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THE METHOD OF UPPER AND LOWER SOLUTIONS FOR SECOND-ORDER NON-HOMOGENEOUS TWO-POINT BOUNDARY-VALUE PROBLEM

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ABSTRACT. This paper studies the existence and uniqueness of solutions for a type of second-order two-point boundary-value problem depending on the first-order derivative through a non-linear term. By constructing a special cone and using the upper and lower solutions method, we obtain the sufficient conditions of the existence and uniqueness of solutions, and a monotone iterative sequence solving the boundary-value problem. An error estimate formula is also given under the condition of a unique solution.

1. INTRODUCTION

In this paper, we study the existence and uniqueness of solutions to the second-order non-homogeneous two-point boundary-value problem

$$\begin{gathered} x''(t) + f(t, x(t), x'(t)) = 0, \quad t \in (0,1), \\ x'(0) = a, \quad x(1) = b, \end{gathered} \tag{1.1}$$

where $f \in C([0,1] \times \mathbb{R}^2, \mathbb{R})$, and a, $b \in \mathbb{R}$.

It is well known that the upper and lower solutions method is an important tool in studying boundary-value problem of ordinary differential equation. Recently, there are numerous results of the problem by means of the method (see the references in this article). We notice that most of these papers study the existence and uniqueness of solutions of the boundary-value problem with nonlinear term $f(t, u)$. The nonlinear term f, however, usually satisfies Nagumo condition when the f depends on the first order derivative (see for example [1, 3, 5, 6]), which weakens the role of the first order derivative term.

In this paper, the nonlinear term f depends on the first order derivative and does not need to satisfy the Nagumo condition. By constructing a special cone and using the upper and lower solutions method, we obtain the sufficient conditions of the existence and uniqueness of solutions, as well as the monotone iterative sequence which is used to solve the boundary-value problem. The error estimate formula is also given under the condition of unique solution. And the method we adopt is new and so are the conclusions we obtain.

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2. Preliminaries

Throughout this paper, we assume that N satisfies the hypothesis

(H1) $0 < N < \frac{\pi}{2}$.

To investigate the boundary-value problem (1.1), we consider the boundary-value problem

$$\begin{gathered} -x''(t) - N^2 x(t) = h(t), \quad t \in (0,1), \\ x'(0) = 0, \quad x(1) = 0, \end{gathered} \tag{2.1}$$

where $h \in C[0,1]$.

Lemma 2.1. *The Green's function of the boundary-value problem*

$$\begin{gathered} -x''(t) - N^2 x(t) = 0, \quad t \in (0,1), \\ x'(0) = 0, \quad x(1) = 0, \end{gathered} \tag{2.2}$$

is

$$G(t,s) = \frac{1}{N\cos N} \begin{cases} \cos(Nt)\sin(N(1-s)), & 0 \le t \le s \le 1, \\ \cos(Ns)\sin(N(1-t)), & 0 \le s \le t \le 1. \end{cases} \tag{2.3}$$

Proof. We look for a Green's function of the form

$$G(t,s) = \begin{cases} A\cos(Nt) + B\sin(Nt), & 0 \le t \le s \le 1, \\ C\cos(Nt) + D\sin(Nt), & 0 \le s \le t \le 1. \end{cases}$$

By the definition and properties of the Green's function and the boundary conditions, it is easy to obtain that

$$\begin{gathered} A = \frac{\sin(N(1-s))}{N\cos N}, \quad B = 0, \\ C = \frac{\sin N \cos(Ns)}{N\cos N}, \quad D = \frac{-\cos(Ns)}{N}. \end{gathered}$$

Hence, the Green's function is as stated in the Lemma. □

It is easy to show that the following lemma holds by means of calculations.

