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BOUNDARY LAYERS FOR TRANSMISSION PROBLEMS WITH SINGULARITIES

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Abstract. We study two-dimensional transmission problems for the Laplace operator for two diffusion coefficients. We describe the boundary layers of this problem and show that the layers appear only in the part where the coefficient is large. The relationship with the singularities of the limit problem is also described.

1. Introduction

We study two-dimensional transmission problems (also called interface problems) for the Laplace operator on polygonal domains consisting of different materials connected via an interface line. Dirichlet boundary conditions on the exterior boundary and standard transmission conditions are imposed. Such problems appear in diffusion problems where the conductivity of the materials are different on some parts of the domain. It is well known that the solutions of such problems have corner singularities due the jump of the coefficients [6, 7, 9, 10, 12, 13, 14]. On the other hand, for a homogeneous medium having a large diffusion coefficient, the solution exhibits boundary layers added to corner singularities. Their relationship and description are well understood nowadays [1, 4, 5, 8, 11]. But to our knowledge, the description of such a phenomenon is not known for transmission problems where only one of the diffusion coefficients is large. Therefore in this paper we study a relatively simple example of a transmission problem that has corner singularities and boundary layers.

For a standard problem

$$-\varepsilon\Delta u_\varepsilon + u_\varepsilon = f \quad \text{in } \Omega, \tag{1.1}$$

when Ω is a polygonal domain of the plane, f is smooth and $\varepsilon > 0$ is a fixed (but small) parameter. An asymptotic expansion of u_ε is well known [1, 4, 8, 11] and may be written as

$$u_\varepsilon = w_\varepsilon + w^{BL} + w^{CL} + r_\varepsilon,$$

where w_ε is the outer expansion, w^{BL} describes the boundary layer, w^{CL} describes the corner layer, and r_ε is a remainder that is estimated as a function of ε in some appropriate norms. Usually the terms w_ε, w^{BL} and w^{CL} are explicit, which means

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that, for numerical purposes for instance, the behaviour of u_ε is fully understood by the behaviour of the terms w_ε, w^{BL} and w^{CL}.

The goal of the present paper is to reproduce a similar but simpler expansion for a transmission problem where on a part Ω_+ of the domain we consider the problem

$$-\varepsilon\Delta u_\varepsilon + u_\varepsilon = f \quad \text{in } \Omega_+,$$

and on the other part Ω_-, the problem

$$-\Delta u_\varepsilon + u_\varepsilon = f \quad \text{in } \Omega_-,$$

with, of course, transmission conditions on the interface. By a simpler expansion, we mean that w_ε, w^{BL}, w^{CL} will be reduced to one term. As we shall see the situation is more complicated than in the standard case of problem (1.1). The main reason is that the solution of the limit problem has singularities in the domain Ω_-. Let us further notice that surprisingly the solution of our problem has only layers in the domain Ω_+.

In this paper, the spaces $H^s(\Omega)$, with $s \geq 0$, are the standard Sobolev spaces in Ω with norm $\|\cdot\|_{s,\Omega}$ and semi-norm $|\cdot|_{s,\Omega}$. The space $H_0^1(\Omega)$ is defined, as usual, by $H_0^1(\Omega) := \{v \in H^1(\Omega)/v = 0 \text{ on } \Gamma\}$. $L^p(\Omega)$, $p > 1$, are the usual Lebesgue spaces with norm $\|\cdot\|_{0,p,\Omega}$ (as usual we drop the index p for $p = 2$). Finally, the notation $a \lesssim b$ means the existence of a positive constant C, which is independent of the quantities a and b under consideration and of the parameter ε, such that $a \leq Cb$.

This paper is organized as follows: In section 2 we start with a one-dimensional problem in order to describe and understand the typical phenomena. Section 3 is devoted to the introduction of the two-dimensional problem and to the (weak) convergence of the solution to the solution of the limit problem. We go on with the description of the boundary and corner layers in section 4, paying a particular attention to the interface layers due to the singularities. Finally in section 5 we give the expansion of the solution of our problem.

2. The one-dimensional case

Let $\varepsilon \in]0,1]$ be a fixed parameter. Consider the following transmission problem in $]-1,1[$:

$$\begin{gathered}
-\varepsilon^2 u_\varepsilon'' + u_\varepsilon = 1 \quad \text{in }]-1,0[,\\
-w_\varepsilon'' + w_\varepsilon = 0 \quad \text{in }]0,1[,\\
u_\varepsilon(-1) = w_\varepsilon(1) = 0,\\
u_\varepsilon(0) - w_\varepsilon(0) = 0,\\
\varepsilon^2 u_\varepsilon'(0) - w_\varepsilon'(0) = 0.
\end{gathered} \tag{2.1}$$

We remark that in this problem the small parameter ε appears only on $]-1,0[$. Consequently the formal limit problem is the non standard transmission problem

$$\begin{gathered}
u_0 = 1 \quad \text{in }]-1,0[,\\
-w_0'' + w_0 = 0 \quad \text{in }]0,1[,\\
u_0(-1) = w_0(1) = 0,\\
u_0(0) - w_0(0) = 0,\\
w_0'(0) = 0.
\end{gathered} \tag{2.2}$$

This limit problem has a solution $w_0 \equiv 0$, but has no solution u_0 since $u_0 = 1$ does not satisfy the boundary condition $u_0(-1) = 0$ and the transmission condition $u_0(0) - w_0(0) = 0$. Therefore, we may expect that u_ε will develop boundary layers at 0 (transmission layer) and at -1 (standard boundary layer). We now justify this formal argument.

The exact solution of this problem (2.1) is

$$\begin{aligned} u_\varepsilon(x) &= \alpha \cosh \frac{x}{\varepsilon} + \beta \sinh \frac{x}{\varepsilon} + 1, \\ w_\varepsilon(x) &= \gamma \cosh x + \delta \sinh x, \end{aligned} \tag{2.3}$$

where α, β, γ and δ are constants (depending on ε) determined in order to check the boundary and transmission conditions. This yields a 4×4 linear system that gives after resolution:

$$\begin{aligned} \alpha + 1 = \gamma, &\quad \beta = \delta/\varepsilon, \\ \gamma = -\delta \tanh 1, &\quad \delta = -\varepsilon\psi(\varepsilon), \end{aligned} \tag{2.4}$$

where the function ψ is

$$\psi(\varepsilon) = \frac{\cosh \frac{1}{\varepsilon} - 1}{\varepsilon \tanh 1 \cosh \frac{1}{\varepsilon} + \sinh \frac{1}{\varepsilon}}.$$

Since one easily sees that $\psi(\varepsilon)$ approaches 1 as ε approaches 0, we deduce that $\delta = -\varepsilon\psi(\varepsilon) \sim -\varepsilon$ as $\varepsilon \to 0$. Due to the identities (2.3) and (2.4), we can show that, as ε approaches 0, $u_\varepsilon \to 1$ and $w_\varepsilon \to 0$, as well as

$$\begin{gathered} u'_\varepsilon(-1) = -\frac{\alpha}{\varepsilon} \sinh \frac{1}{\varepsilon} + \frac{\beta}{\varepsilon} \cosh \frac{1}{\varepsilon} \sim \frac{1}{\varepsilon}, \\ u'_\varepsilon(0) = \frac{\beta}{\varepsilon} \sim \frac{-1}{\varepsilon}, \\ w'_\varepsilon(1) = \gamma \sinh 1 + \delta \cosh 1 \sim \frac{-\varepsilon}{\cosh 1}, \\ w'_\varepsilon(0) = \delta \sim -\varepsilon. \end{gathered}$$

From these equivalences, we may say that w_ε has no layer, while u_ε has a standard boundary layer at -1 and a transmission layer at 0. We also refer to Figure 1 for an illustration of this fact.

