\mathbb{R}^N} |v_n^j|^{2_s^*} dx > 0$. It follows from (3.23) that $\{t_n^j\}$ is bounded. We may assume that $t_n^j \to t_0^j$ as $n \to \infty$, up to a subsequence. Using $v_n^j \in \mathcal{M}_\varepsilon^j$, (3.22) and (3.23), one has

$$\int_{\mathbb{R}^N} \Big(\frac{f(t_n^j v_n^j)}{t_n^j v_n^j} - \frac{f(v_n^j)}{v_n^j}\Big)(v_n^j)^2 dx + ((t_n^j)^{2_s^*-2} - 1) \int_{\mathbb{R}^N} |v_n^j|^{2_s^*} dx + o_n(1) \leq 0. \tag{3.24}$$

If $t_0^j > 1$, letting $n \to \infty$ in (3.24), by (H5) and $\int_{\mathbb{R}^N} |v_n^j|^{2_s^*} dx > 0$, we obtain a contradiction. Therefore

$$0 < t_0^j \leq 1. \tag{3.25}$$

Now, we have

$$\begin{aligned}
I_\varepsilon(v_n^j) &= I_\varepsilon(v_n^j) - \frac{1}{2}\langle I'_\varepsilon(v_n^j), v_n^j\rangle \\
&= \frac{s}{N}\int_{\mathbb{R}^N} |v_n^j|^{2_s^*} dx + \int_{\mathbb{R}^N} \Big(\frac{1}{2} f(v_n^j) v_n^j - F(v_n^j)\Big) dx \\
&\geq \frac{s}{N}\int_{\mathbb{R}^N} |t_n^j v_n^j|^{2_s^*} dx + \int_{\mathbb{R}^N} \Big(\frac{1}{2} f(t_n^j v_n^j) t_n^j v_n^j - F(t_n^j v_n^j)\Big) dx + o_n(1) \\
&= J_{V_\infty}(t_n^j v_n^j) + o_n(1).
\end{aligned}$$

This yields

$$I_\varepsilon(v_n^j) \geq J_{V_\infty}(t_n^j v_n^j) + o_n(1), \quad \langle J'_{V_\infty}(t_n^j v_n^j), t_n^j v_n^j\rangle = 0. \tag{3.26}$$

Let $\widetilde{v}_n^j = t_n^j v_n^j$. Then, $\widetilde{v}_n^j \rightharpoonup 0$ weakly in E_ε. Since the embedding $E_\varepsilon \hookrightarrow H^s(\mathbb{R}^N)$ is continuous, we have $\widetilde{v}_n^j \rightharpoonup 0$ weakly in $H^s(\mathbb{R}^N)$. Applying the Ekeland variational principle ([34]), arguing as in the proof of Lemma 3.2 and taking into account (3.26), we obtain a sequence $\{\overline{v}_n^j\} \subset \mathcal{N}_{V_\infty}$ such that $\|\overline{v}_n^j - \widetilde{v}_n^j\|_{V_\infty} = o_n(1)$, $J_{V_\infty}(\overline{v}_n^j) = J_{V_\infty}(\widetilde{v}_n^j) + o_n(1)$ and $J'_{V_\infty}(\overline{v}_n^j) = o_n(1)$. It follows from (3.26) that

$$b_\varepsilon^j = I_\varepsilon(v_n^j) + o_n(1) \geq J_{V_\infty}(\overline{v}_n^j) + o_n(1), \quad J'_{V_\infty}(\overline{v}_n^j) = o_n(1). \tag{3.27}$$

Moreover, $\overline{v}_n^j \rightharpoonup 0$ weakly in $H^s(\mathbb{R}^N)$.

The concentration-compactness lemma by Lions [22] implies that $\{\overline{v}_n^j\}$ satisfies either vanishing or nonvanishing. If vanishing occurs, using the same argument of Lemma 3.2, with (3.27) and (2.1), we obtain $b_\varepsilon^j > \frac{s}{N} S^{N/(2s)}$. On the other hand, Lemma 3.1 and (2.5) imply $b_\varepsilon^j < \frac{s}{N} S^{N/(2s)}$. Then we obtain a contradiction.