Lemma 2.2. *If (H1) holds. Then: (1)*

$$\frac{\partial G(t,s)}{\partial t} = -\frac{1}{\cos N} \begin{cases} \sin(Nt)\sin(N(1-s)), & 0 \le t < s \le 1, \\ \cos(Ns)\cos(N(1-t)), & 0 \le s < t \le 1, \end{cases}$$

$G(t,s) \ge 0$ *and* $\frac{\partial G(t,s)}{\partial t} \le 0$ *for all* $t, s \in [0,1]$*; (2)*

$$\int_0^1 G(t,s)\,\mathrm{d}s = \frac{\cos(Nt) - \cos N}{N^2 \cos N} \quad \text{for all } t \in [0,1],$$

$$\max_{t\in[0,1]} \int_0^1 G(t,s)\,\mathrm{d}s = \frac{1-\cos N}{N^2\cos N};$$

(3)

$$\int_0^1 \Big(-\frac{\partial G(t,s)}{\partial t}\Big)\,\mathrm{d}s = \frac{\sin(Nt)}{N\cos N} \quad \text{for all } t \in [0,1],$$

$$\max_{t\in[0,1]} \int_0^1 \Big(-\frac{\partial G(t,s)}{\partial t}\Big)\,\mathrm{d}s = \frac{\sin N}{N\cos N}$$

Lemma 2.3. *Suppose $h \in C[0,1]$, $\overline{a}$ and $\overline{b} \in \mathbb{R}$, then the unique solution of the boundary-value problem*

$$\begin{gathered} -x''(t) - N^2 x(t) = h(t), \quad t \in (0,1), \\ x'(0) = \overline{a}, \quad x(1) = \overline{b}, \end{gathered} \tag{2.4}$$

is

$$x(t) = x_0(t) + \int_0^1 G(t,s)h(s)\,\mathrm{d}s$$

where

$$x_0(t) = \frac{1}{N\cos N}[\overline{b}N\cos(Nt) - \overline{a}\sin(N(1-t))] \tag{2.5}$$

Proof. Note that the equation $-x''(t) - N^2x(t) = 0$, has solutions of the form

$$x(t) = c_1\cos(Nt) + c_2\sin(Nt)\,.$$

Using the boundary condition in (2.4), we obtain that the unique solution of the boundary-value problem

$$\begin{gathered} -x''(t) - N^2 x(t) = 0, \quad t \in (0,1), \\ x'(0) = \overline{a}, \quad x(1) = \overline{b}, \end{gathered}$$

is

$$x_0(t) = \frac{1}{N\cos N}[\overline{b}N\cos(Nt) - \overline{a}\sin(N(1-t))]$$

Since $G(t,s)$ is the Green's function of the boundary-value problem (2.2). Then the unique solution of the boundary-value problem (2.4) is

$$x(t) = x_0(t) + \int_0^1 G(t,s)h(s)\,\mathrm{d}s$$

□

From the hypothesis (H1) and the definition of $x_0(t)$, it is easy to see that the following lemma holds.

Lemma 2.4. *If $\overline{b} \geq 0$, $\overline{a} = 0$ and (H1) hold, then $x_0(t) \geq 0$ for all $t \in [0,1]$, where $x_0(t)$ is defined by (2.5).*

In the following, we establish a comparison principle.

Lemma 2.5. *Suppose that $x \in C^2[0,1]$ satisfies*

$$\begin{gathered} -x''(t) - N^2 x(t) \geq 0, \quad t \in (0,1), \\ x'(0) = 0, \quad x(1) \geq 0. \end{gathered}$$

Then $x(t) \geq 0$ for all $t \in [0,1]$.

Proof. Let

$$h(t) = -x''(t) - N^2x(t), \quad \overline{a} = 0, \quad x(1) = \overline{b}\,.$$

Then $h(t) \geq 0$ for all $t \in [0,1]$ and $\overline{b} \geq 0$. Consider the boundary-value problem

$$\begin{gathered} -u''(t) - N^2 u(t) = h(t), \quad t \in (0,1), \\ u'(0) = 0, \quad u(1) = \overline{b}, \end{gathered} \tag{2.6}$$

By Lemma 2.3, this boundary-value problem has the unique solution

$$u(t) = x_0(t) + \int_0^1 G(t,s)h(s)\,\mathrm{d}s$$

Since $x_0(t) \geq 0$ by Lemma 2.4 and $G(t,s) \geq 0$ by Lemma 2.2 for all $t, s \in [0,1]$, we have $u(t) \geq 0$ for all $t \in [0,1]$. It follows from the definition of h that x is a solution of the boundary-value problem (2.6). Hence, $u = x$ which gives $x(t) \geq 0$ for all $t \in [0,1]$. □

Lemma 2.6 ([4, Lemma 1.1.2]). *Let E be partially ordered Banach space, $\{x_n\} \subset E$ is monotone sequence and relatively compact set, then $\{x_n\}$ is convergent.*

Lemma 2.7 ([4, Lemma 1.1.2]). *Let E be partially ordering Banach space, $x_n \preceq y_n$, $(n = 1,2,3\dots)$, if $x_n \to x^*$, $y_n \to y^*$, we have $x^* \preceq y^*$.*

Here the symbol $\preceq$ denotes the partially order in the Banach space E.