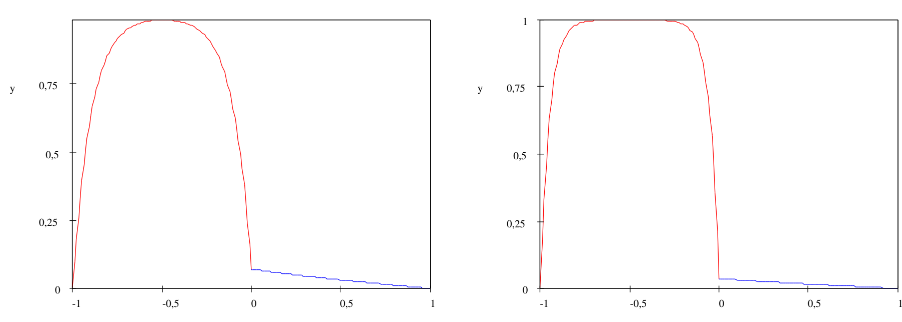

FIGURE 1. Exact solutions for $\varepsilon = 0.1$ (left) and $\varepsilon = 0.05$ (right).

Let us give a more precise result, that will also allow us to underline the fact that the transmission layer at 0 may be seen as a (Dirichlet) boundary layer.

Theorem 2.1. *For any* $\varepsilon \in]0,1]$, *the unique solution* $(u_\varepsilon, w_\varepsilon)$ *of* (2.1) *satisfies*

$$u_\varepsilon(x) = 1 - \chi^b(x)\exp\big(-\frac{\operatorname{dist}(x,-1)}{\varepsilon}\big) - \chi^i(x)\exp\big(-\frac{\operatorname{dist}(x,0)}{\varepsilon}\big) + r_\varepsilon(x), \quad \forall x \in]0,1[, \tag{2.5}$$

where χ^b *and* χ^i *are the two following cut-off functions:*

$$\begin{gathered}\chi^b = 1 \quad \textit{on }]-1,-1+\eta[,\\ \chi^i = 1 \quad \textit{on }]-\eta,\eta[,\\ \operatorname{supp}\chi^b \cap \operatorname{supp}\chi^i = \emptyset.\end{gathered}$$

Moreover,

$$\varepsilon\|r'_\varepsilon\|_{0,]-1,0[} + \|r_\varepsilon\|_{0,]-1,0[} + \|w_\varepsilon\|_{1,]0,1[} \lesssim (\varepsilon e^{\frac{-\eta}{\varepsilon}} + \varepsilon). \tag{2.6}$$

Proof. Let us define the functions $v^b : x \mapsto -\exp\big(-\frac{\operatorname{dist}(x,-1)}{\varepsilon}\big)$, a solution of

$$\begin{gathered}-\varepsilon^2 v^{b\prime\prime} + v^b = 0 \quad \text{in }]-1,0[,\\ v^b(-1) + 1 = 0,\\ v^b(+\infty) = 0,\end{gathered}$$

and $v^i : x \mapsto -\exp\big(-\frac{\operatorname{dist}(x,0)}{\varepsilon}\big)$, a solution of

$$\begin{gathered}-\varepsilon^2 v^{i\prime\prime} + v^i = 0, \quad \text{in }]-1,0[,\\ v^i(0) + 1 = 0,\\ v^i(-\infty) = 0.\end{gathered}$$

Using these two problems and by substitution of (2.5) in (2.1), we see that $(r_\varepsilon, w_\varepsilon)$ is solution of

$$\begin{gathered}-\varepsilon^2 r''_\varepsilon + r_\varepsilon = g_\varepsilon \quad \text{in }]-1,0[,\\ w_\varepsilon - w''_\varepsilon = 0 \quad \text{in }]0,1[,\\ r_\varepsilon(-1) = 0,\\ w_\varepsilon(1) = 0,\\ r_\varepsilon(0) = w_\varepsilon(0),\\ \varepsilon^2 r'_\varepsilon(0) - w'_\varepsilon(0) = -\varepsilon,\end{gathered} \tag{2.7}$$

where

$$g_\varepsilon := \varepsilon^2\left([\chi^b; \frac{d^2}{dx^2}]e^{-\frac{x+1}{\varepsilon}} + [\chi^i; \frac{d^2}{dx^2}]e^{\frac{x}{\varepsilon}}\right),$$

the bracket $[\chi^b; \frac{d^2}{dx^2}]$ being defined as usual,

$$[\chi^b; \frac{d^2}{dx^2}]h = \frac{d^2}{dx^2}(\chi^b h) - \chi^b\frac{d^2}{dx^2}h = h\frac{d^2}{dx^2}\chi^b + 2\frac{d}{dx}\chi^b\frac{d}{dx}h.$$

The variational formulation of this problem is

$$\begin{aligned}\int_{-1}^{0} \varepsilon^2 r_\varepsilon' w'\,dx + \int_0^1 w_\varepsilon' w'\,dx + \int_{-1}^0 r_\varepsilon w\,dx + \int_0^1 w_\varepsilon w\,dx \\ = \int_{-1}^0 g_\varepsilon w\,dx - \varepsilon w(0), \forall w \in H_0^1(]-1,1[).\end{aligned} \tag{2.8}$$

Since this left-hand side is trivially coercive on $H_0^1(]-1,1[)$, by the Lax-Milgram lemma, this problem has a unique solution $r_\varepsilon \in H^1(]-1,0[)$ and $w_\varepsilon \in H^1(]0,1[)$ such that $r_\varepsilon(-1) = w_\varepsilon(1) = 0$, and $r_\varepsilon(0) = w_\varepsilon(0)$ (this means that the function k_ε defined by r_ε on $]-1,0[$ and by w_ε on $]0,1[$ belongs to $H_0^1(]-1,1[)$). Moreover by taking as test function in (2.8) $w = k_\varepsilon$ we obtain

$$\begin{aligned}&\varepsilon^2 \|r_\varepsilon'\|^2_{0,]0,1[} + \|r_\varepsilon\|^2_{0,]-1,0[} + \|w_\varepsilon'\|^2_{0,]0,1[} + \|w_\varepsilon\|^2_{0,]0,1[} \\ &\le \|g_\varepsilon\|_{0,]-1,0[} \|r_\varepsilon\|_{0,]-1,0[} + \varepsilon |w_\varepsilon(0)|.\end{aligned} \tag{2.9}$$

It then remains to estimate the L^2-norm of g_ε. The properties of χ^b and χ^i imply that $\operatorname{supp} g_\varepsilon \subset [-1+\eta, -\eta]$. Since on this interval $e^{-\frac{(x+1)}{\varepsilon}} \le e^{-\frac{\eta}{\varepsilon}}$ and $e^{\frac{x}{\varepsilon}} \le e^{-\frac{\eta}{\varepsilon}}$, we obtain

$$\|g_\varepsilon\|_{0,]-1,0[} \lesssim \varepsilon e^{\frac{-\eta}{\varepsilon}}.$$

On the other hand, the identities (2.3) and (2.4) imply that

$$|w_\varepsilon(0)| \lesssim \varepsilon \quad \forall \varepsilon \in]0,1[.$$

These two estimates in (2.9) yield

$$\varepsilon^2 \|r_\varepsilon'\|^2_{0,]-1,0[} + \|r_\varepsilon\|^2_{0,]-1,0[} + \|w_\varepsilon'\|^2_{0,]0,1[} + \|w_\varepsilon\|^2_{0,]0,1[} \lesssim \varepsilon e^{\frac{-\eta}{\varepsilon}} \|r_\varepsilon\|_{0,]-1,0[} + \varepsilon^2.$$

The desired estimate (2.6) follows from Young's inequality. □

Note that the estimate (2.6) is optimal: Direct calculations yield $\|w_\varepsilon\|_{0,]0,1[} \sim \varepsilon$.