Therefore, nonvanishing occurs. It exists $\{y_n\} \subset \mathbb{R}^N$, $R > 0$ and $\beta > 0$ such that $\int_{B_R(y_n)} |\overline{v}_n^j|^2 dx \geq \beta > 0$. Set $w_n^j = \overline{v}_n^j(x + y_n)$. Since $\{w_n^j\}$ is bounded in $H^s(\mathbb{R}^N)$, we may assume that $w_n^j \rightharpoonup w_0^j \neq 0$ weakly in $H^s(\mathbb{R}^N)$. It follows from (3.27) that

$$b_\varepsilon^j \geq J_{V_\infty}(w_n^j) + o_n(1), \quad J'_{V_\infty}(w_n^j) = o_n(1). \tag{3.28}$$

The weak convergence of $\{w_n^j\}$ implies that $J'_{V_\infty}(w_0^j) = 0$. Set $\widetilde{w}_n^j = w_n^j - w_0^j$. We easily obtain

$$\int_{\mathbb{R}^N} F(w_n^j) - \int_{\mathbb{R}^N} F(w_0^j) = \int_{\mathbb{R}^N} F(\widetilde{w}_n^j) + o_n(1).$$

It follows from the Brezis-Lieb lemma [2] that

$$J_{V_\infty}(w_n^j) = J_{V_\infty}(\widetilde{w}_n^j) + J_{V_\infty}(w_0^j) + o_n(1).$$

This together with (3.28) yields

$$b_\varepsilon^j \geq J_{V_\infty}(\widetilde{w}_n^j) + J_{V_\infty}(w_0^j) + o_n(1), \quad J'_{V_\infty}(w_0^j) = 0. \tag{3.29}$$

We now consider two cases: (i) $\|\widetilde{w}_n^j\|_{V_\infty} \to 0$ as $n \to \infty$, and (ii) $\|\widetilde{w}_n^j\|_{V_\infty} > \alpha$ for some $\alpha > 0$ for large n.

Case (i) If $\|\widetilde{w}_n^j\|_{V_\infty} = o_n(1)$, and $\{y_n\}$ is bounded, then we may assume that $y_n \to y_0 \in \mathbb{R}^N$ up to a subsequence. Consequently, $\overline{v}_n^j \to w_0^j(x - y_0)$ strongly in $H^s(\mathbb{R}^N)$. From $t_n^j \to t_0^j \in (0, 1]$, $\|\overline{v}_n^j - \widetilde{v}_n^j\|_{V_\infty} = o_n(1)$, and $\widetilde{v}_n^j = t_n^j v_n^j$, we obtain $v_n^j \to (t_0^j)^{-1} w_0^j(x - y_0) \neq 0$ strongly in $H^s(\mathbb{R}^N)$ as $n \to \infty$, contradicts $v^j \equiv 0$.

If $\|\widetilde{w}_n^j\|_{V_\infty} = o_n(1)$ and $\{y_n\}$ is unbounded. Without loss of generality, we assume that $|y_n| \to \infty$. It follows from the continuity of Φ that $\Phi(w_n^j) \to \Phi(w_0^j)$ as $n \to \infty$. On the other hand, by $\Phi(v_n^j) \in U^j_{a/\varepsilon}$ and $\|\overline{v}_n^j - \widetilde{v}_n^j\|_{V_\infty} = o_n(1)$, we deduce that

$$\frac{x^j - a}{\varepsilon} - y_n < \Phi(w_n^j) < \frac{x^j + a}{\varepsilon} - y_n$$

for large enough n. It follows from $|y_n| \to \infty$ that $|\Phi(w_n^j)| \to \infty$. It is a contradiction.