Definition. A function $\varphi_0 \in C^2([0,1])$ is said to be a lower solution of boundary-value problem (1.1), if

$$\begin{gathered} -\varphi_0''(t) \leq f(t, \varphi_0(t), \varphi_0'(t)), \\ \varphi_0'(0) = a, \quad \varphi_0(1) \leq b. \end{gathered}$$

A function $\psi_0 \in C^2([0,1])$ is said to be an upper solution of the boundary-value problem (1.1), if

$$\begin{gathered} -\psi_0''(t) \geq f(t, \psi_0(t), \psi_0'(t)), \\ \psi_0'(0) = a, \quad \psi_0(1) \geq b. \end{gathered}$$

3. Existence of solutions of the boundary-value problem

Let $E = C^1[0,1]$ with $\|x\| = \max\{|x|_\infty, |x'|_\infty\}$, where $|x|_\infty = \max_{t\in[0,1]} |x(t)|$. Let

$$P = \{x \in E : x(t) \geq 0 \text{ for all } t \in [0,1], x' \text{ is decreasing and } x'(0) \leq 0\}.$$

Then P is a cone in E and E is a partially ordered Banach space.

Obviously, for any $x \preceq y \in E$ if only and if $y - x \in P$, namely, $x(t) \leq y(t)$ for all $t \in [0,1]$, $y' - x'$ is monotone decreasing and $y'(0) - x'(0) \leq 0$. Then $y'(t) - x'(t) \leq y'(0) - x'(0) \leq 0$ for all $t \in [0,1]$. Therefore

$$x \preceq y \in E \;\Rightarrow\; x(t) \leq y(t) \text{ and } y'(t) - x'(t) \leq 0 \text{ for all } t \in [0,1]. \tag{3.1}$$

For any $\alpha \preceq \beta \in E$, denote $D_0 = [\alpha, \beta] = \{x \in E : \alpha \preceq x \preceq \beta\}$. It is easy to see that D_0 is a bounded set.

Theorem 3.1. *Suppose (H1) holds, and there exist a upper solution ψ_0 and a lower solution φ_0 of boundary-value problem (1.1) such that $\varphi_0 \preceq \psi_0$ and f satisfies:*

- (H2) *$f(t, u_2, v) - f(t, u_1, v) \geq N^2(u_2 - u_1)$ for all $t \in [0,1]$, $\psi_0'(t) \leq v \leq \varphi_0'(t)$ and $\varphi_0(t) \leq u_1 \leq u_2 \leq \psi_0(t)$;*
- (H3) *$f(t, u, v_2) - f(t, u, v_1) \leq 0$ for all $t \in [0,1]$, $\varphi_0(t) \leq u \leq \psi_0(t)$ and $\psi_0'(t) \leq v_1 \leq v_2 \leq \varphi_0'(t)$.*

Then the boundary-value problem (1.1) has a minimal solution φ^* and a maximal solution ψ^* on the ordered interval $[\varphi_0, \psi_0]$. Moreover, the iterative sequences

$$\varphi_n(t) = \overline{x}_0(t) + \int_0^1 G(t,s)(f(s, \varphi_{n-1}(s), \varphi_{n-1}'(s)) - N^2\varphi_{n-1}(s))\,\mathrm{d}s,$$

$$\psi_n(t) = \overline{x}_0(t) + \int_0^1 G(t,s)(f(s,\psi_{n-1}(s),\psi_{n-1}'(s)) - N^2\psi_{n-1}(s))\,\mathrm{d}s$$

converge uniformly on $[0,1]$ *to* φ^* *and* ψ^* *respectively. Here*

$$\overline{x}_0(t) = \frac{1}{N\cos N}[bN\cos(Nt) - a\sin(N(1-t))].$$

Proof. It is easy to see that $x = \varphi_0 = \psi_0$ is the solution of the boundary-value problem (1.1) if $\varphi_0 \equiv \psi_0$. Next we consider $\varphi_0 \not\equiv \psi_0$. Denote $D = [\varphi_0, \psi_0]$. For any $h \in D$, we consider the boundary-value problem

$$\begin{gathered} -x''(t) - N^2x(t) = f(t,h(t),h'(t)) - N^2h(t), \\ x'(0) = a, \quad x(1) = b. \end{gathered} \tag{3.2}$$