The above theorem gives an explicit expansion of u_ε, which also shows that u_ε has two layers (at 0 and -1). It further says that the natural energy norm of the remainder r_ε is of order ε. Finally it says that w_ε has no layer and that its natural energy norm is of order ε.

The goal of the next sections is to show similar results for a polygonal domain on the plane.

3. The two-dimensional problem

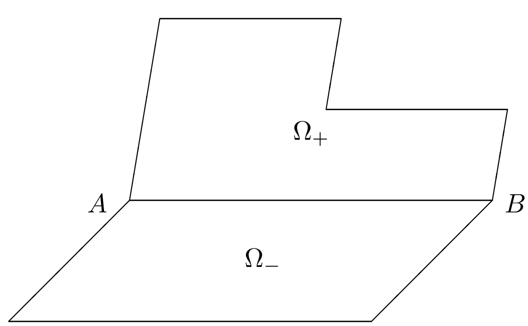

Figure 2. The domains Ω_+ and Ω_-

Let Ω_+ and Ω_- be two polygonal domains of $\mathbb{R}^2$ with respective boundary $\partial\Omega_+$ and $\partial\Omega_-$ having in common a segment $\Sigma = [A, B]$, see Figure 2. Denote by $A_1, A_2, \dots, A_N$ the vertices of $\partial\Omega_+$ enumerated clockwise and so that $A_1 = A$ and $A_2 = B$. Denote further by ω_j the interior angle of Ω_+ at the vertex A_j, for any $j \in \{1, 2, \dots, N\}$ and let φ_j the interior angle of Ω_- at the vertex A_j, $j = 1, 2$.

For further purposes we denote by $\Omega = \Omega_+ \cup \Omega_- \cup \Sigma$. Moreover for a function u defined in Ω, we denote by u_+ (resp. u_-) the restriction of u to Ω_+ (resp. Ω_-).

For $\varepsilon \in]0, 1[$, $f_\pm \in \mathcal{C}^\infty(\bar{\Omega}_\pm)$ and $h \in \mathcal{C}^\infty(\bar{\Sigma})$, we consider the transmission problem in Ω: Find u^ε solution of

$$\begin{gathered}
-\varepsilon^2 \Delta u_+^\varepsilon + u_+^\varepsilon = f_+ \quad \text{in } \Omega_+, \\
-\Delta u_-^\varepsilon + u_-^\varepsilon = f_- \quad \text{in } \Omega_-, \\
u_+^\varepsilon = 0 \quad \text{on } \partial\Omega_+ \setminus \Sigma, \\
u_-^\varepsilon = 0 \quad \text{on } \partial\Omega_- \setminus \Sigma, \\
u_+^\varepsilon - u_-^\varepsilon = 0 \quad \text{on } \Sigma, \\
\varepsilon^2 \frac{\partial u_+^\varepsilon}{\partial \nu} - \frac{\partial u_-^\varepsilon}{\partial \nu} = h \quad \text{on } \Sigma,
\end{gathered} \tag{3.1}$$

where ν denotes the outward normal vector along Σ oriented outside Ω_+. The variational formulation of this problem consists in finding a unique solution $u^\varepsilon \in H_0^1(\Omega)$ of

$$\begin{aligned}
&\int_{\Omega_+} (\varepsilon^2 \nabla u_+^\varepsilon \cdot \nabla v_+ + u_+^\varepsilon v_+) + \int_{\Omega_-} (\nabla u_-^\varepsilon \cdot \nabla v_- + u_-^\varepsilon v_-) \\
&= \int_{\Omega_+} f v + \int_\Sigma h v, \forall v \in H_0^1(\Omega).
\end{aligned} \tag{3.2}$$

Since this left-hand side is a coercive and continuous bilinear form on $H_0^1(\Omega)$, this problem has a unique solution thanks to the Lax-Milgram lemma.

We now look at the limit of the problem and of u^ε as ε goes to zero. As before the formal limit problem is

$$\begin{gathered}
u_+^0 = f_+ \quad \text{in } \Omega_+, \\
-\Delta u_-^0 + u_-^0 = f_- \quad \text{in } \Omega_-, \\
u_+^0 = 0 \quad \text{on } \partial\Omega_+ \setminus \Sigma, \\
u_-^0 = 0 \quad \text{on } \partial\Omega_- \setminus \Sigma, \\
u_+^0 - u_-^0 = 0 \quad \text{on } \Sigma, \\
-\frac{\partial u_-^0}{\partial \nu} = h \quad \text{on } \Sigma.
\end{gathered} \tag{3.3}$$

As in dimension 1, in this limit problem u_-^0 may be seen as the (unique) solution of a mixed Dirichlet-Neumann problem in Ω_-, and since f_+ does not satisfy the Dirichlet boundary condition $f_+ = 0$ on $\partial\Omega_+ \setminus \Sigma$, and $f_+ = u_-^0$ on Σ, the solution u_+^ε should develop boundary layers along $\partial\Omega_+$. This will be proved in details in the remainder of this paper. Let us first state a weak convergence.

Theorem 3.1. *There exists a subsequence of u_ε, still denoted by u_ε, such that the pair $(u_+^\varepsilon, u_-^\varepsilon)$ converges in $L^2(\Omega_+) \times H^1(\Omega_-)$ to (u_+^0, u_-^0) as ε goes to 0, where $u_+^0 = f_+$ and u_-^0 is the unique variational solution of the mixed Dirchlet-Neumann* *problem*

$$\begin{gathered}
-\Delta u_-^0 + u_-^0 = f_- \quad \text{in } \Omega_-,\\
u_-^0 = 0 \quad \text{on } \partial\Omega_- \setminus \Sigma,\\
\frac{\partial u_-^0}{\partial \nu} = -h \quad \text{on } \Sigma.
\end{gathered} \tag{3.4}$$

Before proving this theorem, let us introduce some notation and give a density result. Let us introduce the following bilinear and linear forms:

$$\begin{gathered}
a(u,v) = \int_{\Omega_+} \nabla u_+ \cdot \nabla v_+,\\
b(u,v) = \int_{\Omega_-} \nabla u_- \cdot \nabla v_- + \int_\Omega uv,\\
F(v) = \int_\Omega fv + \int_\Sigma hv.
\end{gathered} \tag{3.5}$$

Let us define the space

$$W = \{w \in L^2(\Omega) : w_- \in H^1(\Omega_-) \text{ and } w_- = 0 \text{ on } \partial\Omega_- \setminus \Sigma\},$$

which is a Hilbert space, equipped with the norm $\|w\|_W^2 = b(w,w)$.