Case (ii) $\|\widetilde{w}_n^j\|_{V_\infty} > \alpha$ for some $\alpha > 0$ for large n. We easily check that $J'_{V_\infty}(\widetilde{w}_n^j) = o_n(1)$. Thus the boundedness of $\{\widetilde{w}_n^j\}$ implies $\langle J'_{V_\infty}(\widetilde{w}_n^j), \widetilde{w}_n^j\rangle = o_n(1)$. By (H5), we can find $\theta_n > 0$ such that $\theta_n \widetilde{w}_n^j \in \mathcal{N}_{V_\infty}$. This together with $f(\widetilde{w}_n^j)\widetilde{w}_n^j \geq 0$ yields

$$\|\widetilde{w}_n^j\|^2_{V_\infty} \geq \theta_n^{2_s^*-2} \int_{\mathbb{R}^N} |\widetilde{w}_n^j|^{2_s^*} dx. \tag{3.30}$$

By (H3) and $\langle J'_{V_\infty}(\widetilde{w}_n^j), \widetilde{w}_n^j\rangle = o_n(1)$, we obtain

$$\|\widetilde{w}_n^j\|^2_{V_\infty} \leq 2(C_{\frac{V_\infty}{2}} + 1) \int_{\mathbb{R}^N} |\widetilde{w}_n^j|^{2_s^*} dx + o_n(1). \tag{3.31}$$

From (3.30), (3.31) and $\|\widetilde{w}_n^j\|_{V_\infty} > \alpha > 0$, we see that θ_n is bounded. Similarly to the proof of (3.24), we obtain $\theta_n \to \theta_0 \in (0, 1]$. Then

$$\begin{aligned}
J_{V_\infty}(\widetilde{w}_n^j) &= J_{V_\infty}(\widetilde{w}_n^j) - \frac{1}{2}\langle J'_{V_\infty}(\widetilde{w}_n^j), \widetilde{w}_n^j\rangle + o_n(1) \\
&= \frac{s}{N} \int_{\mathbb{R}^N} |\widetilde{w}_n^j|^{2_s^*} dx + \int_{\mathbb{R}^N} \Big(\frac{1}{2} f(\widetilde{w}_n^j)\widetilde{w}_n^j - F(\widetilde{w}_n^j)\Big) dx + o_n(1) \\
&\geq \frac{s}{N} \int_{\mathbb{R}^N} |\theta_n \widetilde{w}_n^j|^{2_s^*} dx + \int_{\mathbb{R}^N} \Big(\frac{1}{2} f(\theta_n \widetilde{w}_n^j)\theta_n \widetilde{w}_n^j - F(\theta_n \widetilde{w}_n^j)\Big) dx + o_n(1)
\end{aligned}$$

$$= J_{V_\infty}(\theta_n \widetilde{w}_n^j) + o_n(1).$$

This and (3.29) imply

$$b_\varepsilon^j \geq J_{V_\infty}(\theta_n \widetilde{w}_n^j) + J_{V_\infty}(w_0^j) + o_n(1) \geq 2m_{V_\infty} + o_n(1). \tag{3.32}$$

Since $V_\infty \geq V_0$, by [28, Proposition 4.3], we have $m_{V_\infty} \geq m_{V_0}$. It follows from (3.32) that $b_\varepsilon^j \geq 2m_{V_0} + o_n(1)$. However, Lemma 3.1 implies $b_\varepsilon^j < 2m_{V_0}$ which is a contradiction.

Hence, $v_n^j \rightharpoonup v^j \neq 0$ weakly in E_ε. Now we show that $v_n^j \to v^j \neq 0$ strongly in E_ε. By $v_n^j \rightharpoonup v^j \neq 0$ and $I_\varepsilon'(v_n^j) \to 0$, we obtain $I_\varepsilon'(v^j) = 0$. Let $\widehat{v}_n^j = v_n^j - v^j$. We claim that $\|\widehat{v}_n^j\|_\varepsilon \to 0$ as $n \to \infty$. Otherwise, we assume that there exists $\alpha_0 > 0$ such that $\|\widehat{v}_n^j\|_\varepsilon \geq \alpha_0$. Arguing as in the proof of case (ii) of above, we obtain a contradiction, and this completes the proof. □

Theorem 3.6. *Suppose that* (H1)–(H5) *are satisfied. Then* (2.2) *has at least k positive solutions for small $\varepsilon > 0$.*