By Lemma 2.3, the unique solution of the above boundary-value problem is

$$x(t) = \overline{x}_0(t) + \int_0^1 G(t,s)(f(s,h(s),h'(s)) - N^2h(s))\,\mathrm{d}s := (Qh)(t) \tag{3.3}$$

where

$$\overline{x}_0(t) = \frac{1}{N\cos N}[bN\cos(Nt) - a\sin(N(1-t))]$$

It is clear that x is a solution of boundary-value problem (1.1) if and only if x is a fixed point of Q.

Let $F : D \to C([0,1])$, $(Fh)(t) = f(t,h(t),h'(t)) - N^2h(t)$. Then F is a continuous and bounded operator.

Define $T : C([0,1]) \to C^1([0,1])$, $(Th)(t) = \overline{x}_0(t) + \int_0^1 G(t,s)h(s)\,\mathrm{d}s$. It is obvious that T is a linear completely continuous operator.

Denote $Q = T \circ F$, so $Q : D \to C^1([0,1])$ is continuous and relatively compact, $Q(D)$ is a relatively compact set.

(1) We prove Q is an increasing operator. For any h_1, $h_2 \in D$ and $h_1 \preceq h_2$, by (3.1), we have

$$\varphi_0(t) \le h_1(t) \le h_2(t) \le \psi_0(t) \quad \text{and} \quad \psi_0'(t) \le h_2'(t) \le h_1'(t) \le \varphi_0'(t)$$

for all $t \in [0,1]$. By (H2) and (H3),

$$\begin{aligned} &[f(t,h_2(t),h_2'(t)) - N^2h_2(t)] - [f(t,h_1(t),h_1'(t)) - N^2h_1(t)] \\ &= [f(t,h_2(t),h_2'(t)) - f(t,h_1(t),h_1'(t)))] - N^2(h_2(t) - h_1(t)) \\ &= [f(t,h_2(t),h_2'(t)) - f(t,h_1(t),h_2'(t)))] + [f(t,h_1(t),h_2'(t)) - f(t,h_1(t),h_1'(t)))] \\ &\quad - N^2(h_2(t) - h_1(t)) \\ &\ge N^2(h_2(t) - h_1(t)) - N^2(h_2(t) - h_1(t)) \ge 0 \end{aligned}$$

Therefore, $(Qh_1)(t) \le (Qh_2)(t)$ by Lemma 2.2(1) for all $t \in [0,1]$. Also for all $t \in [0,1]$,

$$\begin{aligned} (Qh_2)''(t) - (Qh_1)''(t) &= -[N^2(Qh_2)(t) + f(t,h_2(t),h_2'(t)) - N^2h_2(t)] \\ &\quad + [N^2(Qh_1)(t) + f(t,h_1(t),h_1'(t)) - N^2h_1(t)] \\ &= N^2[(Qh_1)(t) - (Qh_2)(t)] - [f(t,h_2(t),h_2'(t)) - N^2h_2(t)] \\ &\quad + [f(t,h_1(t),h_1'(t)) - N^2h_1(t)] \\ &\le 0 \end{aligned}$$

Hence, $(Qh_2)'(t) - (Qh_1)'(t)$ is monotonically decreasing for all $t \in [0,1]$.