Lemma 3.2. *$H_0^1(\Omega)$ is dense in W.*

Proof. Let $w \in W$. Since $w_- \in \tilde{H}^{1/2}(\Sigma)$, by [3, Theorem 1.5.2.3] (trace Theorem), there exists $\tilde{w}_+ \in H^1(\Omega_+)$ such that

$$\begin{gathered}
\tilde{w}_+ = w_- \quad \text{on } \Sigma,\\
\tilde{w}_+ = 0 \quad \text{on } \partial\Omega_+ \setminus \Sigma.
\end{gathered}$$

Since $w_+ - \tilde{w}_+$ belongs to $L^2(\Omega_+)$ and since $H_0^1(\Omega_+)$ is dense in $L^2(\Omega_+)$, there exists a sequence of functions $w_+^n \in H_0^1(\Omega_+)$, $n \in \mathbb{N}$ such that

$$\|w_+^n - (w_+ - \tilde{w}_+)\|_{0,\Omega_+} \to 0 \quad \text{as } n \to \infty. \tag{3.6}$$

For all positive integer n, we introduce the function $\tilde{w}^n$ defined in Ω as follows

$$\begin{gathered}
\tilde{w}_+^n = w_+^n + \tilde{w}_+,\\
\tilde{w}_-^n = w_-.
\end{gathered}$$

From the boundary condition satisfied by $\tilde{w}_+$, $\tilde{w}^n$ belongs to $H_0^1(\Omega)$. Moreover from the definition of $\tilde{w}^n$ and owing to (3.6), we have

$$\|\tilde{w}^n - w\|_W = \|w_+^n - (w_+ - \tilde{w}_+)\|_{0,\Omega_+} \to 0.$$

□

Proof of Theorem 3.1. From (3.2) and the definition of u^0, we see that $u^\varepsilon \in H_0^1(\Omega)$ and $u^0 \in W$ are the respective solution of

$$\varepsilon^2 a(u^\varepsilon, v) + b(u^\varepsilon, v) = F(v), \forall v \in H_0^1(\Omega), \tag{3.7}$$

$$b(u^0, w) = F(w), \forall w \in W. \tag{3.8}$$

Step 1. u^ε is weakly convergent to u^0 in W. We first remark that

$$\|u^\varepsilon\|_W^2 = b(u^\varepsilon, u^\varepsilon) \le b(u^\varepsilon, u^\varepsilon) + \varepsilon^2 a(u^\varepsilon, u^\varepsilon).$$

Now taking $v = u^\varepsilon$ in (3.7) and $w = u^\varepsilon$ in (3.8) we obtain

$$\varepsilon^2 a(u^\varepsilon, u^\varepsilon) + b(u^\varepsilon, u^\varepsilon) = b(u^0, u^\varepsilon). \tag{3.9}$$

Using Cauchy-Schwarz's inequality, we directly have

$$|b(u^0, u^\varepsilon)| \le \|u^0\|_W \|u^\varepsilon\|_W.$$

These three properties imply that

$$\|u^\varepsilon\|_W \le \|u^0\|_W. \tag{3.10}$$

Therefore, there exists $w \in W$ and a subsequence of u^ε, still denoted by u^ε, weakly convergent to w in W.

Now for any fixed $v \in H_0^1(\Omega)$, using successively (3.9) and (3.10) we may write

$$\begin{aligned} |a(u^\varepsilon, v)| &\le \|\nabla u_+^\varepsilon\|_{0,\Omega_+} \|\nabla v\|_{0,\Omega_+} \\ &\le \varepsilon^{-1} (\varepsilon^2 \|\nabla u_+^\varepsilon\|_{0,\Omega_+}^2 + b(u^\varepsilon, u^\varepsilon))^{\frac{1}{2}} \|\nabla v\|_{0,\Omega_+} \\ &= \varepsilon^{-1} b(u^0, u^\varepsilon)^{\frac{1}{2}} \|\nabla v\|_{0,\Omega_+} \\ &\le \varepsilon^{-1} \|u^0\|_W^{\frac{1}{2}} \|u^\varepsilon\|_W^{\frac{1}{2}} \|\nabla v\|_{0,\Omega_+} \\ &\le \varepsilon^{-1} \|u^0\|_W \|\nabla v\|_{0,\Omega_+}. \end{aligned}$$

This last estimate implies that

$$\lim_{\varepsilon \to 0} \varepsilon^2 a(u^\varepsilon, v) = 0, \quad \forall v \in H_0^1(\Omega).$$

Therefore, passing to the limit in (3.7), we obtain

$$\lim_{\varepsilon \to 0} b(u^\varepsilon, v) = F(v) = b(u^0, v), \quad \forall v \in H_0^1(\Omega).$$

Since $H_0^1(\Omega)$ is dense in W, we conclude that

$$b(u^0, v) = b(w, v), \quad \forall v \in W.$$

Since $b(\cdot, \cdot)$ is the inner product of W, we deduce that $u^0 = w$.
Step 2. u^ε is strongly convergent to u^0 in W.

$$\begin{aligned} \|u^\varepsilon - u^0\|_W^2 &= b(u^\varepsilon - u^0, u^\varepsilon - u^0) \\ &= b(u^\varepsilon, u^\varepsilon) - b(u^0, u^\varepsilon) - b(u^\varepsilon - u^0, u^0). \end{aligned}$$

Taking into account (3.9), we obtain

$$\|u^\varepsilon - u^0\|_W^2 \le -b(u^\varepsilon - u^0, u^0).$$

Then we have the conclusion, by the weak convergence in W of u^ε to u^0. □

From this Theorem we may see u^0 as the first term of the outer expansion of u^ε. Let us now pass to the description of the boundary layers.

4. Boundary layers

In the sequel let $\mathcal{L}_\varepsilon$ denote the operator $\mathcal{L}_\varepsilon = I - \varepsilon^2\Delta$. In this section, we define in Ω_+, the boundary layer v_j^b along $\Gamma_j = [A_{j-1}, A_j]$, $j = 2, 3, \dots, N$ and the interface layer v^i along Σ, such that if $\mathcal{V}_j$ denote a small neighbourhood of Γ_j, we have

$$\begin{gathered}
\mathcal{L}_\varepsilon(u_+^\varepsilon - f_+ - v_j^b) = \varepsilon^2 O(\varepsilon) \quad \text{in } \mathcal{V}_j \cap \Omega_+, \\
f_+ + v_j^b = 0 \quad \text{on } \Gamma_j
\end{gathered} \tag{4.1}$$

and

$$\begin{gathered}
\mathcal{L}_\varepsilon(u_+^\varepsilon - f_+ - v^i) = \varepsilon^2 O(\varepsilon) \quad \text{in } \mathcal{V}_1 \cap \Omega_+, \\
f_+ + v^i = u_-^0 \quad \text{on } \Sigma,
\end{gathered} \tag{4.2}$$

when $O(\varepsilon)$ denote as usual a function of ε bounded in a neighbourhood of $\varepsilon = 0$. Note that the situation is not the same along Σ due to the lack of regularity of u_-^0 (see below).

4.1. Some notation and definitions.

We denote by (x, y) the Cartesian coordinates of the plane with origin at A_1 and such that $\Gamma_1 \subset \{(x, 0), x > 0\}$. Similarly we denote by (x_j, y_j) the Cartesian coordinates of the plane with origin at A_j and such that $\Gamma_j \subset \{(x_j, 0), x_j > 0\}$.

We now fix two cut-off functions $\chi_j^1, \chi_j^2 \in \mathcal{C}_0^\infty(\mathbb{R})$ satisfying $\operatorname{supp}\chi_j^1 \subset [-a_j, a_j]$, and

$$\chi_j^1(x) = 1 \quad \text{on }]0, l_j[,$$

where l_j is the length of Γ_j, and $\operatorname{supp}\chi_j^2 \subset [-b, b]$, as well as

$$\chi_j^2(y) = 1 \quad \text{on }]-\frac{b}{2}, \frac{b}{2}[,$$

for a sufficiently small fixed $b > 0$.

Now we can introduce the cut-off function along Γ_j by

$$\chi_j^b(x, y) = \chi_j^1(x)\, \chi_j^2(y). \tag{4.3}$$

We finally take $\chi^i = \chi_1^b$.

Now we assume that f_+ is the restriction to Ω_+ of a smooth function $\tilde{f}_+ \in \mathcal{C}^\infty(\mathbb{R}^2)$ and that Ω_+ is convex, i.e., $0 < \omega_j < \pi$, for all $j = 1, \dots, N$. This last assumption is simply made to simplify the construction of corner layers. Using the method of [11], we probably can treat the non convex case.