Proof. For $j = 1, 2, \dots, k$. Lemma 3.4 implies that b_ε^j has a minimizing sequence $\{v_n^j\}$ satisfying $I_\varepsilon(v_n^j) \to b_\varepsilon^j$ and $I_\varepsilon'(v_n^j) \to 0$ as $n \to \infty$. It follows from Proposition 3.5 that $\{v_n^j\}$ satisfying (PS) condition, thai is $v_n^j \to v^j$ strongly in E_ε, up to a subsequence. Then v^j is a nontrivial solution of (2.2). We note that all the arguments above can be repeated word by word, replacing I_ε^+ with the functional

$$I_\varepsilon(v) = \frac{1}{2}\|v\|_\varepsilon^2 - \frac{1}{2_s^*}\int_{\mathbb{R}^N} |v^+|^{2_s^*} dx - \int_{\mathbb{R}^N} F(v^+) dx.$$

In this way we obtain solutions $(v^j)^+$ of the equation

$$(-\Delta)^s v + V(\varepsilon x) v = (v^+)^{2_s^*-1} + f(v^+), \quad x \in \mathbb{R}^N. \tag{3.33}$$

Using $(v^j)^-$ as a test function in (3.33), and integrating by parts, we infer that $(v^j)^- \equiv 0$ and $v^j \geq 0$ in $\mathbb{R}^N$. By a strong maximum principle ([14]), we obtain $v^j > 0$ in $\mathbb{R}^N$.

Since $\{v_n^j\} \subset \mathcal{M}_\varepsilon^j$ and $v_n^j \to v^j$ strongly in E_ε, we have $v^j \in \mathcal{M}_\varepsilon^j \cup \mathcal{O}_\varepsilon^j$, $I_\varepsilon(v^j) = b_\varepsilon^j$ and $I_\varepsilon'(v^j) = 0$. Lemmas 3.1 and 3.2 imply $b_\varepsilon^j < \widetilde{b}_\varepsilon^j$. Hence, we see that $v^j \in \mathcal{M}_\varepsilon^j$. Moreover, $\Phi(v^j) \in U_{a/\varepsilon}^j$ and $U_{a/\varepsilon}^j$ are disjoint, $j = 1, 2, \dots, k$. Therefore, problem (2.2) has at least k distinct positive solutions. □

4. Concentration of positive solutions

In this section, for $j = 1, 2, \dots, k$, v_ε^j are always referred as positive solutions of (2.2) obtained in Theorem 3.6. We shall consider the concentration behavior of v_ε^j as $\varepsilon \to 0$.

Lemma 4.1. *Let $j = 1, 2, \dots, k$. Then there exists a sequence $\{x_\varepsilon^j\} \subset \mathbb{R}^N$ and $R_0, \delta > 0$ such that*

$$\int_{B_{R_0}(x_\varepsilon^j)} |v_\varepsilon^j|^2 dx \geq \delta$$

for small $\varepsilon > 0$.

Proof. Suppose by contradiction that there exists a sequence $\varepsilon_n \to 0$ such that

$$\lim_{n\to\infty} \sup_{x \in \mathbb{R}^N} \int_{B_R(x)} |v_{\varepsilon_n}^j|^2 dx = 0$$

for all $R > 0$. It follows from Lemma 2.1 that $v^j_{\varepsilon_n} \to 0$ strongly in $L^p(\mathbb{R}^N)$ for every $p \in (2, 2^*_s)$. By (H4), we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(v^j_{\varepsilon_n}) dx = 0, \quad \lim_{n\to\infty} \int_{\mathbb{R}^N} f(v^j_{\varepsilon_n}) v^j_{\varepsilon_n} dx = 0.$$

We note that $m_{V_0} \le m_{\varepsilon_n} \le b^j_{\varepsilon_n}$, then by Lemma 3.1 one has $b^j_{\varepsilon_n} \to m_{V_0}$ as $n \to \infty$. Then arguing as in the proof of Lemma 3.2 to prove the vanishing, we can infer that $m_{V_0} > \frac{s}{N} S^{N/(2s)}$, which is impossible according to (2.5). This completes the proof. □

Lemma 4.2. *For $j = 1, 2, \dots, k$ fixed,* $\lim_{\varepsilon\to 0} \varepsilon x^j_\varepsilon = x^j$.