Obviously, $(Qh_2)'(0) - (Qh_1)'(0) = a - a = 0$. So $(Qh_2) - (Qh_1) \in P$, namely $Qh_1 \preceq Qh_2$. We get $Q : D \to C^1([0,1])$ is an increasing operator.
(2) We prove $Q\psi_0 \preceq \psi_0$, $\varphi_0 \preceq Q\varphi_0$. Denote $\psi_1 = Q\psi_0$, since ψ_0 is the upper solution of the boundary-value problem (1.1). Then

$$\begin{gathered} -\psi_0''(t) \geq f(t, \psi_0(t), \psi_0'(t)), \\ \psi_0'(0) = a, \quad \psi_0(1) \geq b. \end{gathered}$$

Let $\psi = \psi_0 - \psi_1$, the definition of Q yields

$$\begin{aligned} & -\psi''(t) - N^2\psi(t) \\ & = -(\psi_0(t) - \psi_1(t))'' - N^2(\psi_0(t) - \psi_1(t)) \\ & = (-\psi_0''(t) - N^2\psi_0(t)) - (-\psi_1''(t) - N^2\psi_1(t)) \\ & \geq (f(t, \psi_0(t), \psi_0'(t)) - N^2\psi_0(t)) - (f(t, \psi_0(t), \psi_0'(t)) - N^2\psi_0(t)) = 0 \end{aligned}$$

for all $t \in [0,1]$ and

$$\begin{gathered} \psi'(0) = \psi_0'(0) - \psi_1'(0) = a - a = 0, \\ \psi(1) = \psi_0(1) - \psi_1(1) \geq b - b = 0. \end{gathered}$$

By Lemma 2.5, we have $\psi(t) \geq 0$ on $[0,1]$. That is $(Q\psi_0)(t) \leq \psi_0(t)$ for all $t \in [0,1]$. Moreover

$$\begin{aligned} & \psi_0''(t) - (Q\psi_0)''(t) \\ & \leq -f(t, \psi_0(t), \psi_0'(t)) + [N^2(Q\psi_0)(t) + f(t, \psi_0(t), \psi_0'(t)) - N^2\psi_0(t)] \\ & = N^2[(Q\psi_0)(t) - \psi_0(t)] \leq 0 \end{aligned}$$

for all $t \in [0,1]$. Hence, $\psi_0'(t) - (Q\psi_0)'(t)$ is monotone decreasing for all $t \in [0,1]$.

Obviously, $\psi_0'(0) - (Q\psi_0)'(0) = a - a = 0$. Therefore, we get $Q\psi_0 \preceq \psi_0$. Similarly, we can prove that $\varphi_0 \preceq Q\varphi_0$. So $\varphi_0 \preceq \varphi_1 \preceq \psi_1 \preceq \psi_0$.
(3) We prove the existence of the minimal solution and the maximal solution of boundary-value problem (1.1). We can repeat step (2) and construct an iterative sequence

$$\begin{gathered} \psi_n = Q\psi_{n-1} = \overline{x}_0(t) + \int_0^1 G(t,s)(f(s, \psi_{n-1}(s), \psi_{n-1}'(s)) - N^2\psi_{n-1}(s))\,\mathrm{d}s, \\ \varphi_n = Q\varphi_{n-1} = \overline{x}_0(t) + \int_0^1 G(t,s)(f(s, \varphi_{n-1}(s), \varphi_{n-1}'(s)) - N^2\varphi_{n-1}(s))\,\mathrm{d}s \end{gathered}$$

for $n = 1, 2, \dots$. We obtain

$$\varphi_0 \preceq \varphi_1 \preceq \varphi_2 \preceq \dots \leq \varphi_n \preceq \dots \preceq \psi_n \preceq \dots \preceq \psi_2 \preceq \psi_1 \preceq \psi_0$$

By $\{\psi_n\}$, $\{\varphi_n\} \subset Q(D)$ and Lemma 2.6 we can show that there exist φ^*, $\psi^* \in D$ such that $\psi_n \to \psi^*$, $\varphi_n \to \varphi^*$ $(n \to \infty)$. By the continuity of Q, we have $\varphi^* = Q\varphi^*$ and $\psi^* = Q\psi^*$ as $n \to \infty$. So φ^* and ψ^* are the fixed points of Q.

In the following, we prove that φ^*, ψ^* are the minimal solution and the maximal solution of boundary-value problem (1.1), respectively.

Assume $z \in D = [\varphi_0, \psi_0]$ is a fixed point of Q, then $\varphi_0 \preceq z \preceq \psi_0$. As Q is an increasing operator, we get $Q\varphi_0 \preceq Qz \preceq Q\psi_0$, that is $\varphi_1 \preceq z \preceq \psi_1$. In a similar way we have $Q\varphi_1 \preceq Qz \preceq Q\psi_1$, that is $\varphi_2 \preceq z \preceq \psi_2$. Repeat it, and we have $\varphi_n \preceq z \preceq \psi_n$ for $n = 3, 4, \dots$.