4.2. Construction of v_j^b.

They are standard, see for instance [4, 5]. For $j = 2, \dots, N$, v_j^b is the unique solution of the problem

$$\begin{gathered}
v_j^b - \varepsilon^2 v_j^{b\prime\prime} = 0 \quad \text{in } y_j > 0, \\
v_j^b = -\tilde{f}_+(x_j, .) \quad \text{at } y_j = 0, \\
v_j^b = 0 \quad \text{at } y_j = +\infty.
\end{gathered}$$

It is explicitly given by

$$v_j^b(x_j, y_j) = -\tilde{f}_+(x_j, 0)\, e^{-y_j/\varepsilon}. \tag{4.4}$$

Since $\omega_j < \pi$, the function $\chi_j^b v_j^b$ is well defined in Ω_+ and satisfies the conditions (4.1). Moreover it has the regularity $\mathcal{C}^\infty(\bar\Omega_+)$ and for any $(x_j, y_j) \in \Omega_+$,

$$\begin{aligned}\mathcal{L}_\varepsilon(\chi_j^b v_j^b)(x_j, y_j) &= (I - \varepsilon^2 \Delta_{(x_j,y_j)})(\chi_j^b v_j^b)(x_j, y_j)\\ &= \varepsilon^2 \chi_j^b(x_j, y_j) e^{-\frac{y_j}{\varepsilon}} (\partial_{x_j})^2 \tilde f_+(x_j, 0) + \varepsilon^2 [\chi_j^b; \Delta_{(x_j,y_j)}] v_j^b,\end{aligned}$$

where we recall that $[\chi; \Delta_{(x_j,y_j)}]v := \chi \Delta_{(x_j,y_j)} v - \Delta_{(x_j,y_j)}(\chi v)$. Since

$$\begin{gathered}|\chi_j^b e^{-\frac{y_j}{\varepsilon}} (\partial_{x_j})^2 f_+(x_j, 0)| \lesssim 1,\\ |[\chi_j^b\ ;\ \Delta_{(x_j,y_j)}] v_j^b| \lesssim \frac{1}{\varepsilon} e^{-\frac{b}{2\varepsilon}},\end{gathered}$$

we deduce that

$$\|\mathcal{L}_\varepsilon(\chi_j^b v_j^b)\|_{0,\Omega_+} \lesssim \varepsilon^2 + \varepsilon e^{-\frac{b}{2\varepsilon}}. \tag{4.5}$$

4.3. **Construction of v^i.** In general the solution u_-^0 of problem (3.4) has only the regularity $u_-^0 \in H^1(\Omega_-)$. Consequently if we proceed as in the previous subsection, namely if we take

$$v^i(x_1, y_1) = (\tilde u_-^0 - \tilde f_+)(x_1, 0)\, e^{-y_1/\varepsilon},$$

the regularity of v^i is not sufficient to obtain an estimate similar to (4.5). To overcome this difficulty, we shall use the decomposition of u_-^0 into a regular part and singular one.

For $j = 1, 2$, we recall that the singular exponents associated with the mixed Dirichlet-Neumann problem near A_j are given by (see [3, 2])

$$\Lambda_j = \{\lambda_k = \frac{\pi}{2\varphi_j} + \frac{k\pi}{\varphi_j}, k \in \mathbb{Z}\}.$$

Let (r_j, θ_j) be the polar coordinates centred at A_j and such that $\theta_j = 0$ on Σ, and $\theta_j = -\omega_j$ on the other edge of Ω_- having A_j as extremity. For $\lambda_k \in \Lambda_j$, we denote

$$S_{j,\lambda_k}(r_j, \theta_j) = r_j^{\lambda_k} \sin \lambda_k(\varphi_j + \theta_j), \quad -\varphi_j < \theta_j < \omega_j. \tag{4.6}$$

Recall that this function satisfies

$$\begin{gathered}\Delta S_{j,\lambda_k} = 0,\\ S_{j,\lambda_k}(r_j, -\varphi_j) = 0,\\ \frac{\partial}{\partial\theta} S_{j,\lambda_k}(r_j, 0) = 0.\end{gathered} \tag{4.7}$$

According to [3, Corollary 4.4.3.8], the solution $u_-^0 \in H^1(\Omega_-)$ of (3.4) admits the decomposition

$$u_-^0 = u_{-,r}^0 + \sum_{j=1,2} \eta_j \sum_{\lambda_k \in \Lambda_j, 0 < \lambda_k < 2} C_{j,\lambda_k} S_{j,\lambda_k}, \tag{4.8}$$

where $u_{-,r}^0 \in H^3(\Omega_- \cap \mathcal{V}_1)$, C_{j,λ_k} are real constants and η_j is a (radial) cut-off function equal to 1 in neighbourhood of A_j and equal to zero outside another neighbourhood of A_j, $j = 1, 2$. Using this expansion, we can define

$$v^i = v_r^i + v_s^i \tag{4.9}$$

where

$$\begin{aligned}v_r^i(x, y) &= (\tilde u_{-,r}^0 - \tilde f_+)(x, 0) e^{-y/\varepsilon},\\ v_s^i(x, y) &= \sum_{j=1,2} \eta_j \sum_{\lambda_k \in \Lambda_j, 0 < \lambda_k < 2} C_{j,\lambda_k} S_{j,\lambda_k} e^{-y/\varepsilon},\end{aligned} \tag{4.10}$$

where $\tilde{u}^0_{-,r}(\cdot)$ is an extension to the real line of $u^0_{-,r}(\cdot,0)$. Since $u^0_{-,r}(\cdot,0)$ belongs to $H^{\frac{5}{2}}(\Sigma)$, this extension may be chosen in $H^{\frac{5}{2}}(\mathbb{R})$ and by the Sobolev embedding Theorem, $v^i_r \in \mathcal{C}^2(\bar{\Omega}_+)$. Therefore, as in the previous subsection, we have

$$\|\mathcal{L}_\varepsilon(\chi^i v^i_r)\|_{0,\Omega_+} \lesssim \varepsilon^2 + \varepsilon e^{-\frac{b}{2\varepsilon}}. \tag{4.11}$$

On the other hand, Leibniz's rule yields

$$\begin{aligned}
&\Delta(\chi^i \eta_j S_{j,\lambda_k} e^{-y/\varepsilon}) \\
&= \chi^i \eta_j S_{j,\lambda_k} \Delta e^{-y/\varepsilon} + 2\nabla(\chi^i \eta_j S_{j,\lambda_k}) \cdot \nabla e^{-y/\varepsilon} + \Delta(\chi^i \eta_j S_{j,\lambda_k}) e^{-y/\varepsilon} \\
&= e^{-y/\varepsilon}\{\frac{1}{\varepsilon^2}\chi^i \eta_j S_{j,\lambda_k} - \frac{2}{\varepsilon}\frac{\partial(\chi^i \eta_j S_{j,\lambda_k})}{\partial y} + \chi^i \eta_j \Delta S_{j,\lambda_k} - [\chi^i \eta_j;\Delta] S_{j,\lambda_k}\},
\end{aligned}$$

and therefore (reminding $\Delta S_{j,\lambda_k} = 0$)

$$\mathcal{L}_\varepsilon(\chi^i \eta_j S_{j,\lambda_k} e^{-y/\varepsilon}) = \varepsilon^2 e^{-y/\varepsilon}\Big(\frac{2}{\varepsilon}\frac{\partial}{\partial y}(\chi^i \eta_j S_{j,\lambda_k}) + [\chi^i \eta_j;\Delta] S_{j,\lambda_k}\Big).$$