Proof. Let $j = 1, 2, \dots, k$. We claim that $\{\varepsilon x^j_\varepsilon\}$ is bounded in $\mathbb{R}^N$. Indeed, if not, there exists a sequence $\varepsilon_n \to 0$ such that $|\varepsilon_n x^j_{\varepsilon_n}| \to \infty$. Note that $I_{\varepsilon_n}(v^j_{\varepsilon_n}) = b^j_{\varepsilon_n}$ and $I'_{\varepsilon_n}(v^j_{\varepsilon_n}) = 0$. By Lemma 3.1, we have $b^j_{\varepsilon_n} = m_{V_0} + o_n(1)$. It follows from (1.5) that

$$m_{V_0} + o_n(1) = b^j_{\varepsilon_n} = I_{\varepsilon_n}(v^j_{\varepsilon_n}) - \frac{1}{2}\langle I'_{\varepsilon_n}(v^j_{\varepsilon_n}), v^j_{\varepsilon_n}\rangle \ge \frac{s}{N} \int_{\mathbb{R}^N} |v^j_{\varepsilon_n}|^{2^*_s} dx,$$

which implies $\int_{\mathbb{R}^N} |v^j_{\varepsilon_n}|^{2^*_s} dx$ is bounded. This together with (H4) and $I'_{\varepsilon_n}(v^j_{\varepsilon_n}) = 0$ yields $\{v^j_{\varepsilon_n}\}$ is bounded in E_ε. Moreover, $\{v^j_{\varepsilon_n}\}$ is bounded in $H^s(\mathbb{R}^N)$. Let $\widetilde{v}^j_n = v^j_{\varepsilon_n}(x + x^j_{\varepsilon_n})$. By Lemma 4.1, we see that $\widetilde{v}^j_n \rightharpoonup \widetilde{v}^j_0 \ne 0$ weakly in $H^s(\mathbb{R}^N)$. From $I'_{\varepsilon_n}(v^j_{\varepsilon_n}) = 0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{(\widetilde{v}^j_n(x) - \widetilde{v}^j_n(y))(\widetilde{v}^j_0(x) - \widetilde{v}^j_0(y))}{|x-y|^{N+2s}}\, dx\, dy + \int_{\mathbb{R}^N} V(\varepsilon_n x + \varepsilon_n x^j_{\varepsilon_n}) \widetilde{v}^j_n \widetilde{v}^j_0 dx \\
&= \int_{\mathbb{R}^N} f(\widetilde{v}^j_n)\widetilde{v}^j_0 dx + \int_{\mathbb{R}^N} |\widetilde{v}^j_n|^{2^*_s - 1} \widetilde{v}^j_0 dx\,.
\end{aligned}$$

Then, by Fatou's lemma,

$$\|\widetilde{v}^j_0\|^2_{V_\infty} \le \int_{\mathbb{R}^N} f(\widetilde{v}^j_0)\widetilde{v}^j_0 dx + \int_{\mathbb{R}^N} |\widetilde{v}^j_0|^{2^*_s} dx\,.$$

By (H5), there exists a unique $t^j > 0$ such that $t^j \widetilde{v}^j_0 \in \mathcal{N}_{V_\infty}$. As in the proof of (3.25), we obtain $t^j \le 1$. Then by $I_{\varepsilon_n}(\widetilde{v}^j_n) = b^j_{\varepsilon_n}$, $I'_{\varepsilon_n}(\widetilde{v}^j_n) = 0$ and Fatou's lemma, as in the argument of (3.8), we have

$$\begin{aligned}
m_{V_0} &\ge \lim_{n\to\infty} I_{\varepsilon_n}(\widetilde{v}^j_n) = \lim_{n\to\infty} \Big(I_{\varepsilon_n}(\widetilde{v}^j_n) - \frac{1}{2}\langle I'_{\varepsilon_n}(\widetilde{v}^j_n), \widetilde{v}^j_n\rangle \Big) \\
&\ge J_{V_\infty}(t^j \widetilde{v}^j_0) \ge m_{V_\infty} \ge m_{V_0}.
\end{aligned}$$