By Lemma 2.7, we can obtain $\varphi^* \preceq z \preceq \psi^*$. Namely, φ^*, ψ^* are the minimal fixed point and the maximal point of Q, respectively. Therefore, φ^*, ψ^* are the minimal solution and maximal solution of boundary-value problem (1.1) in the ordered interval $[\varphi_0, \psi_0]$, respectively. □

4. Uniqueness of solutions of the boundary-value problem

It is easy to show that the following lemma holds.

Lemma 4.1. *If (H1) holds, then* $\sin N > \frac{1-\cos N}{N}$.

Theorem 4.2. *Suppose that the hypotheses of Theorem 3.1 hold, and*

(H4) *There exists a constant M_1 with $0 < M_1 < N \cot N$, such that*
$$f(t,u,v_1) - f(t,u,v_2) \le M_1(v_2 - v_1)$$
for all $t \in [0,1]$, $\varphi_0(t) \le u \le \psi_0(t)$ and $\psi_0'(t) \le v_1 \le v_2 \le \varphi_0'(t)$;

(H5) *There exists a constant M_2 with $N^2 < M_2 < N^2 + N\cot N - M_1$, such that*
$$f(t,u_2,v) - f(t,u_1,v) \le M_2(u_2 - u_1)$$
for all $t \in [0,1]$, $\psi_0'(t) \le v \le \varphi_0'(t)$ and $\varphi_0(t) \le u_1 \le u_2 \le \psi_0(t)$.

Then the boundary-value problem (1.1) *has a unique solution x^* on $[\varphi_0, \psi_0]$ and for any $x_0 \in [\varphi_0, \psi_0]$, iterative sequence*
$$x_n(t) = \overline{x}_0(t) + \int_0^1 G(t,s)(f(s, x_{n-1}(s), x_{n-1}'(s)) - N^2 x_{n-1}(s))\,\mathrm{d}s, \quad n = 1,2,\dots$$
converge uniformly to x^* on [0,1], and its error estimate formula is
$$\|x_n - x^*\| \le 2\Big(\frac{M_1 + M_2 - N^2}{N\cot N}\Big)^n \|\psi_0 - \varphi_0\|, \quad n = 1,2,\dots.$$