From this identity, we deduce that

$$\|\mathcal{L}_\varepsilon(\chi^i v^i_s)\|_{0,\Omega_+} \lesssim \varepsilon^{1+\lambda} + \varepsilon e^{-b/(2\varepsilon)}, \tag{4.12}$$

where $\lambda = \min_{k=1,2} \min\{\lambda_k : \lambda_k \in \Lambda_k\}$. As $v^i = v^i_r + v^i_s$, the estimates (4.11) and (4.12) lead to

$$\|\mathcal{L}_\varepsilon(\chi^i v^i)\|_{0,\Omega_+} \lesssim \varepsilon^{1+\lambda} + \varepsilon e^{-\frac{b}{2\varepsilon}}. \tag{4.13}$$

At this stage if we set

$$U_+ := f_+ + \sum_{j=2}^{N} \chi^b_j v^b_j + \chi^i v^i \quad \text{in } \Omega_+, \tag{4.14}$$

then we may write (since $\mathcal{L}_\varepsilon u^\varepsilon_+ = f_+$)

$$\mathcal{L}_\varepsilon(u^\varepsilon_+ - U_+) = \varepsilon^2 \Delta f_+ - \mathcal{L}_\varepsilon(\sum_{j=2}^{N^+} \chi^b_j v^b_j) - \mathcal{L}_\varepsilon(\chi^i v^i).$$

And by (4.5) and (4.13), we arrive at

$$\|\mathcal{L}_\varepsilon(u^\varepsilon_+ - U_+)\|_{0,\Omega_+} \lesssim \varepsilon^{1+\lambda} + \varepsilon e^{-\frac{b}{2\varepsilon}}. \tag{4.15}$$

At this stage we can say that U_+ approaches u^+_ε in the interior of Ω_+, satisfies the Dirichlet boundary condition in the interior of Γ_j, $j = 2,\dots,N$ and the correct interface condition in the interior of Σ. But the correct boundary/interface conditions are not satisfied near the corners A_j. Therefore, corner correctors have to be introduced.

4.4. **Corner correctors.** For all $j = 1,\dots,N$ consider polar coordinates (r_j,θ_j) centered at A_j and such that $\Gamma_j \subset \{(r_j,0), r_j > 0\}$ and therefore

$$\Gamma_{j-1} \subset \{(r_j\cos\omega_j, r_j\sin\omega_j), r_j > 0\}$$

(here and below the index are considered modulo N, i.e. ${}_0 = {}_N$). Denote

$$\begin{gathered}
S_j = \{(r_j,\theta_j),\ r_j > 0, 0 < \theta_j < \omega_j\}, \\
\tilde{\Gamma}_{j-1} = \{(r_j,\omega_j),\ r_j > 0\}, \\
\tilde{\Gamma}_j = \{(r_j\cos\omega_j, r_j\sin\omega_j),\ r_j > 0\},
\end{gathered}$$

and let $R_j > 0$ be fixed sufficiently small so that

$$\begin{gathered}\operatorname{supp}\chi^b_{j-1} \cap \operatorname{supp}\chi^b_j \cap S_j \subset B(A_j, \frac{R_j}{2}),\\ B(A_j, R_j) \cap B(A_k, R_k) = \emptyset \quad \text{if } k \neq j.\end{gathered}$$

To each vertex A_j we associate a radial cut-off function χ^c_j such that

$$\chi^c_j(r) = \begin{cases} 1 & \text{if } r < \frac{R_j}{2},\\ 0 & \text{if } r > R_j.\end{cases}$$

In the sector S_j, according to the definition of the function U_+ we may write

$$U_+(x, y) = f_+(x, y) + \chi^b_{j-1}(x_{j-1}, y_{j-1})v^b_{j-1}(x_{j-1}, y_{j-1}) + \chi^b_j(x_j, y_j)v^b_j(x_j, y_j), \tag{4.16}$$

where for shortness we write $v^b_1 = v^i$, $\chi^b_1 = \chi^i$. By construction of the boundary layers v^b_j, we then have

$$U_+\big|_{\partial S_j} = \begin{cases} \chi^b_j v^b_j & \text{on } \tilde\Gamma_{j-1},\\ \chi^b_{j-1}v^b_{j-1} & \text{on } \tilde\Gamma_j.\end{cases}$$

Now we introduce the changes of coordinates

$$\begin{aligned}\Psi_j &: (r_j, \theta_j) \longmapsto (x_j, y_j) = (r_j\cos\theta_j, r_j\sin\theta_j),\\ \Phi_j &: (x_j, y_j) \longmapsto (x_{j-1}, y_{j-1}).\end{aligned}$$

Using the fact that $\tilde\Gamma_{j-1}$ (resp. $\tilde\Gamma_j$) is parametrized by $(x_j, y_j) = (r_j\cos\omega_j, r_j\sin\omega_j)$ (resp. $(x_j, y_j) = (r_j, 0)$) and using the definition of v^b_j and v^i, we see that

$$U_+\,|_{\partial S_j} = \begin{cases} g^1_j(r_j)\exp\big(-\frac{r_j\sin\omega_j}{\varepsilon}\big)) & \text{on } \tilde\Gamma_{j-1},\\ g^2_j(r_j)\exp\big(-\frac{r_j\sin\omega_j}{\varepsilon}\big) & \text{on } \tilde\Gamma_j,\end{cases}$$

where, except in the case $j = k = 1$ and $j = k = 2$, the functions g^k_j are smooth, while in the exceptional case, due to (4.9) and (4.10), we have

$$g^1_1(r_1) = g^1_{1,r}(r_1) + g^1_{1,s}(r_1), \tag{4.17}$$

$$g^2_2(r_2) = g^2_{2,r}(r_2) + g^2_{2,s}(r_2), \tag{4.18}$$

$$g^1_{1,r}(r_1) = \chi^i \circ \Psi_1(r_1, \omega_1)v^i_r(r_1\cos\omega_1, 0), \tag{4.19}$$

$$\begin{aligned} g^1_{1,s}(r_1) &= \chi^i \circ \Psi_1(r_1, \omega_1)\eta_1(r_1)\sum_{\lambda_k\in\Lambda_1, 0<\lambda_k<2} C_{1,\lambda_k}S_{1,\lambda_k}(r_1, \omega_1),\\ g^2_{2,r}(r_2) &= \chi^i \circ \Phi_2^{-1} \circ \Psi_2(r_2, \omega_2)v^i_r(-r_2\cos\omega_2 + l_1, 0),\\ g^2_{2,s}(r_2) &= \chi^i \circ \Phi_2^{-1} \circ \Psi_2(r_2, \omega_2)\eta_2(r_2)\sum_{\lambda_k\in\Lambda_2, 0<\lambda_k<2} C_{2,\lambda_k}S_{2,\lambda_k}(r_2, \omega_2).\end{aligned}$$

The boundary condition imposed at v^b_j on Γ_j implies $v^b_j(A_j) = v^b_{j-1}(A_j) = -f_+(A_j)$, $j = 3, \dots, N$. On the other hand $u^0_- \in H^1(\Omega_-)$ and satisfies the Dirichlet condition on $\partial\Omega_- \setminus \Sigma$. By the continuity of u^0_- (due to the expansion (4.8)) we get $u^0_-(A_1) = u^0_-(A_2) = 0$, and consequently $v^b_1(A_j) = -f_+(A_j)$, $j = 1, 2$. All together the next compatibility conditions are satisfied

$$g^1_j(0) = g^2_j(0) \quad \forall j = 1, \dots, N. \tag{4.20}$$

Now we look for explicit functions u_j^c defined in the cone S_j and satisfying the boundary conditions

$$u_j^c = -g_j^1 \quad \text{on } \tilde{\Gamma}_{j-1},$$
$$u_j^c = -g_j^2 \quad \text{on } \tilde{\Gamma}_j.$$