Which implies $t^j = 1$ and $V_\infty = V_0$. Then, it follows from $V(\varepsilon_n x + \varepsilon_n x^j_{\varepsilon_n}) \ge V_0$ that $\widetilde{v}^j_n \to \widetilde{v}^j_0$ strongly in $H^s(\mathbb{R}^N)$. Hence

$$\Phi(\widetilde{v}^j_n) = \Phi(\widetilde{v}^j_0) + o_n(1). \tag{4.1}$$

On the other hand, $\Phi(\widetilde{v}^j_n) + x^j_{\varepsilon_n} = \Phi(v^j_{\varepsilon_n}) \in U^j_{\frac{a}{\varepsilon_n}}$; that is,

$$\frac{x^j + a - \varepsilon_n x^j_{\varepsilon_n}}{\varepsilon_n} < \Phi(\widetilde{v}^j_n) < \frac{x^j + a + \varepsilon_n x^j_{\varepsilon_n}}{\varepsilon_n}.$$

This yields $|\Phi(\widetilde{v}^j_n)| \to \infty$, which contradicts (4.1), and proves $\{\varepsilon x^j_\varepsilon\}$ is bounded in $\mathbb{R}^N$.

Without loss of generality, we assume that $\varepsilon x_\varepsilon^j \to y^j$ as $\varepsilon \to 0$. We are going to show that $y^j = x^j$. Set $\widetilde{v}_\varepsilon^j = v_\varepsilon^j(x + x_\varepsilon^j)$. By Lemma 4.1, we have $\widetilde{v}_\varepsilon^j \rightharpoonup \widetilde{v}_0^j$ weakly in $H^s(\mathbb{R}^N)$. Similarly to the proof of above, we infer that $\widetilde{v}_\varepsilon^j \to \widetilde{v}_0^j$ strongly in $H^s(\mathbb{R}^N)$. This together with the continuity of Φ and $\Phi(v_\varepsilon^j) \in U_{a/\varepsilon}^j$ yields $\varepsilon x_\varepsilon^j \in U_{a/\varepsilon}^j$ for small $\varepsilon > 0$. Therefore, by $V(\varepsilon x_\varepsilon^j) \to V(y^j) = V_0$, we obtain $y^j = x^j$. The proof is complete. $\square$

Lemma 4.3. *For $j = 1, 2, \dots, k$, the function v_ε^j has a maximum point $z_\varepsilon^j \in \mathbb{R}^N$ such that $V(\varepsilon z_\varepsilon^j) \to V(x^j)$ as $\varepsilon \to 0$.*

Proof. Since v_ε^j is a weak solution of (2.2) and satisfies

$$\|\widetilde{v}_\varepsilon^j\|_{\dot{H}^s(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^N} V(\varepsilon x + \varepsilon x_\varepsilon^j)|\widetilde{v}_\varepsilon^j|^2 dx = \int_{\mathbb{R}^N} f(\widetilde{v}_\varepsilon^j)\widetilde{v}_\varepsilon^j dx + \int_{\mathbb{R}^N} |\widetilde{v}_\varepsilon^j|^{2_s^*} dx. \tag{4.2}$$

Using the Morse iterative method, as in the proof [19, Proposition 4.1], we obtain that $\widetilde{v}_\varepsilon^j \in L^\infty(\mathbb{R}^N)$ and there exists $C_1 > 0$ such that $\|\widetilde{v}_\varepsilon^j\|_\infty \le C_1$, and $\widetilde{v}_\varepsilon^j$ vanishes at infinity uniformly for small $\varepsilon > 0$. So $\max_{x\in\mathbb{R}^N} \widetilde{v}_\varepsilon^j$ exists. Then, by (4.2), (2.6) and $V(\varepsilon x + \varepsilon x_\varepsilon^j) \ge V_0$, we obtain

$$\begin{aligned}
\|\widetilde{v}_\varepsilon^j\|_{V_0}^2 &\le 2(C_{\frac{V_0}{2}} + 1)\int_{\mathbb{R}^N} |\widetilde{v}_\varepsilon^j|^{2_s^*} dx \\
&\le 2(C_{\frac{V_0}{2}} + 1)(\max_{x\in\mathbb{R}^N} \widetilde{v}_\varepsilon^j)^{2_s^*-2}\int_{\mathbb{R}^N} |\widetilde{v}_\varepsilon^j|^2 dx \\
&\le 2C(C_{\frac{V_0}{2}} + 1)(\max_{x\in\mathbb{R}^N} \widetilde{v}_\varepsilon^j)^{2_s^*-2}\|\widetilde{v}_\varepsilon^j\|_{V_0}^2.
\end{aligned}$$

Then there exists $C_0 > 0$ independent of ε such that

$$\max_{x\in\mathbb{R}^N} \widetilde{v}_\varepsilon^j \ge C_0 \tag{4.3}$$

for small $\varepsilon > 0$.