Proof. Let φ_n and ψ_n be defined as in Theorem 3.1. According to (H4), (H5), Lemma 2.2 and the assumptions of Theorem 4.2 we have
$$\begin{aligned}
0 &\le \psi_n(t) - \varphi_n(t)\\
&= (Q\psi_{n-1})(t) - (Q\varphi_{n-1})(t)\\
&= \int_0^1 G(t,s)(f(s,\psi_{n-1}(s),\psi_{n-1}'(s)) - N^2\psi_{n-1}(s))\,\mathrm{d}s\\
&\quad - \int_0^1 G(t,s)(f(s,\varphi_{n-1}(s),\varphi_{n-1}'(s)) - N^2\varphi_{n-1}(s))\,\mathrm{d}s\\
&= \int_0^1 G(t,s)\big[f(s,\psi_{n-1}(s),\psi_{n-1}'(s)) - f(s,\varphi_{n-1}(s),\varphi_{n-1}'(s))\\
&\quad + N^2(\varphi_{n-1}(s) - \psi_{n-1}(s))\big]\,\mathrm{d}s\\
&= \int_0^1 G(t,s)[(f(s,\psi_{n-1}(s),\psi_{n-1}'(s)) - f(s,\psi_{n-1}(s),\varphi_{n-1}'(s)))\\
&\quad + (f(s,\psi_{n-1}(s),\varphi_{n-1}'(s)) - f(s,\varphi_{n-1}(s),\varphi_{n-1}'(s)))\\
&\quad + N^2(\varphi_{n-1}(s) - \psi_{n-1}(s))]\,\mathrm{d}s\\
&\le \int_0^1 G(t,s)[M_1(\varphi_{n-1}'(s)) - \psi_{n-1}'(s)) + M_2(\psi_{n-1}(s)) - \varphi_{n-1}(s))\\
&\quad + N^2(\varphi_{n-1}(s) - \psi_{n-1}(s)]\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 G(t,s)[M_1(\varphi'_{n-1}(s)) - \psi'_{n-1}(s)) + (M_2 - N^2)(\psi_{n-1}(s)) - \varphi_{n-1}(s))]\,\mathrm{d}s \\
&\le (M_1 + M_2 - N^2)\int_0^1 G(t,s)\|\psi_{n-1} - \varphi_{n-1}\|\,\mathrm{d}s \\
&\le (M_1 + M_2 - N^2)\|\psi_{n-1} - \varphi_{n-1}\|\frac{1-\cos N}{N^2\cos N}\,.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
0 &\le \varphi'_n(t) - \psi'_n(t) \\
&= (Q\varphi_{n-1})'(t) - (Q\psi_{n-1})'(t) \\
&= -\int_0^1 \frac{\partial G(t,s)}{\partial t}(f(s,\psi_{n-1}(s),\psi'_{n-1}(s)) - N^2\psi_{n-1}(s))\,\mathrm{d}s \\
&\quad + \int_0^1 \frac{\partial G(t,s)}{\partial t}(f(s,\varphi_{n-1}(s),\varphi'_{n-1}(s)) - N^2\varphi_{n-1}(s))\,\mathrm{d}s \\
&= \int_0^1 \Big(-\frac{\partial G(t,s)}{\partial t}\Big)[f(s,\psi_{n-1}(s),\psi'_{n-1}(s)) - f(s,\varphi_{n-1}(s),\varphi'_{n-1}(s)) \\
&\quad + N^2(\varphi_{n-1}(s) - \psi_{n-1}(s))]\,\mathrm{d}s \\
&\le (M_1 + M_2 - N^2)\int_0^1 \Big(-\frac{\partial G(t,s)}{\partial t}\Big)\|\psi_{n-1} - \varphi_{n-1}\|\,\mathrm{d}s \\
&\le (M_1 + M_2 - N^2)\|\psi_{n-1} - \varphi_{n-1}\|\frac{\sin N}{N\cos N}\,.
\end{aligned}
$$

By Lemma 4.1,

$$
\begin{aligned}
\|\psi_n - \varphi_n\| &\le \max\Big\{\frac{\sin N}{N\cos N}, \frac{1-\cos N}{N^2\cos N}\Big\}(M_1 + M_2 - N^2)\|\psi_{n-1} - \varphi_{n-1}\| \\
&= \frac{1}{N\cot N}(M_1 + M_2 - N^2)\|\psi_{n-1} - \varphi_{n-1}\|
\end{aligned}
$$

Using the inequality repeatedly, we have

$$
\|\psi_n - \varphi_n\| \le \Big(\frac{M_1 + M_2 - N^2}{N\cot N}\Big)^n\|\psi_0 - \varphi_0\|
$$

Noting that $0 < \frac{M_1+M_2-N^2}{N\cot N} < 1$, we have

$$
\|\psi_n - \varphi_n\| \to 0, \quad \text{as } n \to \infty.
$$

Since $\psi_n \to \psi^*$, $\varphi_n \to \varphi^*$, there exists the unique $x^* \in \bigcap_{n=1}^{\infty}[\varphi_n, \psi_n]$ such that $\psi_n \to x^*$, $\varphi_n \to x^*$, $(n \to \infty)$. So by Lemma 2.7,

$$
\varphi_n \preceq x^* \preceq \psi_n, \quad x^* \in D.
$$

The monotonicity of Q implies

$$
\varphi_{n+1} = Q\varphi_n \preceq Qx^* \preceq Q\psi_n = \psi_{n+1}.
$$

Let $n \to \infty$ we can show that $x^* \preceq Qx^* \preceq x^*$. So $x^* = Qx^*$. Consequently, x^* is the unique solution of boundary-value problem (1.1). For any $x_0 \in [\varphi_0, \psi_0]$, we have

$$
\|x_n - x^*\| \le \|x_n - \varphi_n\| + \|\varphi_n - x^*\| \le 2\|\psi_n - \varphi_n\| \le 2\Big(\frac{M_1 + M_2 - N^2}{N\cot N}\Big)^n\|\psi_0 - \varphi_0\|
$$