Since the term $g_{1,r}^1$ and $g_{2,r}^2$ are sufficiently smooth (namely $H^{5/2}$), they can be treated as the functions g_j^k, for $j > 2$. As a consequence we split $u_j^c = u_{j,r}^c + u_{j,s}^c$, where $u_{j,s}^c = 0$ for $j \neq 1, 2$ and

$$u_{1,s}^c(r_1, \theta_1) = \begin{cases} 0 & \text{if } \theta_1 = 0, \\ -g_{1,s}^1(r_1) & \text{if } \theta_1 = \omega_1, \end{cases} \tag{4.21}$$

$$u_2^{1c}(r_2, \theta_2) = \begin{cases} 0 & \text{if } \theta_2 = \omega_2, \\ -g_{2,s}^2(r_2) & \text{if } \theta_2 = 0, \end{cases} \tag{4.22}$$

and

$$u_{j,r}^c = -\hat{g}_j^1 \quad \text{on } \tilde{\Gamma}_{j-1}, \tag{4.23}$$

$$u_{j,r}^c = -\hat{g}_j^2 \quad \text{on } \tilde{\Gamma}_j. \tag{4.24}$$

where $\hat{g}_j^k = g_j^k$ except if $j = k = 1$ and $j = k = 2$; in that last cases, we take $\hat{g}_1^1 = g_{1,r}^1$, $\hat{g}_2^2 = g_{2,r}^2$.

For our purpose, we introduce the functions

$$\sigma_{j,\lambda_k}(r_j, \theta_j) = \begin{cases} \frac{r_j^{\lambda_k} \sin(\lambda_k(\varphi_j + \omega_j))}{\omega_j} \theta_j & \text{if } \sin(\lambda_k \omega_j) = 0, \\ \frac{S_{j,\lambda_k}(r_j, \omega_j)}{\sin \lambda_k \omega_j} \sin(\lambda_k \theta_j) & \text{if } \sin(\lambda_k \omega_j) \neq 0, \end{cases}$$

so that it fulfils $\sigma_{j,\lambda_k}(r_j, 0) = 0$ and $\sigma_{j,\lambda_k}(r_j, \omega_j) = S_{j,\lambda_k}(r_j, \omega_j)$. Note that the first choice is also valid in the (generic) case $\sin(\lambda_k \omega_j) \neq 0$, but in this case the second choice gives rise to a harmonic function.

Lemma 4.1. *Let*

$$u_{1,s}^c(r_j, \theta_j) = -\chi^i \circ \Psi_j(r_1, \omega_1) \eta_1(r_1) \sum_{\lambda_k \in \Lambda_1, 0 < \lambda_k < 2} C_{1,\lambda_k} \sigma_{1,\lambda_k}, \tag{4.25}$$

$$u_{2,s}^c(r_j, \theta_j) = -\chi^i \circ \Phi_j^{-1} \circ \Psi_j(r_2, \omega_2) \eta_2(r_2) \sum_{\lambda_k \in \Lambda_2, 0 < \lambda_k < 2} C_{2,\lambda_k} \sigma_{2,\lambda_k}. \tag{4.26}$$

Then they respectively satisfy (4.21) and (4.22) and by setting $\alpha_j = \sin \omega_j$,

$$\|e^{-\frac{\alpha_j r_j}{\varepsilon}} u_{j,s}^c\|_{0,S_j} + \varepsilon \|e^{-\frac{\alpha_j r_j}{\varepsilon}} \nabla u_{j,s}^c\|_{0,S_j} \lesssim \varepsilon^{1+\lambda}, \quad j = 1, 2. \tag{4.27}$$

Moreover

$$\Delta(\chi_j^c e^{-\frac{\alpha_j r_j}{\varepsilon}} u_{1,s}^c(r_j, \theta_j)) \in L^p(S_j),$$

for all $p \in [1, \frac{2}{2-\lambda})$, *where* $\lambda = \min_{k=1,2} \min\{\lambda_k : \lambda_k \in \Lambda_k\}$.

Proof. For simplicity, let us set

$$\hat{\chi}^i(r_1) = \chi^i \circ \Psi_1(r_1, \omega_1) \eta_1(r_1) \quad \text{if } j = 1,$$
$$\hat{\chi}^i(r_2) = \chi^i \circ \Phi_2^{-1} \circ \Psi_2(r_2, \omega_2) \eta_2(r_2) \quad \text{if } j = 2.$$

Since the function $e^{-\frac{r}{\varepsilon}\alpha_j} D^\gamma \sigma_{j,\lambda_k}$ behaves like $e^{-\frac{r}{\varepsilon}\alpha_j} r_j^{\lambda_k-|\gamma|}$ at 0 and at ∞, we have

$$\|\hat{\chi}^i e^{-\frac{r}{\varepsilon}\alpha_j} D^\gamma \sigma_{j,\lambda_k}\|_{0,S_j} \lesssim \|\hat{\chi}^i r^{\lambda_k-|\gamma|} e^{-\frac{r_j}{\varepsilon}\alpha}\|_{0,S_j}. \tag{4.28}$$

For $|\gamma| \le 1 < \lambda_k + 1$, by the scaling $\rho_j = \frac{r_j}{\varepsilon}$, we obtain

$$\begin{aligned} \|\hat{\chi}^i r^{\lambda_k-\gamma} e^{-\frac{r_j}{\varepsilon}\alpha}\|_{0,S_j}^2 &\lesssim \int_0^\infty r^{2(\lambda_k-|\gamma|)} e^{-2\frac{r_j}{\varepsilon}\alpha} r\,dr \\ &= \varepsilon^{2(\lambda_k-|\gamma|+1)} \int_0^\infty \rho^{2(\lambda_k-\gamma)} e^{-2\rho\alpha} \rho\,d\rho \\ &\lesssim \varepsilon^{2(\lambda_k-|\gamma|+1)}. \end{aligned} \tag{4.29}$$

The estimate (4.27) follows directly from (4.28) and (4.29). The regularity of $\Delta(\chi_j^c e^{-\frac{\alpha_j r_j}{\varepsilon}} u_{1,s}^c(r_j,\theta_j)) \in L^p(S_j)$ is proved in a similar manner. $\square$

Lemma 4.2. *There exists $u_{j,r}^c \in H^1(S_j)$ satisfying (4.23) and (4.24) and such that*

$$\|\chi_j^c e^{-\frac{\alpha_j r_j}{\varepsilon}} u_{j,r}^c\|_{0,S_j} + \varepsilon \|\chi_j^c e^{-\frac{\alpha_j r_j}{\varepsilon}} \nabla u_{j,r}^c\|_{0,S_j} \lesssim \varepsilon. \tag{4.30}$$

Moreover

$$\Delta(\chi_j^c e^{-\frac{\alpha_j r_j}{\varepsilon}} u_{1,r}^c(r_j,\theta_j)) \in L^p(S_j),$$

for all $p \in [1,2)$.