Let $y_\varepsilon^j \in \mathbb{R}^N$ be the maximum point of $\widetilde{v}_\varepsilon^j(x)$. By (4.3) and $\lim_{|x|\to\infty} \widetilde{v}_\varepsilon^j(x) = 0$ uniformly for ε, there exists $T^j > 0$ such that $|y_\varepsilon^j| \le T^j$. By the definition of $\widetilde{v}_\varepsilon^j$, we obtain $z_\varepsilon^j := x_\varepsilon^j + y_\varepsilon^j$ is the maximum point of v_ε^j. This together with Lemma 4.2 yields $\varepsilon z_\varepsilon^j \to x^j$ as $\varepsilon \to 0$. Hence, $V(\varepsilon z_\varepsilon^j) \to V(x^j)$ as $\varepsilon \to 0$. $\square$

Now we prove our main result.

Proof of Theorem 1.1. Theorem 3.6 implies problem (2.2) admits at least k distinct positive solutions v_ε^j, $j = 1, \dots, k$. Then, $u_\varepsilon^j(x) = v_\varepsilon^j(\frac{x}{\varepsilon})$ are solutions of problem (1.1). By Lemma 4.3, $\overline{z}_\varepsilon^j = \varepsilon z_\varepsilon^j$ are the maximum points of u_ε^j and satisfy $V(\overline{z}_\varepsilon^j) \to V(x^j)$ as $\varepsilon \to 0$.

Finally, we show the power-type decay of the solutions u_ε^j. Let $\overline{v}_\varepsilon^j(x) = v_\varepsilon^j(x + z_\varepsilon^j)$. Then, $\overline{v}_\varepsilon^j(x)$ are solutions of the problem

$$(-\Delta)^s u + V(\varepsilon x + \varepsilon z_\varepsilon^j)u = f(u) + u^{2_s^*-1}, \quad x \in \mathbb{R}^N.$$

By (H4) and $\lim_{|x|\to\infty} \overline{v}_\varepsilon^j(x) = 0$, we can find an $R > 0$ such that

$$f(\overline{v}_\varepsilon^j) + (\overline{v}_\varepsilon^j)^{2_s^*-1} \le \frac{V_0}{2}\overline{v}_\varepsilon^j, \quad \forall |x| \ge R.$$

Thus

$$(-\Delta)^s \overline{v}_\varepsilon^j + \frac{V_0}{2}\overline{v}_\varepsilon^j \le 0, \tag{4.4}$$

for all $|x| \geq R$ and small $\varepsilon > 0$. By [18, Lemma 4.2], we can choose function $\varphi(x) = C^j|x|^{-(N+2s)}$ such that

$$(-\Delta)^s\varphi + \frac{V_0}{2}\varphi \geq 0, \quad \forall |x| \geq R, \tag{4.5}$$

and $C^j|R|^{-(N+2s)} \geq \overline{v}_\varepsilon^j(x)$ for all $|x| = R$, where C^j is a positive constant. Set $\phi_\varepsilon = \varphi - \overline{v}_\varepsilon^j$. It follows from (4.4) and (4.5) that

$$\begin{gathered}
(-\Delta)^s\phi_\varepsilon + \frac{V_0}{2}\phi_\varepsilon \geq 0, \quad \text{for } |x| \geq R, \\
\phi_\varepsilon(x) \geq 0, \quad \text{for } |x| = R, \\
\lim_{|x|\to\infty} \phi_\varepsilon(x) = 0.
\end{gathered}$$

Applying the maximum principle, we obtain $\phi_\varepsilon(x) \geq 0$ for all $|x| \geq R$. Hence

$$u_\varepsilon^j(x) = v_\varepsilon^j(\frac{x}{\varepsilon}) = \overline{v}_\varepsilon^j(\frac{x - \varepsilon z_\varepsilon^j}{\varepsilon}) \leq C^j\Big|\frac{x - \overline{z}_\varepsilon^j}{\varepsilon}\Big|^{-(N+2s)}.$$

The proof is complete. □

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