where

$$x_n(t) = \overline{x}_0(t) + \int_0^1 G(t,s)(f(s, x_{n-1}(s), x'_{n-1}(s)) - N^2 x_{n-1}(s))\,\mathrm{d}s, \quad n = 1, 2, \dots$$

for all $t \in [0,1]$. □

5. Illustration

In this section, we give an example about the theoretical results. Let $a = 0$, $b = 1$, $N = 1$, $f(t,u,v) = 1 + (1 + \frac{1}{8}t^2)u - \frac{1}{8}v$. Then $f \in C([0,1] \times \mathbb{R}^2, \mathbb{R})$, a, $b \in \mathbb{R}$ and N satisfies the hypothesis (H1). Consider the boundary-value problem

$$\begin{gathered} x''(t) + 1 + (1 + \frac{1}{8}t^2)x(t) - \frac{1}{8}x'(t) = 0, \quad t \in (0,1) \\ x'(0) = 0, \quad x(1) = 1\,. \end{gathered} \tag{5.1}$$

Let $\varphi_0(t) = \int_0^1 G(t,s)\,\mathrm{d}s$, $\psi_0(t) = 4\int_0^1 G(t,s)\,\mathrm{d}s + 1$ for all $t \in [0,1]$, where

$$G(t,s) = \frac{1}{\cos 1}\begin{cases} \cos t \sin(1-s), & 0 \le t \le s \le 1, \\ \cos s \sin(1-t), & 0 \le s \le t \le 1. \end{cases} \tag{5.2}$$

So $\psi'_0(t) = 4\varphi'_0(t)$ and $\psi''_0(t) = 4\varphi''_0(t)$ for all $t \in [0,1]$. By Lemma 2.2, we have

$$\begin{gathered} \max_{t\in[0,1]} \int_0^1 G(t,s)\,\mathrm{d}s = \frac{1}{\cos 1} - 1 \approx 0.8508, \\ \max_{t\in[0,1]} \int_0^1 \big(- \frac{\partial G(t,s)}{\partial t}\big)\,\mathrm{d}s = \tan 1 \approx 1.5574, \end{gathered}$$

and $\varphi'_0(t) \le 0$ for all $t \in [0,1]$. By Lemma 2.1, Lemma 2.2 and Lemma 2.3, we can obtain

$$\begin{gathered} -\varphi''_0(t) = \varphi_0(t) + 1 \le 1 + (1 + \frac{1}{8}t^2)\varphi_0(t) - \frac{1}{8}\varphi'_0(t) = f(t, \varphi_0(t), \varphi'_0(t)), \quad t \in (0,1) \\ \varphi'_0(0) = 0, \quad \varphi_0(1) = 0 < 1 \end{gathered}$$

and

$$\begin{gathered} -\psi''_0(t) = \psi_0(t) + 3 \ge 1 + (1 + \frac{1}{8}t^2)\psi_0(t) - \frac{1}{8}\psi'_0(t) = f(t, \psi_0(t), \psi'_0(t)), \quad t \in (0,1) \\ \psi'_0(0) = 0, \quad \psi_0(1) = 1 \end{gathered}$$

Hence, φ_0, ψ_0 are the lower solution and the upper solution of the boundary-value problem (5.1), respectively, and $\varphi_0 \preceq \psi_0$.

Let $M_1 = \frac{1}{8}$, $M_2 = \frac{9}{8}$. Then $0 < M_1 < \cot 1$ and $1 < M_2 < 1 + \cot 1 - M_1$. Therefore, the boundary-value problem (5.1) satisfies the conditions of Theorem 4.2. Then the boundary-value problem (5.1) has the unique solution x^* on $[\varphi_0, \psi_0]$ and for any $x_0 \in [\varphi_0, \psi_0]$, iterative sequence

$$x_n(t) = \overline{x}_0(t) + \int_0^1 G(t,s)(f(s, x_{n-1}(s), x'_{n-1}(s)) - x_{n-1}(s))\,\mathrm{d}s, \quad n = 1, 2, \dots$$

converge uniformly to x^* on $[0,1]$, and its error estimate formula is

$$\|x_n - x^*\| \le 2\big(\frac{\tan 1}{4}\big)^n \|\psi_0 - \varphi_0\|, \quad n = 1, 2, \dots.$$

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