Proof. We simply take

$$u_{j,r}^c(r,\theta) = (\hat{g}_j^1(r) - \hat{g}_j^2(r))\frac{\theta}{\omega_j} + \hat{g}_j^2(r),$$

which clearly satisfies (4.23) and (4.24). As $\hat{g}_j^1 \in \tilde{H}^{\frac{5}{2}}(\tilde{\Gamma}_{j-1})$, $\hat{g}_j^2 \in \tilde{H}^{\frac{5}{2}}(\tilde{\Gamma}_j)$ and are equal to zero for $r > R_j$, we deduce that $\chi_j^c u_{j,r}^c, \chi_j^c \frac{\partial u_{j,r}^c}{\partial r} \in L^\infty(S_j)$ and

$$\chi_j^c \frac{1}{r}\frac{\partial u_{j,r}^c}{\partial \theta} = \chi_j^c\Big(\frac{\hat{g}_j^1(r) - \hat{g}_j^1(0)}{r}\frac{1}{\omega_j} - \frac{\hat{g}_j^2(r) - \hat{g}_j^2(0)}{r}\frac{1}{\omega_j}\Big) \in L^\infty(S_j).$$

Consequently it holds

$$\|\chi_j^c e^{-\frac{\alpha_j r}{\varepsilon}} u_{j,r}^c\|_{0,S_j} + \varepsilon\|\chi_j^c e^{-\frac{\alpha r}{\varepsilon}} \nabla u_{j,r}^c\|_{0,S_j} \lesssim \|e^{-\frac{\alpha r}{\varepsilon}}\|_{0,S_j}.$$

By the change of variable $\rho = \frac{r}{\varepsilon}$, one has $\|e^{-\frac{\alpha r}{\varepsilon}}\|_{0,S_j} \lesssim \varepsilon$ and the estimate (4.30) follows. The second assertion is proved similarly. $\square$

5. The full decomposition

We are now ready to formulate the main result of this paper.

Theorem 5.1. *Assume that f_+ is the restriction to Ω_+ of a smooth function $\tilde{f}_+ \in \mathcal{C}^\infty(\mathbb{R}^2)$ and that Ω_+ is convex. Write for shortness*

$$U_c = \sum_{k=1}^N \chi_k^c e^{-\sin\omega_k \frac{r_k}{\varepsilon}} u_k^c.$$

Then the unique solution $u^\varepsilon \in H_0^1(\Omega)$ of (3.1) admits the splitting

$$\begin{aligned} u_+^\varepsilon &= f_+ + \sum_{j=2}^N \chi_j^b v_j^b + \chi^i v^i + U_c + r_+^\varepsilon \quad \text{in } \Omega_+, \\ u_-^\varepsilon &= u_-^0 + r_-^\varepsilon \quad \text{in } \Omega_-, \end{aligned} \tag{5.1}$$

where $r^\varepsilon \in H^1_0(\Omega)$ is the variational solution of

$$\begin{aligned}
&\int_{\Omega_+} (\varepsilon^2 \nabla r^\varepsilon_+ \cdot \nabla v_+ + r^\varepsilon_+ v_+) + \int_{\Omega_-} (\nabla r^\varepsilon_- \cdot \nabla v_- + r^\varepsilon_- v_-) \\
&= \int_{\Omega_+} f^\varepsilon v - \int_\Sigma h^\varepsilon v - \int_{\Omega_+} (\varepsilon^2 \nabla U_c \cdot \nabla v_+ + U_c v_+), \quad \forall v \in H^1_0(\Omega),
\end{aligned} \tag{5.2}$$

where $f^\varepsilon = \mathcal{L}_\varepsilon(u^\varepsilon_+ - U_+)$ and $h^\varepsilon = \varepsilon^2 \frac{\partial}{\partial\nu}(f_+ - U_+)$. Moreover,

$$\varepsilon \|\nabla r^\varepsilon_+\|_{0,\Omega_+} + \|r^\varepsilon_+\|_{0,\Omega_+} + \|r^\varepsilon_-\|_{1,\Omega_-} \lesssim \varepsilon. \tag{5.3}$$

Proof. By construction, r^ε clearly belongs to $H^1_0(\Omega)$, and satisfies $\Delta r^\varepsilon_\pm \in L^p(\Omega_\pm)$, for some $p \in (1,2)$. Therefore applying [3, Theorem 1.5.3.11], we may write

$$\begin{aligned}
&\int_{\Omega_+} (\varepsilon^2 \nabla r^\varepsilon_+ \cdot \nabla v_+ + r^\varepsilon_+ v_+) + \int_{\Omega_-} (\nabla r^\varepsilon_- \cdot \nabla v_- + r^\varepsilon_- v_-) \\
&= \int_{\Omega_+} \mathcal{L}_\varepsilon r^\varepsilon_+ v_+ + \langle \varepsilon^2 \frac{\partial r^\varepsilon_+}{\partial\nu} - \frac{\partial r^\varepsilon_-}{\partial\nu}, v\rangle_{\tilde H^{\frac12}(\Sigma)^\star - \tilde H^{\frac12}(\Sigma)}, \quad \forall v \in \mathcal{D}(\Omega).
\end{aligned} \tag{5.4}$$

We remark that the splitting (5.1) means that

$$r^\varepsilon_+ = u^\varepsilon_+ - U_+ - U_c.$$

Since Lemmas 4.1 and 4.2 guarantees that $U_c \in H^1(\Omega_+)$ and $\Delta U_c \in L^p(\Omega_+)$, for some $p \in (1,2)$, again the application of [3, Theorem 1.5.3.11] yields

$$\int_{\Omega_+} \mathcal{L}_\varepsilon U_c v_+ = \int_{\Omega_+} (\varepsilon^2 \nabla U_c \cdot \nabla v_+ + U_c v_+) - \langle \varepsilon^2 \frac{\partial U_c}{\partial\nu}, v\rangle_{\tilde H^{\frac12}(\Sigma)^\star - \tilde H^{\frac12}(\Sigma)}, \quad \forall v \in \mathcal{D}(\Omega).$$

Inserting this expression in (5.4), we obtain (5.2) since $\mathcal{D}(\Omega)$ is dense in $H^1_0(\Omega)$.

Now taking $v = r^\varepsilon$ in (5.2), applying Cauchy-Schwarz's inequality and a trace theorem (in Ω_-), we get

$$\varepsilon\|\nabla r^\varepsilon_+\|_{0,\Omega_+} + \|r^\varepsilon_+\|_{0,\Omega_+} + \|r^\varepsilon_-\|_{1,\Omega_-} \lesssim \|f^\varepsilon\|_{0,\Omega_+} + \|h^\varepsilon\|_{0,\Sigma} + \varepsilon\|\nabla U_c\|_{0,\Omega_+} + \|U_c\|_{0,\Omega_+}. \tag{5.5}$$

The estimate (5.3) follows from this one if we can show that each term of this right-hand side is bounded by ε. The first term is estimate with the help of (4.15). For the second term, due to (4.14), we may write

$$h^\varepsilon = -\varepsilon^2 \frac{\partial}{\partial\nu}(f_+ + \chi^b_N v^b_N + \chi^i v^i).$$

Now by (4.7) we remark that

$$\begin{aligned}
|\frac{\partial}{\partial\nu} f_+| &\lesssim 1, \\
|\frac{\partial}{\partial\nu}(\chi^b_N v^b_N)| = |\frac{\partial \chi^b_N}{\partial\nu} v^b_N + \frac{\partial v^b_N}{\partial\nu}\chi^b_N| &\lesssim \frac{1}{\varepsilon}, \\
|\frac{\partial}{\partial\nu}(\chi^i v^i)| = |\frac{\partial \chi^i}{\partial\nu} v^i + \frac{\partial v_i}{\partial\nu}\chi^i| &\lesssim \frac{1}{\varepsilon}.
\end{aligned}$$

These estimates lead to $\|h^\varepsilon\|_{0,\Sigma} \lesssim \varepsilon$. Finally for the last terms of the right-hand side, using (4.27), (4.30) and Leibniz's rule, we get

$$\varepsilon\|\nabla U_c\|_{0,\Omega_+} + \|U_c\|_{0,\Omega_+} \lesssim \varepsilon.$$

□

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